Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(5): 10049–10066.
DOI: 10.3934/math.2023509
Received: 02 January 2023
Revised: 10 February 2023
Accepted: 20 February 2023
Published: 24 February 2023

Research article

Fixed point results on triple controlled quasi rectangular metric like spaces

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Abstract: In this article, by utilizing the idea of controlled functions, we present a novel notion of triple controlled quasi rectangular metric like spaces and prove Banach fixed point principal in such spaces. A topology in such spaces and its topological properties have been discussed. The result, presented here is a new contribution to the field of fixed point theory. Examples of this new structure are given.

Keywords: fixed point; Cauchy sequence; rectangular metric like space; triple controlled quasi rectangular metric like space
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

Fixed point theory is based on the attempt to solve the equation $\mathscr{L}s = s$, with $\mathscr{L}$ standing for a self mapping on $\mathscr{K}$. Banach [1] developed this theory axiomatically. For extension and improvement of this research field, distinct criteria and different structures on mappings have been proposed by the researchers see [2–9]. In numerous abstract spaces, characterizations of this principle have been proved which have drawn the attention of the researchers who studied it from several perspectives.

Bakhtin [10], initiated a generalization of metric spaces called b-metric spaces. Lately, several generalizations of the b-metric spaces were initiated such as extended b-metric spaces by Kamran et al. [11] and some other can be seen in [12, 13]. In 2018, Mlaiki et al. [14], introduced the concept of controlled metric type spaces (CMTS). Few months later, Abdeljawad et al. in [15], initiated a more general metric type so called double controlled metric type spaces denoted by (DCMTS). In 2020, Mlaiki in [16], introduced a generalization of (DCMTS) so called double controlled metric like spaces denoted by (DCMLS), where he assumed that the self distance is not necessary zero. Another type of extension of metric spaces, where we assume that, we do not necessarily have the symmetry condition of metric function which is called quasi metric space for more details of such spaces, we refer the reader to the following references [17–20]. Haque et al. [21], introduce the concept of double controlled quasi metric like spaces denoted (DCQMLS), where they assume that there is no symmetric condition in the (DCMLS). Further, Branciari [22] in 2000, proposed the concept of rectangular metric space. Then, in 2015, George et al. in [23], generalized rectangular metric spaces to rectangular b-metric spaces. In 2020, Mlaiki et al. in [24], generalized the rectangular b-metric spaces by introducing the controlled rectangular metric spaces. Inspired by the work in [25,26], Haque et al. [27] presented a generalization of controlled rectangular b-metric spaces and partial rectangular metric spaces, so-called controlled rectangular metric-like spaces and prove fixed point results. For more results in rectangular metric space, see [28–32].

The goal of this work is to prove the fixed point theorem for contraction mappings in triple controlled quasi rectangular metric like spaces. The notion of a triple controlled quasi rectangular metric like space that generalizes controlled rectangular metric like spaces and controlled quasi rectangular metric like spaces. In the context of triple controlled quasi rectangular metric like spaces, we have developed a new version of the Banach contraction principle. Additionally, a concrete example is provided to support the outcome. We begin with the following definitions.

Definition 1.1. [22] Suppose that $\mathscr{K}$ is a nonempty set and $d : \mathscr{K} \times \mathscr{K} \to [0, \infty)$. Assume that

d_a : $d(s, l) = 0 \Longleftrightarrow s = l$,
d_b : $d(s, l) = d(l, s)$,
d_c : $d(s, l) \leq d(s, u) + d(u, v) + \delta(v, l)$,

for all $s, l, \in \mathscr{K}$ and for all distinct points $u \neq v \in \mathscr{K} \backslash \{s, l\}$. Then, $(\mathscr{K}, d)$ is called a rectangular metric type space.

As a generalization of rectangular metric spaces, rectangular b-metric space was introduced by George et al. [23].

Definition 1.2. [23] Suppose that $\mathscr{K}$ is a nonempty set and $d_b : \mathscr{K} \times \mathscr{K} \to [0, \infty)$. Assume that $a \geq 1$, we have

b_a : $d_b(s, l) = 0 \Longleftrightarrow s = l$,
b_b : $d_b(s, l) = d_b(l, s)$,
b_c : $d_b(s, l) \leq a[d_b(s, u) + d_b(u, v) + d_b(v, l)]$,

for all $s, l, \in \mathscr{K}$ and for all distinct points $u \neq v \in \mathscr{K} \backslash \{s, l\}$. Then, $(\mathscr{K}, d_b)$ is called a rectangular b-metric space.

In 2020, a new extension to the rectangular metric spaces was defined as follows.

Definition 1.3. [24] Suppose that $\mathscr{K} \neq \phi$ and $\theta : \mathscr{K}^4 \to [1, \infty)$, the function $D : \mathscr{K} \times \mathscr{K} \to [0, \infty)$ is called controlled rectangular b-metric type if

D_a : $D(s, l) = 0 \Longleftrightarrow s = l$,
D_b : $D(s, l) = D(l, s)$,
D_c : $D(s, l) \leq \theta(s, l, u, v)[D(s, u) + D(u, v) + D(u, l)]$,

for all $s, l, \in \mathscr{K}$ and for all distinct points $u, v \in \mathscr{K} \backslash \{s, l\}$. Then, $(\mathscr{K}, D)$ is called a controlled rectangular metric type space.

The concept of controlled rectangular metric like space (CRMLS) is introduced in [27].

Definition 1.4. [27] Let $\mathscr{K} \neq \phi$ and $\alpha : \mathscr{K}^4 \to [1, \infty)$. The function $\delta_a : \mathscr{K} \times \mathscr{K} \to [0, \infty)$ is called controlled rectangular b-metric like if

$\delta_{(i)}$: $\delta_a(s, l) = \delta_a(l, s) = 0$ then $s = l$,
$\delta_{(ii)}$: $\delta_a(s, l) = \delta_a(l, s)$,
$\delta_{(iii)}$: $\delta_a(s, l) \leq \alpha(s, l, u, v)[\delta_a(s, u) + \delta_a(u, v) + \delta_a(u, l)]$,

for all $s, l, \in \mathscr{K}$ and for all distinct points $u, v \in \mathscr{K} \backslash \{s, l\}$. Then, $(\mathscr{K}, \delta_a)$ is called a controlled rectangular metric like space.

Example 1.5. [27] Consider $\mathscr{K} = [0, \infty)$ and $p : [0, +\infty) \times [0, +\infty) \to (1, +\infty)$. Define $\delta_a : \mathscr{K}^2 \to [0, +\infty)$, by

$$\delta_a(s, l) = (s + l)^{p(s,\, l)} \qquad \text{for all} \quad x, y \in \mathscr{K}.$$

Note that $(\mathscr{K}, \delta_a)$ is a controlled rectangular metric like space with

$$\alpha(s, l, u, v) = 2^{p(\max\{s,l\}, \max\{u,v\})-1}.$$

Remark 1.6. In Definition 1.4, if the properties $\delta_{(i)}$ and $\delta_{(iii)}$ are fulfilled only, then the space is called controlled quasi rectangular metric like space.

2. Triple controlled quasi rectangular metric like space

In this section, first of all we define a triple controlled quasi rectangular metric like space. Our new classification, generalize controlled rectangular, controlled quasi rectangular metric like spaces.

Definition 2.1. Assume that $\alpha, \beta, \gamma : \mathscr{K} \times \mathscr{K} \to [1, \infty)$ are three mappings. If $\delta : \mathscr{K} \times \mathscr{K} \to [0, \infty)$ satisfies

δ_a : $\delta(s, l) = \delta(l, s) = 0$ then $s = l$,
δ_b : $\delta(s, l) \leq \alpha(s, u)\delta(s, u) + \beta(u, v)\delta(u, v) + \gamma(v, l)\delta(v, l)$,

for all $s, l, \in \mathscr{K}$ and for all distinct points $u, v \in \mathscr{K} \backslash \{s, l\}$. Then, δ is said to be a triple controlled quasi rectangular metric like by α, β and γ. The pair $(\mathscr{K}, \delta)$ is called triple controlled quasi rectangular metric like space or (simply, TCQRMLS).

Remark 2.2. Any controlled quasi rectangular metric like space is a special case of triple controlled quasi rectangular metric like space but the converse is not true in general.

Example 2.3. Let $\mathscr{K} = \{0, 1, 2, 3\}$. Consider a function $\delta : \mathscr{K} \times \mathscr{K} \to [0, +\infty)$ and $\alpha, \beta, \gamma : \mathscr{K} \times \mathscr{K} \to [1, +\infty)$ are three mappings, defined by (see Tables 1–4).

Table 1. Table for $\delta(s, l)$.

$\delta(s, l)$	0	1	2	3
0	1	1	2	$\frac{7}{2}$
1	2	1	2	1
2	2	2	2	$\frac{1}{4}$
3	6	$\frac{1}{3}$	$\frac{1}{3}$	0

Table 2. For α.

$\alpha(s, l)$	0	1	2	3
0	1	$\frac{4}{3}$	2	$\frac{4}{3}$
1	1	1	1	$\frac{3}{2}$
2	1	1	1	2
3	$\frac{4}{3}$	3	3	1

Table 3. For β.

$\beta(s, l)$	0	1	2	3
0	1	1	2	1
1	1	1	$\frac{3}{2}$	1
2	$\frac{4}{3}$	$\frac{3}{2}$	1	4
3	1	1	1	1

Table 4. For γ.

$\gamma(s, l)$	0	1	2	3
0	1	$\frac{7}{2}$	1	1
1	$\frac{3}{2}$	1	2	2
2	1	1	1	$\frac{9}{2}$
3	2	1	3	1

It is easy to show that $(\mathscr{K}, \delta)$ is triple controlled quasi rectangular metric like space for all pairwise different $s, l, u, v \in \mathscr{K}$.

3. Topological properties of triple controlled quasi rectangular metric like spaces

By the use of TCQRMLS a topology will be defined and its characteristics will be examined. Let $(\mathscr{K}, \delta)$ be a TCQRMLS. Having radius $\lambda > 0$, the right centered ball at $\tilde{s} \in \mathscr{K}$, is the set

$$B_r(\tilde{s}; \lambda) = \{u \in \mathscr{K}, \mid \delta(u, \tilde{s}) - \delta(\tilde{s}, \tilde{s}) \mid < \lambda\},$$

where, the left centered ball at $\tilde{s}$, having radius $\lambda > 0$ is

$$B_l(\tilde{s}; \lambda) = \{u \in \mathscr{K}, \mid \delta(\tilde{s}, u) - \delta(\tilde{s}, \tilde{s}) \mid < \lambda\}.$$

Note that, the open ball in TCQRMLS is not necessarily an open set. Furthermore, let $\mathcal{B}$ be the collection of all subsets $\mathcal{U}$ of $\mathscr{K}$ satisfying the condition that for each $s \in \mathscr{K}$ there exist $\lambda > 0$ such that $B_r(s;\lambda) \subset \mathcal{U}$ or $B_l(s;\lambda) \subset \mathcal{U}$. Then, $\mathcal{B}$ defines a base for the topology on set $\mathscr{K}$ which is not necessarily Hausdorff.

As next, we will investigate convergence on TCQRMLS.

Definition 3.1. Suppose that $(\mathscr{K}, \delta)$ is a TCQRMLS with three mappings α, β and γ.

(i) A sequence $\{\varrho_n\}$ is said to be convergent to some $\varrho \in \mathscr{K}$ if and only if

$$\lim_{n\to+\infty} \delta(\varrho_n, \varrho) = \lim_{n\to+\infty} \delta(\varrho, \varrho_n) = \delta(\varrho, \varrho).$$

(ii) A sequence $\{\varrho_n\}$ is said to be left Cauchy sequence if and only if for all $m > n$, we have $\lim_{m,n\to+\infty} \delta(\varrho_m, \varrho_n)$ exists and is finite.

(iii) A sequence $\{\varrho_n\}$ is said to be right Cauchy sequence if and only if for all $m > n$, we have $\lim_{m,n\to+\infty} \delta(\varrho_n, \varrho_m)$ exists and is finite.

(iv) A sequence is Cauchy sequence if and only if $\{\varrho_n\}$ is left and right Cauchy.

(v) The pair $(\mathscr{K}, \delta)$ is left complete, right complete and complete if and only if each left Cauchy, right Cauchy and Cauchy sequence in $\mathscr{K}$ is convergent respectively.

Remark 3.2. Topology of (TCQRMLS) is not necessarily a Hausdorff topology, so the limit of convergent sequence is not always unique.

Example 3.3. Let $\mathscr{K} = \{0, 1, 2, 3\}$. Consider a function $\delta : \mathscr{K} \times \mathscr{K} \to [0, +\infty)$ and three control functions $\alpha, \beta, \gamma : \mathscr{K} \times \mathscr{K} \to [1, +\infty)$, defined as in Example 2.3. Thus $(\mathscr{K}, \delta)$ is a (TCQRMLS). The constant sequence $(\varrho_n = 1)_{n\in\mathbb{N}}$ is convergent with both 1 and 2 as limits, since

$$\lim_{n\to+\infty} \delta(\varrho_n, 1) = \lim_{n\to+\infty} \delta(1, \varrho_n) = \delta(1, 1) = 1$$

$$\lim_{n\to+\infty} \delta(\varrho_n, 2) = \delta(1, 2) = \delta(2, 1) = \lim_{n\to+\infty} \delta(2, \varrho_n) = \delta(2, 2) = 2.$$

Thus, the limit of a convergent sequence is not always unique.

4. Main result

Theorem 4.1. *Let $(\mathscr{K}, \delta)$ be a complete triple controlled quasi rectangular metric like space where $\alpha, \beta, \gamma : \mathscr{K} \times \mathscr{K} \to [1, \infty)$ are mappings, suppose that $\mathscr{L}$ is a self mapping on $\mathscr{K}$ satisfying the following conditions. If there exists $k \in (0, 1)$ such that.*

$$\delta(\mathscr{L}s, \mathscr{L}l) > 0 \implies \delta(\mathscr{L}s, \mathscr{L}l) \le k\delta(s, l) \quad \forall\ s, l \in \mathscr{K}. \tag{4.1}$$

For, $\varrho_o \in \mathscr{K}$ take $\varrho_n = \mathscr{L}^n \varrho_o,\ n \in \mathbb{N}$. Suppose that.

$$\sup_{m\ge1} \lim_{i\to+\infty} \gamma(\varrho_i, \varrho_m) \frac{\alpha(\varrho_{i+1}, \varrho_{i+2}) + k\beta(\varrho_{i+2}, \varrho_{i+3})}{\alpha(\varrho_i, \varrho_{i+1}) + k\beta(\varrho_{i+1}, \varrho_{i+2})} < \frac{1}{k}. \tag{4.2}$$

We assume that, for $\varrho \in \mathscr{K}$, we have $\lim_{n\to+\infty} \Delta(\varrho_n, \varrho)$, $\lim_{n\to+\infty} \Delta(\varrho, \varrho_n)$, $\lim_{n\to+\infty} \Delta(\varrho_n, \mathscr{L}\varrho)$, $\lim_{n\to+\infty} \Delta(\mathscr{L}\varrho, \varrho_n)$ and $\lim_{n,m\to+\infty} \Delta(\varrho_n, \varrho_m)$ where $\Delta \in \{\alpha, \beta, \gamma\}$, exist and finite for all $n, m \in \mathbb{N}$, $m \ne n$. Then, $\mathscr{L}$ has a fixed point.

Proof. Suppose that $\varrho_o \in \mathscr{K}$. Then, define $\{\varrho_n\}$ as an iterative sequence by

$$\varrho_1 = \mathscr{L}\varrho_o,\ \varrho_2 = \mathscr{L}\varrho_1 = \mathscr{L}^2\varrho_o \ldots \varrho_{n+1} = \mathscr{L}\varrho_n = \mathscr{L}^{n+1}\varrho_o.$$

Obviously, if there exists $n_0 \in \mathbb{N}$ for which $\varrho_{n_0+1} = \varrho_{n_0}$, then $\mathscr{L}\varrho_{n_0} = \varrho_{n_0}$, and the proof is finished. Thus, we suppose that $\varrho_{n+1} \neq \varrho_n$ for every $n_0 \in \mathbb{N}$ Thus, by (4.1), we have

$$\begin{aligned}\delta(\varrho_n, \varrho_{n+1}) = & \ \delta(\mathscr{L}\varrho_{n-1}, \mathscr{L}\varrho_n) \\ \leq & \ k\delta(\varrho_{n-1}, \varrho_n) \\ & \vdots \\ \delta(\varrho_n, \varrho_{n+1}) \leq & \ k^n\delta(\varrho_o, \varrho_1).\end{aligned} \tag{4.3}$$

In the inequality above, if we take $\lim n \to +\infty$, we get

$$\lim_{n\to+\infty} \delta(\varrho_n, \varrho_{n+1}) = 0. \tag{4.4}$$

Similarly we can prove that

$$\lim_{n\to+\infty} \delta(\varrho_{n+1}, \varrho_n) = 0. \tag{4.5}$$

Thus, we have two cases.

Case I: We suppose that for all $n, m \in \mathbb{N}$, $\varrho_n \neq \varrho_m$. Indeed, assume that $\varrho_n = \varrho_m$ for some $n = m + c$ with $c > 0$, and $\mathscr{L}\varrho_n = \mathscr{L}\varrho_m$, then it follows that.

$$\begin{aligned}\delta(\varrho_n, \varrho_{n+1}) = \delta(\varrho_m, \varrho_{m+1}) &= \delta(\mathscr{L}\varrho_{m-1}, \mathscr{L}\varrho_m) \\ &\leq k\delta(\varrho_{m-1}, \varrho_m) \\ &\leq k^2\delta(\varrho_{m-2}, \varrho_{m-1}) \\ &\leq \cdots \\ &\leq k^c\delta(\varrho_m, \varrho_{m+1}) \\ &= k^c\delta(\varrho_n, \varrho_{n+1})\end{aligned}$$

$(1 - k^c)\delta(\varrho_n, \varrho_{n+1}) \leq 0$. This implies that $\delta(\varrho_n, \mathscr{L}\varrho_n) = 0$.
Similarly, $\delta(\mathscr{L}\varrho_n, \varrho_n) = 0$. Thus ϱ_n is a fixed point of $\mathscr{L}$.

Case II: Let $\varrho_n \neq \varrho_m$ for all integers $n \neq m$. Let $n < m \in \mathbb{N}$, and to show that $\{\varrho_n\}$ is a right Cauchy sequence we consider two sub cases:

Subcase I: Assume that $m = n + 2p + 1$ with $p \geq 1$. We get the desired outcome by applying the property δ_{q_2} of TCQRM repeatedly.

$$\begin{aligned}\delta(\varrho_n, \varrho_m) = \delta(\varrho_n, \varrho_{n+2p+1}) & \\ \leq \alpha(\varrho_n, \varrho_{n+1})\delta(\varrho_n, \varrho_{n+1}) &+ \beta(\varrho_{n+1}, \varrho_{n+2})\delta(\varrho_{n+1}, \varrho_{n+2}) + \\ &\gamma(\varrho_{n+2}, \varrho_{n+2p+1})\delta(\varrho_{n+2}, \varrho_{n+2p+1}) \\ \leq \alpha(\varrho_n, \varrho_{n+1})\delta(\varrho_n, \varrho_{n+1}) &+ \beta(\varrho_{n+1}, \varrho_{n+2})\delta(\varrho_{n+1}, \varrho_{n+2}) + \\ \gamma(\varrho_{n+2}, \varrho_{n+2p+1})\Big[\alpha(\varrho_{n+2}, \varrho_{n+3})\delta(\varrho_{n+2}, \varrho_{n+3}) &+ \beta(\varrho_{n+3}, \varrho_{n+4})\delta(\varrho_{n+3}, \varrho_{n+4}) +\end{aligned}$$

$$
\begin{aligned}
&\gamma(\varrho_{n+4},\varrho_{n+2p+1})\delta(\varrho_{n+4},\varrho_{n+2p+1})\Big] \\
&\leq \cdots \\
&\leq \alpha(\varrho_n,\varrho_{n+1})\delta(\varrho_0,\varrho_1)k^n+\beta(\varrho_{n+1},\varrho_{n+2})\delta(\varrho_0,\varrho_1)k^{n+1}+ \\
&\gamma(\varrho_{n+2},\varrho_{n+2p+1})[\alpha(\varrho_{n+2},\varrho_{n+3})k^{n+2}+\beta(\varrho_{n+3},\varrho_{n+4})k^{n+3}]\delta(\varrho_0,\varrho_1)+\cdots \\
&\gamma(\varrho_{n+2},\varrho_{n+2p+1})\gamma(\varrho_{n+4},\varrho_{n+2p+1})\times\cdots\times\gamma(\varrho_{n+2p-2},\varrho_{n+2p+1}) \\
&\times[\alpha(\varrho_{n+2p-2},\varrho_{n+2p-1})k^{n+2p-2}+\beta(\varrho_{n+2p-1},\varrho_{n+2p})k^{n+2p-1}]\delta(\varrho_0,\varrho_1)+ \\
&\gamma(\varrho_{n+2},\varrho_{n+2p+1})\gamma(\varrho_{n+4},\varrho_{n+2p+1})\times\cdots\times\gamma(\varrho_{n+2p},\varrho_{n+2p+1})k^{n+2p}\delta(\varrho_0,\varrho_1) \\
&=\alpha(\varrho_n,\varrho_{n+1})\delta(\varrho_0,\varrho_1)k^n+\beta(\varrho_{n+1},\varrho_{n+2})\delta(\varrho_0,\varrho_1)k^{n+1}+ \\
&\sum_{i=n+2}^{n+2p}\prod_{j=n+2}^{i}\gamma(\varrho_j,\varrho_{n+2p+1})[\alpha(\varrho_i,\varrho_{i+1})k^i+\beta(\varrho_{i+1},\varrho_{i+2})k^{n+2i+1}]\delta(\varrho_0,\varrho_1) \\
&\qquad+\prod_{j=n+2}^{n+2p}\gamma(\varrho_j,\varrho_{n+2p+1})k^{n+2p}\delta(\varrho_0,\varrho_1).
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\delta(\varrho_n,\varrho_m)\leq{}&\alpha(\varrho_n,\varrho_{n+1})\delta(\varrho_0,\varrho_1)k^n+\beta(\varrho_{n+1},\varrho_{n+2})\delta(\varrho_0,\varrho_1)k^{n+1}+ \\
&\sum_{i=n+2}^{n+2p}\prod_{j=n+2}^{i}\gamma(\varrho_j,\varrho_{n+2p+1})[\alpha(\varrho_i,\varrho_{i+1})k^i+\beta(\varrho_{i+1},\varrho_{i+2})k^{i+1}]\delta(\varrho_0,\varrho_1) \\
&+\prod_{j=n+2}^{n+2p}\gamma(\varrho_{n+2j},\varrho_{n+2p+1})k^{n+2p}\delta(\varrho_0,\varrho_1).
\end{aligned}
\tag{4.6}
$$

Since, $\sup_{m\geq 1}\lim_{i+1\to+\infty}\gamma(\varrho_i,\varrho_m)\dfrac{\alpha(\varrho_{i+1},\varrho_{i+2})+k\beta(\varrho_{i+2},\varrho_{i+3})}{\alpha(\varrho_i,\varrho_{i+1})+k\beta(\varrho_{i+1},\varrho_{i+2})}<\frac{1}{k}$. So, the series

$$
\sum_{i=1}^{\infty}\prod_{j=1}^{i}\gamma(\varrho_j,\varrho_{n+2p+1})[\alpha(\varrho_i,\varrho_{i+1})+k\beta(\varrho_{i+1},\varrho_{i+2})]k^i \tag{4.7}
$$

converges by ratio test, which implies that $\delta(\varrho_n,\varrho_{n+2p+1})$ converges as $n\to\infty$.
Now, let

$$
S_n=\sum_{i=1}^{n}\prod_{j=1}^{i}\gamma(\varrho_j,\varrho_{n+2p+1})[\alpha(\varrho_i,\varrho_{i+1})+k\beta(\varrho_{i+1},\varrho_{i+2})]k^i\delta(\varrho_0,\varrho_1). \tag{4.8}
$$

Then, Eq (4.6) takes the following form

$$
\begin{aligned}
\delta(\varrho_n,\varrho_m)\leq{}&\delta(\varrho_0,\varrho_1)[\alpha(\varrho_n,\varrho_{n+1})k^n+\beta(\varrho_{n+1},\varrho_{n+2})k^{n+1}+(S_{m-1}-S_{n+1})] \\
&+\prod_{j=n+2}^{n+2p}\gamma(\varrho_j,\varrho_{n+2p+1})k^{n+2p}\delta(\varrho_0,\varrho_1).
\end{aligned}
\tag{4.9}
$$

Further, if we take limit in inequality (4.9) as $m, n \to +\infty$, we deduce that

$$\lim_{m,n\to+\infty} \delta(\varrho_n, \varrho_m) = 0. \tag{4.10}$$

Subcase II: where $m = n + 2p$, first of all, when $p = 1$ we have

$$\delta(\varrho_n, \varrho_m) = \delta(\varrho_n, \varrho_{n+2}) \le k\delta(\varrho_{n-1}, \varrho_{n+1}) \le k^2\delta(\varrho_{n-2}, \varrho_n) \le \cdots \le k^n\delta(\varrho_0, \varrho_2)$$

which leads us to conclude that $\delta(\varrho_n, \varrho_m) \to 0$, as $n \to \infty$.
When $p > 1$, Similar to subcase I, we have

$$\begin{aligned}
\delta(\varrho_n, \varrho_m) &= \delta(\varrho_n, \varrho_{n+2p}) \\
&\le \alpha(\varrho_n, \varrho_{n+2})\delta(\varrho_n, \varrho_{n+2}) + \beta(\varrho_{n+2}, \varrho_{n+3})\delta(\varrho_{n+2}, \varrho_{n+3}) + \\
&\qquad \gamma(\varrho_{n+3}, \varrho_{n+2p})\delta(\varrho_{n+3}, \varrho_{n+2p}) \\
&\le \alpha(\varrho_n, \varrho_{n+2})\delta(\varrho_n, \varrho_{n+2}) + \beta(\varrho_{n+2}, \varrho_{n+3})\delta(\varrho_{n+2}, \varrho_{n+3}) + \\
&\quad \gamma(\varrho_{n+3}, \varrho_{n+2p})\Big[\alpha(\varrho_{n+3}, \varrho_{n+4})\delta(\varrho_{n+3}, \varrho_{n+4}) + \beta(\varrho_{n+4}, \varrho_{n+5})\delta(\varrho_{n+4}, \varrho_{n+5}) + \\
&\qquad \gamma(\varrho_{n+5}, \varrho_{n+2p})\delta(\varrho_{n+5}, \varrho_{n+2p})\Big] \\
&\le \cdots \\
&\le \alpha(\varrho_n, \varrho_{n+2})\delta(\varrho_0, \varrho_2)k^n + \beta(\varrho_{n+2}, \varrho_{n+3})\delta(\varrho_0, \varrho_1)k^{n+2} + \\
&\quad \gamma(\varrho_{n+3}, \varrho_{n+2p})[\alpha(\varrho_{n+3}, \varrho_{n+4})k^{n+3} + \beta(\varrho_{n+4}, \varrho_{n+5})k^{n+4}]\delta(\varrho_0, \varrho_1) + \cdots \\
&\quad \gamma(\varrho_{n+3}, \varrho_{n+2p})\gamma(\varrho_{n+5}, \varrho_{n+2p}) \times \cdots \times \gamma(\varrho_{n+2p-5}, \varrho_{n+2p}) \\
&\quad \times [\alpha(\varrho_{n+2p-5}, \varrho_{n+2p-4})k^{n+p-5} + \beta(\varrho_{n+2p-4}, \varrho_{n+2p-3})k^{n+p-4}]\delta(\varrho_0, \varrho_1) + \\
&\quad \gamma(\varrho_{n+3}, \varrho_{n+2p})\gamma(\varrho_{n+5}, \varrho_{n+2p}) \times \cdots \times \gamma(\varrho_{n+2p-3}, \varrho_{n+2p}) \\
&\quad \times [\alpha(\varrho_{n+2p-3}, \varrho_{n+2p-2})k^{n+2p-3} + \beta(\varrho_{n+2p-2}, \varrho_{n+2p-1})k^{n+2p-2}]\delta(\varrho_0, \varrho_1) + \\
&\qquad \gamma(\varrho_{n+3}, \varrho_{n+2p})\gamma(\varrho_{n+5}, \varrho_{n+2p}) \times \cdots \times \gamma(\varrho_{n+2p-1}, \varrho_{n+2p})k^{n+2p-1}\delta(\varrho_0, \varrho_1).
\end{aligned}$$

Thus, we conclude

$$\begin{aligned}
\delta(\varrho_n, \varrho_m) &\le \alpha(\varrho_n, \varrho_{n+2})\delta(\varrho_0, \varrho_2)k^n + \beta(\varrho_{n+2}, \varrho_{n+3})\delta(\varrho_0, \varrho_1)k^{n+2} + \\
&\sum_{i=n+3}^{n+2p-1} \prod_{j=n+3}^{i} \gamma(\varrho_j, \varrho_{n+2p})[\alpha(\varrho_i, \varrho_{i+1}) + k\beta(\varrho_{i+1}, \varrho_{i+2})]k^i\delta(\varrho_0, \varrho_1) + \\
&\qquad \prod_{j=n+3}^{n+2p-1} \gamma(\varrho_j, \varrho_{n+2p})k^{n+2p-1}\delta(\varrho_0, \varrho_1).
\end{aligned} \tag{4.11}$$

Since, $\sup_{m\ge1} \lim_{i\to+\infty} \gamma(\varrho_{i+1}, \varrho_m)\dfrac{\alpha(\varrho_{i+1}, \varrho_{i+2}) + k\beta(\varrho_{i+2}, \varrho_{i+3})}{\alpha(\varrho_i, \varrho_{i+1}) + k\beta(\varrho_{i+1}, \varrho_{i+2})} < \frac{1}{k}$. So, the series

$$\sum_{i=1}^{\infty} \prod_{j=1}^{i} \gamma(\varrho_j, \varrho_{n+2p})[\alpha(\varrho_i, \varrho_{i+1}) + k\beta(\varrho_{i+1}, \varrho_{i+2})]k^i \tag{4.12}$$

converges by ratio test, which implies that $\delta(\varrho_n, \varrho_{n+2p})$ for $p > 1$ converges as $n \to \infty$.
Now, let

$$S_q = \sum_{i=1}^{q} \prod_{j=1}^{i} \gamma(\varrho_j, \varrho_{n+2p+1})[\alpha(\varrho_i, \varrho_{i+1}) + k\beta(\varrho_{i+1}, \varrho_{i+2})]k^i \delta(\varrho_0, \varrho_1). \tag{4.13}$$

Then, Eq (4.11) takes the following form

$$\begin{aligned}\delta(\varrho_n, \varrho_m) \leq \delta(\varrho_0, \varrho_2)\alpha(\varrho_n, \varrho_{n+2})k^n + \delta(\varrho_0, \varrho_1)\beta(\varrho_{n+2}, \varrho_{n+3})k^{n+2} + \\ (S_{m-1} - S_{n+2})\delta(\varrho_0, \varrho_1) + \prod_{j=n+3}^{n+2p-1} \gamma(\varrho_j, \varrho_{n+2p})k^{n+2p-1}\delta(\varrho_0, \varrho_1).\end{aligned} \tag{4.14}$$

Similarly, if we take limit in inequality (4.14) as $m, n \to +\infty$, we get

$$\lim_{m,n\to+\infty} \delta(\varrho_m, \varrho_n) = 0. \tag{4.15}$$

Thus, by subcases I and II, we have $\{\varrho_n\}$ is a right Cauchy sequence. Similarly, we can prove same for the left Cauchy sequence. Since $(\mathscr{K}, \delta)$ is complete triple controlled quasi rectangular metric like space. So $\{\varrho_n\}$ converges to $\varrho \in \mathscr{K}$. Thus:

$$\begin{aligned}\lim_{n\to+\infty} \delta(\varrho, \varrho_n) &= \lim_{n\to+\infty} \delta(\varrho_n, \varrho) = \delta(\varrho, \varrho) \\ &= \lim_{m,n\to+\infty} \delta(\varrho_n, \varrho_m) \\ &= \lim_{m,n\to+\infty} \delta(\varrho_m, \varrho_n) = 0.\end{aligned} \tag{4.16}$$

Then, $\delta(\varrho, \varrho) = 0$.
Existence of fixed point: We will now illustrate that ϱ is a fixed point of $\mathscr{L}$. Suppose that $\delta(\varrho, \mathscr{L}\varrho) > 0$, and $\delta(\varrho, \mathscr{L}\varrho) > 0$. Now by the property (δ_{q_2}), we get

$$\begin{aligned}\delta(\mathscr{L}\varrho, \varrho) &\leq \alpha(\mathscr{L}\varrho, \mathscr{L}\varrho_n)\delta(\mathscr{L}\varrho, \mathscr{L}\varrho_n) + \beta(\mathscr{L}\varrho_n, \varrho_n)\delta(\mathscr{L}\varrho_n, \varrho_n) + \gamma(\varrho_n, \varrho)\delta(\varrho_n, \varrho) \\ &\leq \alpha(\mathscr{L}\varrho, \varrho_{n+1})k\delta(\varrho, \varrho_n) + \beta(\varrho_{n+1}, \varrho_n)\delta(\varrho_{n+1}, \varrho_n) + \gamma(\varrho_n, \varrho)\delta(\varrho_n, \varrho).\end{aligned}$$

And,

$$\begin{aligned}\delta(\varrho, \mathscr{L}\varrho) &\leq \alpha(\varrho, \varrho_n)\delta(\varrho, \varrho_n) + \beta(\varrho_n, \mathscr{L}\varrho_n)\delta(\varrho_n, \mathscr{L}\varrho_n) + \gamma(\mathscr{L}\varrho_n, \mathscr{L}\varrho)\delta(\mathscr{L}\varrho_n, \mathscr{L}\varrho) \\ &\leq \alpha(\varrho, \varrho_n)\delta(\varrho, \varrho_n) + \beta(\varrho_n, \varrho_{n+1})\delta(\varrho_n, \varrho_{n+1}) + \gamma(\varrho_{n+1}, \mathscr{L}\varrho)k\delta(\varrho_n, \varrho).\end{aligned}$$

By taking $\lim n \to +\infty$ using (4.4), (4.5),(4.16), we deduce that $\delta(\varrho, \mathscr{L}\varrho) = 0 = \delta(\mathscr{L}\varrho, \varrho)$, Thus $\mathscr{L}\varrho = \varrho$ is a fixed point of $\mathscr{L}$.

□

Corollary 4.2. *Let $(\mathscr{K}, \delta)$ be a complete controlled quasi rectangular metric like space where $\alpha, \mathscr{K} \times \mathscr{K} \to [1, \infty)$ and suppose that $\mathscr{L}$ is a self mapping on $\mathscr{K}$ satisfying the following conditions. If there exists $k \in (0, 1)$ such that.*

$$\sup_{m\geq 1} \lim_{i\to+\infty} \alpha(\varrho_i, \varrho_m) \frac{\alpha(\varrho_{i+1}, \varrho_{i+2}) + k\alpha(\varrho_{i+2}, \varrho_{i+3})}{\alpha(\varrho_i, \varrho_{i+1}) + k\alpha(\varrho_{i+1}, \varrho_{i+2})} < \frac{1}{k}. \tag{4.17}$$

We assume that, for $\varrho \in \mathscr{K}$, we have
$\lim_{n\to+\infty} \alpha(\varrho_n, \varrho)$, $\lim_{n\to+\infty} \alpha(\varrho, \varrho_n)$, $\lim_{n\to+\infty} \alpha(\varrho_n, \mathscr{L}\varrho)$, $\lim_{n\to+\infty} \alpha(\mathscr{L}\varrho, \varrho_n)$ *and* $\lim_{n,m\to+\infty} \alpha(\varrho_n, \varrho_m)$ *exist and finite for all* $n, m \in \mathbb{N}$, $m \neq n$. *Then,* $\mathscr{L}$ *has a fixed point.*

Corollary 4.3. *Let* $(\mathscr{K}, \delta)$ *be a complete triple controlled quasi rectangular metric type space where* $\alpha, \beta, \gamma : \mathscr{K} \times \mathscr{K} \to [1, \infty)$ *are mappings, suppose that* $\mathscr{L}$ *is a self mapping on* $\mathscr{K}$ *satisfying the following conditions. If there exists* $k \in (0, 1)$ *such that*

$$\delta(\mathscr{L}s, \mathscr{L}l) > 0 \implies \delta(\mathscr{L}s, \mathscr{L}l) \leq k\delta(s, l) \quad \forall\ s, l \in \mathscr{K}. \tag{4.18}$$

For, $\varrho_o \in \mathscr{K}$ *take* $\varrho_n = \mathscr{L}^n \varrho_o$, $n \in \mathbb{N}$. *Suppose that*

$$\sup_{m\geq 1} \lim_{i\to+\infty} \gamma(\varrho_i, \varrho_m) \frac{\alpha(\varrho_{i+1}, \varrho_{i+2}) + k\beta(\varrho_{i+2}, \varrho_{i+3})}{\alpha(\varrho_i, \varrho_{i+1}) + k\beta(\varrho_{i+1}, \varrho_{i+2})} < \frac{1}{k}. \tag{4.19}$$

We assume that, for $\varrho \in \mathscr{K}$, we have
$\lim_{n\to+\infty} \Delta(\varrho_n, \varrho)$, $\lim_{n\to+\infty} \Delta(\varrho, \varrho_n)$, $\lim_{n\to+\infty} \Delta(\varrho_n, \mathscr{L}\varrho)$, $\lim_{n\to+\infty} \Delta(\mathscr{L}\varrho, \varrho_n)$ *and* $\lim_{n,m\to+\infty} \Delta(\varrho_n, \varrho_m)$ *where* $\Delta \in \{\alpha, \beta, \gamma\}$, *exist and finite for all* $n, m \in \mathbb{N}$, $m \neq n$. *Then,* $\mathscr{L}$ *has a unique fixed point.*

Proof. The existence of a fixed point follows immediately from Theorem 4.1. To prove the uniqueness, let $\mathscr{L}$ have two distinct fixed points, $\mathscr{Y}_1$ and $\mathscr{Y}_2$. Then, $\mathscr{L}\mathscr{Y}_1 = \mathscr{Y}_1$ and $\mathscr{L}\mathscr{Y}_2 = \mathscr{Y}_2$.
Consider,

$$\delta_q(\mathscr{Y}_1, \mathscr{Y}_2) = \delta_q(\mathscr{L}\mathscr{Y}_1, \mathscr{L}\mathscr{Y}_2) \leq k\delta_q(\mathscr{Y}_1, \mathscr{Y}_2) < \delta_q(\mathscr{Y}_1, \mathscr{Y}_2).$$

Similarly,

$$\delta_q(\mathscr{Y}_2, \mathscr{Y}_1) \leq \delta_q(\mathscr{L}\mathscr{Y}_2, \mathscr{L}\mathscr{Y}_1) \leq k\delta_q(\mathscr{Y}_2, \mathscr{Y}_1) < \delta_q(\mathscr{Y}_2, \mathscr{Y}_1).$$

Which holds, unless $\delta_q(\mathscr{Y}_2, \mathscr{Y}_1) = 0$ and $\delta_q(\mathscr{Y}_1, \mathscr{Y}_2) = 0$. Thus, $\delta_q(\mathscr{Y}_1, \mathscr{Y}_2) = 0$ implies $\mathscr{Y}_1 = \mathscr{Y}_2$. Hence fixed point of $\mathscr{L}$ is unique. □

Theorem 4.4. *Let* $(\mathscr{K}, \delta)$ *be a complete triple controlled quasi rectangular metric like space where* $\alpha, \beta, \gamma : \mathscr{K} \times \mathscr{K} \to (0, \infty)$ *are mappings. Suppose that* $\mathscr{L}$ *is a self mapping on* $\mathscr{K}$ *satisfying the following condition. If there exists* $\lambda \in (0, \frac{1}{2})$ *and* $\delta(\mathscr{L}s, \mathscr{L}l) > 0$, *such that.*

$$\delta(\mathscr{L}s, \mathscr{L}l) \leq \lambda \min[\delta(s, \mathscr{L}s) + \delta(l, \mathscr{L}l), \delta(\mathscr{L}s, s) + \delta(\mathscr{L}l, l)]. \tag{4.20}$$

For, $\varrho_o \in \mathscr{K}$ *take* $\varrho_n = \mathscr{L}^n \varrho_o$, $n \in \mathbb{N}$. *Suppose that.*

$$\sup_{m\geq 1} \lim_{i\to+\infty} \gamma(\varrho_{i+1}, \varrho_m) \frac{\alpha(\varrho_{i+1}, \varrho_{i+2}) + (\frac{\lambda}{1-\lambda})\beta(\varrho_{i+2}, \varrho_{i+3})}{\alpha(\varrho_i, \varrho_{i+1}) + (\frac{\lambda}{1-\lambda})\beta(\varrho_{i+1}, \varrho_{i+2})} < \frac{1-\lambda}{\lambda}. \tag{4.21}$$

We assume that, for $\varrho \in \mathscr{K}$, we have
$\lim_{n\to+\infty} \Delta(\varrho_n, \varrho)$, $\lim_{n\to+\infty} \Delta(\varrho, \varrho_n)$ *and* $\lim_{n,m\to+\infty} \Delta(\varrho_n, \varrho_m)$, *where* $\Delta \in \{\alpha, \beta, \gamma\}$, *exist and finite for all* $m, n \in \mathbb{N}$, $m \neq n$ *such that.*

$$\lim_{n\to+\infty} \Delta(\mathscr{L}\varrho, \mathscr{L}\varrho_n) < \frac{1}{\lambda} \quad or \quad \lim_{n\to+\infty} \Delta(\mathscr{L}\varrho_n, \mathscr{L}\varrho) < \frac{1}{\lambda}. \tag{4.22}$$

Then, $\mathscr{L}$ *has a fixed point.*

Proof. Let $\varrho_o \in \mathscr{K}$ and $\{\varrho_n\}$ is an iterative sequence defined by

$$\varrho_1 = \mathscr{L}\varrho_o,\ \varrho_2 = \mathscr{L}\varrho_1 = \mathscr{L}^2\varrho_o\ \ldots \varrho_{n+1} = \mathscr{L}\varrho_n = \mathscr{L}^{n+1}\varrho_o.$$

Obviously, if there exists $n_0 \in \mathbb{N}$ for which $\varrho_{n_0+1} = \varrho_{n_0}$, then $\mathscr{L}\varrho_{n_0} = \varrho_{n_0}$, and the proof is finished. Thus, we suppose that $\varrho_{n+1} \neq \varrho_n$ for every $n_0 \in \mathbb{N}$. Thus, by (4.20), we have

$$\begin{aligned}
\delta(\varrho_n, \varrho_{n+1}) &= \delta(\mathscr{L}\varrho_{n-1}, \mathscr{L}\varrho_n) \\
&\leq \lambda[\delta(\varrho_{n-1}, \mathscr{L}\varrho_{n-1}) + \delta(\varrho_n, \mathscr{L}\varrho_n)] \\
&\leq \lambda\delta(\varrho_{n-1}, \varrho_n) + \lambda\delta(\varrho_n, \varrho_{n+1}) \\
\delta(\varrho_n, \varrho_{n+1}) &\leq \left(\frac{\lambda}{1-\lambda}\right) \delta(\varrho_{n-1}, \varrho_n) \\
&\leq k\delta(\varrho_{n-1}, \varrho_n) \qquad \text{where } k = \frac{\lambda}{1-\lambda} \qquad \text{and} \quad k \in (0,1) \\
&\leq k^2\delta(\varrho_{n-2}, \varrho_{n-1}) \\
&\vdots \\
\delta(\varrho_n, \varrho_{n+1}) &\leq k^n\delta(\varrho_o, \varrho_1).
\end{aligned} \tag{4.23}$$

Therefore,

$$\lim_{n\to+\infty} \delta(\varrho_n, \varrho_{n+1}) = 0. \tag{4.24}$$

Similarly, we consider

$$\begin{aligned}
\delta(\varrho_{n+1}, \varrho_n) &= \delta(\mathscr{L}\varrho_n, \mathscr{L}\varrho_{n-1}) \\
&\leq \lambda[\delta(\mathscr{L}\varrho_n, \varrho_n) + \delta(\mathscr{L}\varrho_{n-1}, \varrho_{n-1})] \\
&\leq \lambda\delta(\varrho_{n+1}, \varrho_n) + \lambda\delta(\varrho_n, \varrho_{n-1}) \\
\delta(\varrho_{n+1}, \varrho_n) &\leq \left(\frac{\lambda}{1-\lambda}\right) \delta(\varrho_n, \varrho_{n-1}) \\
&\leq k\delta(\varrho_n, \varrho_{n-1}) \\
&\leq k^2\delta(\varrho_{n-1}, \varrho_{n-2}) \\
&\vdots \\
\delta(\varrho_{n+1}, \varrho_n) &\leq k^n\delta(\varrho_1, \varrho_o).
\end{aligned} \tag{4.25}$$

Therefore,

$$\lim_{n\to+\infty} \delta(\varrho_{n+1}, \varrho_n) = 0. \tag{4.26}$$

Now, consider

$$\delta(\varrho_n, \varrho_{n+2}) = \delta(\mathscr{L}\varrho_{n-1}, \mathscr{L}\varrho_{n+1}) \leq \lambda[\delta(\varrho_{n-1}, \varrho_n) + \delta(\varrho_{n+1}, \varrho_{n+2})].$$

Taking limit $n \to +\infty$, and using (4.26), we have

$$\lim_{n\to+\infty} \delta(\varrho_n, \varrho_{n+2}) = 0. \tag{4.27}$$

By using similar method as in Theorem (4.1), we can easily show that $\{\varrho_n\}$ is a Cauchy sequence in $(\mathscr{K}, \delta)$. Since $(\mathscr{K}, \delta)$ is complete triple controlled quasi rectangular metric like space. So $\{\varrho_n\}$ converges to $\varrho \in \mathscr{K}$. Thus,

$$\begin{aligned}\lim_{n\to+\infty} \delta(\varrho, \varrho_n) &= \lim_{n\to+\infty} \delta(\varrho_n, \varrho) \\ &= \delta(\varrho, \varrho) \\ &= \lim_{m,n\to+\infty} \delta(\varrho_n, \varrho_m) \\ &= \lim_{m,n\to+\infty} \delta(\varrho_m, \varrho_n) = 0. \end{aligned} \tag{4.28}$$

Then, $\delta(\varrho, \varrho) = 0$.

Existence of fixed point: Now, we will show that ϱ is a fixed point of $\mathscr{L}$. Suppose that $\delta(\varrho, \mathscr{L}\varrho) > 0$, and $\delta(\mathscr{L}\varrho, \varrho) > 0$. Now by property (δ_{q_2}), we get

$$\begin{aligned}\delta(\mathscr{L}\varrho, \varrho) &\leq \alpha(\mathscr{L}\varrho, \mathscr{L}\varrho_n)\delta(\mathscr{L}\varrho, \mathscr{L}\varrho_n) + \beta(\mathscr{L}\varrho_n, \varrho_n)\delta(\mathscr{L}\varrho_n, \varrho_n) + \gamma(\varrho_n, \varrho)\delta(\varrho_n, \varrho) \\ &\leq \alpha(\mathscr{L}\varrho, \mathscr{L}\varrho_n)\lambda[\delta(\mathscr{L}\varrho, \varrho) + \delta(\mathscr{L}\varrho_n, \varrho_n)] + \beta(\mathscr{L}\varrho_n, \varrho_n)\delta(\varrho_{n+1}, \varrho_n) + \\ &+ \quad \gamma(\varrho_n, \varrho)\delta(\varrho_n, \varrho) \\ &\leq \alpha(\mathscr{L}\varrho, \mathscr{L}\varrho_n)\lambda\delta(\mathscr{L}\varrho, \varrho) + \alpha(\mathscr{L}\varrho, \mathscr{L}\varrho_n)\lambda\delta(\varrho_{n+1}, \varrho_n) + \\ &+ \beta(\mathscr{L}\varrho_n, \varrho_n)\delta(\varrho_{n+1}, \varrho_n) + \gamma(\varrho_n, \varrho)\delta(\varrho_n, \varrho) \\ &\leq \alpha(\mathscr{L}\varrho, \mathscr{L}\varrho_n)\lambda\delta(\mathscr{L}\varrho, \varrho) + [\lambda\alpha(\mathscr{L}\varrho, \mathscr{L}\varrho_n) + \beta(\mathscr{L}\varrho_n, \varrho_n)]\delta(\varrho_{n+1}, \varrho_n) + \\ &+ \gamma(\varrho_n, \varrho)\delta(\varrho_n, \varrho). \end{aligned}$$

By taking $\lim n \to +\infty$ in both sides of the above inequalities using (4.22), (4.26) and (4.28), we deduce that $0 < \delta(\mathscr{L}\varrho, \varrho) < \delta(\mathscr{L}\varrho, \varrho)$. Similarly, we can show that $0 < \delta(\varrho, \mathscr{L}\varrho) < \delta(\varrho, \mathscr{L}\varrho)$, which is a contraction. Hence, $\mathscr{L}\varrho = \varrho$ is a fixed point of $\mathscr{L}$. □

Theorem 4.5. *Let $(\mathscr{K}, \delta)$ be a complete triple controlled quasi rectangular metric like space, where $\alpha, \beta, \gamma : \mathscr{K} \times \mathscr{K} \to (0, \infty)$ are mappings. Suppose that $\mathscr{L}$ is a self mapping on $\mathscr{K}$ satisfying the following condition. If there exists $\lambda \in (0, \frac{1}{3})$ and $\delta(\mathscr{L}s, \mathscr{L}l)$, such that.*

$$\begin{aligned}\delta(\mathscr{L}s, \mathscr{L}l) \leq \lambda \min[&\delta(s, l) + \delta(s, \mathscr{L}s) + \delta(l, \mathscr{L}l), \\ &\delta(s, l) + \delta(\mathscr{L}s, s) + \delta(\mathscr{L}l, l)]. \end{aligned} \tag{4.29}$$

For, $\varrho_o \in \mathscr{K}$ take $\varrho_n = \mathscr{L}^n\varrho_o$, $n \in \mathbb{N}$. Suppose that.

$$\sup_{m\geq 1} \lim_{i\to+\infty} \gamma(\varrho_{i+1}, \varrho_m) \frac{\alpha(\varrho_{i+1}, \varrho_{i+2}) + (\frac{2\lambda}{1-\lambda})\beta(\varrho_{i+2}, \varrho_{i+3})}{\alpha(\varrho_i, \varrho_{i+1}) + (\frac{2\lambda}{1-\lambda})\beta(\varrho_{i+1}, \varrho_{i+2})} < \frac{1-\lambda}{2\lambda}. \tag{4.30}$$

We assume that, for $\varrho \in \mathscr{K}$, we have
$\lim_{n\to+\infty} \Delta(\varrho_n, \varrho)$, $\lim_{n\to+\infty} \Delta(\varrho, \varrho_n)$ and $\lim_{n,m\to+\infty} \Delta(\varrho_n, \varrho_m)$, where $\Delta \in \{\alpha, \beta, \gamma\}$, exist and finite for all $n, m \in \mathbb{N}$, $m \neq n$ such that.

$$\lim_{n\to+\infty} \Delta(\mathscr{L}\varrho, \varrho_n) < \frac{1}{\lambda}, \quad \lim_{n\to+\infty} \Delta(\varrho_n, \mathscr{L}\varrho) < \frac{1}{\lambda}$$

$$\text{and} \quad \lim_{n\to+\infty} \Delta(\varrho_n, \mathscr{L}^2\varrho_n) < \frac{1}{\lambda}. \tag{4.31}$$

Then, $\mathscr{L}$ has a fixed point.

Proof. Let $\varrho_o \in \mathscr{K}$ and $\{\varrho_n\}$ is an iterative sequence defined by

$$\varrho_1 = \mathscr{L}\varrho_o,\ \varrho_2 = \mathscr{L}\varrho_1 = \mathscr{L}^2\varrho_o, \cdots, \varrho_{n+1} = \mathscr{L}\varrho_n = \mathscr{L}^{n+1}\varrho_o.$$

Obviously, if there exists $n_0 \in \mathbb{N}$ for which $\varrho_{n_0+1} = \varrho_{n_0}$, then $\mathscr{L}\varrho_{n_0} = \varrho_{n_0}$, and the proof is finished. Thus, we suppose that $\varrho_{n+1} \neq \varrho_n$ for every $n_0 \in \mathbb{N}$ Thus, by (4.29), we have

$$\begin{aligned}
\delta(\varrho_n, \varrho_{n+1}) &= \delta(\mathscr{L}\varrho_{n-1}, \mathscr{L}\varrho_n) \\
&\leq \lambda[\delta(\varrho_{n-1}, \varrho_n) + \delta(\varrho_{n-1}, \mathscr{L}\varrho_{n-1}) + \delta(\varrho_n, \mathscr{L}\varrho_n)] \\
&\leq \lambda\delta(\varrho_{n-1}, \varrho_n) + \lambda\delta(\varrho_{n-1}, \varrho_n) + \lambda\delta(\varrho_n, \varrho_{n+1}) \\
\delta(\varrho_n, \varrho_{n+1}) &\leq \left(\frac{2\lambda}{1-\lambda}\right) \delta(\varrho_{n-1}, \varrho_n) \\
&\leq k\delta(\varrho_{n-1}, \varrho_n) \qquad \text{where } k = \frac{2\lambda}{1-\lambda} \quad \text{and} \quad k \in (0,1) \\
&\leq k^2\delta(\varrho_{n-2}, \varrho_{n-1}) \\
&\ \vdots \\
\delta(\varrho_n, \varrho_{n+1}) &\leq k^n\delta(\varrho_o, \varrho_1).
\end{aligned} \tag{4.32}$$

Taking, $\lim n \to +\infty$, we get,

$$\lim_{n\to+\infty} \delta(\varrho_n, \varrho_{n+1}) = 0. \tag{4.33}$$

Similarly, we now consider

$$\begin{aligned}
\delta(\varrho_{n+1}, \varrho_n) &= \delta(\mathscr{L}\varrho_n, \mathscr{L}\varrho_{n-1}) \\
&\leq \lambda[\delta(\varrho_n, \varrho_{n-1}) + \delta(\mathscr{L}\varrho_n, \varrho_n) + \delta(\mathscr{L}\varrho_{n-1}, \varrho_{n-1})] \\
&\leq \lambda\delta(\varrho_n, \varrho_{n-1}) + \lambda\delta(\varrho_{n+1}, \varrho_n) + \lambda\delta(\varrho_n, \varrho_{n-1}) \\
\delta(\varrho_{n+1}, \varrho_n) &\leq \left(\frac{2\lambda}{1-\lambda}\right) \delta(\varrho_n, \varrho_{n-1}) \\
&\leq k\delta(\varrho_n, \varrho_{n-1}) \\
&\leq k^2\delta(\varrho_{n-1}, \varrho_{n-2}) \\
&\ \vdots \\
\delta(\varrho_{n+1}, \varrho_n) &\leq k^n\delta(\varrho_1, \varrho_o).
\end{aligned} \tag{4.34}$$

Taking, $\lim n \to +\infty$, we get,

$$\lim_{n\to+\infty} \delta(\varrho_{n+1}, \varrho_n) = 0. \tag{4.35}$$

Now, consider

$$\delta(\varrho_n, \varrho_{n+2}) = \delta(\mathscr{L}\varrho_{n-1}, \mathscr{L}\varrho_{n+1})$$

$$
\begin{aligned}
&\leq \lambda\Big[\delta(\varrho_{n-1},\varrho_{n+1})+\delta(\varrho_{n-1},\mathscr{L}\varrho_{n-1})+\delta(\varrho_{n+1},\mathscr{L}\varrho_{n+1})\Big]\\
&\leq \lambda\Big[\delta(\varrho_{n-1},\varrho_{n+1})+\delta(\varrho_{n-1},\varrho_{n})+\delta(\varrho_{n+1},\varrho_{n+2})\Big]\\
&\leq \lambda\Big[\alpha(\varrho_{n-1},\varrho_{n})\delta(\varrho_{n-1},\varrho_{n})+\beta(\varrho_{n},\varrho_{n+2})\delta(\varrho_{n},\varrho_{n+2})+\\
&\quad \gamma(\varrho_{n+2},\varrho_{n+1})\delta(\varrho_{n+2},\varrho_{n+1})\Big]+\lambda\delta(\varrho_{n-1},\varrho_{n})+\lambda\delta(\varrho_{n+1},\varrho_{n+2})\\
\delta(\varrho_{n},\varrho_{n+2}) &\leq \frac{1}{1-\lambda\beta(\varrho_{n},\mathscr{L}^{2}\varrho_{n})}\Big[\lambda[\alpha(\varrho_{n-1},\varrho_{n})+1]\delta(\varrho_{n-1},\varrho_{n})+\\
&\quad \gamma(\varrho_{n+2},\varrho_{n+1})\delta(\varrho_{n+2},\varrho_{n+1})+\lambda\delta(\varrho_{n+1},\varrho_{n+2})\Big].
\end{aligned}
$$

Taking limit $n\to+\infty$, in the above inequality, we get

$$\lim_{n\to+\infty}\delta(\varrho_{n},\varrho_{n+2})=0. \tag{4.36}$$

By using similar method as in Theorem (4.1), we can easily show that $\{\varrho_n\}$ is a Cauchy sequence in $(\mathscr{K},\delta)$. Since $(\mathscr{K},\delta)$ is complete triple controlled quasi rectangular metric like space. So $\{\varrho_n\}$ converges to $\varrho\in\mathscr{K}$, thus.

$$
\begin{aligned}
\lim_{n\to+\infty}\delta(\varrho,\varrho_{n}) &= \lim_{n\to+\infty}\delta(\varrho_{n},\varrho)\\
&= \delta(\varrho,\varrho)\\
&= \lim_{m,n\to+\infty}\delta(\varrho_{n},\varrho_{m})\\
&= \lim_{m,n\to+\infty}\delta(\varrho_{m},\varrho_{n})=0.
\end{aligned} \tag{4.37}
$$

Then, $\delta(\varrho,\varrho)=0$.

Existence of fixed point: Now, we will prove that ϱ is a fixed point of $\mathscr{L}$. For this, let $\delta(\mathscr{L}\varrho,\varrho)>0$. Now by property (δ_{q_2}), we get

$$
\begin{aligned}
\delta(\mathscr{L}\varrho,\varrho) &\leq \alpha(\mathscr{L}\varrho,\varrho_{n})\delta(\mathscr{L}\varrho,\mathscr{L}\varrho_{n})+\beta(\mathscr{L}\varrho_{n},\varrho_{n})\delta(\mathscr{L}\varrho_{n},\varrho_{n})+\gamma(\varrho_{n},\varrho)\delta(\varrho_{n},\varrho)\\
&\leq \alpha(\mathscr{L}\varrho,\varrho_{n})\lambda[\delta(\varrho,\varrho_{n})+\delta(\mathscr{L}\varrho,\varrho)+\delta(\mathscr{L}\varrho_{n},\varrho_{n})]+\beta(\mathscr{L}\varrho_{n},\varrho_{n})\delta(\varrho_{n+1},\varrho_{n})+\\
&+ \quad \gamma(\varrho_{n},\varrho)\delta(\varrho_{n},\varrho)\\
&\leq \lambda\alpha(\mathscr{L}\varrho,\varrho_{n})\delta(\varrho,\varrho_{n})+\lambda\alpha(\mathscr{L}\varrho,\varrho_{n})\delta(\mathscr{L}\varrho,\varrho)+\lambda\alpha(\mathscr{L}\varrho,\varrho_{n})\delta(\varrho_{n+1},\varrho_{n})+\\
&+ \beta(\varrho_{n+1},\varrho_{n})\delta(\varrho_{n+1},\varrho_{n})+\gamma(\varrho_{n},\varrho)\delta(\varrho_{n},\varrho).
\end{aligned}
$$

By taking $\lim n\to+\infty$ in both sides of the above inequalities and using (4.31), (4.35), (4.37), we deduce that $0<\delta(\mathscr{L}\varrho,\varrho)<\delta(\mathscr{L}\varrho,\varrho)$. Similarly, we can prove that, $0<\delta(\varrho,\mathscr{L}\varrho)<\delta(\varrho,\mathscr{L}\varrho)$, which is a contraction. Hence, $\mathscr{L}\varrho=\varrho$ is a fixed point of $\mathscr{L}$. □

Example 4.6. Let $\mathscr{K}=\mathscr{A}\cup\mathscr{B}$, where $\mathscr{A}=\{\frac{1}{n}:n\in\{3,4,5,6\}\}$ and $\mathscr{B}=[1,2]$. Define $\delta:\mathscr{K}\times\mathscr{K}\to[0,\infty[$ as follows:

$$\delta(s,l)=\delta(l,s)=0 \qquad \text{implies} \qquad s=l.$$

and

$$\begin{cases} \delta(\frac{1}{3}, \frac{1}{4}) = \delta(\frac{1}{4}, \frac{1}{5}) = 0.04 \\ \delta(\frac{1}{3}, \frac{1}{5}) = \delta(\frac{1}{4}, \frac{1}{6}) = 0.09 \\ \delta(\frac{1}{3}, \frac{1}{6}) = \delta(\frac{1}{5}, \frac{1}{6}) = 0.36 \\ \delta(\frac{1}{3}, \frac{1}{3}) = \delta(\frac{1}{5}, \frac{1}{5}) = 0.9 \\ \delta(s, l) = (s - l)^2 \qquad \textit{otherwise.} \end{cases}$$

Then, $(\mathscr{K}, \delta)$ is a complete triple controlled quasi rectangular metric like space with

$$\alpha(s, l) = \begin{cases} \min\{s, l\} & \text{if } s, l \in [1, 2] \\ 1 & \text{otherwise} \end{cases}$$

$$\beta(s, l) = \begin{cases} \max\{s, l\} + 2 & \text{if } s, l \in [1, 2] \\ 4 & \text{otherwise} \end{cases}$$

and

$$\gamma(s, l) = \begin{cases} \max\{s, l\} & \text{if } s, l \in [1, 2] \\ 3 & \text{otherwise} \end{cases}.$$

Note that δ we have the following

(i) $(\mathscr{K}, \delta)$ is neither metric type nor a rectangular metric type space, because

$$\delta(\frac{1}{3}, \frac{1}{3}) \neq 0.$$

(ii) The symmetric property does not holds in $(\mathscr{K}, \delta)$, as

$$\delta(\frac{1}{3}, \frac{1}{4}) \neq \delta(\frac{1}{4}, \frac{1}{3}).$$

(iii) $(\mathscr{K}, \delta)$ is not a quasi metric like space, because

$$\delta(\frac{1}{3}, \frac{1}{6}) = 0.36 > 0.13 = \delta(\frac{1}{3}, \frac{1}{4}) + \delta(\frac{1}{4}, \frac{1}{6}).$$

(iv) $(\mathscr{K}, \delta)$ is not a quasi rectangular metric like space, because

$$\delta(\frac{1}{5}, \frac{1}{6}) = 0.36 > 0.22 = \delta(\frac{1}{5}, \frac{1}{3}) + \delta(\frac{1}{3}, \frac{1}{4}) + \delta(\frac{1}{4}, \frac{1}{6}).$$

(v) $(\mathscr{K}, \delta)$ is not a controlled quasi rectangular metric like space, because

$$\delta(\frac{1}{5}, \frac{1}{6}) = 0.36 > 0.14 = \alpha(\frac{1}{5}, \frac{1}{3})\delta(\frac{1}{5}, \frac{1}{3}) + \alpha(\frac{1}{3}, \frac{1}{4})\delta(\frac{1}{3}, \frac{1}{4}) + \alpha(\frac{1}{4}, \frac{1}{6})\delta(\frac{1}{4}, \frac{1}{6}).$$

Define a mapping $\mathscr{L} : \mathscr{K} \to \mathscr{K}$ by

$$\mathscr{L}(s) = \begin{cases} s^{\frac{1}{2}} & \text{if} \quad s \in [1,2], \\ \\ 1 & \text{if} \quad s \in \mathscr{A}. \end{cases}$$

Then, $\mathscr{L}(s) \in [1,2]$, let $k = \frac{1}{4}$. It can easily be seen that $\delta(\mathscr{L}(s), \mathscr{L}(l)) \leq k\delta(s,l)$.
Further, note that for every s in $\mathscr{K}$,

$$\mathscr{L}^n(s) = \begin{cases} s^{\frac{1}{2n}} & \text{if} \quad s \in [1,2], \\ 1 & \text{if} \quad s \in \mathscr{A}. \end{cases}$$

Thus we obtain

$$\sup_{m\geq 1} \lim_{i\to\infty} \gamma(\varrho_{i+1}, \varrho_m) \frac{\alpha(\varrho_{i+1}, \varrho_{i+2}) + k\beta(\varrho_{i+2}, \varrho_{i+3})}{\alpha(\varrho_i, \varrho_{i+1}) + k\beta(\varrho_{i+1}, \varrho_{i+2})} \leq 3 < 4 = \frac{1}{k}.$$

Also,

$$\lim_{n\to\infty} \gamma(\varrho_n, \varrho) = \lim_{n\to\infty} \gamma(\varrho, \varrho_n) \leq 3 \quad \text{and} \quad \lim_{n,m\to\infty} \gamma(\varrho_n, \varrho_m) \leq 3, \quad \text{for all } m, n \in \mathbb{N}, m \neq n.$$

According to Theorem 4.1, all hypotheses are true. Here, $\varrho = 1$ is the fixed point of $\mathscr{L}$.

5. Conclusions

In this article, we have introduced new type of metric spaces so called, triple controlled quasi rectangular metric like spaces. We have proved the existence and uniqueness of fixed point for self mappings on such spaces that satisfy different type of contractions. Moreover, we have provided some examples to illustrate our results. Our work generalizes many results in the literature.

Acknowledgments

The author N. Mlaiki would like to thank the Prince Sultan University for paying the publication fees for this work through TAS LAB.

Conflict of interest

The authors declare that they have no competing interests.

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