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Research article

Existence of solutions by fixed point theorem of general delay fractional differential equation with p-Laplacian operator

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Abstract: In this manuscript, the main objective is to analyze the existence, uniqueness,(EU) and stability of positive solution for a general class of non-linear fractional differential equation (FDE) with fractional differential and fractional integral boundary conditions utilizing ϕ_p-Laplacian operator. To continue, we will apply Green's function to determine the suggested FDE's equivalent integral form. The Guo-Krasnosel'skii fixed point theorem and the properties of the p-Laplacian operator are utilized to derive the existence results. Hyers-Ulam (HU) stability is additionally evaluated. Further, an application is presented to validate the effectiveness of the result.

Keywords: Riemann-Liouville integral; Caputo's derivative; Green's function; fixed point theorems; Hyres-Ulam stability
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1. Introduction

Fractional calculus (FC) is among the most quickly increasing aspects of research in mathematics. The arbitrary-order generalization of differentiation and integration is FC. Fractional derivatives have been around for nearly three decades. FC is now widely used in the research and engineering to represent an effective and wide range of real-world issues. Biophysics, aerodynamics, capacitor theory, control theory, biology, polymer rheology, geophysics, and many more domains use FDEs. The monograph [1–3] provides a mathematical perspective and the principles of FC and FDEs. The theory of FDEs and its applications benefited from the contributions of many mathematicians, engineers, and physicists. To address a broad range of natural processes and phenomena, several types of fractional order derivative operators have been developed. More information can be found at [4, 5].

The EU of solutions of FDEs with regard to fractional boundary value conditions has attracted a lot of interest. Existence results for singular FDEs for mixed and Dirichlet problems was discussed by Agarwal et al. [6, 7]. EU for FDEs with Caputo derivative with bounded delay was analyzed by Baleanu et al. [8]. For instance, see [9–11]. For more information see [12–14]. Also, using a variety of mathematical approaches, several authors have recently investigated FDEs with the operator, p-Laplacian. Positive solution for a FDE with integral boundary conditions; At resonance, a boundary value problem for a FDE; and In Banach spaces, solutions of FDE with the operator, p-Laplacian is discussed in [15–17] respectively. For more information, see [18].

Also, the stability of FDEs has recently been proven to have a large interset. Stability of the solutions of FDEs is one of the qualitative properties of FDEs. Besides that, many scientists have become interested in the Ulam-Hyers stability concerns. The Hyers-Ulam stability refers to the fact that an FDE's exact solution is extremely close to the approximate solution of the differential equation (DE), and the error is small enough to be calculated. For ABC-FDEs with p-Laplacian operator containing spatial singularity, EU of solutions, as well as Hyers-Ulam stability, were examined in [19]. For much more information on stability analysis, read the articles [22, 23]. The stability and existence results for non-linear FDE with singularity have been discussed by Alkhazzan et al. [27]. EU findings for delay DE was discussed in [28]. Existence results and Hyers Ulam stability analysis for FDE using fixed point theorem was also examined in [29]. The problem we have proposed is broader and more complex than the ones previously discussed. For more details, and useful works see [30–37]. For more results, one may refer [38–45].

To the aimed to contribute, there seems to be no article in the literature which then examines the aforementioned problem. According to the literature, no attempt has been made to investigate the p-Laplacian operator in the context of delay fractional DEs. As a direct consequence, there has been no analysis of the operator, p-Laplacian in the context of the general class of non-linear delay FDE in the literature. We also discuss the Green's function properties. As a result of the previous work, we look at the Hyers-Ulam stability along with uniqueness and existence and results for non-linear FDEs that involves the ϕ_p^*-Laplacian operator:

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}^{\varsigma}\phi_p^*[\mathfrak{D}^{\varrho}(y(\omega) - \vartheta(\omega, y(\omega)))] = -\Lambda(\omega)\psi^*(\omega, y(\omega-\mu)), \quad \omega \in [0,1], \\ \phi_p^*[\mathfrak{D}^{\varrho}y(\omega) - \vartheta(\omega, y(\omega))]^{(i)}\big|_{\omega=0} = 0, \text{ for } i = 0, 1, 2, \cdots m-1, \\ y(\omega)\big|_{\omega=0} = 0, \quad \mathfrak{D}^{\nu}[y(\omega)]\big|_{\omega=1} = 0, \ 1 < \nu \le 2, \\ I^{k-\varrho}[y(\omega) - \vartheta(\omega, y(\omega))]\big|_{\omega=0} = 0, \ k = 3, \cdots, m. \end{cases} \tag{1.1}$$

Here ${}^{\mathcal{C}}\mathfrak{D}^{\varsigma}$, $\mathfrak{D}^{\varrho}$ stands for fractional derivatives of order ς and ϱ in Caputo's and Riemann-Liouville sense respectively, ψ^* is continuous function, $\Lambda(\omega)$ is non-vanishing function and

$$\|\Lambda(\omega)\| \le \max_{t\in[0,1]} |\Lambda(\omega)| < +\infty.$$

The orders $m-1 < \varrho, \ \varsigma \le m$, where $m \in \{3, 4, 5, \cdots\}$, $\mu > 0$ and $\psi^* \in \mathcal{L}[0,1]$ and

$$\phi_p^*(y) = |y|^{p-2}y$$

symbolises the p-Laplacian operator and satisfies the

$$\frac{1}{p}+\frac{1}{q}=1,\ (\phi_p^*)^{-1}=\phi_q^*.$$

This is how the paper is structured. The remaining of the work is split into four sections. Sections 1 and 2 contain the introducton and fundamental definitions as well as the desired lemmas. Section 3 discusses the properties of Green's functions. Section 4 contains the Existence and uniqueness results. Section 5 discusses HU stability and Section 6 illustrates an application. We'll go over certain definitions, relevant theorems and auxiliary technical lemmas that are necessary to get to the main outcome. Section 7 includes conclusions.

2. Preliminaries

Definition 2.1. [1] The Riemann-Liouville (RL) fractional integral for an integrable and real valued continuous function, ψ of fractional order $\delta \in \mathbb{R}$ is expressed as on $(0,+\infty)$:

$$I^{\delta}\psi(\mathfrak{z})=\frac{1}{\Gamma(\delta)}\int_0^{\mathfrak{z}}(\mathfrak{z}-\mathfrak{s})^{\delta-1}\psi^*(\mathfrak{s})d\mathfrak{s},\ \ \delta>0.$$

Definition 2.2. [1] The Caputo's derivative of fractional order $\delta \in \mathbb{R}$ ($\delta > 0$), for ψ defined on $(0,+\infty)$ which is k-times continuously differentiable real valued function, is defined by:

$${}^{C}\mathfrak{D}^{\delta}\psi(\mathfrak{z})=\frac{1}{\Gamma(\mathsf{k}-\delta)}\int_0^{\mathfrak{z}}(\mathfrak{z}-\mathfrak{s})^{\mathsf{k}-\delta-1}\psi^{\mathfrak{k}}(\mathfrak{s})d\mathfrak{s},\ \ \mathsf{k}-1<\delta<\mathsf{k},\ \ \mathsf{k}=[\delta]+1,$$

where $[\delta]$ is used for the greatest-integer part, such that the integral is well defined on $(0,+\infty)$.

Lemma 2.1. *[2] Let $\varrho \in (n-1,n]$, $\psi(\omega) \in C^{n-1}$, then*

$$\mathcal{I}^{\varrho}\mathfrak{D}^{\varrho}\psi(\mathfrak{z})=\psi(\mathfrak{z})+d_0+d_1\mathfrak{z}+d_2\mathfrak{z}^2+d_3\mathfrak{z}^3+\cdots+d_{n-1}\mathfrak{z}^{n-1},$$

for $d_j \in \mathbb{R}$; $j=0,1,2,3,\cdots,n-1$.

Theorem 2.1. [25,26] Let us consider a Banach space, φ^* and a cone $\mathcal{M}^* \in \varphi^*$. Assuming $\mathcal{S}_1^*, \mathcal{S}_2^*$ are two bounded subset of φ^* such that $0 \in \mathcal{S}_1^*$, $\overline{\mathcal{S}_1^*} \subset \mathcal{S}_2^*$. Then, an operator $\mathcal{T}^*$: $\mathcal{M}^* \cap (\overline{\mathcal{S}_2^*} \setminus \mathcal{S}_1^*) \longrightarrow \mathcal{M}^*$, is completely continuous and satisfies the following:

$$(\mathcal{E}_1^*)\ \ \|\mathcal{T}^*y\| \geq \|y\|\ \ if\ \ y \in \mathcal{M}^* \cap \partial\mathcal{S}_1^*\ \ and\ \ \|\mathcal{T}^*y\| \leq \|y\|\ \ if\ \ y \in \mathcal{M}^* \cap \partial\mathcal{S}_2^*,$$

or

$$(\mathcal{E}_2^*)\ \ \|\mathcal{T}^*y\| \leq \|y\|\ \ if\ y \in \mathcal{M}^* \cap \partial\mathcal{S}_1^*\ \ and\ \ \|\mathcal{T}^*y\| \geq \|y\|\ \ if\ y \in \mathcal{M}^* \cap \partial\mathcal{S}_2^*,$$

has a fixed point in $\mathcal{M}^* \cap (\overline{\mathcal{S}_2^*} \setminus \mathcal{S}_1^*)$ (Guo-Krasnosel'skii Theorem [25,26]).

Lemma 2.2. *[20,21] The following conditions hold true, for the operator p-Laplacian ϕ_p^*:*

(1) *If* $|\gamma_1|, |\gamma_2| \geq \varrho > 0,\ 1 < p \leq 2,\ \gamma_1\gamma_2 > 0,$ *then*

$$|\phi_p^*(\gamma_1)-\phi_p^*(\gamma_2)| \leq (p-1)\varrho^{p-2}|\gamma_1-\gamma_2|.$$

(2) *If* $p > 2, |\gamma_1|, |\gamma_2| \leq \varrho^* > 0,$ *then*

$$|\phi_p^*(\gamma_1)-\phi_p^*(\gamma_2)| \leq (p-1)(\varrho^*)^{p-2}|\gamma_1-\gamma_2|.$$

3. Green's function and its properties

Theorem 3.1. Assuming $\psi^* \in \mathcal{C}[0,1]$ satisfies the given FDE (1.1) with ϕ_p^*. Then for $\varsigma, \varrho \in (3, n]$ for positive integer $n \geq 4$, the solution of the given FDE with the operator, ϕ_p^* has a solution equivalent to

$$y(\omega) = \vartheta(\omega, y(\omega)) + \int_0^1 \mathcal{W}^{\varrho}(\omega, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))d\varepsilon\Big)dx, \tag{3.1}$$

where $\mathcal{W}^{\varrho}(\omega, x)$, the Green's function is given by

$$\mathcal{W}^{\varrho}(\omega, x) = \begin{cases} \dfrac{-(\omega-x)^{\varrho-1}}{\Gamma(\varrho)} + \omega^{\varrho-1}\dfrac{(1-x)^{\varrho-\nu-1}}{\Gamma(\varrho)}, & 0 < x \leq \omega < 1, \\ \omega^{\varrho-1}\dfrac{(1-x)^{\varrho-\nu-1}}{\Gamma(\varrho)}, & 0 < \omega \leq x < 1. \end{cases} \tag{3.2}$$

Proof. Utilizing I^{ς}, the fractional integral operator on (1.1) and by the virtue of Lemma 2.1, the above Eq (1.1) gets the form

$$\phi_p^*({}^c\mathfrak{D}^{\varrho}[y(\omega) - \vartheta(\omega, y(\omega))]) = -I^{\varsigma}[\psi^*(\omega, y(\omega-\mu))] + C_1 + C_2\omega + C_3\omega^2 + \cdots + C_m\omega^{m-1}. \tag{3.3}$$

From the condition

$$\phi_p^*({}^c\mathfrak{D}^{\varrho}[y(\omega) - \vartheta(\omega, y(\omega))])\Big|_{\omega=0} = 0, \implies C_1 = C_2 = C_3 \cdots C_m = 0.$$

Using arbitrary constants values, then (3.3) becomes

$$\phi_p^*({}^c\mathfrak{D}^{\varrho}[y(\omega) - \vartheta(\omega, y(\omega))]) = -I^{\varsigma}[\psi^*(\omega, y(\omega-\mu))]. \tag{3.4}$$

Using the p-Laplacian operator further, (3.4) we obtain the form

$${}^c\mathfrak{D}^{\varrho}[y(\omega) - \vartheta(\omega, y(\omega))] = -\phi_q^*(I^{\varsigma}[\psi^*(\omega, y(\omega-\mu))]). \tag{3.5}$$

Applying the fractional integral operator I^{ϱ} to (3.5) again and imposing Lemma 2.1, (3.5) becomes

$$y(\omega) - \vartheta(\omega, y(\omega)) = -I^{\varrho}\Big(\phi_q^*(I^{\varsigma}[\psi^*(\omega, y(\omega-\mu))])\Big) + h_1\omega^{\varrho-1} + h_2\omega^{\varrho-2} + h_3\omega^{\varrho-3} + \cdots + h_m\omega^{\varrho-m}, \tag{3.6}$$

where $h_j \in \mathbb{R}$ for $j = 1, 2, \cdots, \varrho - m$.

Making use of boundary conditions, $y(0) = 0$ and

$$I^{k-\varrho}[y(\omega) - \vartheta(\omega, y(\omega))]\Big|_{\omega=0} = 0, \quad k = 2, 3, \cdots, m,$$

for $j = 2, 3, 4, \cdots, \varrho - m$, in (3.6), $\implies h_j = 0$ and

$$\mathfrak{D}^{\nu}[y(\omega)]\Big|_{\omega=1} = 0, \quad 1 < \nu \leq 2,$$

implies that

$$h_1 = \frac{\Gamma(\varrho-\delta)}{\Gamma(\varrho)} I^{\varrho-\delta}\Big(\phi_q^*(I^{\varsigma}[\Lambda(\omega)\psi^*(\omega, y(\omega-\mu))])\Big)\Big|_{\omega=1}.$$

Putting constants values h_i in (3.6), we have

$$y(\omega) = \vartheta(\omega, y(\omega)) + \int_0^1 \mathcal{W}^{\varrho}(\omega, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))d\varepsilon\Big)dx, \tag{3.7}$$

where $\mathcal{W}^{\varrho}(\omega, x)$ is defined in (3.2).

Lemma 3.1. *The function, $\mathcal{W}^{\varrho}(\omega, x)$ in (3.2) fulfils the following given considerations:*
($\mathcal{P}_1$) $\mathcal{W}^{\varrho}(\omega, x) > 0, \quad \forall\ x, \omega \in (0, 1)$;
($\mathcal{P}_2$) *The Green function is non-decreasing and* $\mathcal{W}^{\varrho}(1, x) = \max_{\omega \in [0,1]} \mathcal{W}^{\varrho}(\omega, x)$;
($\mathcal{P}_3$) $\mathcal{W}^{\varrho}(1, x) \geq \omega^{\varrho-1} \max_{\omega \in [0,1]} \mathcal{W}^{\varrho}(\omega, x)$, *for* $\omega, x \in (0, 1)$.

Proof. The two cases will be used to prove ($\mathcal{P}_1$) $\forall\ \omega, x \in (0, 1)$.
Case 1. For $0 < x \leq \omega < 1$, then $x \leq \frac{x}{\omega}$. Subsequently,

$$\begin{aligned} 1 - x &\geq 1 - \frac{x}{\omega}, \\ (1 - x)^{\varrho-\delta-1} &\geq (1 - \frac{x}{\omega})^{\varrho-1}, \\ \omega^{\varrho-1} \frac{(1 - x)^{\varrho-\delta-1}}{\Gamma(\varrho)} &\geq \frac{\omega^{\varrho-1}(1 - \frac{x}{\omega})^{\varrho-1}}{\Gamma(\varrho)}, \\ \omega^{\varrho-1} \frac{(1 - x)^{\varrho-\delta-1}}{\Gamma(\varrho)} &\geq \frac{(\omega - x)^{\varrho-1}}{\Gamma(\varrho)}, \\ \frac{-(\omega - x)^{\varrho-1}}{\Gamma(\varrho)} &+ \omega^{\varrho-1} \frac{(1 - x)^{\varrho-\delta-1}}{\Gamma(\varrho)} \geq 0, \\ \mathcal{W}^{\varrho}(\omega, x) = \frac{-(\omega - x)^{\varrho-1}}{\Gamma(\varrho)} &+ \omega^{\varrho-1} \frac{(1 - x)^{\varrho-\delta-1}}{\Gamma(\varrho)} \geq 0. \end{aligned} \tag{3.8}$$

Case 2. When $\omega \leq x < 1$ then $\omega^{\varrho-1}, (1 - \jmath)^{\varrho-\delta-1} \geq 0$. Consequently,

$$\mathcal{W}^{\varrho}(\omega, x) = \omega^{\varrho-1} \frac{(1 - x)^{\varrho-\delta-1}}{\Gamma(\varrho)} \geq 0. \tag{3.9}$$

From (3.8) and (3.9), $\mathcal{W}^{\varrho}(\omega, x) > 0, \ \forall x, \omega \in (0, 1)$. For proving ($\mathcal{P}_2$), suppose that $\forall x, \omega \in (0, 1)$.
Case 3. For $x \leq \omega$. As $\varrho > 3$, then

$$\begin{aligned} (1 - x)^{\varrho-\delta-1} &> (1 - x)^{\varrho-2}, \\ \frac{\omega^{\varrho-2}(1 - x)^{\varrho-\delta-1}}{\Gamma(\varrho - 1)} &> \frac{\omega^{\varrho-2}(1 - x)^{\varrho-2}}{\Gamma(\varrho - 1)}, \\ \frac{-\omega^{\varrho-2}(1 - \frac{x}{\omega})^{\varrho-2}}{\Gamma(\varrho - 1)} &+ \frac{\omega^{\varrho-2}(1 - x)^{\varrho-\delta-1}}{\Gamma(\varrho - 1)} > 0, \\ \frac{\partial}{\partial \omega} \mathcal{W}^{\varrho}(\omega, x) = \frac{\omega^{\varrho-2}(1 - x)^{\varrho-\delta-1}}{\Gamma(\varrho - 1)} &- \frac{(\omega - x)^{\varrho-2}}{\Gamma(\varrho - 1)} > 0, \ \text{ for } \jmath \leq \omega < 1. \end{aligned} \tag{3.10}$$

Case 4. When $\omega \leq x$, we find that

$$\frac{\partial}{\partial \omega} \mathcal{W}^{\varrho}(\omega, x) = \frac{\omega^{\varrho-2}(1 - x)^{\varrho-\delta-1}}{\Gamma(\varrho - 1)} > 0. \tag{3.11}$$

From Eqs (3.10) and (3.11), it shows that $\frac{\partial}{\partial \omega} \mathcal{W}^{\varrho}(\omega, x) > 0$, for all $x, \omega \in (0, 1)$, subsequently, $\frac{\partial}{\partial \omega} \mathcal{W}^{\varrho}(\omega, x)$ be non-decreasing function.

Therefore, as a result we have for $t \geq x$,

$$\max_{\omega\in[0,1]} \mathscr{W}^{\varrho}(\omega, x) = \frac{-(1-x)^{\varrho-1}}{\Gamma(\varrho)} + \frac{(1-x)^{\varrho-\delta-1}}{\Gamma(\varrho)} = \mathscr{W}^{\varrho}(1, x), \tag{3.12}$$

and for $x \geq \omega$

$$\max_{\omega\in[0,1]} \mathscr{W}^{\varrho}(\omega, x) = \frac{(1-x)^{\varrho-\delta-1}}{\Gamma(\varrho)} = \mathscr{W}^{\varrho}(1, x). \tag{3.13}$$

To prove $(\mathscr{P}_3)$, suppose

Case 5. When $x \leq \omega \Longrightarrow x \leq \frac{x}{\omega}$, then

$$\begin{aligned} \mathscr{W}^{\varrho}(\omega, x) &= \frac{-(\omega-x)^{\varrho-1}}{\Gamma(\varrho)} + \omega^{\varrho-1}\frac{(1-x)^{\varrho-\delta-1}}{\Gamma(\varrho)} \\ &= \frac{-\omega^{\varrho-1}(1-\frac{x}{\omega})^{\varrho-1}}{\Gamma(\varrho)} + \omega^{\varrho-1}\frac{(1-x)^{\varrho-\delta-1}}{\Gamma(\varrho)} \\ &\geq \frac{-\omega^{\varrho-1}(1-x)^{\varrho-1}}{\Gamma(\varrho)} + \omega^{\varrho-1}\frac{(1-x)^{\varrho-\delta-1}}{\Gamma(\varrho)} \\ &= \omega^{\varrho-1}\Big(\frac{-(1-x)^{\varrho-1}}{\Gamma(\varrho)} + \frac{(1-x)^{\varrho-\delta-1}}{\Gamma(\varrho)}\Big) \\ &= \omega^{\varrho-1}\mathscr{W}^{\varrho}(1, x). \end{aligned} \tag{3.14}$$

Case 6. When $\omega \leq x < 1$, we obtain

$$\mathscr{W}^{\varrho}(\omega, x) = \omega^{\varrho-1}\frac{(1-x)^{\varrho-\delta-1}}{\Gamma(\varrho)} \geq \omega^{\varrho-1}\Big[\frac{(1-x)^{\varrho-\delta-1}}{\Gamma(\varrho)}\Big] = \omega^{\varrho-1}\mathscr{W}^{\varrho}(1, x). \tag{3.15}$$

Thus, by Eqs (3.14) and (3.15), condition $\mathscr{P}_3$ is proved.

4. Main results of existence and uniqueness

4.1. Existence results

Consider $\varphi^* = \mathscr{C}[0, 1]$, Banach space which contains all real valued continuous functions and endowed with

$$|y| = \max_{\omega\in[0,1]}\{|y(\omega)| : y \in \varphi^*\}$$

and are defined on $[0, 1]$. Assume that $\mathcal{M}^* \in \varphi^*$is a positive, and of the type

$$\mathcal{M}^* = \{y \in \varphi^* : y(\omega) \geq \omega^{\varrho}|y|, \omega \in [0, 1]\}$$

cone of functions.

Let

$$x^*(h) = \{y \in \mathcal{M}^* : |y| < h\}, \partial x^*(h) = \{y \in \mathcal{M}^* : |y| = h\}.$$

By applying Theorem 3.1, the alternate form of given problem (1.1) is

$$y(\omega) = \mathscr{v}(\omega, y(\omega)) + \int_0^1 \mathscr{W}^{\varrho}(\omega, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))d\varepsilon]\Big)dx. \tag{4.1}$$

Let us start by defining an operator $\mathcal{T}^*: \mathcal{M}^* \setminus \{0\} \to \varphi^*$ assosiated with problem (1.1), in order for

$$\mathcal{T}^* y(\omega) = \mathfrak{v}(\omega, y(\omega)) + \int_0^1 \mathcal{W}^{\varrho}(\omega, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx. \quad (4.2)$$

The solution of our problem given by Eq (1.1) can be a fixed point $y(\omega)$ of $\mathcal{T}^*$ using Theorem 3.1 that is,

$$y(\omega) = \mathcal{T}^* y(\omega). \quad (4.3)$$

The following suppositions are essential to reach the existence outcome:

($\mathcal{B}_1$) Continuous and non-vanishing function; $\Lambda(\omega) : (0,1) \longrightarrow \mathbb{R}^+$ with

$$\|\Lambda(\omega)\| \leq \max_{t\in[0,1]} |\Lambda(\omega)| < +\infty.$$

($\mathcal{B}_2$) $\psi^*(\omega, y(\omega)), \mathfrak{v}(\omega, y(\omega))$: $(0,1)\times(0,+\infty) \longrightarrow \mathbb{R}^+$ are continuous.

($\mathcal{B}_3$)

$$|\mathfrak{v}(\omega, y(\omega))| \leq \phi_p^*(g_1|y(\omega)|^{l_1} + \aleph_1),\ \forall y \in \varphi^*,$$

$$|\psi^*(\omega, y(\omega))| \leq \phi_p^*(g_2|y(\omega)|^{l_2} + \aleph_2),\ \forall y \in \varphi^*.$$

Where $g_1, g_2, \aleph_1, \aleph_2$ are positive constants and $l_1, l_2 \in [0,1]$.

($\mathcal{B}_4$)

$$|\mathfrak{v}(\omega, y(\omega)) - \mathfrak{v}(\omega, \upsilon(\omega))| \leq \mathcal{L}_1(|y(\omega) - \upsilon(\omega)|),\ \forall \mathcal{L}_1 > 0,\ y, \upsilon \in \varphi^*,$$

$$|\psi^*(\omega, y(\omega-\mu)) - \psi^*(\omega, \upsilon(\omega-\mu))| \leq \mathcal{L}_2(|y(\omega) - \upsilon(\omega)|),\ \forall \mathcal{L}_2 > 0,\ y, \upsilon \in \varphi^*.$$

Theorem 4.1. Let us consider that conditions ($\mathcal{B}_1$)–($\mathcal{B}_3$) are satisfied. Then, $\mathcal{T}^*$ is completely continuous.

Proof. Utilizing (4.2) and Lemma 3.1, we obtain for any $y \in (\overline{\mathcal{S}^*{}_2(h)}) \setminus x_1^*(h))$,

$$\begin{aligned}\mathcal{T}^* y(\omega) &= \mathfrak{v}(\omega, y(\omega)) + \int_0^1 \mathcal{W}^{\varrho}(\omega, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx \\ &\leq \mathfrak{v}(\omega, y(\omega)) + \int_0^1 \mathcal{W}^{\varrho}(1, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx,\end{aligned} \quad (4.4)$$

$$\begin{aligned}\mathcal{T}^* y(\omega) &= \mathfrak{v}(\omega, y(\omega)) + \int_0^1 \mathcal{W}^{\varrho}(\omega, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx \\ &\geq \mathfrak{v}(\omega, y(\omega)) + \omega^{\varrho-1}\int_0^1 \mathcal{W}^{\varrho}(1, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1} \\ &\times [\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx.\end{aligned} \quad (4.5)$$

From (4.4) and (4.5), we arrive at

$$\omega^{\varrho-1}|\mathcal{T}^*y(\omega)| \leq \mathcal{T}^*y(\omega), \quad \omega \in [0,1]. \tag{4.6}$$

As a result, $\mathcal{T}^*$: $(\overline{\mathcal{S}^*{}_2(h)}) \setminus x_1^*(h)) \to \mathcal{M}^*$.

We then prove: $|\mathcal{T}^*y_n(\omega) - \mathcal{T}^*y(\omega)| \to 0$ as $n \to \infty$ to demonstrate that $\mathcal{T}^*$ map is a continuous map, let us contrive

$$\begin{aligned}
|\mathcal{T}^*y_n(\omega) - \mathcal{T}^*y(\omega)| = \Big\| & \upsilon(\omega, y_n(\omega)) - \upsilon(\omega, y(\omega)) \\
& + \int_0^1 \mathcal{W}^{\varrho}(\omega, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon, y_n(\varepsilon-\mu))]d\varepsilon\Big)dx \\
& - \int_0^1 \mathcal{W}^{\varrho}(\omega, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx\Big\| \\
& \leq \int_0^1 |\mathcal{W}^{\varrho}(\omega, x)|\Big\|\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon, y_n(\varepsilon-\mu))]d\varepsilon\Big)dx \\
& - \phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx\Big\|.
\end{aligned} \tag{4.7}$$

By means of function continuity, υ, ψ^*, we now have $|\mathcal{T}^*y_n(\omega) - \mathcal{T}^*y(\omega)| \to 0$ as $n \to \infty$. This implies, $\mathcal{T}^*$ map is a continuous map. Here, we need to exemplify that T is uniformly bounded on $(\overline{\mathcal{S}^*{}_2(h)}) \setminus x^*{}_1(h)$.

By (4.2) and using $(\mathcal{B}_1), (\mathcal{B}_3)$, for $\omega \in [0,1]$, we have got

$$\begin{aligned}
|\mathcal{T}^*y| &= \Big\|\upsilon(\omega, y(\omega)) + \int_0^1 \mathcal{W}^{\varrho}(\omega, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1} \\
&\quad \times [\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx\Big\| \\
&\leq \Big\|\upsilon(\omega, y(\omega))\Big\| + \int_0^1 \Big|\mathcal{W}^{\varrho}(1, x)\Big|\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1} \\
&\quad \times \|\Lambda\|\phi_p^*(g_2|y(\varepsilon)|^{l_2} + \aleph_2)d\varepsilon\Big)dx \\
&\leq \Big[\frac{1}{\Gamma(\varrho+1)} + \frac{1}{\Gamma(\varrho)\Gamma(\varrho-\delta)}\Big]\Big[\frac{1}{\Gamma(\varsigma+1)}\Big]^{q-1} \\
&\quad \times \|\Lambda\|^{q-1}(g_2|y(\varepsilon)|^{l_1} + \aleph_2) + (g_1|y(\varepsilon)|^{l_2} + \aleph_1) \\
&= \Theta\|\Lambda\|^{q-1}(g_2|y(\varepsilon)|^{l_1} + \aleph_2) + (g_1|y(\varepsilon)|^{l_2} + \aleph_1),
\end{aligned} \tag{4.8}$$

where

$$\Theta = \Big[\frac{1}{\Gamma(\varrho+1)} + \frac{1}{\Gamma(\varrho)\Gamma(\varrho-\delta)}\Big]\Big[\frac{1}{\Gamma(\varsigma+1)}\Big]^{q-1}.$$

$\mathcal{T}^*$ is uniformly bounded by operator $\mathcal{T}^*$, as a result of this. The equicontinuity of the operator $\mathcal{T}^*$ is used to demonstrate that operator $\mathcal{T}^*$ is compact.

When $0 < \omega_1 < \omega_2 < 1$, we proceed

$$
\begin{aligned}
|\mathcal{T}^*y(\omega_2) - \mathcal{T}^*y(\omega_1)| \leq\ & \left\|\upsilon(\omega_1, y(\omega_1)) - \upsilon(\omega_2, y(\omega_2))\right\| \\
& + \Big\| \int_0^1 \mathcal{W}^{\varrho}(\omega_2, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx \\
& - \int_0^1 \mathcal{W}^{\varrho}(\omega_1, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx\Big\| \\
& \leq \left\|\upsilon(\omega_1, y(\omega_1)) - \upsilon(\omega_2, y(\omega_2))\right\| \\
& + \int_0^1 \left|\mathcal{W}^{\varrho}(\omega_2, x) - \mathcal{W}^{\varrho}(\omega_1, x)\right|\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1} \\
& \times (\|\Lambda\|\phi_p^*(g_2|y(\varepsilon)|^{l_2} + \aleph_2))d\varepsilon\Big)dx \\
& \leq \Big(\frac{\left|\omega_2{}^{\varrho} - \omega_1{}^{\varrho}\right|}{\Gamma(\varrho+1)} + \frac{\left|\omega_2{}^{\varrho-1} - \omega_1{}^{\varrho-1}\right|}{\Gamma(\varrho)\Gamma(\varrho-\delta)}\Big)\Big[\frac{1}{\Gamma(\varsigma+1)}\Big]^{q-1} \\
& \times \|\Lambda\|^{q-1}(g_2|y(\varepsilon)|^{l_2} + \aleph_2)),
\end{aligned}
$$

$$
|\mathcal{T}^*y(\omega_2) - \mathcal{T}^*y(\omega_1)| \to 0 \ \text{ as } \ (\omega_2 - \omega_1) \to 0. \tag{4.9}
$$

As a result, $\mathcal{T}^*$ is equicontinuous operator on $(\overline{\mathcal{S}^*{}_2(h)}) \setminus x_1^*(h)$, and $\mathcal{T}^*$ is compact according to the Arzela Ascoli theorem on $(\overline{\mathcal{S}^*{}_2(h)}) \setminus x_1^*(h)$.

Consequently, all the conditions of Theorem 3.1's [25] got satisfied. Furthermore,

$$
\mathcal{T}^* : (\overline{\mathcal{S}^*{}_2(h)}) \setminus x_1^*(h)) \to \mathcal{M}^*
$$

is completely continuous.

Let us evaluate the height functions for $\psi^*(\omega, y(\omega))$, for $h > 0$ and $\forall\ \omega \in [0, 1]$ now.

$$
\begin{cases}
\phi^*_{\max}(\omega, h) = \max\limits_{\omega\in[0,1]}\{\psi^*(\omega, y(\omega-\mu)) : \omega^{\varrho-1}h \leq y \leq h\} \leq \overline{\mathsf{M}} < +\infty, \\
\phi^*_{\min}(\omega, h) = \min\limits_{\omega\in[0,1]}\{\psi^*(\omega, y(\omega-\mu)) : \omega^{\varrho-1}h \leq y \leq h\} \geq \overline{\mathsf{m}} > -\infty.
\end{cases} \tag{4.10}
$$

Theorem 4.2. Suppose that assumptions $(\mathcal{B}_1)$–$(\mathcal{B}_3)$ hold and $\exists a, b \in \mathbb{R}^+$ such that

$$
(\mathcal{U}_1)\ \ a \leq |\upsilon(\omega, y(\omega))| + \int_0^1 \mathcal{W}^{\varrho}(1, x)\phi_q^*(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\phi^*_{\min}(\varepsilon, a)]d\varepsilon)dx < +\infty,
$$

and

$$
|\upsilon(\omega, y(\omega))| + \int_0^1 \mathcal{W}^{\varrho}(1, x)\phi_q^*(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\phi^*_{\max}(\varepsilon, b)]d\varepsilon)dx \leq b,
$$

or

$$
(\mathcal{U}_2)\ \ |\upsilon(\omega, y(\omega))| + \int_0^1 \mathcal{W}^{\varrho}(1, x)\phi_q^*(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\phi^*_{\max}(\varepsilon, a)]d\varepsilon)dx < a,
$$

and

$$
b \leq |\upsilon(\omega, y(\omega))| + \int_0^1 \mathcal{W}^{\varrho}(1, x)\phi_q^*(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\phi^*_{\min}(\varepsilon, b)]d\varepsilon)dx < +\infty.
$$

Then the given Eq (1.1) has a positive solution $y \in \mathcal{M}^*$ & $a \leq |y| \leq b$.

Proof. First, let's look at $(\mathcal{U}_1)$. If $y \in \partial\mathcal{S}^*(a)$ then $\forall\omega \in [0,1]$, $\omega^{\varrho-1}a \leq y \leq a$, and $|y| = a$. Using (4.10), $\phi^*_{\min}(\omega, a) \leq \psi^*(\omega, y(\omega))$, we proceed

$$
\begin{aligned}
|\mathcal{T}^* y| &= \max_{\omega\in[0,1]} \Big\| \upsilon(\omega, y(\omega)) + \int_0^1 \mathcal{W}^{\varrho}(\omega, x)\phi^*_q\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1} \\
&\times [\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx\Big\| \\
&\geq \Big\|\upsilon(\omega, y(\omega))\Big\| + \omega^{\varrho-1}\int_0^1 \Big|\mathcal{W}^{\varrho}(1,x)\Big|\phi^*_q\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1} \\
&\times [\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx \\
&\geq \Big\|\upsilon(\omega, y(\omega))\Big\| + \omega^{\varrho-1}\int_0^1 \Big|\mathcal{W}^{\varrho}(1,x)\Big|\phi^*_q\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1} \\
&\times [\Lambda(\varepsilon)\phi^*_{\min}(\varepsilon, a)]d\varepsilon\Big)dx \geq a \\
&= |y|.
\end{aligned} \tag{4.11}
$$

Now, $\forall\ \omega \in [0,1]$, $\omega^{\varrho-1}b \leq y \leq b$.

If $y \in \partial\mathcal{S}^*(b)$ then $|y| = b$. Using (4.10), we have $\psi^*(\omega, y(\omega)) \leq \phi^*_{\max}(\omega, b)$. This implies

$$
\begin{aligned}
|\mathcal{T}^* y| &= \max_{\omega\in[0,1]} \Big\| \upsilon(\omega, y(\omega)) + \int_0^1 \mathcal{W}^{\varrho}(\omega, x)\phi^*_q\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1} \\
&\times [\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx\Big\| \\
&\leq \Big\|\upsilon(\omega, y(\omega))\Big\| + \int_0^1 \Big|\mathcal{W}^{\varrho}(1,x)\Big|\phi^*_q\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1} \\
&\times [\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx \\
&\leq \Big\|\upsilon(\omega, y(\omega))\Big\| + \omega^{\varrho-1}\int_0^1 \Big|\mathcal{W}^{\varrho}(1,x)\Big|\phi^*_q\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1} \\
&\times [\Lambda(\varepsilon)\phi^*_{\min}(\varepsilon, a)]d\varepsilon\Big)dx \leq b \\
&= |y|.
\end{aligned} \tag{4.12}
$$

Thus using Theorem 2.2, $\mathcal{T}^*$ has a fixed point, i.e. $y \in (\overline{\mathcal{S}^*(b)}) \setminus x^*(a)$. Theorem 3.1 and Lemma 3.1 are used, for $\omega \in (0,1)$ & $a \leq |y| \leq b$, we have $y(\omega) \geq \omega^{\varrho-1}|y(\omega)| \geq a\omega^{\varrho-1} > 0$. Therefore, the solution $y(\omega)$ is positive.

$$
\begin{aligned}
\frac{\partial}{\partial\omega}y(\omega) &= \frac{\partial}{\partial\omega}\mathcal{T}^* y(\omega) \\
&= \frac{\partial}{\partial\omega}\upsilon(\omega, y(\omega)) + \int_0^1 \frac{\partial}{\partial\omega}\mathcal{W}^{\varrho}(\omega, x)\phi^*_q\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1} \\
&\times [\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))]d\varepsilon\Big)dx \\
&> 0.
\end{aligned} \tag{4.13}
$$

4.2. Uniqueness of solution

Theorem 4.3. Let us consider assumptions $(\mathcal{B}_1), (\mathcal{B}_2)$ and $(\mathcal{B}_4)$ are satisfied. Then, for the given Eq (1.1) on $[0, 1]$, there is a unique solution, if

$$\Delta^* = \mathcal{L}_2(q-1)\varrho^{q-2}\Big[\frac{1}{\Gamma(\varrho+1)} + \frac{1}{\Gamma(\varrho)\Gamma(\varrho-\delta)}\Big]\Big[\frac{1}{\Gamma(\varsigma+1)}\Big]^{q-1}\|\Lambda(\varepsilon)\|^{q-1} \leq 1. \tag{4.14}$$

Proof. We show that the result for $p \geq 2$ is unique.

By (4.10) and $\forall\ \omega \in [0, 1]$,

$$\begin{aligned} I^{\varsigma}[\psi^*(\omega, y(\omega-\mu))] &= \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-x)^{\varsigma-1}\psi^*(x, y(x-\mu))\mathscr{d}x \\ &\leq \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-x)^{\varsigma-1}\overline{\mathsf{M}}\mathscr{d}x \leq \frac{\overline{\mathsf{M}}}{\Gamma(\varsigma+1)}. \end{aligned} \tag{4.15}$$

For each $y \in (\overline{\mathcal{S}^*(h)}) \setminus x^*(h)$, and using (4.15) we get

$$\begin{aligned} |\mathcal{T}^*y - \mathcal{T}^*v| &= \max_{\omega\in[0,1]}\Big\|\mathsf{v}(\omega, y(\omega)) + \int_0^1 \mathscr{W}^{\varrho}(\omega, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1} \\ &\times \Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))\mathscr{d}\varepsilon\Big)\mathscr{d}x - \mathsf{v}(\omega, y(\omega)) \\ &- \int_0^1 \mathscr{W}^{\varrho}(\omega, x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}\Lambda(\varepsilon)\psi^*(\varepsilon, v(\varepsilon-\mu))\mathscr{d}\varepsilon\Big)\mathscr{d}x\Big\| \\ &\leq \int_0^1 \big|\mathscr{W}^{\varrho}(1, x)\big|\Big\|\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}\Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))\mathscr{d}\varepsilon\Big)\mathscr{d}x \\ &- \phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}\Lambda(\varepsilon)\psi^*(\varepsilon, v(\varepsilon-\mu))\mathscr{d}\varepsilon\Big)\mathscr{d}x\Big\| \\ &\leq (q-1)\varrho^{q-2}\|\Lambda(\varepsilon)\|^{q-1}\int_0^1 \big|\mathscr{W}^{\varrho}(1, x)\big|\Big\|\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1} \\ &\times \Lambda(\varepsilon)\psi^*(\varepsilon, y(\varepsilon-\mu))\mathscr{d}\varepsilon\Big)\mathscr{d}x \\ &- \Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}\Lambda(\varepsilon)\psi^*(\varepsilon, v(\varepsilon-\mu))\mathscr{d}\varepsilon\Big)\mathscr{d}x\Big\| \\ &\leq \mathcal{L}_2(q-1)\varrho^{q-2}\Big[\frac{1}{\Gamma(\varrho+1)} + \frac{1}{\Gamma(\varrho)\Gamma\varrho-\delta}\Big]\Big[\frac{1}{\Gamma(\varsigma+1)}\Big]^{q-1}\|\Lambda(\varepsilon)\|^{q-1} \\ &= \Delta^*\ \forall\omega \in [0, 1], \end{aligned}$$

but in (4.14) assumed that $\Delta^* < 1$. $\mathcal{T}^*$ is a contraction map, as evidenced by this. According to the Banach mapping principle of contraction, $\mathcal{T}^*$ thus possesses a unique fixed point. As a result, on $[0, 1]$ the provided equation (1.1) has unique solution.

5. HU stability

Definition 5.1. [24] If there is a positive constant Δ^*, the I.E. (4.1) is stated as Hyers-Ulam stable if it fulfills the following conditions for some fixed positive constant $\gamma^* > 0$.

If,

$$\Big\|y(\omega)-\mathfrak{v}(\omega,y(\omega))+\int_0^1 \mathscr{W}^{\varrho}(\omega,x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon,y(\varepsilon))]d\varepsilon\Big)dx\Big\|\leq\gamma^*. \tag{5.1}$$

Then $\exists\mathfrak{w}(\omega)$, a continuous function that fulfils the following equation:

$$\mathfrak{w}(\omega)=\mathfrak{v}(\omega,y(\omega))+\int_0^1 \mathscr{W}^{\varrho}(\omega,x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon,(\mathfrak{w}(\varepsilon))]d\varepsilon\Big)dx, \tag{5.2}$$

with

$$|y(\omega)-\mathfrak{w}(\omega)|\leq\Delta^*\gamma^*. \tag{5.3}$$

Theorem 5.1. For $p>2$, if $(\mathscr{R}_1),(\mathscr{R}_2)$ and $(\mathscr{R}_4)$ are satisfied, the FDE (1.1) with the non-linear p-Laplacian opertor is HU stable.

Proof. For problem (1.1), we show that Eq (4.1), with suppositions $(\mathscr{R}_1)$, $(\mathscr{R}_2)$, and $(\mathscr{R}_4)$ is HU stable. Consequently, we have

$$\begin{aligned}
|y(\omega)-\mathfrak{w}(\omega)| &= \max_{\omega\in[0,1]}\Big\|\mathfrak{v}(\omega,y(\omega))+\int_0^1 \mathscr{W}^{\varrho}(\omega,x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}\\
&\times[\Lambda(\varepsilon)\psi^*(\varepsilon,y(\varepsilon-\mu))]d\varepsilon\Big)dx-\mathfrak{v}(\omega,\mathfrak{w}(\omega))\\
&-\int_0^1 \mathscr{W}^{\varrho}(\omega,x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon,\mathfrak{w}(\varepsilon-\mu))]d\varepsilon\Big)dx\Big\|\\
&\leq\int_0^1 \Big|\mathscr{W}^{\varrho}(\omega,x)\Big|\Big\|\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon,y(\varepsilon-\mu))]d\varepsilon\Big)dx\\
&-\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\psi^*(\varepsilon,\mathfrak{w}(\varepsilon-\mu))]d\varepsilon\Big)dx\Big\|\\
&\leq(q-1)\varrho^{q-2}\|\Lambda(\varepsilon)\|^{q-1}\int_0^1 \Big|\mathscr{W}^{\varrho}(\omega,x)\Big|\Big\|\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}\\
&\times(\psi^*(\varepsilon,y(\varepsilon-\mu))-\psi^*(\varepsilon,\mathfrak{w}(\varepsilon-\mu))]d\varepsilon\Big)dx\\
&\leq\mathscr{L}_2(q-1)\varrho^{q-2}\Big[\frac{1}{\Gamma(\varrho+1)}+\frac{1}{\Gamma(\varrho)\Gamma(\varrho-\delta)}\Big]\Big[\frac{1}{\Gamma(\varsigma+1)}\Big]^{q-1}\\
&\times\|\Lambda(\varepsilon)\|^{q-1}|y(\omega)-\mathfrak{w}(\omega)|\\
&=\Delta^*|y(\omega)-\mathfrak{w}(\omega)|,\quad\forall\omega\in[0,1].
\end{aligned} \tag{5.4}$$

As a conclusion, the Eq (4.1) is Hyers-Ulam stable by using (5.4). So, the FDE (1.1) has achieved HU stability.

6. Application

Furthermore, we give an illustration of our findings using an application.

Consider the following FDE

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}^{\varsigma}\phi_p^*[\mathfrak{D}^{\varrho}(y(\omega)-\upsilon(\omega,y(\omega)))] = -\Lambda(\omega)\psi^*(\omega,y(\omega-\mu)), \quad \omega\in[0,1],\\ \phi_p^*[\mathfrak{D}^{\varrho}y(\omega)-\upsilon(\omega,y(\omega))]^{(i)}\big|_{\omega=0}=0, \quad \text{for } i=0,1,2,\cdots m-1,\\ y(\omega)\big|_{\omega=0}=0, \quad \mathfrak{D}^{\nu}[y(\omega)]\big|_{\omega=1}=0, \quad 1<\nu\le 2,\\ I^{k-\varrho}[y(\omega)-\upsilon(\omega,y(\omega))]\big|_{\omega=0}=0, \quad k=3,\cdots,m, \end{cases} \tag{6.1}$$

where $\omega\in[0,1]$,

$$\psi(\omega,y(\omega-\mu)) = y^{3/2}(\omega)+\frac{1-\mu}{\sqrt{y(\omega)}},$$

$q=2.5, \varrho=4, \varsigma=3.5, \nu=1.5, \mu=0.3$,

$$\begin{aligned} \upsilon(\omega,y(\omega)) &= 0.003,\\ \Lambda(\omega) &= \frac{\omega}{\sqrt{3-3\omega}},\\ \psi^*(\omega,y(\omega)) &= y^{3/2}(\omega)+\frac{1}{\sqrt{y(\omega)}},\\ \psi^*\in C((0,1) &\times (0,+\infty)),[0,+\infty), \Lambda\in C((0,1),[0,+\infty)). \end{aligned}$$

Height functions are given by

$$\begin{cases} \phi^*_{\max}(\omega,h) = \max\limits_{\omega\in[0,1]}\{y^{3/2}+\dfrac{1-\mu}{\sqrt{y(\omega)}} : \omega^3 h\le y\le h\} \le \omega^{9/2}h^{3/2}+\dfrac{0.7}{\sqrt{h\omega^3}},\\ \phi^*_{\min}(\omega,h) = \min\limits_{\omega\in[0,1]}\{y^{3/2}+\dfrac{1-\mu}{\sqrt{y(\omega)}} : \omega^{\varrho-1} h\le y\le h\} \ge \omega^{9/2}h^{3/2}+\dfrac{1}{\sqrt{h}}. \end{cases} \tag{6.2}$$

Then, $\omega\in(0,1)$, we have

$$\begin{aligned} &0.003+\int_0^1 \mathcal{W}^{\varrho}(\omega,x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\phi^*_{\min}(\varepsilon,a)]d\varepsilon\Big)dx\\ &\quad= 0.003+\int_0^1 \mathcal{W}^{\varrho}(1,x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\phi^*_{\min}(\varepsilon,10^{-4})]d\varepsilon\Big)dx\\ &\quad\ge 0.003+\int_0^1 \mathcal{W}^{4}(1,x)\phi_{5/2}^*\Big(\frac{1}{\Gamma(7/2)}\int_0^x (x-\varepsilon)^{5/2}\frac{\varepsilon}{\sqrt{3-3\varepsilon}}(10^{-6}\varepsilon^{9/2}+100)d\varepsilon\Big)dx\\ &\quad= 0.0019006\ge 10^{-4}.\\ &0.003+\int_0^1 \mathcal{W}^{\varrho}(\omega,x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\phi^*_{\max}(\varepsilon,b)]d\varepsilon\Big)dx\\ &\quad= 0.003+\int_0^1 \mathcal{W}^{\varrho}(1,x)\phi_q^*\Big(\frac{1}{\Gamma(\varsigma)}\int_0^x (x-\varepsilon)^{\varsigma-1}[\Lambda(\varepsilon)\phi^*_{\max}(\varepsilon,1)]d\varepsilon\Big)dx\\ &\quad\le 0.003+\int_0^1 \mathcal{W}^{4}(1,x)\phi_{5/2}^*\Big(\frac{1}{\Gamma(7/2)}\int_0^x (x-\varepsilon)^{5/2}\frac{\varepsilon}{\sqrt{3-3\varepsilon}}(1+\frac{1}{\sqrt{\varepsilon^3}})d\varepsilon\Big)dx\\ &\quad= 0.00960413<1. \end{aligned}$$

Therefore, all the results of the above proved theorems holds, which implies solution exists and $\dfrac{1}{10000}\le\|y^*\|\le 1$.

7. Conclusions

The focus of this paper is on two fascinating and crucial elements of FC: EU and Hyers-Ulam stability. Green's function was used to convert the proposed problem into an I.E. in order to achieve this objective. The Green's function was then tested on (0,1) to see if it was increasing or decreasing, and whether it was positive or negative. Following that, Guo-Krasnoselskii's fixed point theorem is used to construct existence results for general FDE including the operator, p-Laplacian. Following that, the proposed problem's Hyers-Ulam stability was taken into account. Further, a significant example is provided to represent the conclusion. The proposed problem for multiplicity findings can also be studied using other mathematical methodologies by the readers. For future research work, for the proposed problem, we suggest qualitative properties/analysis and evaluation of the issue for multiplicity and exponential stability of the solution of the FDEs.

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Competing interests

There does not exist any kind of competing interest.

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