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Research article

Existence of fuzzy fixed points of set-valued fuzzy mappings in metric and fuzzy metric spaces

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Abstract: A contemporary fuzzy technique is employed in the current study to generalize some established and recent findings. For researchers, fixed point (FP) procedures are highly advantageous and appealing mechanisms. Discovering fuzzy fixed points of fuzzy mappings (FM) meeting Nadler's type contraction in complete fuzzy metric space (FMS) and Ćirić type contraction in complete metric spaces (MS) is the core objective of this research. The outcomes are backed up by example and applications that highlight these findings. There are also preceding conclusions that are given as corollaries from the relevant literature. In this mode, numerous consequences exist in the significant literature are extended and combined by our findings.

Keywords: fuzzy metric space; fuzzy fixed point; fuzzy mapping; Hausdorff metric space; Hausdorff fuzzy metric space
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1. Introduction and preliminaries

An ongoing aspect of social existence is anonymity. In this area, accurate computations or hypotheses are not relevant. For human intelligence, this value error is particularly sticky. Many other mathematical ideas, like fuzzy sets (FS), soft sets, intuitionistic sets etc. have been developed as practical solutions to this problem. The fuzzy logics were created using a group structure with hazy knowledge. Due to fuzzy sets's adaptability in handling unreliability, it is even fantastically terrific for humanistic logic that is based on correct truth and limitless information. This idea is certainly a cornerstone of classical sets since it gives greater room for wrong knowledge to be used, which leads to better answers for a variety of problems. When faced with extremely constrained options like yes or no, these firms create favorable models. The ability to examine the benefits and drawbacks of false ideas is another important quality of this knowledge.

The branch of mathematics connected with fuzzy set theory is known as fuzzy mathematics. In 1965, Zadeh [1] is the first to mention the concept of fuzzy logic. The affiliation of an element to the set in the theory of fuzzy logic is given as a number from the interval $[0,1]$, unlike the theory of classical logic, where an element either belongs to the set or not. Zadeh has been studying the theory of FS to address the issue of indeterminacy because uncertainty is a crucial component of a genuine problem.

In the discipline of mathematical analysis, the fixed point (FP) principles offer excellent conditions for approximation the solutions of differential and integral equations with both linear and nonlinear. Analysis, geometry, and topology are remarkably linked in the framework of FP theory, making it a valuable and essential technique for analyzing non-linear phenomena. The FP paradigm is intensively used in both applied and pure mathematics. Across many numerous domains, encompassing biology, engineering, non-linear programming, economics, game theory, theory of differential equations, etc., FP techniques frequently prove to be advantageous.

Fuzzy logic is one of the many perspectives that may be used to understand theory of FP in fuzzy metric spaces (FMS). Heilpern [2] introduced the theory of fuzzy mapping (FM) and established a theorem on FP for FM in metric linear space, which serves as a fuzzy generalization of Banach's contraction principle [3]. This sparked the interest of numerous authors to investigate various contractions conditions using FM.

The subject of Hausdorff distance is essential to several areas of computer science and mathematics, such as fractals, image processing, and optimization theory. Lopez and Romaguern [4] applied the concept of Hausdorff metric space in fuzzy setting and introduced Hausdorff fuzzy metric spaces. This allowed researchers to investigate the “fixed point theory” of multivalued mappings in spaces with fuzzy metrics form.

Every metric, in a very normal and modest way, generates a FMS. The theory of FP is being evolved as a crucial area of interest in the core of non-linear analysis and FS theory within the framework of FMS.

In 1975, Kramosil and Michalek [5] developed the idea of FMS, opening the door for further development of analysis in such environments. George and Veeramani [6] improved fuzzy metric spaces to become Hausdorff spaces. However, it seems that Kramosil and Michhlek’s analysis of fuzzy metric spaces offers a route for very smoothing machinery to produce FP theorems, especially for contractive type maps. Grabiec [7] was the next from among a number (at least four) of formulations of FMS. In fuzzy mathematics, fixed point theorems are emerging with fervent hope and firm confidence. Since then, numerous attempts to develop FP theorems in fuzzy mathematics have been made (see, for instance, [7–16]). Numerous fixed point and common fixed point results in FMS and Hausdorff metric spaces can be found in literature (see [17–21] and references therein). Literature shows that a lot of valuable and practical work is done in fuzzy set, rough set, soft set, intuitionistic set theories in several ways of decision making, decision models, pattern classifications and in other fields (see [22–29]).

The structure of paper is as follows:

First of all, some basic concepts are recalled, the motivation behind this action is to facilitate the readers to have comprehensive knowledge about the fundamental definitions, examples and lemmas that are necessary to understand our established results. All these essentials are collected from previous research articles exist in the literature.

In Section 2, existence theorems regarding fuzzy FPs of FMs satisfying $\acute{C}iri\acute{c}$ type contractions are obtained in the framework of complete metric spaces. The result is equipped with an interesting example and an application. Further, some previous results are given as corollaries of our results.

Section 3 deals with some theoretical results. In this study, we have established fuzzy FPs of set-valued FM by using a contraction in the setting of complete FMS. The obtained results are furnished an example and applications. Previous results are given in the form of corollaries of obtained results.

Note: i) $CB(\Omega)$ denotes the family of all closed and bounded subsets of metric space (Ω, d);

ii) $\acute{K}(\Omega)$ denotes the family of all compact subsets of fuzzy metric space $(\Omega, F, *)$.

Hausdorff Metric Space. ([16]) Let (Ω, d) be a MS. Hausdorff metric H on $CB(\Omega)$ induced by d is defined as $H(A, B) = max\{\sup_{\mu \in A} d(\mu, B), \sup_{\nu \in B} d(A, \nu)\}$ for all $A, B \in CB(\Omega)$, where

$$d(\mu, B) = inf\{d(\mu, \eta) : \eta \in B\}.$$

Lemma 1. ([13]) Let $G, K \in CB(\Omega)$. If $\mu \in G$ then, $d(\mu, K) \leq H(G, K)$ for all $\mu \in G$.

Lemma 2. ([13]) Let $P, Q \in CB(\Omega)$ and $0 < \sigma \in \mathbb{R}$. Then, for $i \in P$, there exists $\zeta \in Q$ such that

$$d(i, \zeta) \leq H(P, Q) + \sigma.$$

Lemma 3. ([13]) If $P, Q \in CB(\Omega)$ with $H(P, Q) < \varepsilon$, then for all $\mu \in P$ there exists $\upsilon \in Q$ such that $d(\mu, \upsilon) < \varepsilon$.

Lemma 4. ([13]) For $\mu \in \Omega$ *and* $P \in CB(\Omega)$, $d(\mu, P) \leq d(\mu, \nu)$ for all $\nu \in P$.

Fuzzy Set. ([1]) In the fuzzy theory, fuzzy set *A* of universe *X* is defined by function μ_A: X → [0, 1] called the membership function of set *A*

where $\mu_A(x) = 1$ if x is totally in A;
$\mu_A(x) = 0$ if x is not in A;
$0 < \mu_A(x) < 1$ if x is partly in A.

This definition of set allows a continuum of possible choices. For any element x of universe X, membership function $\mu_A(x)$ equals the degree to which x is an element of set A. This degree, a value between 0 and 1, represents the degree of membership, also called membership value, of element x in set A.

The α-cut of fuzzy set A is defined as:

$$[A]_\alpha = \{u \in X : A(u) \geq \alpha\}; \alpha \in (0, 1].$$

Fuzzy Mapping. ([16]) Let Ψ_1 be any set and Ψ_2 be a metric space. A function $g: \Psi_1 \to F(\Psi_2)$ is called a FM. A FM g is a FS on $\Psi_1 \times \Psi_2$ with membership function $g(x)(y)$. The image $g(x)(y)$ is the grade of membership of y in $g(x)$.

Fuzzy Fixed Point. ([13]) Suppose (Ψ, d) is a MS and $T: \Psi \to F(\Psi)$. A point $z \in \Psi$ is a fuzzy FP of T if $z \in [Tz]_\alpha$ for some $\alpha \in (0, 1]$.

Common Fuzzy Fixed Point. ([16]) Consider a MS (Ψ, d) and $T_1, T_2: \Psi \to F(\Psi)$. A point $z \in \Psi$ is a fuzzy common FP of T_1 and T_2 if $z \in [T_1 z]_{\alpha_{T_1}} \cap [T_2 z]_{\alpha_{T_2}}$ for some $\alpha_{T_1}, \alpha_{T_2} \in (0, 1]$.

Ćirić Type Contraction for Fuzzy Mappings. Let (Ω, d) be a complete MS and $G: \Omega \to F(\Omega)$ be a fuzzy map. Let $[G(u)]_\alpha$ *and* $[G(v)]_\alpha$ be non-empty closed and bounded subsets of Ω, the condition

$$H([[G(u)]_\alpha, [G(v)]_\alpha) \leq \alpha d(u, v) + \beta\,[d(u, [G(u)]_\alpha) + d(v, [G(v)]_\alpha)] + \gamma[d(u, [G(v)]_\alpha) +$$

$$d(v, [G(u)]_{\alpha})],$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$ is Ćirić type contraction for FM.

Triangular Norm. ([17]) A map $*$ from [0,1]×[0,1] to [0,1] is called continuous triangular norm (t-norm) or a conjunction, if following conditions are fulfilled for all $\varrho, \sigma, \varsigma, \tau \in [0,1]$:

(1) Symmetry: $\varrho * \sigma = \sigma * \varrho$;

(2) Monotonicity: $\varrho * \sigma \leq \varsigma * \tau, if\ \varrho \leq \varsigma\ and\ \sigma \leq \tau$;

(3) Associativity: $\big(\varrho * (\sigma * \varsigma)\big) = \big((\varrho * \sigma) * \varsigma\big)$;

(4) Boundary condition: $1 * \varrho = \varrho$.

The following are three basic t-norms:

(1) $\varrho * \sigma = min(\varrho, \sigma)$;

(2) $\varrho * \sigma = \varrho\sigma$;

(3) $\varrho * \sigma = max(\varrho + \sigma - 1, 0)$.

Fuzzy Metric Space. ([6]) The triple $(\Omega, M, *)$ is known as FMS if Ω is an arbitrary set, $*$ is t-norm and M is a FS on $\Omega \times \Omega \times [0, \infty)$ s.t $\forall\, \xi, \eta, \zeta \in \Omega$ and $\mu, \nu \geq 0$ we have:

$(M1)$ $M(\xi, \eta, 0) = 0$;

$(M2)$ $M(\xi, \eta, \mu) = 1, \forall \mu > 0$ iff $\xi = \eta$;

$(M3)$ $M(\xi, \eta, \mu) = M(\eta, \xi, \mu)$

$(M4)$ $M(\xi, \zeta, \mu + \nu) \geq M(\xi, \eta, \mu) * M(\eta, \zeta, \nu)$;

$(M5)$ $M(\xi, \eta, .) : (0, \infty) \to [0, 1]$ is continuous.

Example 1. Let (Ω, d) be a MS. Define $M: \Omega \times \Omega \times \mathbb{R}^+ \cup \{0\} \to [0, 1]$ as

$$M(\eta, \lambda, \mu) = \frac{min\,\{\eta,\ \gamma\} + \mu}{max\,\{\eta,\ \gamma\} + \mu},$$

for all $\eta, \lambda \in \Omega$ and $\mu \geq 0$ is a fuzzy MS.

Example 2. Let (Ω, d) be a bounded MS with $d\,(u, v) < \kappa$ (for all $u, v \in \Omega$, where κ is fixed constant in $(0, \infty)$) and $G: \mathbb{R}^+ \to (\kappa, \infty)$ be an increasing continuous function. Define a function $M: \Omega^2 \times (0, \infty) \to [0, 1]$ as

$M\,(u, v, \lambda) = 1 - \frac{d(u,v)}{G(\lambda)}$, for all $u, v \in \Omega$, and $\lambda > 0$.

Then $(\Omega, M, *)$ is a FMS on Ω where $*$ is a Lukasiewicz t-norm.

Example 3. Let (Ω, d) be a MS. Define $\mu * \nu = \mu\nu\ (or\ \mu * \nu = min\{\mu, \nu\})$ for all $\mu, \nu \in [0,1]$. Then, one can define a fuzzy metric F by $F(\xi, \eta, \sigma) = \frac{\sigma}{\sigma + d(\xi, \eta)}$ for all $\xi, \eta \in \Omega$ and $\sigma \geq 0$.

Example 4. Let Ω be a non-empty set, $f: \Omega \to \mathbb{R}^+$ be a one-one function and $g: \mathbb{R}^+ \to [0, \infty)$ be an increasing continuous function. For fixed $\alpha, \beta > 0$, define M: $\Omega^2 \times (0, \infty) \to [0, 1]$ as

$M(u, v, \lambda) = \left(\frac{min\,\{f(u), f(v)\}^{\alpha} + g(\lambda)}{max\,\{f(u), f(v)\}^{\alpha} + g(\lambda)}\right)^{\beta}$, for all $u, v \in \Omega$ and $\lambda > 0$.

Then $(\Omega, M, *)$ is a FMS on Ω where $*$ is the product t-norm.

Convergent Sequence in Fuzzy Metric Space. ([7]) Let $(\Omega, M, *)$ be a FMS. A sequence $\{\omega_n\}$ in Ω is said to be convergent to a point $\omega \in \Omega$ if $\lim_{n\to\infty} M\,(\omega_n, \omega, \mu) = 1$ for all $\mu > 0$.

Cauchy Sequence in Fuzzy Metric Space. ([7]) Let $(\Omega, M, *)$ be a fuzzy MS. A sequence $\{\omega_n\}$ in a FMS $(\Omega, M, *)$ is said to be Cauchy sequence if for every $\varepsilon \in (0,1)$ and $\mu > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$M(\omega_n, \omega_m, \mu) > 1 - \varepsilon, for\ all\ n, m \geq n_0.$$

Complete Fuzzy Metric Space. ([7]) A FMS in which every Cauchy sequence is convergent is called complete.

Hausdorff Fuzzy Metric. ([4]) Let $(\Omega, F, *)$ be a FMS. Hausdorff FM H_F on $\acute{K}(\Omega) \times \acute{K}(\Omega) \times (0, \infty)$ to $[0,1]$ is defined as:
$H_F(A, B, \rho) = min\{\inf_{i \in A}(\sup_{j \in B} F(i, j, \rho)), \inf_{j \in B}(\sup_{i \in A} F(i, j, \rho))\}$, for all $A, B \in \acute{K}(\Omega)$ and $\rho > 0$, where $\acute{K}(X)$ is the collection of all non-empty compact subsets of Ω.
Lemma 5. ([4]) Let $(\Omega, F, *)$ be a complete FMS, such that $(\acute{K}(\Omega), H_F, *)$ is a Hausdorff fuzzy MS on $\acute{K}(\Omega)$. Then for all $S, G \in \acute{K}(X)$, for all $u \in S$ and for $\lambda > 0$, there exist $v_u \in G$ satisfies

$$F(u, G, t) = F(u, v_u, \lambda).$$

Then, $H_F(S, G, \lambda) \leq F(u, v_u, \lambda)$.
Lemma 6. ([14]) Let $(\Omega, F, *)$ be a complete FMS, if there exist $\sigma \in (0,1)$ such that $F(\xi, \eta, \sigma\lambda) \geq F(\xi, \eta, \lambda)$ for all $\xi, \eta, \in \Omega$ and $\lambda \in (0, \infty)$, then $\eta = \xi$.
Lemma 7. ([14]) Let $(\Omega, F, *)$ be a FMS. Then, for each $i \in \Omega, B \in \acute{K}(\Omega)$ and for $\tau > 0$ there exists $j_0 \in B$ such that

$$F(i, j_0, \tau) = F(i, B, \tau).$$

Where $\acute{K}(X)$ is the collection of all non-empty compact subsets of Ω.
Lemma 8. ([14]) Let B be any non-empty subset of a FMS $(\Omega, F, *)$, for $\omega \in \Omega$ and $\tau > 0$ then,

$$F(\omega, B, \tau) = sup\{F(\omega, \mu, \tau) : \mu \in B\}.$$

2. Fuzzy fixed points of fuzzy mappings in metric spaces

2.1. Fuzzy fixed points for $\acute{C}iri\acute{c}$ type contraction

In this section, we apply the Hasudorff metric for fuzzy sets to find the fuzzy fixed points of fuzzy mapping that meet a rational inequality. These results are free from the conditions of approximate quantity for G(x) and linearity for Ω.
Theorem 2.1. Let (Ω, d) be a complete MS and $G: \Omega \to F(\Omega)$ be a FM. Suppose for all $a, b \in \Omega$ there exists $\alpha \in (0,1]$ and $[G(a)]_\alpha$ and $[G(b)]_\alpha)$ be non-empty closed and bounded subsets of **Ω** such that

$$\begin{aligned} &H\,([G(a)]_\alpha, [G(b)]_\alpha) \\ &\leq \rho d\,(a, b) + \beta[d(a, [G(a)]_\alpha) + d(b, [G(b)]_\alpha)] \\ &+\gamma[d(a, [G(b)]_\alpha) + d(b, [G(a)]_\alpha)], \end{aligned} \tag{2.1}$$

for $\rho, \beta, \gamma > 0$ and $\rho + 2\beta + 2\gamma < 1$. Then G has a FP in **Ω** i.e there exists $u \in \Omega$ such that $u \in [G(u)]_\alpha$.
Proof. Since $\rho + 2\beta + 2\gamma < 1$, so $\left(\frac{\rho+\beta+\gamma}{1-\beta-\gamma}\right) < 1$. Consider $\lambda = \left(\frac{\rho+\beta+\gamma}{1-\beta-\gamma}\right)$.
Let $a_0 \in \Omega$ and $[G(a_0)]_\alpha \neq 0$ be a closed and bounded subset of Ω.
Let $a_1 \in [G(a_0)]_\alpha. Since\ G(a_1) \neq \emptyset$ a closed and bounded subset of Ω, using Lemma 2, there exists $a_2 \in [G(a_1)]_\alpha$ such that

$$d\,(a_1, a_2) \leq H\,[G(a_0)]_\alpha, [G(a_1)]_\alpha) \,+\, \lambda.$$

Now $[G(a_2)]_\alpha \neq \emptyset$ are also closed and bounded subset of Ω. By using Lemma 2, there exist $a_3 \in$

$[G(a_2)]_\alpha$ such that

$$d\,(a_2, a_3) \leq H\,([G(a_1)]_\alpha, [G(a_2)]_\alpha) + \lambda^2.$$

Similarly, for $a_n \in [G(a_{n-1})]_\alpha$, we can choose $a_{n+1} \in [G(a_{n-1})]_\alpha$ such that

$$d\,(a_n, a_{n+1}) \leq H\,([G(a_{n-1})]_\alpha, [G(a_n)]_\alpha\,) + \lambda^n.$$

Now,

$$d\,(a_1, a_2) \leq H\,([G(a_0)]_\alpha, [G(a_1)]_\alpha) + \lambda,$$

using (2.1) we get

$$d\,(a_1, a_2) \leq \rho d(a_0, a_1) + \beta[d(a_0, [G(a_0)]_\alpha) + d(a_1, [G(a_1)]_\alpha)] + \gamma[d(a_0, [G(a_1)]_\alpha) + d(a_1, [G(a_0)]_\alpha)] + \lambda,$$

$$d\,(a_1, a_2) \leq \rho d\,(a_0, a_1) + \beta d(a_0, a_1) + \beta d\,(a_1, a_2) + \gamma d\,(a_0, a_2) + \gamma d(a_1, a_1) + \lambda,$$

$$d\,(a_1, a_2) \leq \rho d\,(a_0, a_1) + \beta d(a_0, a_1) + \beta d\,(a_1, a_2) + \gamma d\,(a_0, a_2) + \lambda.$$

Using triangular inequality we get

$$d\,(a_1, a_2) \leq \rho d\,(a_0, a_1) + \beta d(a_0, a_1) + \beta d\,(a_1, a_2) + \gamma d\,(a_0, a_1) + \gamma d\,(a_1, a_2) + \lambda,$$

$$(1 - \beta - \gamma)\, d\,(a_1, a_2) \leq (\rho + \beta + \gamma) d\,(a_0, a_1) + \lambda,$$

$$d\,(a_1, a_2) \leq \left(\frac{\rho + \beta + \gamma}{1 - \beta - \gamma}\right) d\,(a_0, a_1) + \left(\frac{\lambda}{1 - \beta - \gamma}\right).$$

Thus,

$$d\,(a_1, a_2) \leq \lambda d\,(a_0, a_1) + \left(\frac{\lambda}{1-\beta-\gamma}\right). \tag{2.2}$$

Now,

$$d\,(a_2, a_3) \leq H([G(a_1)]_\alpha, [G(a_2)]_\alpha) + \lambda^2,$$

using (2.1) we get

$$d(a_2, a_3) \leq \rho d(a_1, a_2) + \beta[d(a_1, [G(a_1)]_\alpha) + d(a_2, [G(a_2)]_\alpha)] + \gamma[d(a_1, [G(a_2)]_\alpha) + d(a_2, [G(a_1)]_\alpha)] + \lambda^2,$$

$$d(a_2, a_3) \leq \rho d(a_1, a_2) + \beta[d(a_1, a_2\,) + d(a_2, a_3)] + \gamma[d(a_1, a_3) + d(a_2, a_2)] + \lambda^2,$$

again using triangular inequality we get

$$d(a_2, a_3) \leq \rho d(a_1, a_2) + \beta[d(a_1, a_2\,) + d(a_2, a_3)] + \gamma d(a_1, a_2\,) + \gamma d(a_2, a_3\,) + \lambda^2,$$

$$(1 - \beta - \gamma) d(a_2, a_3) \leq (\rho + \beta + \gamma) d(a_1, a_2) + \lambda^2,$$

$$d\,(a_2, a_3) \leq \left(\frac{\rho + \beta + \gamma}{1 - \beta - \gamma}\right) d\,(a_1, a_2) + \left(\frac{\lambda^2}{1 - \beta - \gamma}\right),$$

$$d\,(a_2, a_3) \leq \lambda d\,(a_1, a_2) + \left(\frac{\lambda^2}{1 - \beta - \gamma}\right).$$

Using (2.2) we get

$$d\,(a_2,a_3)\ \le\ \lambda\left[\lambda d\,(a_0,a_1)+\left(\frac{\lambda}{1-\beta-\gamma}\right)\right]+\left(\frac{\lambda^2}{1-\beta-\gamma}\right),$$

$$d\,(a_2,a_3)\ \le \lambda^2 d\,(a_0,a_1)+\left(\frac{\lambda^2}{1-\beta-\gamma}\right)+\left(\frac{\lambda^2}{1-\beta-\gamma}\right),$$

$$d\,(a_2,a_3)\ \le \lambda^2 d\,(a_0,a_1)+\left(\frac{2\lambda^2}{1-\beta-\gamma}\right). \tag{2.3}$$

Now,

$$d\,(a_3,a_4)\le H\,([G(a_2)]_\alpha,[G(a_3)]_\alpha)+\lambda^3,$$

applying (2.1) we get

$$d\,(a_3,a_4)\le \rho d(a_2,a_3)+\ \beta[d(a_2,[G(a_2)]_\alpha)+d(a_3,[G(a_3)]_\alpha)]$$
$$+\gamma[d(a_2,[G(a_3)]_\alpha)+d(a_3,[G(a_2)]_\alpha)]+\lambda^3,$$

$$d\,(a_3,a_4)\le \rho d(a_2,a_3)+\beta[d(a_2,a_3)+d(a_3,a_4)]+\gamma[d(a_2,a_4)+d(a_3,a_3)]+\lambda^3.$$

Since $d(a_3,a_3)=0$, this implies that

$$d\,(a_3,a_4)\le \rho d(a_2,a_3)+\beta[d(a_2,a_3)+d(a_3,a_4)]+\gamma d(a_2,a_4)+\lambda^3.$$

Again using triangular inequality we get

$$d\,(a_3,a_4)\le \rho d(a_2,a_3)+\beta[d(a_2,a_3)+d(a_3,a_4)]+\gamma d(a_2,a_3)+\gamma d(a_3,a_4)+\lambda^3,$$

$$(1-\beta-\gamma)d\,(a_3,a_4)\le (\rho+\beta+\gamma)d(a_2,a_3)+\lambda^3,$$

$$d\,(a_3,a_4)\ \le\ \left(\frac{\rho+\beta+\gamma}{1-\beta-\gamma}\right)d(a_2,a_3)+\left(\frac{\lambda^3}{1-\beta-\gamma}\right).$$

Using (2.3) we get

$$d\,(a_3,a_4)\ \le\lambda\left[\lambda^2 d\,(a_0,a_1)+\left(\frac{2\lambda^2}{1-\beta-\gamma}\right)\right]+\left(\frac{\lambda^3}{1-\beta-\gamma}\right),$$

$$d\,(a_3,a_4)\ \le\lambda^3 d\,(a_0,a_1)+\left(\frac{2\lambda^3}{1-\beta-\gamma}\right)+\left(\frac{\lambda^3}{1-\beta-\gamma}\right),$$

$$d\,(a_3,a_4)\ \le\lambda^3 d\,(a_0,a_1)+\left(\frac{3\lambda^3}{1-\beta-\gamma}\right).$$

So,

$$d\,(a_n,a_{n+1})\ \le\lambda^n d\,(a_0,a_1)+\left(\frac{n\lambda^n}{1-\beta-\gamma}\right). \tag{2.4}$$

Let $m,n\in\mathbb{N}$ with $m>n$

$$d\,(a_n,a_m)\le d(a_n,a_{n+1})+d(a_{n+1},a_{n+2})+\cdots+d(a_{m-1},a_m),$$

applying (2.4) we get

$$d\,(a_n, a_m) \ \le \lambda^n d\,(a_0, a_1) + \left(\frac{n\lambda^n}{1-\beta-\gamma}\right) + \lambda^{n+1} d\,(a_0, a_1) + \left(\frac{(n+1)\lambda^{n+1}}{1-\beta-\gamma}\right) + \cdots$$
$$+ \lambda^{m-1} d\,(a_0, a_1) + \left(\frac{(m-1)\lambda^{m-1}}{1-\beta-\gamma}\right),$$

$$d\,(a_n, a_m) \ \le \lambda^n d\,(a_0, a_1)(1 + \lambda + \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5 + \cdots + \lambda^{m-n-1}) + \sum_{i=n}^{m-1} \frac{i\lambda^i}{1-\beta-\gamma},$$

$$d\,(a_n, a_m) \ \le \lambda^n d\,(a_0, a_1) \left(\frac{1-\lambda^{m-n}}{1-\lambda}\right) + \sum_{i=n}^{m-1} \frac{i\lambda^i}{1-\beta-\gamma}.$$

When $m, n \to \infty$ then right hand side becomes zero. So,

$$d\,(a_n, a_m) = 0.$$

Thus, $\{a_n\}$ is a Cauchy sequence in complete MS. Therefore, there exist $\mu \in \Omega$ such that $a_n \to \mu$. Now,

$$d(\mu, [G(\mu)]_\alpha) \le [d(\mu, a_n) + d(a_n, [G(\mu)]_\alpha)],$$
$$d(\mu, [G(\mu)]_\alpha) \le [d(\mu, a_n) + H([G(a_{n-1})]_\alpha, [G(\mu)]_\alpha)],$$

using (2.1) we get

$$d(\mu, [G(\mu)]_\alpha) \le d(\mu, a_n) + \rho d(a_{n-1}, \mu) + \beta[d(a_{n-1}, [G(a_{n-1})]_\alpha) + d(\mu, [G(\mu)]_\alpha)]$$
$$+ \gamma[d(a_{n-1}, [G(\mu)]_\alpha) + d(\mu, [G(a_{n-1})]_\alpha)],$$
$$d(u, [G(u)]_\alpha) \le d(\mu, a_n) + \rho d(a_{n-1}, \mu) + \beta[d(a_{n-1}, a_n) + d(\mu, [G(\mu)]_\alpha)]$$
$$+ \gamma[d(a_{n-1}, [G(\mu)]_\alpha) + d(\mu, a_n)].$$

As n approaches to ∞ then,

$$d(\mu, [G(\mu)]_\alpha) \le d(\mu, \mu) + \rho\, d(\mu, \mu) + \beta[d(\mu, \mu) + d(\mu, [G(\mu)]_\alpha)] + \gamma[d(\mu, [G(\mu)]_\alpha) + d(\mu, \mu)].$$

Since $d(\mu, \mu) = 0$. So, $(1-\beta-\gamma)d(\mu, [G(\mu)]_\alpha) \le 0$.
As $\rho + 2\beta + 2\gamma \ < \ 1$, this implies that $\rho + \beta + \gamma < 1 - \ \beta - \lambda$, therefore, $1 - \ \beta - \lambda \ne 0$.
So only possibility is

$$d(\mu, [G(\mu)]_\alpha) = 0.$$

This implies that $\mu \in [G(\mu)]_\alpha$. Thus, μ is a FP of G.

Example 2.2. Let $\Omega = [0, 2]$ be a usual MS which is complete and $J\colon \Omega \to F(\Omega)$ be a FM such that $J(w) \in F(\Omega)$, where $w \in \Omega$ and $J(w)\colon \Omega \to [0,1]$ is a function defined as

$$Jw(t) = \begin{cases} \frac{1}{2}, & 0 \le t \le \frac{1}{2} \\ \frac{1}{3}, & \frac{1}{2} < t < 1 \\ 0, & 1 \le t \le 2 \end{cases}.$$

Taking $\alpha = \frac{1}{2}$ we define

$$[Jw]_{\frac{1}{2}} = \left\{t : Jw(t) \geq \frac{1}{2}\right\},$$

$$[Jw]_{\frac{1}{2}} = \left[0, \frac{1}{2}\right] \text{ and } [Jv]_{\frac{1}{2}} = \left[0, \frac{1}{2}\right].$$

Now,

$$H([Jw]_{\frac{1}{2}}, [Jv]_{\frac{1}{2}}) = \max\{\sup_{x\in[Jw]_{\frac{1}{2}}} d\left(x, [Jv]_{\frac{1}{2}}\right), \sup_{y\in[Jv]_{\frac{1}{2}}} d\left(x, [Jw]_{\frac{1}{2}}\right)\},$$

$$H([Jw]_{\frac{1}{2}}, [Jv]_{\frac{1}{2}}) = 0.$$

$$d(w, v) = |w - v|.$$

$$d\left([Jv]_{\frac{1}{2}}, v\right) = \begin{cases} 0 \ \ if\ v \in [Jv]_{\frac{1}{2}} \\ \text{Otherwise non zero} \end{cases}.$$

$$d\left([Jw]_{\frac{1}{2}}, w\right) = \begin{cases} 0 \ \ if\ w \in [Jw]_{\frac{1}{2}} \\ \text{Otherwise non zero} \end{cases}.$$

$$d\left([Jw]_{\frac{1}{2}}, v\right) = \begin{cases} 0 \ \ if\ v \in [Jw]_{\frac{1}{2}} \\ \text{Otherwise non zero} \end{cases}.$$

$$d\left([Jv]_{\frac{1}{2}}, w\right) = \begin{cases} 0 \ \ if\ w \in [Jv]_{\frac{1}{2}} \\ \text{Otherwise non zero} \end{cases}.$$

Take $\rho = \frac{1}{10}, \beta = \frac{1}{4}$ and $\gamma = \frac{1}{4}$, we get

$$H([Jw]_{\frac{1}{2}}, [Jv]_{\frac{1}{2}}) \leq \frac{1}{10} d(x, y) + \frac{1}{4}\left[d\left([Jv]_{\frac{1}{2}}, v\right) + d\left([Jw]_{\frac{1}{2}}, w\right)\right]$$

$$+ \frac{1}{4}\left[d\left([Jw]_{\frac{1}{2}}, v\right) + d\left([Jv]_{\frac{1}{2}}, w\right)\right].$$

$$0 \leq \frac{1}{10}|x - y| + \frac{1}{4}\left[d\left([Jv]_{\frac{1}{2}}, v\right) + d\left([Jw]_{\frac{1}{2}}, w\right)\right] + \frac{1}{4}\left[d\left([Jw]_{\frac{1}{2}}, v\right) + d\left([Jv]_{\frac{1}{2}}, w\right)\right].$$

So, for all $w, v \in \Omega$ the conditions of Theorem 2.1, are satisfied. Hence J has FP in Ω.

Corollary 2.3. Let (Ω, d) be a complete MS and $G: \Omega \to F(\Omega)$ be an FM. Suppose for all $a, b \in \Omega$ there exists $\alpha \in (0,1]$ and $[G(a)]_\alpha$, $[G(b)]_\alpha)) \in CB(\Omega)$ such that

$$H\left([G(a)]_\alpha, [G(b)]_\alpha\right) \leq \rho d\left(a, b\right) + \beta[d(a, [G(a)]_\alpha) + d(b, [G(b)]_\alpha)]$$

for $\rho, \beta > 0$ and $\rho + 2\beta < 1$. Then G has a FP in Ω.

Corollary 2.4. Let (Ω, d) be a complete MS and $G: \Omega \to F(\Omega)$ be an FM. Suppose for all $a, b \in \Omega$ there exists $\alpha \in (0,1]$ and $[G(a)]_\alpha$, $[G(b)]_\alpha) \in CB(\Omega)$ such that

$$H\left([G(a)]_\alpha, [G(b)]_\alpha\right) \leq \rho d\left(a, b\right) + \gamma[d(a, [G(b)]_\alpha) + d(b, [G(a)]_\alpha)]$$

for $\rho, \gamma > 0$ and $\rho + 2\gamma < 1$. Then G has an FP in Ω.

Corollary 2.5. Let (Ω, d) be a complete MS and $G: \Omega \to F(\Omega)$ be an FM. Suppose for all $a, b \in \Omega$ there exists $\alpha \in (0,1]$ and $[G(a)]_\alpha$, $[G(b)]_\alpha)) \in CB(\Omega)$ such that

$$H\left([G(a)]_\alpha, [G(b)]_\alpha\right) \leq \rho d\,(a,b)$$

for $0 < \rho < 1$. Then G has an FP in **Ω**.

Corollary 2.6. Let (Ω, d) be a complete MS and $G: \Omega \to F(\Omega)$ be an FM. Suppose for all $a, b \in \Omega$ there exists $\alpha \in (0,1]$ and $[G(a)]_\alpha$, $[G(b)]_\alpha)) \in CB(\Omega)$ such that

$$H\left([G(a)]_\alpha, [G(b)]_\alpha\right) \leq \beta[d(a,[G(a)]_\alpha) + d(b,[G(b)]_\alpha)]$$

for $0 < 2\beta < 1$. Then G has an FP in **Ω**.

Corollary 2.7. Let (Ω, d) be a complete MS and $G: \Omega \to F(\Omega)$ be an FM. Suppose for all $a, b \in \Omega$ there exist $\alpha \in (0,1]$ and $[G(a)]_\alpha$, $[G(b)]_\alpha) \in CB(\Omega)$ such that

$$H\left([G(a)]_\alpha, [G(b)]_\alpha\right) \leq \gamma[d(a,[G(b)]_\alpha) + d(b,[G(a)]_\alpha)]$$

for $0 < 2\gamma < 1$. Then G has an FP in **Ω**.

2.2. Application

As an application of the fuzzy fixed point result of the previous section we obtain fixed points of multivalued mappings (see, [30]).

Theorem. Let (Ω, d) be a complete MS and $A: \Omega \to CB(\Omega)$ be a multi-valued mapping. Suppose for all $a, b \in \Omega$ $A(a)$ and $A(b)$ be non-empty closed and bounded subsets of **Ω** such that

$$H\left(A(a), A(b)\right) \leq \rho d\,(a,b) + \beta[d(a,A(a)) + d(b,A(b))] + \gamma[d(a,A(b)) + d(b,A(a))]$$

for $\rho, \beta, \gamma > 0$ and $\rho + 2\beta + 2\gamma < 1$. Then G has a FP in **Ω** i.e., there exists $u \in \Omega$ such that $u \in A(u)$.

Proof. Consider an arbitrary mapping $S: \Omega \longrightarrow (0,1]$ and a fuzzy mapping $G: \Omega \to F(\Omega)$ defined by

$$G(x)(t) = \begin{cases} Sx & t \in Ax \\ 0 & t \notin Ax. \end{cases}$$

Then for $x \in \Omega$,

$$[Gx]_\alpha = \{t: G(x)(t) \geq \alpha\} = Ax.$$

Therefore, Theorem 2.1 can be applied to obtain $u \in \Omega$ such that $u \in [Gu^*]_\alpha = Au^*$.

Corollary. ([30]) Let (Ω, d) be a complete MS and $A: \Omega \to CB(\Omega)$ be a multi-valued mapping. Suppose for all $a, b \in \Omega$ $A(a)$ and $A(b)$ be non-empty closed and bounded subsets of **Ω** such that

$$H\left(A(a), A(b)\right) \leq \rho d\,(a,b)$$

for $\rho > 0$ and $\rho < 1$. Then G has an FP in **Ω** i.e., there exists $u \in \Omega$ such that $u \in A(u)$.

Proof. By setting $\beta = 0$ and $\gamma = 0$ in above theorem, we can find the required result.

3. Fuzzy fixed points of fuzzy mappings in fuzzy metric spaces

This section deals with the existence theorems for fixed point of fuzzy mappings satisfying Nadler's type contractions in complete fuzzy metric space. An example and applications are incorporated to demonstrate the obtained results.

Theorem 3.1. Let $(\Omega, F, *)$ be a complete FMS and $S: \Omega: \to F(\Omega)$ be an FM satisfying these conditions:

$$a)\ \lim_{\lambda \to \infty} F(i,j,\lambda) = 1, \tag{3.1}$$

$$\text{b)}\ H_F\left([Si]_{\alpha(i)}, [Sj]_{\alpha(j)}, k\lambda\right) \geq \mu(i,j,\lambda), \tag{3.2}$$

where,

$$\mu(i,j,\lambda) = min\left\{\frac{(F(j,[Sj]_{\alpha(j)},\lambda)[1+F(i,[Si]_{\alpha(i)},\lambda)]}{[1+F(i,j,\lambda\,)]}, F(i,j,\lambda)\right\}$$

for all $i,j \in \Omega, \alpha \in (0,1]$ and $\kappa \in (0,1)$ such that $[Si]_{\alpha(i)}$ and $[Sj]_{\alpha(j)}$ are compact subsets of Ω. Then, S has an FP.

Proof. Let i_0 be any point in Ω. We construct a sequence $\{i_n\}$ of points in Ω as follows:
For $i_1 \in \Omega$, consider that $i_1 \in [Si_0]_{\alpha(i_0)}$, by using Lemma 5, we can choose $i_2 \in [Si_1]_{\alpha(i_1)}$ such that

$$F(i_1,i_2,\lambda) \geq H_F([Si_0]_{\alpha(i_0)},[Si_1]_{\alpha(i_1)},\lambda), \text{for all } \lambda > 0.$$

By induction we can write $i_{n+1} \in [Si_n]_{\alpha(i_n)}$, for all $n \in \mathbb{N}$, satisfying

$$F(i_n,i_{n+1},\lambda) \geq H_F([Si_{n-1}]_{\alpha(i_{n-1})},[Si_n]_{\alpha(i_n)},\lambda), \text{for all } \lambda > 0.$$

Now,

$$F(i_2,i_3,\lambda) \geq H_F([Si_1]_{\alpha(i_1)},[Si_2]_{\alpha(i_2)},\lambda).$$

By using inequality (3.2) we get

$$F(i_2,i_3,\lambda) \geq \mu\left(i_1,i_2,\frac{\lambda}{\kappa}\right), \tag{3.3}$$

where,

$$\mu\left(i_1,i_2,\frac{\lambda}{\kappa}\right) = min\left\{\frac{F\left(i_2,[Si_2]_{\alpha(i_2)},\frac{\lambda}{\kappa}\right)\left[1+F\left(i_1,[Si_1]_{\alpha(i_1)},\frac{\lambda}{\kappa}\right)\right]}{[1+[F\left(i_1,i_2,\frac{\lambda}{\kappa}\right)]}, F\left(i_1,i_2,\frac{\lambda}{\kappa}\right)\right\},$$

$$\mu\left(i_1,i_2,\frac{\lambda}{\kappa}\right) = min\left\{\frac{F\left(i_2,i_3,\frac{\lambda}{\kappa}\right)\left[1+F\left(i_1,i_2,\frac{\lambda}{\kappa}\right)\right]}{[1+[F\left(x_1,x_2,\ \frac{t}{k}\right)]}, F\left(i_1,i_2,\frac{\lambda}{\kappa}\right)\right\}$$
$$= min\left\{F\left(i_2,i_3,\frac{\lambda}{\kappa}\ \right), F\left(i_1,i_2,\frac{\lambda}{\kappa}\right)\right\}.$$

If $F\left(i_1,i_2,\frac{\lambda}{\kappa}\right) \geq F\left(i_2,i_3,\frac{\lambda}{\kappa}\right)$ then, by (3.3), we have

$$F(i_2,i_3,\lambda) \geq F\left(i_2,i_3,\frac{\lambda}{\kappa}\right).$$

So, by the Lemma 6 nothing left to prove. Now, if we have

$$F(i_2,i_3,\frac{\lambda}{\kappa}) \geq F\left(i_1,i_2,\frac{\lambda}{\kappa}\right),$$

then, again by Lemma 5, we have $F(i_2,i_3,\lambda) \geq F(i_1,i_2,\frac{\lambda}{\kappa})$

$$F(i_2,i_3,\lambda) \geq H_F([Si_0]_{\alpha(i_0)},[Si_1]_{\alpha(i_1)},\frac{\lambda}{\kappa})$$
$$F(i_2,i_3,\lambda) \geq \mu\left(i_0,i_1,\frac{\lambda}{\kappa^2}\right), \tag{3.4}$$

where,

$$\mu\left(i_0, i_1, \frac{\lambda}{\kappa^2}\right) = min\left\{\frac{F\left(i_1, [Si_1]_{\alpha(i_1)}, \frac{\lambda}{\kappa^2}\right)\left[1 + F\left(i_0, [Si_0]_{\alpha(i_0)}, \frac{\lambda}{\kappa^2}\right)\right]}{[1 + [F\left(i_0, i, \frac{t}{k^2}\right)]}, F\left(i_0, i_1, \frac{\lambda}{\kappa^2}\right)\right\},$$

$$\mu\left(i_0, i_1, \frac{\lambda}{\kappa^2}\right) = min\left\{\frac{F\left(i_1, i_2, \frac{\lambda}{\kappa^2}\right)\left[1 + F\left(i_0, i_1, \frac{\lambda}{\kappa^2}\right)\right]}{[1 + [F\left(i_0, i_1, \frac{t}{k^2}\right)]}, F\left(i_0, i_1, \frac{\lambda}{\kappa^2}\right)\right\},$$

$$= min\left\{F\left(i_1, i_2, \frac{\lambda}{\kappa^2}\right), F\left(i_0, i_1, \frac{\lambda}{\kappa^2}\right)\right\}.$$

If,

$$F\left(i_0, i_1, \frac{\lambda}{\kappa^2}\right) \geq F(i_1, i_2, \frac{\lambda}{\kappa^2}),$$

then, again by Lemma 6, nothing left to prove. If,

$$F\left(i_1, i_2, \frac{\lambda}{\kappa^2}\right) \geq F\left(i_0, i_1, \frac{\lambda}{\kappa^2}\right),$$

then, by (3.4) we have

$$F(i_2, i_3, t) \geq F\left(i_0, i_1, \frac{\lambda}{\kappa^2}\right).$$

Consequently,

$$F(i_n, i_{n+1}, \lambda) \geq F\left(i_0, i_1, \frac{\lambda}{\kappa^n}\right). \tag{3.5}$$

Now, for $m > n$, that is $m = n + p$ we have

$$F(i_n, i_{n+p}, \lambda) \geq F\left(i_n, i_{n+1}, \frac{\lambda}{p}\right) * \ldots * F\left(x_{n+p-1}, x_{n+p}, \frac{\lambda}{p}\right), (p - times)$$

by using (3.5), we get

$$F(i_n, i_{n+p}, \lambda) \geq F(i_0, i_1, \frac{\lambda}{p\kappa^n}) *\cdots* F(i_0, i_1, \frac{\lambda}{p\kappa^{n+p-1}}).$$

Now, taking $\lim\limits_{n\to\infty}$ and using (3.1) we have,

$$\lim_{n\to\infty} F\left(i_n, i_{n+p}, \lambda\right) = 1.$$

Hence, $\{i_n\}$ is a Cauchy sequence in Ω. So, by completeness there exists $z \in \Omega$ such that $i_n \to z$. Now, we claim that z is an FP of S.
Consider,

$$F\left(z, [Sz]_{\alpha(z)}, \lambda\right) \geq F(z, i_{n+1}, (1-k)\lambda) * F\left(i_{n+1}, [Sz]_{\alpha(z)}, \kappa\lambda\right),$$

$$F(z, [Sz]_{\alpha(z)}, \lambda) \geq F(z, i_{n+1}, (1-k)\lambda) * H_F(\left[Si_n\right]_{\alpha(i_n)}, [Sz]_{\alpha(z)}, \kappa\lambda\right),$$

$$F\left(z, [Sz]_{\alpha(z)}, \lambda\right) \geq F(z, i_{n+1}, (1-k)\lambda) * \mu(i_n, z, \lambda), \tag{3.6}$$

where

$$\mu(i_n, z, \lambda) = min\left\{\frac{(F(z,\ [Sz]_{\alpha(z),}\ \lambda)[1 + F(i_n, [Si_n]_{\alpha(i_n)}, \lambda]}{[1 + F(i_n, z,\ \lambda)]}, F(i_n, z, \lambda)\right\},$$

$$\mu(i_n, z, \lambda) = min\left\{\frac{(F(z,\ [Sz]_{\alpha(z),}\ \lambda)[1 + F(i_n, i_{n+1}, \lambda)\ ]}{[1 + F(i_n, z,\ \lambda)]}, F(i_n, z, \lambda)\right\}.$$

Taking $\lim_{n\to\infty}$ in above inequality, we get

$$\mu(z, z, \lambda\ ) = min\{F(z, [Sz]_{\alpha(z)}, \lambda), 1\}.$$

If,

$$F(z, [Sz]_{\alpha(z)}, \lambda) \geq 1,$$

then, we get z is the fuzzy fixed point for *S*. If

$$F(z, [Sz]_{\alpha(z)}, \lambda) < 1,$$

then, by using (3.6) we have,

$$F(z, [Sz]_{\alpha(z)}, \lambda) \geq F(z, i_{n+1}, (1-k)\lambda) * F(z, [Sz]_{\alpha(z)}, \lambda).$$

Now, taking $\lim_{n\to\infty}$, we get $z \in [Sz]_{\alpha(z)}$. Hence z is a fuzzy FP of S.

Example. Let (Ω, d) be a bounded MS with $d\ (i, j) < \lambda$ and $\Omega = [1, 3]$ (for all $i, j \in \Omega$, where λ is fixed constant in $(0, \infty)$) and $G: \mathbb{R}^+ \to (\lambda, \infty)$ be an increasing continuous function defined as $G(\lambda) = \lambda + 2$.

Define a function $F: \Omega^{\,2} \times (0, \infty) \to [0, 1]$ as

$$F\ (i, j, \lambda) = 1 - \frac{d(i,j)}{G(\lambda)} \text{ for all } i, j \in \Omega \text{ and } \lambda > 0.$$

Then $(\Omega, F, *)$ is a complete fuzzy metric space, where $*$ is a Lukasiewicz t-norm.

Define a fuzzy map $S: \Omega \to F(\Omega)$ as

$$S(i)(t) = \begin{cases} \frac{1}{2}, & 1 \leq t \leq \frac{3}{2} \\ \frac{1}{3}, & \frac{3}{2} < t < 2 \\ 0, & 2 \leq t \leq 3 \end{cases}.$$

Now for $\alpha = \frac{1}{2}$,

$$[Si]_{\frac{1}{2}} = \left\{t: Si(t) \geq \frac{1}{2}\right\} = \left[1, \frac{3}{2}\right]$$

$$[Sj]_{\frac{1}{2}} = \left\{t: Sj(t) \geq \frac{1}{2}\right\} = \left[1, \frac{3}{2}\right].$$

It is to be noted that

$$\lim_{\lambda\to\infty} F(i, j, \lambda) = 1 - \frac{d(i,j)}{G(\lambda)} = 1$$

and

$$H_F\left([Si]_{\frac{1}{2}},[Sj]_{\frac{1}{2}},k\lambda\right)=min\{\inf_{i\in[Si]_{\frac{1}{2}}}(\sup_{j\in[Sj]_{\frac{1}{2}}}F(i,j,k\lambda)),\inf_{j\in[Sj]_{\frac{1}{2}}}(\sup_{i\in[Si]_{\frac{1}{2}}}F(i,j,k\lambda))\}=0$$

for all $[Si]_{\frac{1}{2}},[Sj]_{\frac{1}{2}}\in \acute{K}(\Omega)$, $k=\frac{1}{2}$ and $\lambda>0$.
We also find

$$\mu(i,j,\lambda)=min\left\{\frac{(F(j,[Sj]_{\frac{1}{2}},\lambda)[1+F(i,[Si]_{\frac{1}{2}},\lambda)]}{[1+F(i,j,\lambda)]},F(i,j,\lambda)\right\}=0.$$

Thus, all the conditions of Theorem 3.1 are satisfied. So S has a fuzzy fixed point in fuzzy metric space.
Corollary 3.2. Let $(\Omega,F,*)$ be a FMS and $S:\Omega:\to F(\Omega)$ be an FM satisfying these conditions:

$$(a)\ \lim_{\lambda\to\infty}F(i,j,\lambda)=1,$$

$$\text{(b)}\ H_F([Si]_{\alpha(i)},[Sj]_{\alpha(j)},k\lambda)\geq F(i,j,\lambda)$$

for all $i,j\in\Omega,\alpha\in(0,1]$ and $\kappa\in(0,1)$ such that $[Si]_{\alpha(i)}$ and $[Sj]_{\alpha(j)}$ are compact subsets of Ω. Then, S has an FP.
Definition 3.3. Let us define,

$$\Gamma=\{\omega:[0,1]\to[0,1]\}$$

is a collection of all continuous function such that $\omega(1)=1,\omega(0)=0$, $\omega(\upsilon)>\upsilon$ for all $0<\upsilon<1$.
Theorem 3.4. Let $(\Omega,F,*)$ be a complete FMS and $S:\Omega\to F(\Omega)$ be an FM satisfying these conditions:

$$\text{a)}\ \lim_{\lambda\to\infty}F(p,q,\lambda)=1,$$

$$\text{b)}\ H_F\big([Sp]_{\alpha(p)},[Sq]_{\alpha(q)},\kappa\lambda\big)\geq\omega\{\mu(p,q,\lambda)\},$$

where,

$$\mu(p,q,\lambda)=min\left\{\frac{(F(q,\ [Sq]_{\alpha(q)},\lambda)[1+F(p,[Sp]_{\alpha(p)},\lambda)]}{[1+F(p,q,\lambda)]},F(p,q,\lambda)\right\}$$

for all $p,q\in\Omega,\alpha\in(0,1]$, $\kappa\in(0,1)$ and $\omega\in\Gamma$ such that $[Sp]_{\alpha(p)}$ and $[Sq]_{\alpha(q)}$ are compact subsets of Ω. Then, S has an FP.
Proof. Using Definition 3.3, we get $\omega(\upsilon)>\upsilon$ for all $0<\upsilon<1$.
Thus,

$$H_F\big([Sp]_{\alpha(p)},[Sq]_{\alpha(q)},\kappa\lambda\big)\geq\omega\{\mu(p,q,\lambda)\}\geq\mu(p,q,\lambda).$$

Now, using Theorem 3.1, we get the desired result.
Corollary 3.5. Let $(\Omega,F,*)$ be a complete FMS and $S:\Omega\to F(\Omega)$ be an FM satisfying these conditions:

$$\text{a)}\ \lim_{\lambda\to\infty}F(p,q,\lambda)=1,$$

$$\text{b)}\ H_F\big([Sp]_{\alpha(p)},[Sq]_{\alpha(q)},\kappa\lambda\big)\geq\omega F(p,q,\lambda)$$

for all $p,q\in\Omega,\alpha\in(0,1]$, $\kappa\in(0,1)$ and $\omega\in\Gamma$ such that $[Sp]_{\alpha(p)}$ and $[Sq]_{\alpha(q)}$ are compact subsets of Ω. Then, S has an FP.

Applications:
Let us define a function $\vartheta: [0, \infty) \rightarrow [0, \infty)$ as

$$\vartheta(\lambda) = \int_0^\lambda \rho(\lambda) d\lambda \ \forall \lambda > 0,$$

be a non-decreasing and continuous function. Moreover for each $\delta > 0, \rho(\delta) > 0$. Also $\rho(\lambda) = 0$ if and only if $\lambda = 0$.

Theorem 3.6. Let $(\Omega, F, *)$ be a complete FMS and $S: \Omega \rightarrow F(\Omega)$ be an FM satisfying these conditions:

a) $\lim_{\lambda \to \infty} F(p, q, \lambda) = 1,$

b) $\int_0^{H_F([Sp]_{\alpha(p)}, [Sq]_{\alpha(q)}, \kappa\lambda)} \rho(\lambda) d\lambda \geq \int_0^{\mu(p,q,\lambda)} \rho(\lambda) d\lambda,$

where,

$$\mu(p, q, \lambda) = min\left\{\frac{(F(q, [Sq]_{\alpha(q)}, \lambda)[1 + F(p, [Sp]_{\alpha(p)}, \lambda)]}{[1 + F(p, q, \lambda)]}, F(p, q, \lambda)\right\}$$

for all $p, q \in \Omega, \rho(\lambda) \in [0, \infty), \alpha \in (0, 1]$ and $\kappa \in (0, 1)$ such that $[Sp]_{\alpha(p)}$ and $[Sq]_{\alpha(q)}$ are compact subsets of Ω. Then S has an FP.

Proof. Let us take $\rho(\lambda) = 1$ and using Theorem 3.1, we get the desired result.

Theorem 3.7. Let $(\Omega, F, *)$ be a complete FMS and $S: \Omega \rightarrow F(\Omega)$ be an FM satisfying these conditions:

$a)$ $\lim_{\lambda \to \infty} F(p, q, \lambda) = 1,$

b) $\int_0^{H_F([Sp]_{\alpha(p)}, [Sq]_{\alpha(q)}, \kappa\lambda)} \rho(\lambda) d\lambda \geq \omega\left\{\int_0^{\mu(p,q,\lambda)} \rho(\lambda) d\lambda\right\},$

where,

$$\mu(p, q, \lambda) = min\left\{\frac{(F(q, [Sq]_{\alpha(q)}, \lambda)[1 + F(p, [Sp]_{\alpha(p)}, \lambda)]}{[1 + F(p, q, \lambda)]}, F(p, q, \lambda)\right\}$$

for all $p, q \in \Omega, \rho(\lambda) \in [0, \infty), \omega \in \Gamma, \alpha \in (0, 1]$ and $k \in (0, 1)$ such that $[Sx]_{\alpha(x)}$ and $[Sy]_{\alpha(y)}$ are compact subsets of Ω. Then S has an FP.

Proof. Using Definition 3.3, we get $\omega(\upsilon) > \upsilon$ for all $0 < \upsilon < 1$.
Taking $\rho(\lambda) = 1$ and using Theorem 3.4, we get the desired result.

4. Conclusions

In order to demonstrate the existence and uniqueness of solutions to distinct mathematical models, fixed point theorems are crucial tools. Results that identify fixed points of self and nonself nonlinear operators in a metric space are widely published in the last 40 years. Among various developments of fuzzy sets theory, a progressive development has been made to find the fuzzy analogues of fixed point results of the classical fixed point theorems. In this research, we employ two generalized contractive conditions, i.e., Ćirić type contraction and Nadler's type contraction incorporating rational expressions in the setting of metric and fuzzy metric spaces respectively to study fuzzy fixed point theorems for fuzzy set valued mappings. Completion is compulsory for both spaces in order to ensure the existence of fuzzy fixed points. Examples and applications that emphasis and support our obtained results are integrated. From the pertinent literature, there are additional previous conclusions that are provided as corollaries. Since Fuzzy mappings are generalized form of multi-valued mappings, so in this way many fixed point results exist in the relevant literature have been generalized by our obtained results.

5. Possible future developments

We conclude this paper by indicating, in the form of open questions, some directions for further investigation and work.

(1) Can the condition of $\rho + 2\beta + 2\gamma < 1$ in Theorem 2.1 be relaxed?

(2) If the answer to 1 is yes, then what hypotheses is needed to guarantee the existence of fixed point of G?

(3) Whether the condition $H_F([Si]_{\alpha(i)}, [Sj]_{\alpha(j)}, k\lambda) \leq \mu(i, j, \lambda)$, in Theorem 3.1 can be applied to ensure the existence of fixed point.

(4) Can the concept offered in this article be extended to more than one mappings?

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Conflict of interest

The authors declare the there is no conflict of interest regarding the publications.

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