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Research article

Communicable disease model in view of fractional calculus

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Abstract: The COVID-19 pandemic still gains the attention of many researchers worldwide. Over the past few months, China faced a new wave of this pandemic which increases the risk of its spread to the rest of the world. Therefore, there has become an urgent demand to know the expected behavior of this pandemic in the coming period. In this regard, there are many mathematical models from which we may obtain accurate predictions about the behavior of this pandemic. Such a target may be achieved via updating the mathematical models taking into account the memory effect in the fractional calculus. This paper generalizes the power-law growth model of the COVID-19. The generalized model is investigated using two different definitions in the fractional calculus, mainly, the Caputo fractional derivative and the conformable derivative. The solution of the first-model is determined in a closed series form and the convergence is addressed. At a specific condition, the series transforms to an exact form. In addition, the solution of the second-model is evaluated exactly. The results are applied on eight European countries to predict the behavior/variation of the infected cases. Moreover, some remarks are given about the validity of the results reported in the literature.

Keywords: fractional differential equation; conformable; initial value problem; series solution; exact solution
Mathematics Subject Classification: 34A08

1. Introduction

In the past decades, numerous mathematical models have been suggested to explore the progress of various communicable and non-communicable diseases. The literature is rich of many mathematical models of communicable diseases such as Li and Zou [1] (with a spatially continuous domain), Siettos and Russo [2], and Jenness et al. [3]. Beside, the authors [4] analyzed a fractional order model for smoking as a non-communicable disease. Nowadays, the COVID-19 still attracting the interest of many researchers. Several mathematical models have been suggested to describe this pandemic such as Shaikh et al. [5], Abajo [6], Gepreel et al. [7], and Xenikos and Asimakopoulos [8]. In [9], an improved SIR model was formulated to describe the epidemic dynamics of the COVID-19 in China.

On the other hand, several models have been proposed for studying this epidemic in other countries such as India [10, 11], Bulgaria [12], Italy [13], and Pakistan [14]. In addition, one can find in [15–23] another set of mathematical models which have been developed to investigate a number of communicable diseases. Some of the above discussed mathematical models are basically governed by differential equations with integer orders. However, the formulated models using classical derivatives can be generalized by incorporating a non-integer order in view of fractional calculus (FC). The FC has been widely applied in the literature to generalize many scientific problems in different areas of applications, such as the projectile dynamics [24,25], the Ambartsumian equation in astrophysics [26], and the chlorine transport used in water treatment [27]. Other interesting applications of the FC in biology and different fields can be found in [28–34].

So, the present work focuses on generalizing the dynamic scaling model [8] of the diffusion growth phase of the COVID-19 epidemic in Europe. The authors [8] showed that the fatality cases grow follow a power law in all European countries. However, the difference among countries is the value of the power-law exponent. The model describes the observed statistical distribution and contributed to the discussion on basic assumptions for homogeneous mixing or for a network perspective in epidemiological studies of COVID-19. The proposed power-law growth model [8] can be generalized by means of the FC as

$$ {}_{0}^{C}D_{t}^{\alpha}N(t) = \left(\frac{t}{\tau}\right)^{\gamma-1}\left(N^{*} - N(t)\right), \quad \alpha \in (0,1], \tag{1.1}$$

where α is the order of the fractional derivative in the Caputo sense ${}_{0}^{C}D_{t}^{\alpha}N(t) = \frac{d^{\alpha}N}{dt^{\alpha}}$ and $N = N(t)$ is the infected cases at the moment t. The magnitude N^{*} is an estimation of the total susceptible population where the growth phase of the epidemic in this model is in the limit $n < N(t) < N^{*}$, n denotes the fatality cases. For all European countries, the power-law exponent lies in the range $3.5 < \gamma < 8$. Also, the power-law growth model of COVID-19 can be further generalized using the conformable-derivative (CD). The CD has been utilized in the literature to analyze/generalize some physical/engineering models (see [35–38]). In this case, the CD-model for COVID-19 is

$$ {}_{0}^{CD}D_{t}^{\alpha}N(t) = \left(\frac{t}{\tau}\right)^{\gamma-1}\left(N^{*} - N(t)\right), \quad \alpha \in (0,1]. \tag{1.2}$$

The above two models are subjected to the following initial condition (IC) [8]:

$$ N(0) = 0. \tag{1.3}$$

As $\alpha \to 1$, i.e., for classical derivative with respect to t, both the CFD-model (1.1) and the CD-model (1.2) reduces to the ordinary/classical version:

$$\frac{dN}{dt} = \left(\frac{t}{\tau}\right)^{\gamma-1} (N^* - N(t)), \quad N(0) = 0. \tag{1.4}$$

The parameter τ introduced in Eqs (1.2) and (1.4) should be larger than all values of t. It has physical meaning of the upper limit in time, where the "memory effect" presumed in the present model is valid [8]. The objective of this work is to obtain the solutions of the present CFD-model and the CD-model for COVID-19. The scenario of the paper is as follows. In Section 2, basic concepts of the CFD and the CD are presented. Section 3 is devoted to determine the solution of the CFD-model in a closed series form. Proof of convergence of such series is addressed in Section 4. Beside, the solution of the CD-model is established in an exact form through Section 5. Properties/behaviors of the solutions for the two models are discussed in Section 6. In addition, some numerical results and various plots will be extracted in Section 7 to reveal the expected future for the diversity of COVID-19. Moreover, the difference in results between the CFD-model and the CD-model is to be explained and interpreted. It will also be shown that the solutions of models (1.1) and (1.2) reduce to the corresponding solution of the classical/ordinary model (1.4) as $\alpha \to 1$. Furthermore, the validity of the published solution in [8] for the classical model (1.4) will be examined in Section 7. The paper is concluded in Section 8.

2. Concepts of the CFD&CD

The Riemann-Liouville fractional integral of order α is defined as [24, 26]:

$${}_0I_t^{\alpha} N(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{N(\mu)}{(t-\mu)^{1-\alpha}} d\mu, \qquad \alpha > 0, \quad t > 0. \tag{2.1}$$

Let $\alpha \neq 0$ denotes the order of the derivative in such a way that $m - 1 < \alpha \leq m$. Then, the Caputo fractional derivative of a function $N(t)$ is defined by [24, 26]

$${}_0^C D_t^{\alpha} N(t) = \frac{d^{\alpha} N(t)}{dt^{\alpha}} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\mu)^{m-\alpha-1} N^{(m)}(\mu) d\mu, & \text{if } m-1 < \alpha < m, \\ \frac{d^m N(t)}{dt^m}, & \text{if } \alpha = m. \end{cases} \tag{2.2}$$

In order to solve physical models, it is required to use effective definitions in the FC so that the physical initial conditions can be implemented. These demands are achieved through the following relation between the fractional operators ${}_0I_t^{\alpha}$ and ${}_0^C D_t^{\alpha}$:

$${}_0I_t^{\alpha} \left({}_0^C D_t^{\alpha} N(t)\right) = N(t) - N(0), \quad \alpha \in (0, 1]. \tag{2.3}$$

A fundamental property of the operator ${}_0I_t^{\alpha}$ is

$${}_0I_t^{\alpha} t^r = \frac{\Gamma(r+1)}{\Gamma(r+\alpha+1)} t^{r+\alpha}, \quad r \geq -1. \tag{2.4}$$

Also, we have

$${}_0^{CD} D_t^{\alpha} t^r = \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha}. \tag{2.5}$$

These properties are essential to construct the solution in a series form when solving fractional differential equations. The CD of arbitrary order α, $0 < \alpha \leq 1$, of a function $N(t)$: $[0, \infty) \to \mathbb{R}$ is defined as [35–38]

$$ {}_{0}^{CD}D_t^{\alpha} N(t) = t^{1-\alpha} \frac{dN}{dt}. \tag{2.6}$$

This is also a basic property of the operator ${}_{0}^{CD}D_t^{\alpha}$. Usually, it can be used to convert the fractional differential Eq (1.2) into a classical differential equation.

3. Solution for the CFD-model

In this section, the solution of the CFD-model (1.1) is determined via a straightforward approach. Let us assume that $\beta = \gamma - 1$. Accordingly, the CFD-model (1.1) becomes

$$ {}_{0}^{C}D_t^{\alpha} N(t) = \left(\frac{t}{\tau}\right)^{\beta} (N^* - N(t)). \tag{3.1}$$

The present approach is mainly based on integrating both sides of Eq (3.1) as follows:

$$ {}_{0}I_t^{\alpha} \left({}_{0}^{C}D_t^{\alpha} N(t) \right) = {}_{0}I_t^{\alpha} \left(\left(\frac{t}{\tau}\right)^{\beta} N^* \right) - {}_{0}I_t^{\alpha} \left(\left(\frac{t}{\tau}\right)^{\beta} N(t) \right), \tag{3.2}$$

which yields

$$ N(t) = N(0) + \frac{N^*}{\tau^{\beta}}\, {}_{0}I_t^{\alpha} \left(t^{\beta}\right) - \frac{1}{\tau^{\beta}}\, {}_{0}I_t^{\alpha} \left(t^{\beta} N(t)\right), \tag{3.3}$$

or

$$ N(t) = \frac{N^*}{T}\, {}_{0}I_t^{\alpha} \left(t^{\beta}\right) - \frac{1}{T}\, {}_{0}I_t^{\alpha} \left(t^{\beta} N(t)\right), \qquad T = \tau^{\beta}. \tag{3.4}$$

Let us assume the solution of the last equation in the ADM-series form [39]:

$$ N(t) = \sum_{n=0}^{\infty} N_n(t). \tag{3.5}$$

The ADM was widely applied to solve various scientific models [40–45]. Moreover, the convergence of the ADM has been proven in [46, 47]. Implementing the above assumption leads to the recurrence-scheme [39]:

$$ \begin{aligned} N_0(t) &= \frac{N^*}{T}\, {}_{0}I_t^{\alpha} \left(t^{\beta}\right) = \frac{N^*}{T} \left(\frac{\Gamma(\beta+1) t^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} \right), \\ N_n(t) &= -\frac{1}{T}\, {}_{0}I_t^{\alpha} \left(t^{\beta} N_{n-1}(t)\right), \qquad n \geq 1. \end{aligned} \tag{3.6}$$

For $n = 1$, we have

$$ \begin{aligned} N_1(t) &= -\frac{1}{T}\, {}_{0}I_t^{\alpha} \left(t^{\beta} N_0(t)\right), \\ &= -\frac{N^*}{T^2} \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} \right) {}_{0}I_t^{\alpha} \left(t^{2\beta+\alpha}\right), \end{aligned}$$

$$= -\frac{N^*}{T^2}\left(\frac{\Gamma(\beta+1)\Gamma(2\beta+\alpha+1)}{\Gamma(\beta+\alpha+1)\Gamma(2\beta+2\alpha+1)}\right)t^{2\beta+2\alpha}. \tag{3.7}$$

Also, for $n = 2$, one can get

$$\begin{aligned} N_2(t) &= -\frac{1}{T}\,{}_0I_t^{\alpha}\left(t^{\beta}N_1(t)\right), \\ &= \frac{N^*}{T^3}\left(\frac{\Gamma(\beta+1)\Gamma(2\beta+\alpha+1)}{\Gamma(\beta+\alpha+1)\Gamma(2\beta+2\alpha+1)}\right){}_0I_t^{\alpha}\left(t^{3\beta+2\alpha}\right), \\ &= \frac{N^*}{T^3}\left(\frac{\Gamma(\beta+1)\Gamma(2\beta+\alpha+1)\Gamma(3\beta+2\alpha+1)}{\Gamma(\beta+\alpha+1)\Gamma(2\beta+2\alpha+1)\Gamma(3\beta+3\alpha+1)}\right)t^{3\beta+3\alpha}. \end{aligned} \tag{3.8}$$

Similarly, we can obtain $N_3(t)$ as

$$N_3(t) = -\frac{N^*}{T^4}\left(\frac{\Gamma(\beta+1)\Gamma(2\beta+\alpha+1)\Gamma(3\beta+2\alpha+1)\Gamma(4\beta+3\alpha+1)}{\Gamma(\beta+\alpha+1)\Gamma(2\beta+2\alpha+1)\Gamma(3\beta+3\alpha+1)\Gamma(4\beta+4\alpha+1)}\right)t^{4\beta+4\alpha}. \tag{3.9}$$

Higher order components can be evaluated recurrently, however, a unified formula for all components can be derived. Regarding, the above four components can be rewritten as

$$N_0(t) = \frac{N^*}{T}\left(\frac{\prod_{k=0}^{0}\Gamma\left((k+1)\beta+k\alpha+1\right)}{\prod_{k=0}^{0}\Gamma\left((k+1)(\beta+\alpha)+1\right)}t^{\beta+\alpha}\right), \tag{3.10}$$

$$N_1(t) = -\frac{N^*}{T^2}\left(\frac{\prod_{k=0}^{1}\Gamma\left((k+1)\beta+k\alpha+1\right)}{\prod_{k=0}^{1}\Gamma\left((k+1)(\beta+\alpha)+1\right)}t^{2(\beta+\alpha)}\right), \tag{3.11}$$

$$N_2(t) = \frac{N^*}{T^3}\left(\frac{\prod_{k=0}^{2}\Gamma\left((k+1)\beta+k\alpha+1\right)}{\prod_{k=0}^{2}\Gamma\left((k+1)(\beta+\alpha)+1\right)}t^{3(\beta+\alpha)}\right), \tag{3.12}$$

$$N_3(t) = -\frac{N^*}{T^3}\left(\frac{\prod_{k=0}^{3}\Gamma\left((k+1)\beta+k\alpha+1\right)}{\prod_{k=0}^{3}\Gamma\left((k+1)(\beta+\alpha)+1\right)}t^{4(\beta+\alpha)}\right). \tag{3.13}$$

In view of these expressions, we have the unified formula:

$$N_n(t) = (-1)^n\frac{N^*}{T^{n+1}}\left(\frac{\prod_{k=0}^{n}\Gamma\left((k+1)\beta+k\alpha+1\right)}{\prod_{k=0}^{n}\Gamma\left((k+1)(\beta+\alpha)+1\right)}t^{(n+1)(\beta+\alpha)}\right), \quad n \geq 0. \tag{3.14}$$

Therefore,

$$N(t) = N^*\sum_{n=0}^{\infty}\frac{(-1)^n}{T^{n+1}}\left(\frac{\prod_{k=0}^{n}\Gamma\left((k+1)\beta+k\alpha+1\right)}{\prod_{k=0}^{n}\Gamma\left((k+1)(\beta+\alpha)+1\right)}t^{(n+1)(\beta+\alpha)}\right), \tag{3.15}$$

i.e.,

$$N(t) = N^*\sum_{n=0}^{\infty}\left(\frac{(-1)^n\prod_{k=0}^{n}\Gamma\left((k+1)(\beta+\alpha)+1-\alpha\right)}{\prod_{k=0}^{n}\Gamma\left((k+1)(\beta+\alpha)+1\right)}\left(\frac{t^{\beta+\alpha}}{T}\right)^{n+1}\right). \tag{3.16}$$

On using $T = \tau^{\beta}$, the above solution takes the form:

$$N(t) = N^*\sum_{n=0}^{\infty}\left(\frac{(-1)^n\prod_{k=0}^{n}\Gamma\left((k+1)(\beta+\alpha)+1-\alpha\right)}{\prod_{k=0}^{n}\Gamma\left((k+1)(\beta+\alpha)+1\right)}\left(\left(\frac{t}{\tau}\right)^{\beta}t^{\alpha}\right)^{n+1}\right). \tag{3.17}$$

Inserting $\beta = \gamma - 1$ into the last equation, yields

$$N(t) = N^* \sum_{n=0}^{\infty} \left(\frac{(-1)^n \prod_{k=0}^{n} \Gamma\left((k+1)(\gamma+\alpha-1)+1-\alpha\right)}{\prod_{k=0}^{n} \Gamma\left((k+1)(\gamma+\alpha-1)+1\right)} \left(\left(\frac{t}{\tau}\right)^{\gamma-1} t^{\alpha} \right)^{n+1} \right), \tag{3.18}$$

which is a closed form series solution for the CFD-model (1.1). The convergence of this series is to be addressed in the next section. In addition, it will be shown later that the above closed form series solution transforms to a certain function as $\alpha \to 1$ (the ordinary version). Furthermore, it reduces to the following exact solution at a specific value of $\gamma = 1 - \alpha$:

$$N(t) = \frac{N^* \tau^{\alpha} \Gamma(1-\alpha)}{1 + \tau^{\alpha} \Gamma(1-\alpha)}. \tag{3.19}$$

Moreover, some observations regarding the solution reported in [8] for the model (1.4) will be declared.

4. Convergence analysis

Theorem 1. *For $\gamma > 1$, the series solution (3.18) converges for all $t \in \mathbb{R}$.*

Proof. Let us rewrite the series solution (3.18) as

$$N(t) = N^* \sum_{n=0}^{\infty} \rho_n(t), \tag{4.1}$$

where

$$\rho_n(t) = c_n T_1^{n+1}, \quad c_n = \frac{(-1)^n \prod_{k=0}^{n} \Gamma\left((k+1)\nu + 1 - \alpha\right)}{\prod_{k=0}^{n} \Gamma\left((k+1)\nu + 1\right)},$$
$$\nu = \gamma + \alpha - 1, \quad T_1 = \left(\frac{t}{\tau}\right)^{\gamma-1} t^{\alpha}. \tag{4.2}$$

Implementing the relation $\nu = \gamma + \alpha - 1$, one can write the coefficients c_n in the form:

$$c_n = \frac{(-1)^n \prod_{k=0}^{n} \Gamma\left(k\nu + \gamma\right)}{\prod_{k=0}^{n} \Gamma\left((k+1)\nu + 1\right)}. \tag{4.3}$$

Applying the ratio test, then

$$\lim_{n\to\infty} \left| \frac{\rho_{n+1}(t)}{\rho_n(t)} \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} T_1 \right| = \left| \left(\frac{t}{\tau}\right)^{\gamma-1} t^{\alpha} \right| \lim_{n\to\infty} \left| \frac{\Gamma\left((n+1)\nu + \gamma\right)}{\Gamma\left((n+2)\nu + 1\right)} \right|. \tag{4.4}$$

Since $0 < \alpha \leq 1$, then $\gamma - 1 < \nu \leq \gamma$. For $\gamma > 1$, we have

$$\lim_{n\to\infty} \left| \frac{\Gamma\left((n+1)\nu + \gamma\right)}{\Gamma\left((n+2)\nu + 1\right)} \right| = 0, \tag{4.5}$$

and thus,

$$\lim_{n\to\infty} \left| \frac{\rho_{n+1}(t)}{\rho_n(t)} \right| = 0. \tag{4.6}$$

Therefore, the series solution (3.18) converges for all $t \in \mathbb{R}$ such that $\tau \neq 0$. □

5. Solution of conformable-model

In this section, the solution of the CD-model (1.2) is evaluated in exact form. As a first step, the substitution $\beta = \gamma - 1$ is used to express the CD-model (1.2) in the form:

$$ {}_0^{CD}D_t^{\alpha} N(t) = \left(\frac{t}{\tau}\right)^{\beta} (N^* - N(t)) . \tag{5.1} $$

Employing the definition of the CD in Eq (2.6), then the differential Eq (5.1) becomes

$$ t^{1-\alpha}\frac{dN}{dt} = \left(\frac{t}{\tau}\right)^{\beta} (N^* - N(t)) . \tag{5.2} $$

Using the assumption $T = \tau^{\beta}$, the last equation can be rewritten as

$$ \frac{dN}{dt} = \frac{t^{\alpha+\beta-1}}{T} (N^* - N(t)) . \tag{5.3} $$

This is a first-order linear differential equation which can be easily solved under the given IC using the separation of variables method. Accordingly, the solution reads

$$ N(t) = N^* \left[1 - e^{-\frac{t^{\alpha+\beta}}{(\alpha+\beta)T}}\right], \tag{5.4} $$

i.e.,

$$ N(t) = N^* \left[1 - e^{-\left(\frac{t}{\tau}\right)^{\beta} \frac{t^{\alpha}}{\alpha+\beta}}\right]. \tag{5.5} $$

Substituting $\beta = \gamma - 1$ leads to

$$ N(t) = N^* \left[1 - e^{-\left(\frac{t}{\tau}\right)^{\gamma-1} \frac{t^{\alpha}}{\alpha+\gamma-1}}\right]. \tag{5.6} $$

It is shown in the next section that the solution (5.6) for the CD-model reduces to the corresponding one of the classical/ordinary version (1.4) as $\alpha \to 1$.

6. Solution of the ordinary versions: $\alpha \to 1$

In this section, we show that the obtained solutions in the previous sections for the CFD-model and the CD-model agree with the result of the ordinary-model as $\alpha \to 1$. Let us begin with the solution of the CFD-model when $\alpha \to 1$. In this case, we obtain from Eq (3.18) that

$$ N(t) = N^* \sum_{n=0}^{\infty} \left(\frac{(-1)^n \prod_{k=0}^{n} \Gamma((k+1)\gamma)}{\prod_{k=0}^{n} \Gamma((k+1)\gamma+1)} \left(\left(\frac{t}{\tau}\right)^{\gamma-1} t \right)^{n+1} \right). \tag{6.1} $$

Using the equality:

$$ \begin{aligned} \prod_{k=0}^{n} \Gamma((k+1)\gamma+1) &= \prod_{k=0}^{n} (k+1)\gamma \prod_{k=0}^{n} \Gamma((k+1)\gamma), \\ &= (n+1)!\gamma^{n+1} \prod_{k=0}^{n} \Gamma((k+1)\gamma), \end{aligned} \tag{6.2} $$

into Eq (6.1) with simplification, then

$$N(t) = N^* \sum_{n=0}^{\infty} \frac{(-1)^n \left(\left(\frac{t}{\tau}\right)^{\gamma} t\right)^{n+1}}{(n+1)!\gamma^{n+1}}, \tag{6.3}$$

which is equivalent to

$$N(t) = N^* \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \left(\left(\frac{t}{\tau}\right)^{\gamma} t\right)^{n}}{n!\gamma^{n}}. \tag{6.4}$$

The last equation implies

$$N(t) = -N^* \sum_{n=1}^{\infty} \frac{-\left(\left(\frac{t}{\tau}\right)^{\gamma} \frac{t}{\gamma}\right)^{n}}{n!}, \tag{6.5}$$

or

$$N(t) = N^* \left[1 - e^{-\left(\frac{t}{\tau}\right)^{\gamma} \frac{t}{\gamma}}\right]. \tag{6.6}$$

In addition, the solution of the CD-model gives, as $\alpha \to 1$, the following form:

$$N(t) = N^* \left[1 - e^{-\left(\frac{t}{\tau}\right)^{\gamma-1} \frac{t}{\gamma}}\right], \tag{6.7}$$

which agrees with the expression (6.6). This solution satisfies the initial condition $N(0) = 0$. Beside, it can be easily checked via a direct substitution into the governing Eq (1.4).
Remark. In [8], the authors introduced the expression:

$$N(t) = N^* \left[1 - e^{-\left(\frac{t}{\tau}\right)^{\gamma} \frac{1}{\gamma}}\right], \tag{6.8}$$

as a solution for the classical/ordinary model (1.4). A first sight on this expression, one can find that the exponent in (6.8) is different than ours in (6.6) or (6.7). In fact, the solution reported in [8] missed some terms, beside, it dosen't satisfy the model (1.4).

7. Results and discussion

This section reports some numerical results about the validity, the convergence, and the accuracy of the obtained series solution for the CFD-model. Beside, this section investigates the behaviors of the obtained solutions for the CFD-model and the CD-model for the purpose of comparisons. The data for eight European countries are considered to extract the current results. In [8], the values of γ for eight European countries are estimated as $\gamma = 3.6$ (Greece), $\gamma = 5.2$ (Sweden), $\gamma = 6.1$ (Germany), $\gamma = 6.5$ (UK), $\gamma = 6.7$ (Italy), $\gamma = 6.8$ (France), $\gamma = 6.9$ (Netherlands), and $\gamma = 7.6$ (Belgium).

7.1. Convergence of the solution: CFD-model

This part of the discussion aims to report some results on the convergence of the series solution (3.18) for the CFD-model (1.1). The m-term approximations of this closed form series solution may be written as

$$\Phi_m(t) = N^* \sum_{n=0}^{m-1} c_n T_1^{n+1}, \tag{7.1}$$

where

$$c_n = \frac{(-1)^n \prod_{k=0}^{n} \Gamma\left((k+1)\nu + 1 - \alpha\right)}{\prod_{k=0}^{n} \Gamma\left((k+1)\nu + 1\right)}, \quad \nu = \gamma + \alpha - 1, \quad T_1 = \left(\frac{t}{\tau}\right)^{\gamma-1} t^{\alpha}. \tag{7.2}$$

Figures 1–4 indicate the convergence of the sequence of the approximations $\{\Phi_m(t)\}$ at some selected values of m (the number of terms taken to approximate the infected cases $N(t)$). These figures use the data of the first four European countries listed above. It can be seen that the sequence of the approximations $\{\Phi_m(t)\}$ converges in the case of Greece ($\gamma = 3.6$, Figure 1) in the most domain of the first 100 days, however, the convergence rate is better for the other three countries, mainly Sweden ($\gamma = 5.2$, Figure 2), Germany ($\gamma = 6.1$, Figure 3), and UK ($\gamma = 6.5$, Figure 4) in which the convergence includes the whole domain, i.e., $t \in [0, 100]$. A conclusion here is that the convergence rate, for the implemented approximations $\Phi_m(t)$, $m = 60, 64, 68, 72$, is better for those countries with γ-values greater than 3.6. Of course, one can increase the number of terms m to reach the best convergence rate for the eight countries, may be $m = 100$ is adequate to achieve such target. This point is confirmed/discussed in the next section through estimating the residuals of the obtained CDF-solution.

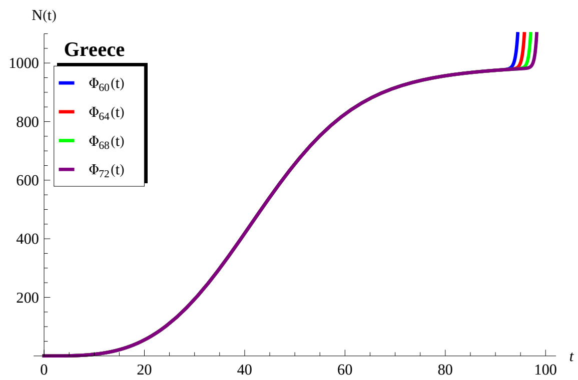

Figure 1. Convergence of the m-term approximate series solution $\Phi_m(t)$ for the CFD-model (1.1) vs t at different values of m at $N^* = 1000$, $\tau = 100$, $\alpha = 3/4$, and $\gamma = 3.6$ (Greece).

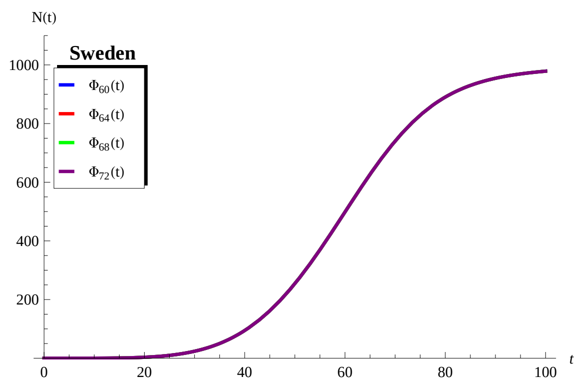

Figure 2. Convergence of the m-term approximate series solution $\Phi_m(t)$ for the CFD-model (1.1) vs t at different values of m at $N^* = 1000$, $\tau = 100$, $\alpha = 3/4$, and $\gamma = 5.2$ (Sweden).

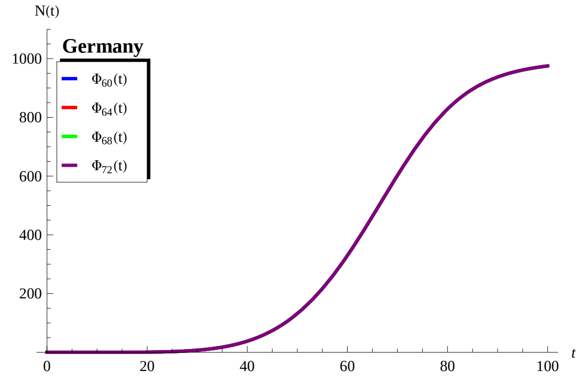

Figure 3. Convergence of the m-term approximate series solution $\Phi_m(t)$ for the CFD-model (1.1) vs t at different values of m at $N^* = 1000$, $\tau = 100$, $\alpha = 3/4$, and $\gamma = 6.1$ (Germany).

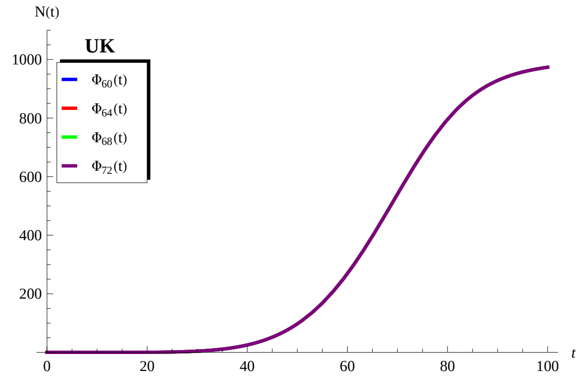

Figure 4. Convergence of the m-term approximate series solution $\Phi_m(t)$ for the CFD-model (1.1) vs t at different values of m at $N^* = 1000$, $\tau = 100$, $\alpha = 3/4$, and $\gamma = 6.5$ (UK).

7.2. Accuracy of the solution: CFD-model

In this section, the accuracy of the series solution for the CFD-model is estimated by means of the residuals given by

$$RE_m(t) = N^* \left| \sum_{n=0}^{m-1} \left(d_n \frac{t^{\nu(n+1)-\alpha}}{\tau^{(\gamma-1)(n+1)}} + c_n \left(\frac{t}{\tau}\right)^{(\gamma-1)(n+2)} t^{\alpha(n+1)} \right) - \left(\frac{t}{\tau}\right)^{\gamma-1} \right|, \tag{7.3}$$

where d_n is

$$d_n = \frac{c_n \Gamma\left(\nu(n+1)+1\right)}{\Gamma\left(\nu(n+1)+1-\alpha\right)}. \tag{7.4}$$

The residuals $RE_{100}(t)$ ($m = 100$) is plotted in Figures 5 and 6 for Greece ($\gamma = 3.6$) and Sweden ($\gamma = 5.2$), respectively. It is seen in these figures that the accuracy achieved in Figure 6 is much better than the corresponding accuracy in Figure 5 due to the increase in the value of γ. This conclusion is also confirmed in Figure 7 for the other six countries in which the values of γ are greater than those in Figures 5 and 6. These results reveal that the current approximation using 100 terms of the CDF-solution is sufficient to accurately modeling the number of the infected cases for the considered eight European countries.

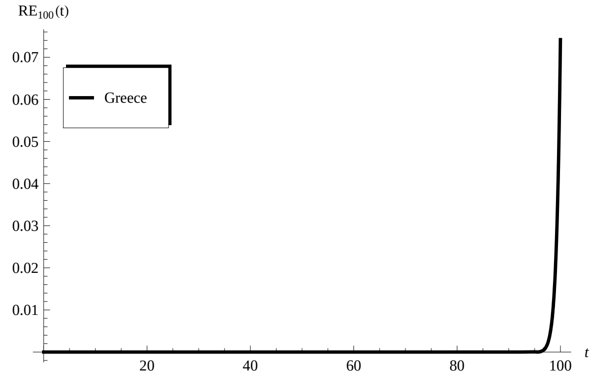

Figure 5. The residual $RE_{100}(t)$ for the CFD-model (1.1) at $N^* = 1000$, $\tau = 100$, $\alpha = 3/4$, and $\gamma = 3.6$ (Greece).

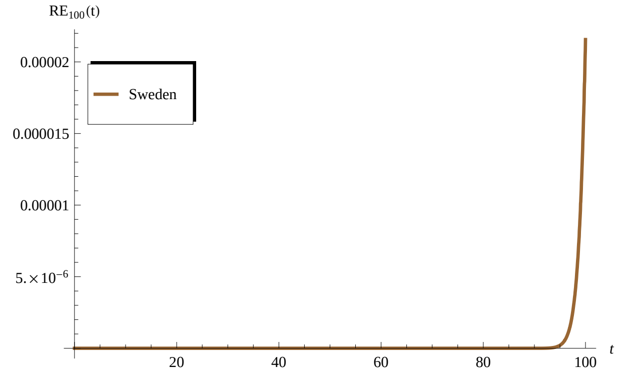

Figure 6. The residual $RE_{100}(t)$ for the CFD-model (1.1) at $N^* = 1000$, $\tau = 100$, $\alpha = 3/4$, and $\gamma = 5.2$ (Sweden).

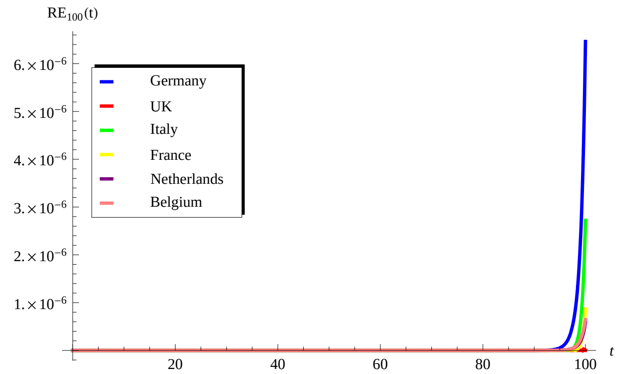

Figure 7. The residual $RE_{100}(t)$ for the CFD-model (1.1) at $N^* = 1000$, $\tau = 100$, $\alpha = 3/4$, $\gamma = 6.1$ (Germany), $\gamma = 6.5$ (UK), $\gamma = 6.7$ (Italy), $\gamma = 6.8$ (France), $\gamma = 6.9$ (Netherlands), and $\gamma = 7.6$ (Belgium).

7.3. Behavior of the solution: CD-model

In a previous section, the solution of the CD-model was obtained in exact form. This exact solution is easily verified by direct substitution into the governing equation of the CD-model. In addition, the behavior of the exact CD-solution is displayed in Figure 8 for the considered eight countries. Although the behavior of the infected cases $N(t)$ in view of the CD-model looks similar to the CFD-model, the time is doubled so that $N(t) \to N^*$ is satisfied. In addition, Figure 9 shows the impact of the fractional-order α on the variation of the infected cases $N(t)$ for the CD-model at $N^* = 1000$, $\tau = 200$, $\gamma = 3.6$ (Greece). Similar results can be also obtained for the impact of α on the behavior of $N(t)$ for the rest of the European countries. It can be observed that the number of the infected cases increases with increasing the order α of the CD-model.

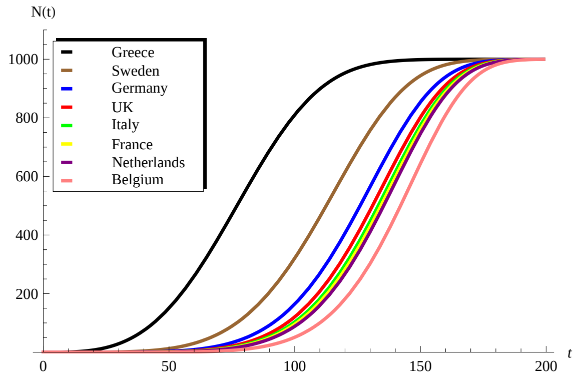

Figure 8. Behavior of the infected cases $N(t)$ in view of the CD-model at $N^* = 1000$, $\tau = 200$, $\alpha = 3/4$.

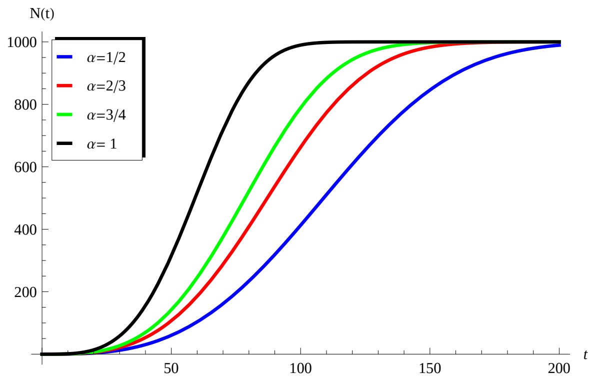

Figure 9. Impact of the fractional-order α on the behavior of the infected cases $N(t)$ for the CD-model at $N^* = 1000$, $\tau = 200$, $\gamma = 3.6$ (Greece).

8. Conclusions

In this paper, the power-law growth model for COVID-19 was generalized using the Caputo fractional derivative (CFD-model) and the conformable derivative (CD-model). A closed form series solution was determined for the CFD-model. Beside, the convergence of the series solution was proved theoretically. It is shown that the obtained series solution transforms to a certain function as the fractional order tends to unity. In addition, the solution of the CD-model was evaluated in an exact form. The obtained results were invested to predict the number of the infected cases for eight European countries. The behavior/variation of the infected cases was depicted in several plots for the eight European countries and the results were analyzed. It was declared that the solutions of present generalized models reduce to the corresponding solution of the original model as the fractional-order tends to unity. The main advantage of this work is that it gives accurate solution which can be implemented for further comparisons with other COVID-19 models. Finally, the given remarks on the reported in the literature may reflex the effectiveness of the current analysis.

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Conflict of interest

The authors declare that they have no competing interests.

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