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Research article

Existence and uniqueness for a coupled system of fractional equations involving Riemann-Liouville and Caputo derivatives with coupled Riemann-Stieltjes integro-multipoint boundary conditions

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Abstract: Recently, coupled systems of fractional differential equations play a central role in the modelling of many systems in e.g., financial economics, ecology, and many more. This study investigates the existence and uniqueness of solutions for a nonlinear coupled system of fractional differential equations involving Riemann-Liouville and Caputo derivatives with coupled Riemann-Stieltjes integro-multipoint boundary conditions. The main tools are known fixed point theorems, namely, Leray-Schauder alternative, Banach fixed point theorem, and the Krasnoselskii fixed point theorem. The new system, which can be considered as a generalized version of many previous fascinating systems, is where the article's novelty lies. Examples are presented to illustrate the results. In this way, we generalize several earlier results.

Keywords: coupled system of fractional differential equations; integro-multipoint boundary conditions; fixed point theorems
Mathematics Subject Classification: 26A33, 34A08, 34A12, 47H10

1. Introduction

Numerous phenomena are mathematically described using fractional order differential and integral operators. The fundamental advantage of these operators is that they are nonlocal. This makes it possible to describe the components and procedures used throughout the phenomenon's history. As a result, fractional-order models are more precise and useful than their equivalents in integer order. Because fractional calculus (FC) techniques are frequently used in a number of real-world applications, numerous scholars established this significant branch of mathematical analysis (see e.g., [1–9]).

Recent studies on fractional differential equations (FDEs) with various boundary conditions (BCs) have been carried out by several researchers. Nonlocal nonlinear fractional-order boundary value issues, especially, have received a lot of attention. In the work of Bitsadze and Samarski (see [10]), when nonlocal conditions were first presented, they were used to describe physical occurrences that occurred within a specific domain's bounds. Due to a blood vessel's shifting form throughout the vessel, it is difficult to defend the assumption of a circular cross section in computational fluid dynamics analyses of blood flow problems. To overcome this issue, integral BCs have been introduced. Additionally, ill-posed parabolic backward problems are resolved using integral BCs. The mathematical models of bacterial self-regularization also depend heavily on integral BCs. In mathematical models of bacterial self-regularization, integral BCs are also essential.

In the mathematical modelling of a number of practical issues, coupled systems of FDEs represent the main tools. Examples include fractional dynamics, chaos, financial economics, ecology, and bio-engineering, etc (see e.g., [1–9]), also see the recent interesting results in e.g., [11–14]. The study of fractional differential systems has been a well-liked and significant field of science, supplemented by many types of BCs. The advancement of this topic has been aided by several researchers who have published countless outputs. Modern functional analysis techniques greatly aid in obtaining existence (Exs.) and uniqueness (Unq.) findings for these issues. We recommend the reader study a number of papers for some recent research on fractional or sequential FDEs with nonlocal integral BCs (e.g., [15, 16]).

In [17], by using fixed point theorems (FPTs), the authors looked into the possibility of solving an initial value problem (IVP) involving a sequential FDE. In [18], using the method of upper and lower solutions and the monotone iterative technique, the Exs. and Unq. results for a periodic boundary value issue of nonlinear sequential FDEs were discovered. Since they contain multipoint and integral BCs as special examples, Riemann-Stieltjes BCs are highly general (see [19]). The astronomer T. J. Stieltjes generalization of the Riemann integral, the Riemann-Stieltjes integral, has potential uses in probability theory (see e.g., [20, 21]).

Banach and Schaefer FPTs have been employed in [22] (see also e.g., [23–25]) to study the Exs. and Unq. of solutions for a coupled system of nonlinear fractional integro-differential equations (Int-DifEqn.) involving Riemann-Liouville integrals with several continuous functions.

$$\begin{cases} D^{\alpha}U(\rho) = f_1(\rho, U(\rho), V(\rho)) + \sum_{i=1}^{m}\int_0^{\rho}\frac{(\rho-\lambda)^{\alpha_i-1}}{\Gamma(\alpha_i)}\varphi_i(\lambda)g_i(\lambda, U(\lambda), V(\lambda))d\lambda, \\ D^{\beta}V(\rho) = f_2(\rho, U(\rho), V(\rho)) + \sum_{i=1}^{m}\int_0^{\rho}\frac{(\rho-\lambda)^{\beta_i}}{\Gamma(\beta_i)}\phi_i(\lambda)h_i(\lambda, U(\lambda), V(\lambda))d\lambda, \\ U(0) = a > 0, V(0) = b > 0, \rho \in [0, 1], \end{cases}$$

where D^{α}, D^{β} denote the Caputo fractional derivatives (CFD), $0 < \alpha, \beta < 1; \alpha_i; \beta_i$ are nonnegative real numbers, φ_i and ϕ_i are some continuous functions. It should be remarked that the authors in [26] considered the short-memory which can be considered in some work.

The authors in [27] investigated a boundary value problem of coupled systems of nonlinear Riemann-Liouvillle fractional Int-DifEqn. supplemented with nonlocal Riemann-Liouvillle fractional Int-Dif. BCs. The results obtained by using some standard FPTs (with $\rho \in [0, T]$, $1 < \alpha, \beta \leq 2$)

$$\begin{cases} D^{\alpha} U(\rho) = A(\rho, U(\rho), V(\rho), (\phi_1 U)(\rho), (\psi_1 V)(\rho)), \\ D^{\beta} V(\rho) = B(\rho, U(\rho), V(\rho), (\phi_2 U)(\rho), (\psi_2 V)(\rho)), \end{cases}$$

with the following coupled Riemann-Liouville Int-Dif. BCs (with $0 < \eta < T$, $0 < \sigma < T$)

$$\begin{cases} D^{\alpha-2} u(0^+) = 0, \;\; D^{\alpha-1} u(0^+) = \nu I^{\alpha-1} \upsilon(\eta), \\ D^{\beta-2} u(0^+) = 0, \;\; D^{\beta-1} \upsilon(0^+) = \mu I^{\beta-1} u(\sigma), \end{cases}$$

where the Riemann-Liouville derivatives is denoted by $D^{(.)}$, and $I^{(.)}$ denotes the Riemann-Liouville integral of fractional order (.), and $f, g : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ are given continuous functions,ν, μ are real constants, and $\phi_i, \psi_i, i = 1, 2$ are given operators.

For a nonlinear coupled system of Liouville-Caputo type fractional Int-DifEqn. with non-local discrete and integral BCs, the Exs. and Unq. of solutions have been studied in [28]. The Exs. results are obtained by usng Leray-Schauder FPT, while the Unq. results by the concept of Banach FPT.

$$\begin{cases} {}^C D^q x(r_1) = \mathbb{A}(r_1, x(r_1), y(r_1))), \\ {}^C D^p y(r_1) = \mathbb{B}(r_1, x(r_1), y(r_1))), \\ x'(0) = \alpha \int_0^{\xi} x'(r_2) dr_2, \;\; x(1) = \beta \int_0^1 g(x'(r_2)) dr_2, \\ y'(0) = \alpha_1 \int_0^{\theta} y'(r_2) dr_2, \;\; y(1) = \beta_1 \int_0^1 g(y'(r_2)) dr_2, \\ r_1 \in [0, 1], 1 < q, p \leq 2, 0 \leq \xi, \theta \leq 1, \end{cases}$$

where ${}^C D^q, {}^C D^p$ denote the Caputo fractional derivatives (CFDs) of order $q, p, \mathbb{A}, \mathbb{B} : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and $\alpha, \beta, \alpha_1, \beta_1$ are real constants.

In [29], the authors discussed the FDEs with integral and ordinary-fractional flux BCs

$$\begin{cases} {}^C D^{p_1} x(\kappa) = F(s, x(\kappa), y(\kappa))), \\ {}^C D^{p_2} y(\kappa) = G(s, x(\kappa), y(\kappa))), \\ x(0) + x(1) = a \int_0^1 x(r_2) dr_2, \;\; x'(0) = b {}^C D^{q_1} x(1), \\ y(0) + y(1) = z \int_0^1 y(r_2) dr_2, \;\; y'(0) = b_1^C D^{e_1} y(1), \\ s \in [0, 1], 1 < p_1, p_2 \leq 2, 0 \leq q_, e_1 \leq 1, \end{cases}$$

where ${}^C D^{p_1}, {}^C D^{p_2}, {}^C D^{q_1}, {}^C D^{e_1}$ denote the CFDs of order $p_1, p_2, F, G : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and a, z, b, b_1 are real constants. The Exs. results have been analyzed in [30] for coupled system of FDEs (with $\mu \in (0, 1), 1 < \alpha, \beta < 2, 0 < \eta < 1$)

$$\begin{cases} D^{\alpha} U(\mu) = A(\mu, V(\mu), D^p V(\mu))), \\ D^{\beta} V(\mu) = B(\mu, U(\mu), D^q U(\mu))), \\ U(0) = 0, \;\; U(1) = \gamma U(\eta), \\ V(0) = 0, \;\; V(1) = \gamma V(\eta), \end{cases}$$

where $D^{\cdot}$ denotes the Riemann-Liouville FDs of order $(\cdot)$, $A, B : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$, are given continuous functions, and γ is a real constant.

Exs. of solutions for nonlinear coupled Caputo fractional Int-DifEqn has been investigated in [16],

$$\begin{cases} {}^C D^{\alpha} u(\rho) = f(\rho, u(\rho), v(\rho), {}^C D^{\zeta_1} \upsilon(\rho), I^{\xi} \upsilon(\rho)), \ \rho \in [0, T] := U, \\ {}^C D^{\beta} \upsilon(\rho) = g(\rho, u(\rho), u(\rho), {}^C D^{\iota_1} \upsilon(\rho), I^{\varsigma} \upsilon(\rho)), \ \rho \in [0, T] := U, \end{cases}$$

with nonlocal integral and multi-point BCs

$$\begin{cases} U(0) = \psi_1(V), \ U'(0) = \varepsilon_1 \int_0^{\nu_1} V'(\theta) d\theta, \ U''(0) = 0, \cdots, U^{n-2}(0) = 0, \\ U(T) = \lambda_1 \int_0^{\delta_1} V(\theta) d\theta + \mu_1 \sum_{j=1}^{k-2} w_j V(\theta_j), \\ V(0) = \psi_2(V), \ V'(0) = \varepsilon_2 \int_0^{\nu_2} U'(\theta) d\theta, \ V''(0) = 0, \cdots, V^{n-2}(0) = 0, \\ V(T) = \lambda_2 \int_0^{\delta_2} U(\theta) d\theta + \mu_2 \sum_{j=1}^{k-2} w_j U(\varphi_j), \end{cases}$$

where ${}^C D^{\alpha}, {}^C D^{\beta}, {}^C D^{\zeta_1}, {}^C D^{\iota_1}$ are the Caputo FDs of order $n-1 < \alpha, \beta < n$, $0 < \zeta_1, \iota_1 < 1$, I^{ξ}, I^{ς} are the Riemann-Liouville fractional integrals (FI) of order $\xi, \varsigma > 0$.

In this paper, we investigate the Exs. and Unq. of solutions for the following nonlinear coupled system of FDEs involving Riemann-Liouville and Caputo derivatives with coupled Riemann-Stieltjes integro-multipoint boundary conditions

$$\begin{cases} {}^{RL} D^{\mathfrak{f}_1} \left[({}^c D^{\mathfrak{h}_1} + \alpha_1) x(t) + \beta_1 I^{\mathfrak{s}_1} H(t, x(t), y(t)) \right] = \phi(t, x(t), y(t)), \ 1 < \mathfrak{h}_1, \mathfrak{f}_1 \leq 2, \ t \in [\mathfrak{q}, \mathfrak{p}], \\ {}^{RL} D^{\mathfrak{f}_2} \left[({}^c D^{\mathfrak{h}_2} + \alpha_2) y(t) + \beta_2 I^{\mathfrak{s}_2} \mathfrak{U}(t, x(t), y(t)) \right] = \psi(t, x(t), y(t)), \ 1 < \mathfrak{h}_2, \mathfrak{f}_2 \leq 2, \ t \in [\mathfrak{q}, \mathfrak{p}], \end{cases} \tag{1.1}$$

with coupled non-conjugate Riemann-Stieltjes integro-multipoint BCs:

$$\begin{cases} x(\mathfrak{q}) = \sum_{i=1}^{\tau-2} \eta_i y(\xi_i) + \int_{\mathfrak{q}}^{\mathfrak{p}} y(\kappa) d\Lambda(\kappa), \ x'(\mathfrak{q}) = 0, \ x(\mathfrak{p}) = 0, \ x'(\mathfrak{p}) = 0, \\ y(\mathfrak{q}) = \sum_{i=1}^{\tau-2} \eta_i x(\xi_i) + \int_{\mathfrak{q}}^{\mathfrak{p}} x(\kappa) d\Lambda(\kappa), \ y'(\mathfrak{q}) = 0, \ y(\mathfrak{p}) = 0, \ y'(\mathfrak{p}) = 0, \end{cases} \tag{1.2}$$

where ${}^c D^{\mathfrak{a}}$ denotes the Caputo fractional differential operator of order $\mathfrak{a}$ with $(\mathfrak{a} = \mathfrak{h}_1, \mathfrak{h}_2)$, ${}^{Rl} D^{\mathfrak{b}}$ denotes the Riemann-Liouville fractional differential operator of order $\mathfrak{b}$ with $(\mathfrak{b} = \mathfrak{f}_1, \mathfrak{f}_2)$, with $\mathfrak{h}_1 + \mathfrak{f}_1 > 3$, $\mathfrak{h}_2 + \mathfrak{f}_2 > 3$, $I^{\mathfrak{s}_1}$, $I^{\mathfrak{s}_2}$ are Riemann-Liouville FI of order $\mathfrak{s}_1$, $\mathfrak{s}_2 > 1$, α_i, $\beta_i \in \mathbb{R}$, $i = 1, 2$, $H, \phi, \mathfrak{U}, \psi : [\mathfrak{q}, \mathfrak{p}] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions, Λ is a function of bounded variation, $\mathfrak{q} < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < \mathfrak{p}, \eta_j \in \mathbb{R}$, $j = 1, 2, \cdots n-2$. It should be remarked that some fundamental assumptions for orders of fractional derivatives are postulated in our study and potential relaxation of this limitations can be considered in some further study. The main contribution of this article can be seen as follows:

(1) A generalization of the results obtained in [16].
(2) A generalization of the results obtained in [29].
(3) A generalization of the results obtained in [30].

Here we emphasize that the present work is motivated by a recent work [31]. Next section recalls some basic definitions of FC and present an auxiliary lemma. In section 3, we discuss the existence of solutions for the given problem while the uniqueness results is presented in section 4, section 5 shows examples that illustrate our results, and section 6 concludes our work.

2. Preliminary material

Now, we recall some basic definitions of fractional calculus.

Definition 2.1. *[8]* For $\beta > 0$, the Riemann-Liouville FI of order β for $\vartheta \in L_1[\mathfrak{q}, \mathfrak{p}]$, existing almost everywhere on $[\mathfrak{q}, \mathfrak{p}]$, (with $-\infty < \mathfrak{q} < \mathfrak{p} < \infty$) is defined by

$$I^\beta \vartheta(t) = \int_{\mathfrak{q}}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \vartheta(s) ds,$$

where Γ denotes the Euler gamma function.

Definition 2.2. *[8]* For, $\beta \in (n-1, n], n \in \mathbb{N}$, and $g \in AC^n[\mathfrak{q}, \mathfrak{p}]$, the Riemann-Liouville and CFDs of order β are respectively defined by

$${}^{RL}D^\beta \vartheta(t) = \frac{d^n}{dt^n} \int_{\mathfrak{q}}^{t} \frac{(t-s)^{n-\beta-1}}{\Gamma(n-\beta)} \vartheta(s) ds \quad \text{and} \quad {}^{c}D^\beta \vartheta(r) = \int_{\mathfrak{q}}^{r} \frac{(r-s)^{n-\beta-1}}{\Gamma(n-\beta)} \vartheta^{(n)}(s) ds.$$

Lemma 2.1. *For $m-1 < \beta \leq m,\ t \in [\mathfrak{q}, \mathfrak{p}]$, the general solution of the FDE ${}^cD^\beta x(b) = 0$, is*

$$x(b) = \mathfrak{r}_0 + \mathfrak{r}_1(b-\mathfrak{q}) + \mathfrak{r}_2(b-\mathfrak{q})^2 + ... + \mathfrak{r}_{m-1}(b-\mathfrak{q})^{m-1},$$

$\mathfrak{r}_i \in \mathbb{R},\ i = 0, 1, ..., m-1$. *Moreover,*

$$(I^\beta\, {}^cD^\beta x)(b) = x(b) + \sum_{i=0}^{m-1} \mathfrak{r}_i (b-\mathfrak{q})^i.$$

Lemma 2.2. *[8] For $\beta > 0$ and $x \in C(\mathfrak{q}, \mathfrak{p}) \cap L(\mathfrak{q}, \mathfrak{p})$, the general solution of $({}^{RL}D^\beta x)(b) = 0$ is*

$$x(b) = \sigma_1(b-\mathfrak{q})^{\beta-1} + \sigma_2(b-\mathfrak{q})^{\beta-2} + \cdots + \sigma_{m-1}(b-\mathfrak{q})^{\beta-m-1} + \sigma_m(b-\mathfrak{q})^{\beta-m},$$

where $\sigma_j \in \mathbb{R},\ j = 1, 2, \cdots, m$, and

$$\begin{aligned}(I^\beta\, {}^{RL}D^\beta x)(b) &= x(b) + \sigma_1(b-\mathfrak{q})^{\beta-1} + \sigma_2(b-\mathfrak{q})^{\beta-2} + \cdots + \sigma_{m-1}(b-\mathfrak{q})^{\beta-m-1} + \sigma_m(b-\mathfrak{q})^{\beta-m} \\ &= x(b) + \sum_{j=1}^{m} \sigma_i (b-\mathfrak{q})^{\beta-j}.\end{aligned}$$

On the other hand, $({}^{RL}D^\beta I^\beta x)(b) = x(b)$.

See also Lemma A.1 in Appendix. A for more details.

3. Existence results

Denote by $\mathfrak{X}^* = \{x(t) | x(t) \in C([\mathfrak{q}, \mathfrak{p}], \mathbb{R})\}$ as the Banach space (BSp.) of all functions (continuous) from $[\mathfrak{q}, \mathfrak{p}]$ into $\mathbb{R}$ equipped with the norm $\|x\| = \sup_{t \in [\mathfrak{q}, \mathfrak{p}]} |x(t)|$. Obviously $(\mathfrak{X}^*, \|.\|)$ is a BSp. and as a result, the product space $(\mathfrak{X}^* \times \mathfrak{X}^*, \|.\|)$ is a BSp. with the norm $\|(r, s)\| = \|r\| + \|s\|$ for $(r, s) \in \mathfrak{X}^* \times \mathfrak{X}^*$. By Lemma A.1, we define an operator $\mathcal{A} : \mathfrak{X}^* \times \mathfrak{X}^* \to \mathfrak{X}^* \times \mathfrak{X}^*$ as

$$\mathcal{A}(x, y)(t) := (\mathcal{A}_1(x, y)(t), \mathcal{A}_2(x, y)(t)), \tag{3.1}$$

where

$$\mathcal{A}_1(x,y)(t) = -\alpha_1 \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa - \beta_1 \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H(\kappa, x(\kappa), y(\kappa))d\kappa$$
$$+ \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{h}_1+\mathfrak{k}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{k}_1)} \phi(\kappa, x(\kappa), y(\kappa))d\kappa + \mathcal{V}_1(t)\Big[\alpha_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa$$
$$+ \beta_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H(\kappa, x(\kappa), y(\kappa))d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{k}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{k}_1)} \phi(\kappa, x(\kappa), y(\kappa))d\kappa\Big]$$
$$+ \mathcal{V}_2(t)\Big[\alpha_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-2}}{\Gamma(\mathfrak{h}_1-1)} x(\kappa)d\kappa + \beta_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-2}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1-1)} H(\kappa, x(\kappa), y(\kappa))d\kappa$$
$$- \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{k}_1-2}}{\Gamma(\mathfrak{h}_1+\mathfrak{k}_1-1)} \phi(\kappa, x(\kappa), y(\kappa))d\kappa\Big] + \mathcal{V}_3(t)\Big[-\alpha_1 \sum_{i=1}^{n-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa$$
$$- \beta_1 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H(\kappa, x(\kappa), y(\kappa))d\kappa$$
$$+ \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1+\mathfrak{k}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{k}_1)} \phi(\kappa, x(\kappa), y(\kappa))d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(-\alpha_1 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(u)du$$
$$- \beta_1 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H(u, x(u), y(u))du + \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1+\mathfrak{k}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{k}_1)} \phi(u, x(u), y(u))du\Big)d\Lambda(\kappa)\Big]$$
$$+ \mathcal{V}_4(t)\Big[\alpha_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(\kappa)d\kappa$$
$$+ \beta_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}(\kappa, x(\kappa), y(\kappa))d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{k}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{k}_2)} \psi(\kappa, x(\kappa), y(\kappa))d\kappa\Big]$$
$$+ \mathcal{V}_5(t)\Big[\alpha_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-2}}{\Gamma(\mathfrak{h}_2-1)} y(\kappa)d\kappa + \beta_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2-1)} \mathfrak{U}(\kappa, x(\kappa), y(\kappa))d\kappa$$
$$- \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{k}_2-2}}{\Gamma(\mathfrak{h}_2+\mathfrak{k}_2-1)} \psi(\kappa, x(\kappa), y(\kappa))d\kappa\Big] + \mathcal{V}_6(t)\Big[-\alpha_2 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(\kappa)d\kappa$$
$$- \beta_2 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}(\kappa, x(\kappa), y(\kappa))d\kappa$$
$$+ \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2+\mathfrak{k}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{k}_2)} \psi(\kappa, x(\kappa), y(\kappa))d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(-\alpha_2 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(u)du \qquad (3.2)$$
$$- \beta_2 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}(u, x(u), y(u))du + \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2+\mathfrak{k}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{k}_2)} \psi(u, x(u), y(u))du\Big)d\Lambda(\kappa)\Big]\Big],$$

$$
\begin{aligned}
\mathcal{A}_2(x,y)(t) &= -\alpha_2 \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(\kappa)d\kappa - \beta_2 \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}(\kappa, x(\kappa), y(\kappa))d\kappa \\
&+ \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \psi(\kappa, x(\kappa), y(\kappa))d\kappa + \mathcal{W}_1(t)\Big[\alpha_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa \\
&+ \beta_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H(\kappa, x(\kappa), y(\kappa))d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \phi(\kappa, x(\kappa), y(\kappa))d\kappa\Big] \\
&+ \mathcal{W}_2(t)\Big[\alpha_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-2}}{\Gamma(\mathfrak{h}_1-1)} x(\kappa)d\kappa + \beta_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-2}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1-1)} H(\kappa, x(\kappa), y(\kappa))d\kappa \\
&- \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-2}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)} \phi(\kappa, x(\kappa), y(\kappa))d\kappa\Big] + \mathcal{W}_3(t)\Big[-\alpha_1 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa \\
&- \beta_1 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H(\kappa, x(\kappa), y(\kappa))d\kappa \\
&+ \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \phi(\kappa, x(\kappa), y(\kappa))d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(-\alpha_1 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(u)du \\
&- \beta_1 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H(u, x(u), y(u))du + \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \phi(u, x(u), y(u))du\Big)d\Lambda(\kappa)\Big] \\
&+ \mathcal{W}_4(t)\Big[\alpha_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(\kappa)d\kappa \\
&+ \beta_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}(\kappa, x(\kappa), y(\kappa))d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \psi(\kappa, x(\kappa), y(\kappa))d\kappa\Big] \\
&+ \mathcal{W}_5(t)\Big[\alpha_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-2}}{\Gamma(\mathfrak{h}_2-1)} y(\kappa)d\kappa + \beta_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2-1)} \mathfrak{U}(\kappa, x(\kappa), y(\kappa))d\kappa \\
&- \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)} \psi(\kappa, x(\kappa), y(\kappa))d\kappa\Big] + \mathcal{W}_6(t)\Big[-\alpha_2 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(\kappa)d\kappa \\
&- \beta_2 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}(\kappa, x(\kappa), y(\kappa))d\kappa \\
&+ \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \psi(\kappa, x(\kappa), y(\kappa))d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(-\alpha_2 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(u)du \qquad (3.3) \\
&- \beta_2 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}(u, x(u), y(u))du + \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \psi(u, x(u), y(u))du\Big)d\Lambda(\kappa)\Big],
\end{aligned}
$$

and $\mathcal{V}_i(t)(i = 1, \cdots, 6)$ and $\mathcal{W}_j(t),\ j = 1, ..., 6$ are given by (A.4) and (A.5) respectively. From now on, we impose that $H, \phi, \mathfrak{U}, \psi : [\mathfrak{q}, \mathfrak{p}]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions satisfying the following condition:
$(\mathcal{H}_1)$ For all $t \in [\mathfrak{q}, \mathfrak{p}],\ \mathsf{x}, \mathsf{y} \in \mathbb{R}$, $\exists$ real constants $\varpi_i,\ \varepsilon_i,\ n_i,\ m_i \geq 0\ (i = 1, 2),\ \varpi_0,\ \varepsilon_0, n_0, m_0 > 0$:

$$|H(t, \mathsf{x}, \mathsf{y})| \leq \varpi_0 + \varpi_1|\mathsf{x}| + \varpi_2|\mathsf{y}|,$$

$$|\phi(t, \mathsf{x}, \mathsf{y})| \leq \varepsilon_0 + \varepsilon_1|\mathsf{x}| + \varepsilon_2|\mathsf{y}|,$$

$$|\mathfrak{U}(t, \mathsf{x}, \mathsf{y})| \leq n_0 + n_1|\mathsf{x}| + n_2|\mathsf{y}|,$$

$$|\psi(t, \mathsf{x}, \mathsf{y})| \leq m_0 + m_1|\mathsf{x}| + m_2|\mathsf{y}|.$$

For simplicity, we use the following notations:

$$\mathcal{F}_0 = |\alpha_1|\Big\{\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1}}{\Gamma(\mathfrak{h}_1+1)} + \widetilde{\mathcal{V}}_1\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1}}{\Gamma(\mathfrak{h}_1+1)} + \widetilde{\mathcal{V}}_2\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} + \widetilde{\mathcal{V}}_3\Big(\sum_{i=1}^{\tau-2}|\eta_i|\frac{(\xi_i-\mathfrak{q})^{\mathfrak{h}_1}}{\Gamma(\mathfrak{h}_1+1)} + \int_{\mathfrak{q}}^{\mathfrak{p}}\frac{(\kappa-\mathfrak{q})^{\mathfrak{h}_1}}{\Gamma(\mathfrak{h}_1+1)}d\Lambda(\kappa)\Big)\Big\},$$

$$\begin{aligned}
\mathcal{F}_1 &= |\beta_1|\Big\{\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{s}_1+\mathfrak{h}_1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1+1)} + \widetilde{\mathcal{V}}_1\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{s}_1+\mathfrak{h}_1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1+1)} + \widetilde{\mathcal{V}}_2\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} + \widetilde{\mathcal{V}}_3\Big(\sum_{i=1}^{\tau-2}|\eta_i|\frac{(\xi_i-\mathfrak{q})^{\mathfrak{s}_1+\mathfrak{h}_1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1+1)} \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}}\frac{(\kappa-\mathfrak{q})^{\mathfrak{s}_1+\mathfrak{h}_1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1+1)}d\Lambda(\kappa)\Big)\Big\}, \\
\mathcal{F}_2 &= \Big\{\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1+1)} + \widetilde{\mathcal{V}}_1\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1+1)} + \widetilde{\mathcal{V}}_2\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} + \widetilde{\mathcal{V}}_3\Big(\sum_{i=1}^{\tau-2}|\eta_i|\frac{(\xi_i-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1+1)} \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}}\frac{(\kappa-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1+1)}d\Lambda(\kappa)\Big)\Big\}, \\
\mathcal{F}_3 &= |\alpha_2|\Big\{\widetilde{\mathcal{V}}_4\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2}}{\Gamma(\mathfrak{h}_2+1)} + \widetilde{\mathcal{V}}_5\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} + \widetilde{\mathcal{V}}_6\Big(\sum_{i=1}^{\tau-2}|\eta_i|\frac{(\xi_i-\mathfrak{q})^{\mathfrak{h}_2}}{\Gamma(\mathfrak{h}_2+1)} + \int_{\mathfrak{q}}^{\mathfrak{p}}\frac{(\kappa-\mathfrak{q})^{\mathfrak{h}_2}}{\Gamma(\mathfrak{h}_2+1)}d\Lambda(\kappa)\Big)\Big\}, \\
\mathcal{F}_4 &= |\beta_2|\Big\{\widetilde{\mathcal{V}}_4\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{s}_2+\mathfrak{h}_2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2+1)} + \widetilde{\mathcal{V}}_5\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} + \widetilde{\mathcal{V}}_6\Big(\sum_{i=1}^{\tau-2}|\eta_i|\frac{(\xi_i-\mathfrak{q})^{\mathfrak{s}_2+\mathfrak{h}_2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2+1)} \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}}\frac{(\kappa-\mathfrak{q})^{\mathfrak{s}_2+\mathfrak{h}_2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2+1)}d\Lambda(\kappa)\Big)\Big\}, \\
\mathcal{F}_5 &= \Big\{\widetilde{\mathcal{V}}_4\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2+1)} + \widetilde{\mathcal{V}}_5\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} + \widetilde{\mathcal{V}}_6\Big(\sum_{i=1}^{\tau-2}|\eta_i|\frac{(\xi_i-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2+1)} \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}}\frac{(\kappa-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2+1)}d\Lambda(\kappa)\Big)\Big\}, \qquad (3.4)
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_0 &= |\alpha_1|\Big\{\widetilde{\mathcal{W}}_1\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1}}{\Gamma(\mathfrak{h}_1+1)} + \widetilde{\mathcal{W}}_2\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} + \widetilde{\mathcal{W}}_3\Big(\sum_{i=1}^{\tau-2}|\eta_i|\frac{(\xi_i-\mathfrak{q})^{\mathfrak{h}_1}}{\Gamma(\mathfrak{h}_1+1)} + \int_{\mathfrak{q}}^{\mathfrak{p}}\frac{(\kappa-\mathfrak{q})^{\mathfrak{h}_1}}{\Gamma(\mathfrak{h}_1+1)}d\Lambda(\kappa)\Big)\Big\}, \\
\mathcal{G}_1 &= |\beta_1|\Big\{\widetilde{\mathcal{W}}_1\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{s}_1+\mathfrak{h}_1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1+1)} + \widetilde{\mathcal{W}}_2\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} + \widetilde{\mathcal{W}}_3\Big(\sum_{i=1}^{\tau-2}|\eta_i|\frac{(\xi_i-\mathfrak{q})^{\mathfrak{s}_1+\mathfrak{h}_1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1+1)}
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\kappa-\mathfrak{q})^{\mathfrak{s}_1+\mathfrak{h}_1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1+1)} d\Lambda(\kappa)\Big)\Big\},\\
\mathcal{G}_2 &= \Big\{\widetilde{\mathcal{W}}_1 \frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1+1)} + \widetilde{\mathcal{W}}_2 \frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} + \widetilde{\mathcal{W}}_3 \Big(\sum_{i=1}^{\tau-2} |\eta_i| \frac{(\xi_i-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1+1)}\\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\kappa-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1+1)} d\Lambda(\kappa)\Big)\Big\},\\
\mathcal{G}_3 &= |\alpha_2| \Big\{\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2}}{\Gamma(\mathfrak{h}_2+1)} + \widetilde{\mathcal{W}}_4 \frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2}}{\Gamma(\mathfrak{h}_2+1)} + \widetilde{\mathcal{W}}_5 \frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} + \widetilde{\mathcal{W}}_6 \Big(\sum_{i=1}^{\tau-2} |\eta_i| \frac{(\xi_i-\mathfrak{q})^{\mathfrak{h}_2}}{\Gamma(\mathfrak{h}_2+1)}\\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\kappa-\mathfrak{q})^{\mathfrak{h}_2}}{\Gamma(\mathfrak{h}_2+1)} d\Lambda(\kappa)\Big)\Big\},\\
\mathcal{G}_4 &= |\beta_2| \Big\{\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{s}_2+\mathfrak{h}_2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2+1)} + \widetilde{\mathcal{W}}_4 \frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{s}_2+\mathfrak{h}_2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2+1)} + \widetilde{\mathcal{W}}_5 \frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} + \widetilde{\mathcal{W}}_6 \Big(\sum_{i=1}^{\tau-2} |\eta_i| \frac{(\xi_i-\mathfrak{q})^{\mathfrak{s}_2+\mathfrak{h}_2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2+1)}\\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\kappa-\mathfrak{q})^{\mathfrak{s}_2+\mathfrak{h}_2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2+1)} d\Lambda(\kappa)\Big)\Big\},\\
\mathcal{G}_5 &= \Big\{\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2+1)} + \widetilde{\mathcal{W}}_4 \frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2+1)} + \widetilde{\mathcal{W}}_5 \frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} + \widetilde{\mathcal{W}}_6 \Big(\sum_{i=1}^{\tau-2} |\eta_i| \frac{(\xi_i-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2+1)}\\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\kappa-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2+1)} d\Lambda(\kappa)\Big)\Big\}, \qquad (3.5)
\end{aligned}
$$

where $\widetilde{\mathcal{V}}_i = \sup_{t\in[\mathfrak{q},\mathfrak{p}]} |\mathcal{V}_i(t)|,\ i = 1, ..., 6$ and $\widetilde{\mathcal{W}}_j = \sup_{t\in[\mathfrak{q},\mathfrak{p}]} |\mathcal{W}_j(t)|,\ j = 1, ..., 6,$

$$\mathbf{O}_0 = (\mathcal{F}_1 + \mathcal{G}_1)\varpi_0 + (\mathcal{F}_2 + \mathcal{G}_2)\varepsilon_0 + (\mathcal{F}_4 + \mathcal{G}_4)n_0 + (\mathcal{F}_5 + \mathcal{G}_5)m_0, \qquad (3.6)$$

$$\mathbf{O}_1 = (\mathcal{F}_0 + \mathcal{G}_0) + (\mathcal{F}_1 + \mathcal{G}_1)\varpi_1 + (\mathcal{F}_2 + \mathcal{G}_2)\varepsilon_1 + (\mathcal{F}_4 + \mathcal{G}_4)|n_1 + (\mathcal{F}_5 + \mathcal{G}_5)m_1, \qquad (3.7)$$

$$\mathbf{O}_2 = (\mathcal{F}_1 + \mathcal{G}_1)\varpi_2 + (\mathcal{F}_2 + \mathcal{G}_2)\varepsilon_2 + (\mathcal{F}_3 + \mathcal{G}_3) + (\mathcal{F}_4 + \mathcal{G}_4)n_2 + (\mathcal{F}_5 + \mathcal{G}_5)m_2, \qquad (3.8)$$

$$\mathbf{O} = \max\{\mathbf{O}_1, \mathbf{O}_2\}. \qquad (3.9)$$

Now we introduce our Exs. results. In the first method we use Leray-Schauder alternative to show the Exs. of solution for the systems (1.1) and (1.2).

Lemma 3.1. *(Leray-Schauder alternative [32]): Let a completely continuous operator* $\mathbf{S} : \mathcal{J} \longrightarrow \mathcal{J}$*. Assume that* $\mathcal{E}(\mathbf{S}) = \{y \in \mathcal{J} : y = \lambda \mathbf{S}(y),\ 0 < \lambda < 1\}$*. Then:*

(1) the set $\mathcal{E}(\mathbf{S})$ *is unbounded, or*
(2) $\mathbf{S}$ *has at lest one FP.*

Theorem 3.1. *If continuous functions* $H, \phi, \mathfrak{U}, \psi : [\mathfrak{q}, \mathfrak{p}] \times \mathbb{R}^2 \to \mathbb{R}$ *satisfying* $(\mathcal{H}_1)$*. Then the systems (1.1) and (1.2) has at least one solution on* $[\mathfrak{q}, \mathfrak{p}]$ *if* $\mathbf{O} < 1$*, where* $\mathbf{O}$ *is given by (3.9).*

Proof. We start by proving that the operator $\mathcal{A} : \mathfrak{X}^* \times \mathfrak{X}^* \to \mathfrak{X}^* \times \mathfrak{X}^*$ is completely continuous. Since the functions $H, \phi, \mathfrak{U}$ and ψ, are continuous, then the operator $\mathcal{A}$ is continuous.

Let $\mathcal{P} \subset \mathfrak{X}^* \times \mathfrak{X}^*$ be bounded. Then $\exists$ constants $\zeta_i > 0 (i = 1, ..., 4)$: $|H(t, x(t), y(t))| \leq \zeta_1,\ |\phi(t, x(t), y(t))| \leq \zeta_2,\ |\mathfrak{U}(t, x(t), y(t))| \leq \zeta_3,\ |\psi(t, x(t), y(t))| \leq \zeta_4,\ \forall (x, y) \in \mathcal{P}$. Then, for any $(x, y) \in \mathcal{P}$, we have

$$
\begin{aligned}
|\mathcal{A}_1(x,y)(t)| \leq\ & |\alpha_1| \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} |x(\kappa)| d\kappa + |\beta_1| \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} \zeta_1 d\kappa \\
&+ \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \zeta_2 d\kappa + |\mathcal{V}_1(t)| \Big[|\alpha_1| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} |x(\kappa)| d\kappa \\
&+ |\beta_1| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} \zeta_1 d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \zeta_2 d\kappa \Big] \\
&+ |\mathcal{V}_2(t)| \Big[|\alpha_1| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-2}}{\Gamma(\mathfrak{h}_1-1)} |x(\kappa)| d\kappa + |\beta_1| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-2}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1-1)} \zeta_1 d\kappa \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-2}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)} \zeta_2 d\kappa \Big] + |\mathcal{V}_3(t)| \Big[|\alpha_1| \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} |x(\kappa)| d\kappa \\
&+ |\beta_1| \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} \zeta_1 d\kappa + \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \zeta_2 d\kappa \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(|\alpha_1| \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} |x(u)| du + |\beta_1| \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} \zeta_1 du \\
&+ \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \zeta_2 du \Big) d\Lambda(\kappa) \Big] + |\mathcal{V}_4(t)| \Big[|\alpha_2| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} |y(\kappa)| d\kappa \\
&+ |\beta_2| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \zeta_3 d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \zeta_4 d\kappa \Big] \\
&+ |\mathcal{V}_5(t)| \Big[|\alpha_2| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-2}}{\Gamma(\mathfrak{h}_2-1)} |y(\kappa)| d\kappa + |\beta_2| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2-1)} \zeta_3 d\kappa \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)} \zeta_4 d\kappa \Big] + |\mathcal{V}_6(t)| \Big[|\alpha_2| \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} |y(\kappa)| d\kappa \\
&+ |\beta_2| \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \zeta_3 d\kappa + \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \zeta_4 d\kappa \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(|\alpha_2| \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} |y(u)| du + |\beta_2| \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \zeta_3 du \\
&+ \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \zeta_4 du \Big) d\Lambda(\kappa) \Big] \\
\leq\ & \mathcal{F}_0 |x(t)| + \mathcal{F}_1 \zeta_1 + \mathcal{F}_2 \zeta_2 + \mathcal{F}_3 |y(t)| + \mathcal{F}_4 \zeta_3 + \mathcal{F}_5 \zeta_4,
\end{aligned}
$$

which implies that,

$$\|\mathcal{A}_1(x,y)\| \;\leq\; \mathcal{F}_0\|x\| + \mathcal{F}_1\zeta_1 + \mathcal{F}_2\zeta_2 + \mathcal{F}_3\|y\| + \mathcal{F}_4\zeta_3 + \mathcal{F}_5\zeta_4.$$

Similarly, we can get

$$\|\mathcal{A}_2(x,y)\| \;\leq\; \mathcal{G}_0\|x\| + \mathcal{G}_1\zeta_1 + \mathcal{G}_2\zeta_2 + \mathcal{G}_3\|y\| + \mathcal{G}_4\zeta_3 + \mathcal{G}_5\zeta_4.$$

Hence, the operator $\mathcal{A}$ is uniformly bounded, since $\|\mathcal{A}(x,y)\| \leq (\mathcal{F}_0 + \mathcal{G}_0)\|x\| + (\mathcal{F}_1 + \mathcal{G}_1)\zeta_1 + (\mathcal{F}_2 + \mathcal{G}_2)\zeta_2 + (\mathcal{F}_3 + \mathcal{G}_3)\|y\| + (\mathcal{F}_4 + \mathcal{G}_4)\zeta_3 + (\mathcal{F}_5 + \mathcal{G}_5)\zeta_4$.

Next, we show that $\mathcal{A}$ is equicontinuous. For $t_1, t_2 \in [\mathfrak{q}, \mathfrak{p}]$ with $t_1 < t_2$, we obtain

$$\begin{aligned}
&\Big|\mathcal{A}_1(x,y)(t_2) - \mathcal{A}_1(x,y)(t_1)\Big| \leq |\alpha_1|\Bigg[\Bigg|\int_{\mathfrak{q}}^{t_1} \frac{[(t_2-\kappa)^{\mathfrak{h}_1-1} - (t_1-\kappa)^{\mathfrak{h}_1-1}]}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa\Bigg| \\
&+ \Bigg|\int_{t_1}^{t_2} \frac{(t_2-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa\Bigg|\Bigg] + |\beta_1|\Bigg[\Bigg|\int_{\mathfrak{q}}^{t_1} \frac{[(t_2-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1} - (t_1-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}]}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H(\kappa, x(\kappa), y(\kappa))d\kappa\Bigg| \\
&+ \Bigg|\int_{t_1}^{t_2} \frac{(t_2-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H(\kappa, x(\kappa), y(\kappa))d\kappa\Bigg|\Bigg] \\
&+ \Bigg|\int_{\mathfrak{q}}^{t_1} \frac{[(t_2-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1} - (t_1-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}]}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \phi(\kappa, x(\kappa), y(\kappa))d\kappa\Bigg| \\
&+ \Bigg|\int_{t_1}^{t_2} \frac{(t_2-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \phi(\kappa, x(\kappa), y(\kappa))d\kappa\Bigg| + \Big|\mathcal{V}_1(t_2) - \mathcal{V}_1(t_1)\Big|\Bigg[|\alpha_1| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} |x(\kappa)|d\kappa \\
&+ |\beta_1| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} |H(\kappa, x(\kappa), y(\kappa))|d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} |\phi(\kappa, x(\kappa), y(\kappa))|d\kappa\Bigg] \\
&+ \Big|\mathcal{V}_2(t_2) - \mathcal{V}_2(t_1)\Big|\Bigg[|\alpha_1| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-2}}{\Gamma(\mathfrak{h}_1-1)} |x(\kappa)|d\kappa + |\beta_1| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-2}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1-1)} |H(\kappa, x(\kappa), y(\kappa))|d\kappa \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-2}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)} |\phi(\kappa, x(\kappa), y(\kappa))|d\kappa\Bigg] + \Big|\mathcal{V}_3(t_2) - \mathcal{V}_3(t_1)\Big|\Bigg[|\alpha_1| \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} |x(\kappa)|d\kappa \\
&+ |\beta_1| \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} |H(\kappa, x(\kappa), y(\kappa))|d\kappa \\
&+ \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} |\phi(\kappa, x(\kappa), y(\kappa))|d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(|\alpha_1| \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} |x(u)|du \\
&+ |\beta_1| \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} |H(u, x(u), y(u))|du + \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} |\phi(u, x(u), y(u))|du\Big)d\Lambda(\kappa)\Bigg]
\end{aligned}$$

$$+ \Big|\mathcal{V}_4(t_2) - \mathcal{V}_4(t_1)\Big|\Bigg[|\alpha_2| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} |y(\kappa)|d\kappa + |\beta_2| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} |\mathfrak{U}(\kappa, x(\kappa), y(\kappa))|d\kappa$$

$$
\begin{aligned}
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)}|\psi(\kappa,x(\kappa),y(\kappa))|d\kappa\Big] + \Big|\mathcal{V}_5(t_2)-\mathcal{V}_5(t_1)\Big|\Big[|\alpha_2|\int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-2}}{\Gamma(\mathfrak{h}_2-1)}|y(\kappa)|d\kappa \\
&+ |\beta_2|\int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2-1)}|\mathfrak{U}(\kappa,x(\kappa),y(\kappa))|d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)}|\psi(\kappa,x(\kappa),y(\kappa))|d\kappa\Big] \\
&+ \Big|\mathcal{V}_6(t_2)-\mathcal{V}_6(t_1)\Big|\Big[|\alpha_2|\sum_{i=1}^{\tau-2}|\eta_i|\int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)}|y(\kappa)|d\kappa \\
&+ |\beta_2|\sum_{i=1}^{\tau-2}|\eta_i|\int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)}|\mathfrak{U}(\kappa,x(\kappa),y(\kappa))|d\kappa \\
&+ \sum_{i=1}^{\tau-2}|\eta_i|\int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)}|\psi(\kappa,x(\kappa),y(\kappa))|d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}}\Big(|\alpha_2|\int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)}|y(u)|du \\
&+ |\beta_2|\int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)}|\mathfrak{U}(u,x(u),y(u))|du + \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)}|\psi(u,x(u),y(u))|du\Big)d\Lambda(\kappa)\Big] \\
&\le \frac{|\alpha_1|\|x\|}{\Gamma(\mathfrak{h}_1+1)}\Big(|(t_2-\mathfrak{q})^{\mathfrak{h}_1}-(t_1-\mathfrak{q})^{\mathfrak{h}_1}| + 2(t_2-t_1)^{\mathfrak{h}_1}\Big) \\
&+ \frac{|\beta_1|\|\zeta_1}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1+1)}\Big(|(t_2-\mathfrak{q})^{\mathfrak{s}_1+\mathfrak{h}_1}-(t_1-\mathfrak{q})^{\mathfrak{s}_1+\mathfrak{h}_1}| + 2(t_2-t_1)^{\mathfrak{s}_1+\mathfrak{h}_1}\Big) \\
&+ \frac{\zeta_2}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1+1)}\Big(|(t_2-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1}-(t_1-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1}| + 2(t_2-t_1)^{\mathfrak{h}_1+\mathfrak{f}_1}\Big) \\
&+ \Big|\mathcal{V}_1(t_2)-\mathcal{V}_1(t_1)\Big|\Big[|\alpha_1|\int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)}|x(\kappa)|d\kappa + |\beta_1|\int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)}\zeta_1 d\kappa \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)}\zeta_2 d\kappa\Big] + \Big|\mathcal{V}_2(t_2)-\mathcal{V}_2(t_1)\Big|\Big[|\alpha_1|\int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-2}}{\Gamma(\mathfrak{h}_1-1)}|x(\kappa)|d\kappa \\
&+ |\beta_1|\int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-2}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1-1)}\zeta_1 d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-2}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)}\zeta_2 d\kappa\Big] \\
&+ \Big|\mathcal{V}_3(t_2)-\mathcal{V}_3(t_1)\Big|\Big[|\alpha_1|\sum_{i=1}^{\tau-2}|\eta_i|\int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)}|x(\kappa)|d\kappa \\
&+ |\beta_1|\sum_{i=1}^{\tau-2}|\eta_i|\int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)}\zeta_1 d\kappa + \sum_{i=1}^{\tau-2}|\eta_i|\int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)}\zeta_2 d\kappa \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}}\Big(|\alpha_1|\int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)}|x(u)|du + |\beta_1|\int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)}\zeta_2 du \\
&+ \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)}|\zeta_3 du\Big)d\Lambda(\kappa)\Big] \\
&+ \Big|\mathcal{V}_4(t_2)-\mathcal{V}_4(t_1)\Big|\Big[|\alpha_2|\int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)}|y(\kappa)|d\kappa + |\beta_2|\int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)}\zeta_3 d\kappa
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)}\zeta_4 d\kappa\Big] + \Big|\mathcal{V}_5(t_2)-\mathcal{V}_5(t_1)\Big|\Big[|\alpha_2|\int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-2}}{\Gamma(\mathfrak{h}_2-1)}|y(\kappa)|d\kappa \\
&+ |\beta_2|\int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2-1)}\zeta_3 d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)}\zeta_4 d\kappa\Big] \\
&+ \Big|\mathcal{V}_6(t_2)-\mathcal{V}_6(t_1)\Big|\Big[|\alpha_2|\sum_{i=1}^{\tau-2}|\eta_i|\int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)}|y(\kappa)|d\kappa \\
&+ |\beta_2|\sum_{i=1}^{\tau-2}|\eta_i|\int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)}\zeta_3 d\kappa + \sum_{i=1}^{\tau-2}|\eta_i|\int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)}\zeta_4 d\kappa \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}}\Big(|\alpha_2|\int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)}|y(u)|du \\
&+ |\beta_2|\int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)}\zeta_3 du + \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)}\zeta_4 du\Big)d\Lambda(\kappa)\Big].
\end{aligned}
$$

Similarly, we can find that $\|\mathcal{A}_2(x,y)-\mathcal{A}_2(x,y)\| \to 0$ independent of x and y as $t_2 \to t_1$. Therefore, the operator $\mathcal{A}(x,y)$ is equicontinuous. As a consequence of our steps together with the Arzela-Ascoli theorem, the operator $\mathcal{A}$ is completely continuous. Next, we prove that the set $\mathcal{E} = \{(x,y) \in \mathfrak{X}^* \times \mathfrak{X}^* | (x,y) = \sigma\mathcal{A}(x,y), 0 \le \sigma \le 1\}$ is bounded. Take $(x,y) \in \mathcal{E}$, then $(x,y) = \sigma\mathcal{A}(x,y)$ and $\forall\ t \in [\mathfrak{q},\mathfrak{p}]$, we have

$$x(t) = \sigma\mathcal{A}_1(x,y)(t),\ y(t) = \sigma\mathcal{A}_2(x,y)(t).$$

In consequence, we have

$$
\begin{aligned}
|x(t)| &\le \mathcal{F}_0|x| + \mathcal{F}_1(\varpi_0+\varpi_1|x|+\varpi_2|y|) + \mathcal{F}_2(\varepsilon_0+\varepsilon_1|x|+\varepsilon_2|y|) \\
&+ \mathcal{F}_3|y| + \mathcal{F}_4(n_0+n_1|x|+n_2|y|) + \mathcal{F}_5(m_0+m_1|x|+m_2|y|),
\end{aligned}
$$

which yields

$$
\begin{aligned}
\|x\| &\le \mathcal{F}_0\|x\| + \mathcal{F}_1(\varpi_0+\varpi_1\|x\|+\varpi_2\|y\|) + \mathcal{F}_2(\varepsilon_0+\varepsilon_1\|x\|+\varepsilon_2\|y\|) \\
&+ \mathcal{F}_3\|y\| + \mathcal{F}_4(n_0+n_1\|x\|+n_2\|y\|) + \mathcal{F}_5(m_0+m_1\|x\|+m_2\|y\|).
\end{aligned} \tag{3.10}
$$

In a similar manner, we can find that

$$
\begin{aligned}
\|y\| &\le \mathcal{G}_0\|x\| + \mathcal{G}_1(\varpi_0+\varpi_1\|x\|+\varpi_2\|y\|) + \mathcal{G}_2(\varepsilon_0+\varepsilon_1\|x\|+\varepsilon_2\|y\|) \\
&+ \mathcal{G}_3\|y\| + \mathcal{G}_4(n_0+n_1\|x\|+n_2\|y\|) + \mathcal{G}_5(m_0+m_1\|x\|+m_2\|y\|).
\end{aligned} \tag{3.11}
$$

From (3.10) and (3.11) together with notations (3.6)–(3.9) lead to

$$
\begin{aligned}
\|x\|+\|y\| &\le [(\mathcal{F}_1+\mathcal{G}_1)\varpi_0 + (\mathcal{F}_2+\mathcal{G}_2)\varepsilon_0 + (\mathcal{F}_4+\mathcal{G}_4)n_0 + (\mathcal{F}_5+\mathcal{G}_5)m_0] \\
&+ [(\mathcal{F}_0+\mathcal{G}_0) + (\mathcal{F}_1+\mathcal{G}_1)|\varpi_1 + (\mathcal{F}_2+\mathcal{G}_2)\varepsilon_1 + (\mathcal{F}_4+\mathcal{G}_4)n_1 \\
&+ (\mathcal{F}_5+\mathcal{G}_5)m_1]\|x\| + [(\mathcal{F}_1+\mathcal{G}_1)\varpi_2 + (\mathcal{F}_2+\mathcal{G}_2)\varepsilon_2 + (\mathcal{F}_3+\mathcal{G}_3) \\
&+ (\mathcal{F}_4+\mathcal{G}_4)n_2 + (\mathcal{F}_5+\mathcal{G}_5)m_2]\|y\|.
\end{aligned}
$$

Which implies,

$$\begin{aligned}\|(x,y)\| &\le \mathbf{O}_0 + \max\{\mathbf{O}_1 + \mathbf{O}_2\}\|(x,y)\| \\ &\le \mathbf{O}_0 + \mathbf{O}\|(x,y)\|,\end{aligned}$$

consequently,

$$\|(x,y)\| \le \frac{\mathbf{O}_0}{1-\mathbf{O}}.$$

This prove that the set $\mathcal{E}$ is bounded. Thus, by Lemma 3.1, the operator $\mathcal{A}$ has at least one FP. Therefore, the systems (1.1) and (1.2) has at least one solution on $[\mathfrak{q}, \mathfrak{p}]$.

Next results are based on Krasnoselskii FPTs. We assume continuous functions $H, \phi, \mathfrak{U}, \psi : [\mathfrak{q}, \mathfrak{p}] \times \mathbb{R}^2 \to \mathbb{R}$ satisfying the condition:

($\mathcal{H}_2$) For all $t \in [\mathfrak{q}, \mathfrak{p}]$ and $\mathrm{x}_j, \mathrm{y}_j \in \mathbb{R} (j = 1, 2)$, $\exists\, L_i,\ i = 1, ..., 4$:

$$|H(t, \mathrm{x}_1, \mathrm{y}_1) - H(t, \mathrm{x}_2, \mathrm{y}_2)| \le L_1(|\mathrm{x}_1 - \mathrm{x}_2| + |\mathrm{y}_1 - \mathrm{y}_2|),$$

$$|\phi(t, \mathrm{x}_1, \mathrm{y}_1) - \phi(t, \mathrm{x}_2, \mathrm{y}_2)| \le L_2(|\mathrm{x}_1 - \mathrm{x}_2| + |\mathrm{y}_1 - \mathrm{y}_2|),$$

$$|\mathfrak{U}(t, \mathrm{x}_1, \mathrm{y}_1) - \mathfrak{U}((t, \mathrm{x}_2, \mathrm{y}_2)| \le L_3(|\mathrm{x}_1 - \mathrm{x}_2| + |\mathrm{y}_1 - \mathrm{y}_2|),$$

$$|\psi(t, \mathrm{x}_1, \mathrm{y}_1) - \psi(t, \mathrm{x}_2, \mathrm{y}_2)| \le L_4(|\mathrm{x}_1 - \mathrm{x}_2| + |\mathrm{y}_1 - \mathrm{y}_2|);$$

For simplicity, we introduce the following notations:

$$\mathfrak{N} = \Delta_1 + \Delta_2, \tag{3.12}$$

$$\overline{\mathfrak{N}} = \overline{\Delta}_1 + \overline{\Delta}_2, \tag{3.13}$$

$$\Delta_1 = \mathcal{F}_0 + L_1\mathcal{F}_1 + L_2\mathcal{F}_2, \tag{3.14}$$

$$\Delta_2 = \mathcal{F}_3 + L_3\mathcal{F}_4 + L_4\mathcal{F}_5, \tag{3.15}$$

$$\Delta_3 = \mathcal{Q}_0 + L_1\mathcal{Q}_1 + L_2\mathcal{Q}_2, \tag{3.16}$$

$$\overline{\Delta}_1 = \mathcal{G}_0 + L_1\mathcal{G}_1 + L_2\mathcal{G}_2, \tag{3.17}$$

$$\overline{\Delta}_2 = \mathcal{G}_3 + L_3\mathcal{G}_4 + L_4\mathcal{G}_5, \tag{3.18}$$

$$\overline{\Delta}_3 = \mathcal{Q}_3 + L_3\mathcal{Q}_4 + L_4\mathcal{Q}_5, \tag{3.19}$$

where

$$\begin{aligned}\mathcal{Q}_0 &= \mathcal{F}_0 - |\alpha_1|\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1}}{\Gamma(\mathfrak{h}_1+1)},\ \mathcal{Q}_1 = \mathcal{F}_1 - |\beta_1|\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{s}_1+\mathfrak{h}_1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1+1)},\ \mathcal{Q}_2 = \mathcal{F}_2 - \frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1+1)}, \\ \mathcal{Q}_3 &= \mathcal{G}_3 - |\alpha_2|\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2}}{\Gamma(\mathfrak{h}_2+1)},\ \mathcal{Q}_4 = \mathcal{G}_4 - |\beta_2|\frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{s}_2+\mathfrak{h}_2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2+1)},\ \mathcal{Q}_5 = \mathcal{G}_5 - \frac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2+1)},\end{aligned} \tag{3.20}$$

and $\mathcal{F}_i,\ \mathcal{G}_i\ (i = 0, ..., 5)$ are given by (3.4) and (3.5).

Lemma 3.2. *(Krasnoselskii) Let $\mathcal{B} \neq \emptyset$ be a closed, bounded, convex subset of a BSp. $\mathcal{K}$. Let operators $\mathcal{M}_1, \mathcal{M}_2$:$\mathcal{B} \to \mathcal{K}$:*

- (a) *$\mathcal{M}_1 z_1 + \mathcal{M}_2 z_2 \in \mathcal{B}$ where $z_1, z_2 \in \mathcal{B}$;*
- (b) *$\mathcal{M}_1$ is compact and continuous;*
- (c) *$\mathcal{M}_2$ is a contraction mapping.*

Then $\exists\, z \in \mathcal{B}$: $z = \mathcal{M}_1 z + \mathcal{M}_2 z$.

4. Uniqueness of solution

Here we prove the Unq. result of solution for the systems (1.1) and (1.2) by applying Banach's FPTs.

For simplicity we use the following notations:

$$\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2, \ \ \mathfrak{B}_1 = \mathfrak{Z}_1\mathcal{F}_1 + \mathfrak{Z}_2\mathcal{F}_2, \ \ \mathfrak{B}_2 = \mathfrak{Z}_3\mathcal{F}_4 + \mathfrak{Z}_4\mathcal{F}_5, \tag{4.1}$$

$$\overline{\mathfrak{B}} = \overline{\mathfrak{B}}_1 + \overline{\mathfrak{B}}_2, \ \ \overline{\mathfrak{B}}_1 = \mathfrak{Z}_1\mathcal{G}_1 + \mathfrak{Z}_2\mathcal{G}_2, \ \ \overline{\mathfrak{B}}_2 = \mathfrak{Z}_3\mathcal{G}_4 + \mathfrak{Z}_4\mathcal{G}_5, \tag{4.2}$$

$$\begin{aligned} \mathfrak{Z}_1 &= \sup_{t\in[\mathfrak{q},\mathfrak{p}]} |H(t,0,0)| < \infty, \ \ \mathfrak{Z}_2 = \sup_{t\in[\mathfrak{q},\mathfrak{p}]} |\phi(t,0,0,)| < \infty, \ \ \mathfrak{Z}_3 = \sup_{t\in[\mathfrak{q},\mathfrak{p}]} |\mathfrak{U}(t,0,0,)| < \infty, \\ \mathfrak{Z}_4 &= \sup_{t\in[\mathfrak{q},\mathfrak{p}]} |\psi(t,0,0,)| < \infty. \end{aligned} \tag{4.3}$$

Theorem 4.1. *Let the condition $(\mathcal{H}_2)$ holds. Then (1.1) and (1.2) has a unique solution on $[\mathfrak{q}, \mathfrak{p}]$ if*

$$\mathfrak{N} + \overline{\mathfrak{N}} < 1, \tag{4.4}$$

where $\mathfrak{N}$ and $\overline{\mathfrak{N}}$ are given by (3.12) and (3.13) respectively.

Proof. Setting $\mathfrak{S} > \dfrac{\mathfrak{B} + \overline{\mathfrak{B}}}{1 - \mathfrak{N} - \overline{\mathfrak{N}}}$, where $\mathfrak{N}, \overline{\mathfrak{N}}, \mathfrak{B}$ and $\overline{\mathfrak{B}}$ are given by (3.12), (3.13), (4.1) and (4.2) respectively. We show that $\mathcal{A}S_{\mathfrak{S}} \subset S_{\mathfrak{S}}$, where $S_{\mathfrak{S}} = \{(x,y) \in \mathfrak{X}^* \times \mathfrak{X}^* : \|(x,y)\| \leq \mathfrak{S}\}$, and the operator $\mathcal{A}$ is given by (3.1).

By assumption $(\mathcal{H}_2)$ together with (4.3), for $(x,y) \in S_{\mathfrak{S}}, \ e \in [\mathfrak{q}, \mathfrak{p}]$, we have

$$|H(e,x(e),y(e))| \leq |H(e,x(e),y(e)) - H(e,0,0)| + |H(e,0,0)| \leq L_1(\|x\| + \|y\|) + \mathfrak{Z}_1 \leq L_1\mathfrak{S} + \mathfrak{Z}_1,$$

$$|\phi(e,x(e),y(e))| \leq |\phi(e,x(e),y(e)) - \phi(e,0,0)| + |\phi(e,0,0)| \leq L_2(\|x\| + \|y\|) + \mathfrak{Z}_2 \leq L_2\mathfrak{S} + \mathfrak{Z}_2.$$

$$|\mathfrak{U}(e,x(e),y(e))| \leq |\mathfrak{U}(e,x(e),y(e)) - \mathfrak{U}(e,0,0)| + |\mathfrak{U}(e,0,0)| \leq L_3(\|x\| + \|y\|) + \mathfrak{Z}_3 \leq L_3\mathfrak{S} + \mathfrak{Z}_3,$$

$$|\psi(e,x(e),y(e))| \leq |\psi(e,x(e),y(e)) - \psi(e,0,0)| + |\psi(e,0,0)| \leq L_4(\|x\| + \|y\|) + \mathfrak{Z}_4 \leq L_4\mathfrak{S} + \mathfrak{Z}_4.$$

By using (3.12) and (4.1), we obtain

$$
\begin{aligned}
|\mathcal{A}_1(x,y)(e)| &\leq \|x\|\mathcal{F}_0 + (L_1\mathfrak{S} + \mathfrak{Z}_1)\mathcal{F}_1 + (L_2\mathfrak{S} + \mathfrak{Z}_2)\mathcal{F}_2 \\
&+ \|y\|\mathcal{F}_3 + (L_3\mathfrak{S} + \mathfrak{Z}_3)\mathcal{F}_4 + (L_4\mathfrak{S} + \mathfrak{Z}_4)\mathcal{F}_5 \\
&\leq \Big(\mathcal{F}_0 + L_1\mathcal{F}_1 + L_2\mathcal{F}_2 + |\mathcal{F}_3 + L_3\mathcal{F}_4 + L_4\mathcal{F}_5\Big)\mathfrak{S} \\
&+ \Big(\mathfrak{Z}_1\mathcal{F}_1 + \mathfrak{Z}_2\mathcal{F}_2 + \mathfrak{Z}_3\mathcal{F}_4 + \mathfrak{Z}_4\mathcal{F}_5\Big) \\
&= (\Delta_1 + \Delta_2)\mathfrak{S} + (\mathfrak{B}_1 + \mathfrak{B}_2) \\
&= \mathfrak{N}\mathfrak{S} + \mathfrak{B},
\end{aligned}
$$

hence,

$$
\|\mathcal{A}_1(x,y)\| \leq \mathfrak{N}\mathfrak{S} + \mathfrak{B}. \tag{4.5}
$$

In the same way, by using (3.13) and (4.2), we obtain

$$
\begin{aligned}
|\mathcal{A}_2(x,y)(e)| &\leq \Big(\mathcal{G}_0 + L_1\mathcal{G}_1 + L_2\mathcal{G}_2 + \mathcal{G}_3 + L_3\mathcal{G}_4 + L_4\mathcal{G}_5\Big)\mathfrak{S} \\
&+ \Big(\mathfrak{Z}_1|\mathcal{G}_1 + \mathfrak{Z}_2\mathcal{G}_2 + |\mathfrak{Z}_3\mathcal{G}_4 + \mathfrak{Z}_4\mathcal{G}_5\Big) \\
&= (\overline{\Delta}_1 + \overline{\Delta}_2)\mathfrak{S} + (\overline{\mathfrak{B}}_1 + \overline{\mathfrak{B}}_2) \\
&= \overline{\mathfrak{N}}\mathfrak{S} + \overline{\mathfrak{B}},
\end{aligned}
$$

which lead to

$$
\|\mathcal{A}_2(x,y)\| \leq \overline{\mathfrak{N}}\mathfrak{S} + \overline{\mathfrak{B}}. \tag{4.6}
$$

Consequently, from (4.5) and (4.6) we get

$$
\begin{aligned}
\|\mathcal{A}(x,y)\| &\leq (\mathfrak{N}\mathfrak{S} + \mathfrak{B}) + (\overline{\mathfrak{N}}\mathfrak{S} + \overline{\mathfrak{B}}) \\
&\leq (\mathfrak{N} + \overline{\mathfrak{N}})\mathfrak{S} + (\mathfrak{B} + \overline{\mathfrak{B}}) \leq \mathfrak{S}.
\end{aligned}
$$

Therefore, $\mathcal{A}S_{\mathfrak{S}} \subset S_{\mathfrak{S}}$. Now, for any $(x_1,y_1),(x_2,y_2) \in \mathfrak{X}^* \times \mathfrak{X}^*$, $e \in [\mathfrak{q},\mathfrak{p}]$ and by using conditions $(\mathcal{H}_2)$, (3.12) and (3.13), we get

$$
\begin{aligned}
&\|\mathcal{A}_1(x_1,y_1) - \mathcal{A}_1(x_2,y_2)\| = \sup_{t\in[\mathfrak{q},\mathfrak{p}]} |\mathcal{A}_1(x_1,y_1)(e) - \mathcal{A}_1(x_2,y_2)(e)| \\
\leq\ & \sup_{e\in[\mathfrak{q},\mathfrak{p}]} \Bigg\{ |\alpha_1| \int_{\mathfrak{q}}^{e} \frac{(e-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} |x_1(\kappa) - x_2(\kappa)| d\kappa \\
+\ & |\beta_1| \int_{\mathfrak{q}}^{e} \frac{(e-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} \Big| H(\kappa, x_1(\kappa), y_1(\kappa)) - H(\kappa, x_2(\kappa), y_2(\kappa)) \Big| d\kappa \\
+\ & \int_{\mathfrak{q}}^{e} \frac{(e-\kappa)^{\mathfrak{h}_1+\mathfrak{k}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{k}_1)} \Big| \phi(\kappa, x_1(\kappa), y_1(\kappa)) - \phi(\kappa, x_2(\kappa), y_2(\kappa)) \Big| d\kappa \\
+\ & |\mathcal{V}_1(e)| \Big[|\alpha_1| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} |x_1(\kappa) - x_2(\kappa)| d\kappa
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad |\beta_1| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} \Big| H(\kappa, x_1(\kappa), y_1(\kappa)) - H(\kappa, x_2(\kappa), y_2(\kappa)) \Big| d\kappa \\
&+ \quad \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \Big| \phi(\kappa, x_1(\kappa), y_1(\kappa)) - \phi(\kappa, x_2(\kappa), y_2(\kappa)) \Big| d\kappa \Big] \\
&+ \quad |\mathcal{V}_2(e)| \Big[|\alpha_1| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-2}}{\Gamma(\mathfrak{h}_1-1)} |x_1(\kappa) - x_2(\kappa)| d\kappa \\
&+ \quad |\beta_1| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-2}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1-1)} \Big| H(\kappa, x_1(\kappa), y_1(\kappa)) - H(\kappa, x_2(\kappa), y_2(\kappa)) \Big| d\kappa \\
&+ \quad \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-2}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)} \Big| \phi(\kappa, x_1(\kappa), y_1(\kappa)) - \phi(\kappa, x_2(\kappa), y_2(\kappa)) \Big| d\kappa \Big] \\
&+ \quad |\mathcal{V}_3(e)| \Big[|\alpha_1| \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} |x_1(\kappa) - x_2(\kappa)| d\kappa \\
&+ \quad |\beta_1| \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} \Big| H(\kappa, x_1(\kappa), y_1(\kappa)) - H(\kappa, x_2(\kappa), y_2(\kappa)) \Big| d\kappa \\
&+ \quad \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \Big| \phi(\kappa, x_1(\kappa), y_1(\kappa)) - \phi(\kappa, x_2(\kappa), y_2(\kappa)) \Big| d\kappa \\
&+ \quad \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(|\alpha_1| \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} |x_1(u) - x_2(u)| du \\
&+ \quad |\beta_1| \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} \Big| H(u, x_1(u), y_1(u)) - H(u, x_2(u), y_2(u)) \Big| du \\
&+ \quad \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \Big| \phi(u, x_1(u), y_1(u)) - \phi(u, x_2(u), y_2(u)) \Big| du \Big) d\Lambda(\kappa) \Big] \\
&+ \quad |\mathcal{V}_4(e)| \Big[|\alpha_2| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} |y_1(\kappa) - y_2(\kappa)| d\kappa \\
&+ \quad |\beta_2| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \Big| \mathfrak{U}(\kappa, x_1(\kappa), y_1(\kappa)) - \mathfrak{U}(\kappa, x_2(\kappa), y_2(\kappa)) \Big| d\kappa \\
&+ \quad \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \Big| \psi(\kappa, x_1(\kappa), y_1(\kappa)) - \psi(\kappa, x_2(\kappa), y_2(\kappa)) \Big| d\kappa \Big] \\
&+ \quad |\mathcal{V}_5(e)| \Big[|\alpha_2| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-2}}{\Gamma(\mathfrak{h}_2-1)} |y_1(\kappa) - y_2(\kappa)| d\kappa \\
&+ \quad |\beta_2| \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2-1)} \Big| \mathfrak{U}(\kappa, x_1(\kappa), y_1(\kappa)) - \mathfrak{U}(\kappa, x_2(\kappa), y_2(\kappa)) \Big| d\kappa \\
&+ \quad \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)} \Big| \psi(\kappa, x_1(\kappa), y_1(\kappa)) - \psi(\kappa, x_2(\kappa), y_2(\kappa)) \Big| d\kappa \Big] \\
&+ \quad |\mathcal{V}_6(e)| \Big[|\alpha_2| \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} |y_1(\kappa) - y_2(\kappa)| d\kappa
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad |\beta_2| \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \Big|\mathfrak{U}(\kappa, x_1(\kappa), y_1(\kappa)) - \mathfrak{U}(\kappa, x_2(\kappa), y_2(\kappa))\Big| d\kappa \\
&+ \quad \sum_{i=1}^{\tau-2} |\eta_i| \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \Big|\psi(\kappa, x_1(\kappa), y_1(\kappa)) - \psi(\kappa, x_2(\kappa), y_2(\kappa))\Big| d\kappa \\
&+ \quad \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(|\alpha_2| \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} |y_1(u) - y_2(u)| du \\
&+ \quad |\beta_2| \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \Big|\mathfrak{U}(u, x_1(u), y_1(u)) - \mathfrak{U}(u, x_1(u), y_1(u))\Big| du \\
&+ \quad \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \Big|\psi(u, x_1(u), y_1(u)) - \psi(u, x_2(u), y_2(u))\Big| du \Big) d\Lambda(\kappa) \Big] \Bigg\} \\
&\leq \quad \Bigg\{ \mathcal{F}_0 \|x_1 - x_2\| + L_1 \mathcal{F}_1 \Big(\|x_1 - x_2\| + \|y_1 - y_2\| \Big) + L_2 \mathcal{F}_2 \Big(\|x_1 - x_2\| + \|y_1 - y_2\| \Big) \\
&+ \quad \mathcal{F}_3 \|y_1 - y_2\| + L_3 \mathcal{F}_4 \Big(\|x_1 - x_2\| + \|y_1 - y_2\| \Big) + L_4 \mathcal{F}_5 \Big(\|x_1 - x_2\| + \|y_1 - y_2\| \Big) \Bigg\} \\
&\leq \quad \Big(\mathcal{F}_0 + L_1 \mathcal{F}_1 + L_2 \mathcal{F}_2 + \mathcal{F}_3 + L_3 \mathcal{F}_4 + L_4 \mathcal{F}_5 \Big) \Big(\|x_1 - x_2\| + \|y_1 - y_2\| \Big) \\
&= \quad \Big(\Delta_1 + \Delta_2 \Big) \Big(\|x_1 - x_2\| + \|y_1 - y_2\| \Big) \\
&= \quad \mathfrak{R} \Big(\|x_1 - x_2\| + \|y_1 - y_2\| \Big).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\|\mathcal{A}_2(x_1, y_1) - \mathcal{A}_2(x_2, y_2)\| &= \sup_{e \in [\mathfrak{q}, \mathfrak{p}]} |\mathcal{A}_2(x_1, y_1)(e) - \mathcal{A}_2(x_2, y_2)(e)| \\
&\leq \Big(\overline{\Delta}_1 + \overline{\Delta}_2 \Big) \Big(\|x_1 - x_2\| + \|y_1 - y_2\| \Big) \\
&= \overline{\mathfrak{R}} \Big(\|x_1 - x_2\| + \|y_1 - y_2\| \Big).
\end{aligned}
$$

Consequently, we obtain

$$
\|\mathcal{A}(x_1, y_1) - \mathcal{A}(x_2, y_2))\| \leq (\mathfrak{R} + \overline{\mathfrak{R}})(\|x_1 - x_2\| + \|y_1 - y_2\|),
$$

which implies that $\mathcal{A}$ is a contraction operator by the assumption (4.4). Hence, by Banach's FPT, the operator $\mathcal{A}$ has a unique FP, which is the unique solution of systems (1.1) and (1.2) on $[\mathfrak{q}, \mathfrak{p}]$.

5. Examples

This section presents examples that illustrate our results.

Example 5.1. Assume the coupled system of FDEs given by

$$
\begin{cases}
{}^{RL}D^{19/11}\Big[({}^{c}D^{39/21} + \frac{1}{414})x(t) + \frac{4}{407} I^{8/3} H(t, x(t), y(t))\Big] = \phi(t, x(t), y(t)), \\
{}^{RL}D^{29/17}\Big[({}^{c}D^{38/23} + \frac{3}{880})x(t) + \frac{1}{336} I^{16/5} \mathfrak{U}(t, x(t), y(t))\Big] = \psi(t, x(t), y(t)), \ t \in [-2, -1],
\end{cases}
\tag{5.1}
$$

with the BCs

$$\begin{cases} x'(-2) = 0,\ x(-1) = 0,\ x'(-1) = 0,\ x(-2) = \sum_{i=1}^{3} \eta_i y(\xi_i) + \int_{-2}^{-1} y(\kappa) d\Lambda(\kappa), \\ y'(-2) = 0,\ y(-1) = 0,\ y'(-1) = 0,\ y(-2) = \sum_{i=1}^{3} \eta_i x(\xi_i) + \int_{-2}^{-1} x(\kappa) d\Lambda(\kappa). \end{cases} \tag{5.2}$$

where $\mathfrak{q} = -2$, $\mathfrak{p} = -1$, $\mathfrak{f}_1 = \frac{19}{11}$, $\mathfrak{h}_1 = \frac{39}{21}$, $\mathfrak{f}_2 = \frac{29}{17}$, $\mathfrak{h}_2 = \frac{38}{23}$, $\mathfrak{s}_1 = \frac{8}{3}$, $\mathfrak{s}_2 = \frac{16}{5}$, $\alpha_1 = \frac{1}{414}$, $\beta_1 = \frac{4}{407}$, $\alpha_2 = \frac{3}{880}$, $\beta_2 = \frac{1}{336}$, $\xi_1 = \frac{-7}{4}$, $\xi_2 = \frac{-3}{2}$, $\xi_3 = \frac{-5}{4}$, $\eta_1 = -3$, $\eta_2 = \frac{9}{4}$, $\eta_3 = \frac{5}{2}$

$$\begin{aligned} H(t, x(t), y(t)) &= \frac{1}{\ln(5)} + \frac{\sin x(t)}{933} + \frac{y(t)}{(t^2 + 649)}, \\ \phi(t, x(t), y(t)) &= \frac{1}{66} + \frac{x(t)}{(t^8 + 22)^2} + \frac{y(t)|x(t)|}{800(1 + |x(t)|)}, \\ \mathfrak{U}(t, x(t), y(t)) &= \frac{2y(t)}{23(1 + y(t))} + \frac{\sin(2\pi x(t))}{900\pi} + \frac{y(t)}{\sqrt{t^4 + 2400}}, \end{aligned}$$

and

$$\psi(t, x(t), y(t)) = \frac{1}{312 + t^3} + \frac{\sin x(t)|\tan^{-1} y(t)|}{57\pi} + \frac{y(t)}{12(\sqrt[4]{t^2 + 6560})}.$$

Using the given data, we have that $\mathcal{F}_0 \simeq 0.016050$, $\mathcal{F}_1 \simeq 0.002735$, $\mathcal{F}_2 \simeq 1.04237$, $\mathcal{F}_3 \simeq 0.0294380$, $\mathcal{F}_4 \simeq 0.000408$, $\mathcal{F}_5 \simeq 1.26577$, $\mathcal{G}_0 \simeq 0.017124$, $\mathcal{G}_1 \simeq 0.002247$, $\mathcal{G}_2 \simeq 0.918958$, $\mathcal{G}_3 \simeq 0.245213$, $\mathcal{G}_4 \simeq 0.006572$, $\mathcal{G}_5 \simeq 16.2678$.

Clearly,

$$|H(t, x(t), y(t))| \le \frac{1}{\ln(5)} + \frac{1}{933}\|x\| + \frac{1}{650}\|y\|, \quad |\phi(t, x(t), y(t))| \le \frac{1}{66} + \frac{1}{529}\|x\| + \frac{1}{800}\|y\|,$$

$$|\mathfrak{U}(t, x(t), y(t))| \le \frac{2}{23} + \frac{1}{450}\|x\| + \frac{1}{49}\|y\|, \quad |\psi(t, x(t), y(t))| \le \frac{1}{313} + \frac{1}{114}\|x\| + \frac{1}{108}\|y\|,$$

with $\varpi_0 = \frac{1}{\ln(5)}$, $\varpi_1 = \frac{1}{933}$, $\varpi_2 = \frac{1}{650}$, $\varepsilon_0 = \frac{1}{66}$, $\varepsilon_1 = \frac{1}{529}$ $\varepsilon_2 = \frac{1}{800}$, $n_0 = \frac{2}{23}$, $n_1 = \frac{1}{450}$, $n_2 = \frac{1}{49}$, $m_0 = \frac{1}{313}$, $m_1 = \frac{1}{114}$, and $m_2 = \frac{1}{108}$. Using (3.7) and (3.8), we find that $\mathbf{O}_1 \simeq 0.190707$, $\mathbf{O}_2 \simeq 0.439600$ and $\mathbf{O} = \max\{\mathbf{O}_1, \mathbf{O}_2\} \simeq 0.439600 < 1$. Therefore, by Theorem 3.1, the problems (5.1) and (5.2) have at least one solution on $[-2, -1]$.

Example 5.2. Consider the system (5.1) with the coupled BCs (5.2) and

$$\begin{aligned} H(a, x(a), y(a)) &= e^{-2a} \cos 2a + \frac{1}{70}\Big(\sin x(a) + y(a)\Big), \quad a \in [-2, -1], \\ \phi(a, x(a), y(a)) &= 30a^5 + \frac{1}{510}\Big(\frac{|x(a)|}{1 + |x(a)|} + \cos y(a)\Big), \quad a \in [-2, -1], \\ \mathfrak{U}(a, x(a), y(a)) &= \frac{1}{4\sqrt{a^6 + 6399}}\Big(x(a) + \tan^{-1} y(a)\Big), \quad a \in [-2, -1], \end{aligned}$$

$$\psi(a,x(a),y(a)) \;=\; 2\sec a+\frac{1}{1800}\Big(\sin^2 x(a)+\frac{2|y(a)|}{1+|y(a)|}\Big),\;\; a\in[-2,-1].$$

Clearly,

$$\begin{aligned}
|H(a,x_1,y_1)-H(a,x_2,y_2)| &\le \frac{1}{70}(\|x_1-x_2\|+\|y_1-y_2\|),\\
|\phi(a,x_1,y_1)-\phi(a,x_2,y_2)| &\le \frac{1}{510}(\|x_1-x_2\|+\|y_1-y_2\|),\\
|\mathfrak{U}(a,x_1,y_1)-\mathfrak{U}((a,x_2,y_2)| &\le \frac{1}{320}(\|x_1-x_2\|+\|y_1-y_2\|),\\
|\psi(a,x_1,y_1)-\psi(a,x_2,y_2)| &\le \frac{1}{900}(\|x_1-x_2\|+\|y_1-y_2\|).
\end{aligned}$$

Using the given data in Example (5.1), we find that $\mathfrak{N}+\overline{\mathfrak{N}}\simeq 0.331246<1$. Thus, in view of Theorem 4.1 the problem (5.1) has a unique solution on $[-2,-1]$.

6. Conclusions

We managed to employ Leray-Schauder alternative, Banach, and the Krasnoselskii fixed point theory to study the Existence and Uniqueness of solutions for a nonlinear coupled system of fractional differential equations involving Riemann-Liouville and Caputo derivatives with coupled Riemann-Stieltjes integro-multipoint boundary conditions. The system under study is a generalized version of many recent studied system. We used some examples to illustrate the results. Potential future work could be to investigate our results based on other fractional derivates such as, e.g., Abu-Shady-Kaabar fractional derivative, Katugampola derivative, and conformable derivative.

Conflict of interest

The authors declare that there are no competing interests.

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A. Appendix A

Lemma A.1. *Let $H, \Phi, U, \Psi \in C(\mathfrak{q}, \mathfrak{p}) \cap L(\mathfrak{q}, \mathfrak{p})$, the solution of the linear system of FDEs:*

$$\begin{cases} {}^{RL}D^{\mathfrak{k}_1}\Big[({}^{c}D^{\mathfrak{h}_1} + \alpha_1)x(t) + \beta_1 I^{\mathfrak{s}_1} H^*(t)\Big] = \Phi(t), \ 1 < \mathfrak{h}_1, \mathfrak{k}_1 \leq 2, \ t \in (\mathfrak{q}, \mathfrak{p}), \\ {}^{RL}D^{\mathfrak{k}_2}\Big[({}^{c}D^{\mathfrak{h}_2} + \alpha_2)y(t) + \beta_2 I^{\mathfrak{s}_2} \mathfrak{U}^*(t)\Big] = \Psi(t), \ 1 < \mathfrak{h}_2, \mathfrak{k}_2 \leq 2, \ t \in (\mathfrak{q}, \mathfrak{p}), \end{cases} \tag{A.1}$$

with the BCs (1.2) is equivalent to the system:

$$\begin{aligned} x(t) &= -\alpha_1 \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa - \beta_1 \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H^*(\kappa)d\kappa + \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{h}_1+\mathfrak{k}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{k}_1)} \Phi(\kappa)d\kappa \\ &+ \mathcal{V}_1(t)\Big[\alpha_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa + \beta_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H^*(\kappa)d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{k}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{k}_1)} \Phi(\kappa)d\kappa\Big] \\ &+ \mathcal{V}_2(t)\Big[\alpha_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-2}}{\Gamma(\mathfrak{h}_1-1)} x(\kappa)d\kappa + \beta_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-2}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1-1)} H^*(\kappa)d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{k}_1-2}}{\Gamma(\mathfrak{h}_1+\mathfrak{k}_1-1)} \Phi(\kappa)d\kappa\Big] \\ &+ \mathcal{V}_3(t)\Big[-\alpha_1 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa - \beta_1 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H^*(\kappa)d\kappa \\ &+ \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1+\mathfrak{k}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{k}_1)} \Phi(\kappa)d\kappa \\ &+ \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(-\alpha_1 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(u)du - \beta_1 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H^*(u)du \\ &+ \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1+\mathfrak{k}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{k}_1)} \Phi(u)du\Big)d\Lambda(\kappa)\Big] + \mathcal{V}_4(t)\Big[\alpha_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(\kappa)d\kappa \\ &+ \beta_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}^*(\kappa)d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{k}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{k}_2)} \Psi(\kappa)d\kappa\Big] + \mathcal{V}_5(t)\Big[\alpha_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-2}}{\Gamma(\mathfrak{h}_2-1)} y(\kappa)d\kappa \\ &+ \beta_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2-1)} \mathfrak{U}^*(\kappa)d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{k}_2-2}}{\Gamma(\mathfrak{h}_2+\mathfrak{k}_2-1)} \Psi(\kappa)d\kappa\Big] \\ &+ \mathcal{V}_6(t)\Big[-\alpha_2 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(\kappa)d\kappa - \beta_2 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}^*(\kappa)d\kappa \\ &+ \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2+\mathfrak{k}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{k}_2)} \Psi(\kappa)d\kappa \\ &+ \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(-\alpha_2 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(u)du - \beta_2 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}^*(u)du \end{aligned}$$

$$+ \quad \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa - u)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2 + \mathfrak{f}_2)} \Psi(u)du\Big)d\Lambda(\kappa)\Big], \tag{A.2}$$

$$\begin{aligned}
y(t) &= -\alpha_2 \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(\kappa)d\kappa - \beta_2 \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}^*(\kappa)d\kappa + \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \Psi(\kappa)d\kappa \\
&+ \mathcal{W}_1(t)\Big[\alpha_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa + \beta_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H^*(\kappa)d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \Phi(\kappa)d\kappa\Big] \\
&+ \mathcal{W}_2(t)\Big[\alpha_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-2}}{\Gamma(\mathfrak{h}_1-1)} x(\kappa)d\kappa + \beta_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-2}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1-1)} H^*(\kappa)d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-2}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)} \Phi(\kappa)d\kappa\Big] \\
&+ \mathcal{W}_3(t)\Big[-\alpha_1 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa - \beta_1 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H^*(\kappa)d\kappa \\
&+ \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \Phi(\kappa)d\kappa \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(-\alpha_1 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(u)du - \beta_1 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H(u)du \\
&+ \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \Phi(u)du\Big)d\Lambda(\kappa)\Big] + \mathcal{W}_4(t)\Big[\alpha_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(\kappa)d\kappa \\
&+ \beta_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}^*(\kappa)d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \Psi(\kappa)d\kappa\Big] + \mathcal{W}_5(t)\Big[\alpha_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-2}}{\Gamma(\mathfrak{h}_2-1)} y(\kappa)d\kappa \\
&+ \beta_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2-1)} \mathfrak{U}^*(\kappa)d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)} \Psi(\kappa)d\kappa\Big] \\
&+ \mathcal{W}_6(t)\Big[-\alpha_2 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(\kappa)d\kappa - \beta_2 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}^*(\kappa)d\kappa \\
&+ \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \Psi(\kappa)d\kappa \\
&+ \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(-\alpha_2 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(u)du - \beta_2 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}^*(u)du \\
&+ \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \Psi(u)du\Big)d\Lambda(\kappa)\Big],
\end{aligned} \tag{A.3}$$

where

$$\mathcal{V}_i(t) = \mathfrak{B}_1(t)\rho_i + \mathfrak{Q}_1(t)\omega_i + \epsilon_i, \; i = 1, ..., 6, \tag{A.4}$$

$$\mathcal{W}_j(t) \;=\; \mathfrak{B}_2(t)\tau_j + \mathfrak{Q}_2(t)\lambda_j + \delta_j,\ j = 1, ..., 6, \tag{A.5}$$

$$\mathfrak{B}_\ell(t) = \frac{(t-\mathfrak{q})^{\mathfrak{h}_\ell+\mathfrak{k}_\ell-1}\Gamma(\mathfrak{k}_\ell)}{\Gamma(\mathfrak{h}_\ell+\mathfrak{k}_\ell)},\ \mathfrak{Q}_\ell(t) = \frac{(t-\mathfrak{q})^{\mathfrak{h}_\ell+\mathfrak{k}_\ell-2}\Gamma(\mathfrak{k}_\ell-1)}{\Gamma(\mathfrak{h}_\ell+\mathfrak{k}_\ell-1)},\ \ell = 1, 2, \tag{A.6}$$

$$\begin{cases} \rho_1 = \dfrac{(A_4A_7^2 - A_4)\mu_3 + A_4A_7\mu_1}{\sigma},\ \rho_2 = \dfrac{(-A_2A_7^2 + A_6A_7 + A_2)\mu_3 + (-A_2A_7 + A_6)\mu_1}{\sigma}, \\ \rho_3 = \dfrac{A_4A_7\mu_3 + A_4\mu_1}{\sigma},\ \rho_4 = \dfrac{-A_4\mu_1}{\sigma},\ \rho_5 = \dfrac{A_4\mu_2}{\sigma},\ \rho_6 = \dfrac{A_4\mu_3}{\sigma}, \end{cases} \tag{A.7}$$

$$\begin{cases} \omega_1 = \dfrac{-(A_3A_7^2 - A_3)\mu_3 - A_3A_7\mu_1}{\sigma},\ \omega_2 = \dfrac{(A_1A_7^2 - A_5A_7 - A_1)\mu_3 + (A_1A_7 - A_5)\mu_1}{\sigma}, \\ \omega_3 = \dfrac{-(A_3A_7\mu_3 + A_3\mu_1)}{\sigma},\ \omega_4 = \dfrac{A_3\mu_1}{\sigma},\ \omega_5 = \dfrac{-A_3\mu_2}{\sigma},\ \omega_6 = \dfrac{-A_3\mu_3}{\sigma}, \end{cases} \tag{A.8}$$

$$\begin{cases} \epsilon_1 = \dfrac{\nu_1A_7\mu_3 + \nu_1\mu_1}{\sigma},\ \epsilon_2 = \dfrac{-(\nu_2A_7\mu_3 + \nu_2\mu_1)}{\sigma}, \\ \epsilon_3 = \dfrac{-(\nu_3A_7\mu_3 + \nu_3\mu_1)}{\sigma},\ \epsilon_4 = \dfrac{\nu_3\mu_1}{\sigma},\ \epsilon_5 = \dfrac{-\nu_3\mu_2}{\sigma},\ \epsilon_6 = \dfrac{-\nu_3\mu_3}{\sigma}, \end{cases} \tag{A.9}$$

$$\begin{cases} \tau_1 = \dfrac{-B_4\nu_1}{\sigma},\ \tau_2 = \dfrac{B_4\nu_2}{\sigma},\ \tau_3 = \dfrac{B_4\nu_3}{\sigma},\ \tau_4 = \dfrac{(A_7^2B_4 - B_4)\nu_3 + A_7B_4\nu_1}{\sigma}, \\ \tau_5 = \dfrac{(-A_7^2B_2 + A_7B_6 + B_2)\nu_3 - (A_7B_2 - B_6)\nu_1}{\sigma},\ \tau_6 = \dfrac{A_7B_4\nu_3 + B_4\nu_1}{\sigma}, \end{cases} \tag{A.10}$$

$$\begin{cases} \lambda_1 = \dfrac{B_3\nu_1}{\sigma},\ \lambda_2 = \dfrac{-B_3\nu_2}{\sigma},\ \lambda_3 = \dfrac{-B_3\nu_3}{\sigma},\ \lambda_4 = \dfrac{-(A_7^2B_3 - B_3)\nu_3 - A_7B_3\nu_1}{\sigma}, \\ \lambda_5 = \dfrac{(A_7^2B_1 - A_7B_5 - B_1)\nu_3 + (A_7B_1 - B_5)\nu_1}{\sigma},\ \lambda_6 = \dfrac{-A_7B_3\nu_3 - B_3\nu_1}{\sigma}, \end{cases} \tag{A.11}$$

$$\begin{cases} \delta_1 = \dfrac{\mu_3\nu_1}{\sigma},\ \delta_2 = \dfrac{-\mu_3\nu_2}{\sigma},\ \delta_3 = \dfrac{-\mu_3\nu_3}{\sigma},\ \delta_4 = \dfrac{\mu_1A_7\nu_3 + \mu_1\nu_1}{\sigma}, \\ \delta_5 = \dfrac{-\mu_2A_7\nu_3 - \mu_2\nu_1}{\sigma},\ \delta_6 = \dfrac{-\mu_3A_7\nu_3 - \mu_3\nu_1}{\sigma}, \end{cases} \tag{A.12}$$

$$\mu_1 \;=\; B_3B_6 - B_4B_5,\ \mu_2 = B_1B_6 - B_2B_5,\ \mu_3 = B_1B_4 - B_2B_3, \tag{A.13}$$

$$\nu_1 \;=\; A_3A_6 - A_4A_5,\ \nu_2 = A_1A_6 - A_2A_5,\ \nu_3 = A_1A_4 - A_2A_3, \tag{A.14}$$

$$\begin{cases} A_1 &= \dfrac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-1}\Gamma(\mathfrak{f}_1)}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)},\ A_2 = \dfrac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-2}\Gamma(\mathfrak{f}_1-1)}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)}, \\ A_3 &= \dfrac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-2}\Gamma(\mathfrak{f}_1)}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)},\ A_4 = \dfrac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-3}\Gamma(\mathfrak{f}_1-1)}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-2)}, \\ A_5 &= \displaystyle\sum_{i=1}^{\tau-2}\eta_i\frac{(\xi_i-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-1}\Gamma(\mathfrak{f}_1)}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} + \int_{\mathfrak{q}}^{\mathfrak{p}}\frac{(\kappa-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-1}\Gamma(\mathfrak{f}_1)}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)}d\Lambda(\kappa), \\ A_6 &= \displaystyle\sum_{i=1}^{\tau-2}\eta_i\frac{(\xi_i-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-2}\Gamma(\mathfrak{f}_1-1)}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)} + \int_{\mathfrak{q}}^{\mathfrak{p}}\frac{(\kappa-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-2}\Gamma(\mathfrak{f}_1-1)}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)}d\Lambda(\kappa), \\ A_7 &= \displaystyle\sum_{i=1}^{\tau-2}\eta_i + \int_{\mathfrak{q}}^{\mathfrak{p}}d\Lambda(\kappa), \end{cases} \tag{A.15}$$

$$\begin{cases} B_1 &= \dfrac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-1}\Gamma(\mathfrak{f}_2)}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)},\ B_2 = \dfrac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-2}\Gamma(\mathfrak{f}_2-1)}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)}, \\ B_3 &= \dfrac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-2}\Gamma(\mathfrak{f}_2)}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)},\ B_4 = \dfrac{(\mathfrak{p}-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-3}\Gamma(\mathfrak{f}_2-1)}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-2)}, \\ B_5 &= \displaystyle\sum_{i=1}^{\tau-2}\eta_i\frac{(\xi_i-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-1}\Gamma(\mathfrak{f}_2)}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} + \int_{\mathfrak{q}}^{\mathfrak{p}}\frac{(\kappa-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-1}\Gamma(\mathfrak{f}_2)}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)}d\Lambda(\kappa), \\ B_6 &= \displaystyle\sum_{i=1}^{\tau-2}\eta_i\frac{(\xi_i-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-2}\Gamma(\mathfrak{f}_2-1)}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)} + \int_{\mathfrak{q}}^{\mathfrak{p}}\frac{(\kappa-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-2}\Gamma(\mathfrak{f}_2-1)}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)}d\Lambda(\kappa), \end{cases} \tag{A.16}$$

and it is assumed that

$$\sigma = (\nu_3 A_7^2 + \nu_1 A_7 - \nu_3)\mu_3 + (\nu_3 A_7 + \nu_1)\mu_1 \neq 0, \tag{A.17}$$

Proof. Solving the FDE (A.1) in a standard manner and using Lemmas 2.1 and 2.2, we get

$$\begin{aligned} x(t) &= -\alpha_1\int_{\mathfrak{q}}^{t}\frac{(t-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)}x(\kappa)d\kappa - \beta_1\int_{\mathfrak{q}}^{t}\frac{(t-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)}H^*(\kappa)d\kappa + \int_{\mathfrak{q}}^{t}\frac{(t-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)}\Phi(\kappa)d\kappa \\ &+ c_1\frac{(t-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-1}\Gamma(\mathfrak{f}_1)}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} + c_2\frac{(t-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-2}\Gamma(\mathfrak{f}_1-1)}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)} + c_3 + c_4(t-\mathfrak{q}), \end{aligned} \tag{A.18}$$

$$\begin{aligned} x'(t) &= -\alpha_1\int_{\mathfrak{q}}^{t}\frac{(t-\kappa)^{\mathfrak{h}_1-2}}{\Gamma(\mathfrak{h}_1-1)}x(\kappa)d\kappa - \beta_1\int_{\mathfrak{q}}^{t}\frac{(t-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-2}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1-1)}H^*(\kappa)d\kappa + \int_{\mathfrak{q}}^{t}\frac{(t-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-2}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)}\Phi(\kappa)d\kappa \\ &+ c_1\frac{(t-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-2}\Gamma(\mathfrak{f}_1)}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)} + c_2\frac{(t-\mathfrak{q})^{\mathfrak{h}_1+\mathfrak{f}_1-3}\Gamma(\mathfrak{f}_1-1)}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-2)} + c_4. \end{aligned} \tag{A.19}$$

$$y(t) = -\alpha_2\int_{\mathfrak{q}}^{t}\frac{(t-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)}y(\kappa)d\kappa - \beta_2\int_{\mathfrak{q}}^{t}\frac{(t-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)}\mathfrak{U}^*(\kappa)d\kappa + \int_{\mathfrak{q}}^{t}\frac{(t-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)}\Psi(\kappa)d\kappa$$

$$+ \quad b_1 \frac{(t-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-1}\Gamma(\mathfrak{f}_2)}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} + b_2 \frac{(t-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-2}\Gamma(\mathfrak{f}_2-1)}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)} + b_3 + b_4(t-\mathfrak{q}), \tag{A.20}$$

$$\begin{aligned} y'(t) &= -\alpha_2 \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{h}_2-2}}{\Gamma(\mathfrak{h}_2-1)} y(\kappa)d\kappa - \beta_2 \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2-1)} \mathfrak{U}^*(\kappa)d\kappa + \int_{\mathfrak{q}}^{t} \frac{(t-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)} \Psi(\kappa)d\kappa \\ &+ \quad b_1 \frac{(t-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-2}\Gamma(\mathfrak{f}_2)}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)} + b_2 \frac{(t-\mathfrak{q})^{\mathfrak{h}_2+\mathfrak{f}_2-3}\Gamma(\mathfrak{f}_2-1)}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-2)} + b_4. \end{aligned} \tag{A.21}$$

$c_i, b_i \in \mathbb{R}, i = 1, \cdots, 4$ are some unknown arbitrary constants.

Using the BCs (1.2) in Eqs (A.18)–(A.21), together with notations (A.15) and (A.16) , we obtain $c_4 = 0,\ b_4 = 0$, and a system of equations in $c_i,\ b_i (i = 1, 2, 3)$ given by

$$\begin{cases} A_1c_1 + A_2c_2 + c_3 &= K_1, \\ B_1b_1 + B_2b_2 + b_3 &= E_1, \\ A_3c_1 + A_4c_2 &= K_2, \\ B_3b_1 + B_4b_2 &= E_2, \\ c_3 - B_5b_1 - B_6b_2 - A_7b_3 &= E_3, \\ b_3 - A_5c_1 - A_6c_2 - A_7c_3 &= K_3, \end{cases} \tag{A.22}$$

where $A_i\ (i = 1, ..., 7),\ B_j\ (j = 1, ..., 6)$ are given by (A.15) and (A.16) and $K_i, E_i, i = 1, 2, 3$, are defined by

$$\begin{aligned} K_1 &= \alpha_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa + \beta_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H^*(\kappa)d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \Phi(\kappa)d\kappa, \\ K_2 &= \alpha_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1-2}}{\Gamma(\mathfrak{h}_1-1)} x(\kappa)d\kappa + \beta_1 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-2}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1-1)} H^*(\kappa)d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-2}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1-1)} \Phi(\kappa)d\kappa, \\ K_3 &= -\alpha_1 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(\kappa)d\kappa - \beta_1 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H^*(\kappa)d\kappa \\ &+ \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \Phi(\kappa)d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(-\alpha_1 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1-1}}{\Gamma(\mathfrak{h}_1)} x(u)du \\ &- \beta_1 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_1+\mathfrak{h}_1-1}}{\Gamma(\mathfrak{s}_1+\mathfrak{h}_1)} H^*(u)du + \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_1+\mathfrak{f}_1-1}}{\Gamma(\mathfrak{h}_1+\mathfrak{f}_1)} \Phi(u)du \Big) d\Lambda(\kappa), \\ E_1 &= \alpha_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(\kappa)d\kappa + \beta_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}^*(\kappa)d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \Psi(\kappa)d\kappa, \\ E_2 &= \alpha_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2-2}}{\Gamma(\mathfrak{h}_2-1)} y(\kappa)d\kappa + \beta_2 \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-2}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2-1)} \mathfrak{U}^*(\kappa)d\kappa - \int_{\mathfrak{q}}^{\mathfrak{p}} \frac{(\mathfrak{p}-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-2}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2-1)} \Psi(\kappa)d\kappa, \\ E_3 &= -\alpha_2 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(\kappa)d\kappa - \beta_2 \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}^*(\kappa)d\kappa \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{\tau-2} \eta_i \int_{\mathfrak{q}}^{\xi_i} \frac{(\xi_i-\kappa)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \Psi(\kappa)d\kappa + \int_{\mathfrak{q}}^{\mathfrak{p}} \Big(-\alpha_2 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2-1}}{\Gamma(\mathfrak{h}_2)} y(u)du \\
&- \beta_2 \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{s}_2+\mathfrak{h}_2-1}}{\Gamma(\mathfrak{s}_2+\mathfrak{h}_2)} \mathfrak{U}^*(u)du + \int_{\mathfrak{q}}^{\kappa} \frac{(\kappa-u)^{\mathfrak{h}_2+\mathfrak{f}_2-1}}{\Gamma(\mathfrak{h}_2+\mathfrak{f}_2)} \Psi(u)du \Big) d\Lambda(\kappa),
\end{aligned} \tag{A.23}
$$

Solving the system (A.22) for $c_i,\ b_i (i = 1, 2, 3)$, we find that

$$c_1 = \rho_1 K1 + \rho_2 K_2 + \rho_3 K_3 + \rho_4 E_1 + \rho_5 E_2 + \rho_6 E_3, \tag{A.24}$$

$$c_2 = \omega_1 K1 + \omega_2 K_2 + \omega_3 K_3 + \omega_4 E_1 + \omega_5 E_2 + \omega_6 E_3, \tag{A.25}$$

$$c_3 = \epsilon_1 K1 + \epsilon_2 K_2 + \epsilon_3 K_3 + \epsilon_4 E_1 + \epsilon_5 E_2 + \epsilon_6 E_3, \tag{A.26}$$

$$b_1 = \tau_1 K1 + \tau_2 K_2 + \tau_3 K_3 + \tau_4 E_1 + \tau_5 E_2 + \tau_6 E_3, \tag{A.27}$$

$$b_1 = \lambda_1 K1 + \lambda_2 K_2 + \lambda_3 K_3 + \lambda_4 E_1 + \lambda_5 E_2 + \rho_6 E_3, \tag{A.28}$$

$$b_1 = \delta_1 K1 + \delta_2 K_2 + \delta_3 K_3 + \delta_4 E_1 + \delta_5 E_2 + \delta_6 E_3, \tag{A.29}$$

where $\rho_i,\ \omega_i,\ \epsilon_i,\ \tau_i,\ \lambda_i$ and δ_i $(i = 1, ..., 6)$ are given by (A.7)–(A.12) respectively.

Inserting the values of $c_1, c_2, c_3, c_4, b_1, b_2, b_3$ and b_4 in (A.18) and (A.20), we get (A.2) and (A.3). The converse follows by direct computation. This completes the proof.