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Research article

Counting the number of dissociation sets in cubic graphs

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Abstract: Let G be a graph. A dissociation set of G is a subset of vertices that induces a subgraph with vertex degree at most 1. The dissociation polynomial of G is $D_G(\lambda) = \sum_{D\in\mathcal{D}(G)} \lambda^{|D|}$, where $\mathcal{D}(G)$ is the set of all dissociation sets of G. In this paper, we prove that for any cubic graph G and any $\lambda \in (0, 1]$,

$$\frac{1}{|V(G)|} \ln D_G(\lambda) \leq \frac{1}{4} \ln D_{K_4}(\lambda)$$

with equality if and only if G is a disjoint union of copies of the complete graph K_4. When $\lambda = 1$, the value of $D_G(\lambda)$ is exactly the number of dissociation sets of G. Hence, for any cubic graph G on n vertices, $|\mathcal{D}(G)| \leq |\mathcal{D}(K_4)|^{n/4} = 11^{n/4}$.

Keywords: extremal graph theory; counting; cubic graphs; dissociation sets; graph polynomials
Mathematics Subject Classification: 05A17, 05C31, 05C69

1. Introduction

All graphs considered in this paper are simple, undirected and labeled. Let G be a graph. A subset of vertices of G is called a dissociation set if it induces a subgraph with vertex degree at most 1. The empty set is also thought to be a dissociation set of G. Let $\mathcal{D}(G)$ be the set of all dissociation sets of G and $|\mathcal{D}(G)|$ be the total number of dissociation sets of G. The dissociation polynomial of G is $D_G(\lambda) = \sum_{D\in\mathcal{D}(G)} \lambda^{|D|}$.

The concept of dissociation sets was introduced by Yannakakis [7] in 1981, and has been studied extensively in the last four decades. It is also a natural generalization of the well known independent set. Compared with the independent set, the study of dissociated set is more difficult; for example, the problem of finding a maximum dissociation set is NP-hard in bipartite graphs, while the problem of finding a maximum independent set is polynomially solvable in bipartite graphs.

The extremal problems of counting the number of a given graph substructure of a graph of a given type has got lots of attention in the last two decades [1,4–6,8]. In 2017, Davies et al. [2] introduced a novel technique called the occupancy method and used this method to prove tight upper bounds on the independence polynomial and matching polynomial of d-regular graphs. The occupancy method has also been applied to other counting problems [2,3,6].

In this paper, we use the occupancy method to give a tight upper bound on the dissociation polynomial of cubic graphs, and answer the question of which cubic graphs have the largest number of dissociation sets.

We first introduce a probability distribution over all dissociation sets in G, parameterized by a real number $\lambda > 0$, where each dissociation set D is chosen with probability,

$$\Pr[D] = \frac{\lambda^{|D|}}{\sum_{D\in\mathcal{D}(G)} \lambda^{|D|}} = \frac{\lambda^{|D|}}{D_G(\lambda)}.$$

We call the probability distribution the dissociation probability model. The dissociation occupancy fraction of the dissociation probability model, denoted by $\beta_G(\lambda)$, is the expected fraction of vertices of G contained in a random dissociation set D chosen from the dissociation probability model. Specifically,

$$\begin{aligned}\beta_G(\lambda) &= \frac{1}{|V(G)|}\sum_{v\in G}\Pr[v\in D] = \frac{1}{|V(G)|}\frac{\sum_{D\in\mathcal{D}}|D|\lambda^{|D|}}{D_G(\lambda)}\\ &= \frac{1}{|V(G)|}\cdot\frac{\lambda\cdot(D_G(\lambda))'}{D_G(\lambda)} = \lambda\cdot\left(\frac{1}{|V(G)|}\ln D_G(\lambda)\right)'. \qquad (1.1)\end{aligned}$$

By (1.1) and the fact that $D_G(0) = 1$, we have

$$\frac{1}{|V(G)|}\ln D_G(\lambda) = \int_0^\lambda \frac{\beta_G(t)}{t}dt. \qquad (1.2)$$

The main contribution of this work is to prove a tight upper bound on the dissociation occupancy fractions of cubic graphs for $\lambda \in (0, 1]$.

Theorem 1.1. *For any cubic graph G and any* $\lambda \in (0, 1]$*,*

$$\beta_G(\lambda) \le \beta_{K_4}(\lambda),$$

with equality if and only if G is a disjoint union of copies of the complete graph K_4.

By (1.2), we can directly obtain the following corollary:

Corollary 1.1. *For any cubic graph G and any* $\lambda \in (0, 1]$*,*

$$\frac{1}{|V(G)|}\ln D_G(\lambda) \le \frac{1}{4}\ln D_{K_4}(\lambda)$$

with equality if and only if G is a disjoint union of copies of the complete graph K_4.

The value of $D_G(1)$ is exactly the total number of dissociation sets of G. Note that $D_{G\cup H}(\lambda) = D_G(\lambda) \cdot D_H(\lambda)$, where $G \cup H$ is a disjoint union of two graphs G and H. It follows from Corollary 1.1 that a disjoint union of $n/4$ copies of the complete graph K_4 has the most dissociation sets of all cubic graphs on n vertices. Hence, for any cubic graph G on n vertices,

$$|\mathcal{D}(G)| \leq |\mathcal{D}(K_4)|^{n/4} = 11^{n/4}.$$

2. Proof of Theorem 1.1

The dissociation polynomial of the complete graph K_4 is

$$D_{K_4}(\lambda) = 1 + 4\lambda + 6\lambda^2,$$

and its dissociation occupancy fraction is

$$\beta_{K_4}(\lambda) = \frac{1}{4} \cdot \frac{\lambda\,(D_{K_4}(\lambda))'}{D_{K_4}(\lambda)} = \frac{\lambda + 3\lambda^2}{1 + 4\lambda + 6\lambda^2}.$$

Let G be a cubic graph. We choose a vertex, v, uniformly from $V(G)$ at random, and a dissociation set D from the dissociation probability model. We say that the vertex, v, is occupied if $v \in D$, and is otherwise unoccupied. The i-th neighborhood of v, denoted by $N_i(v)$, is the set of vertices of G each of which is at distance i from v. Clearly, $N_1(v) = N(v)$.

We divide the neighborhood $N(v)$ of v into three vertex sets, A_0, A_1, and A_2, as shown in Figure 1, where the black vertices represent the vertices belonging to the dissociation set D. A vertex $u \in N(v)$ is called externally uncovered if none of the vertices in $N(u) \cap N_2(v)$ are in D. The set A_2 consists of vertices of $N(v)$ that are externally uncovered. A vertex $u \in N(v) \setminus A_2$ is called *partly externally covered* if only one vertex in $(N(u) \cap N_2(v)) \cup (N_2(u) \cap N_3(v))$ is in D, and the set A_1 consists of vertices in $N(v) \setminus A_2$ that are partly externally covered. Let $A_0 := N(v) \setminus (A_1 \cup A_2)$, where every vertex of A_0 is called an externally covered vertex. Let $A_1' = (\cup_{u\in A_1} N(u)) \cap N_2(v) \cap D$.

It is worth pointing out that, although we have sampled a dissociation set D of G, it is best to think of the information about which vertices in $N(v) \cup \{v\}$ belong to D as having not been revealed.

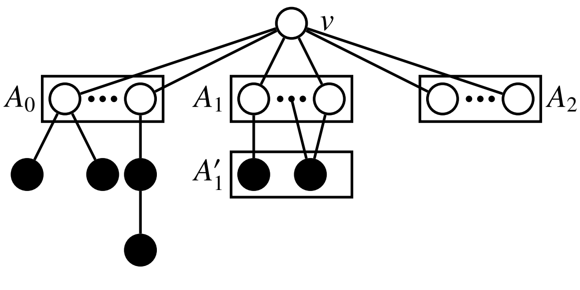

Figure 1. Divide the neighborhood $N(v)$ of v into three vertex sets A_0, A_1, and A_2.

Then, we define a local view of the subgraph induced by $\{v\} \cup A_1 \cup A_2 \cup A_1'$ and record the implementation of the local view as a configuration C, while denoting the dissociation polynomial of C by $D_C(\lambda)$. Let H be the subgraph induced by $A_1 \cup A_2$ and define $D_H(\lambda)$ as the dissociation polynomial of H for the given configuration C. Let a_i $(i = 0, 1, 2)$ be the size of the set A_i $(i = 0, 1, 2)$, and a_1' be the size of the set A_1'. Clearly, $a_1' \leq a_1$, $a_1 + a_2 \leq 3$. We write $C = C_i(a_1', a_1, a_2)$ for a local view of v with respect to D.

It is easy to check that, for cubic graphs, there is a total of 34 configurations up to symmetries, which are pictured in Figure 2.

Let $\mathcal{C}$ be the set of all possible configurations C. Note that $C_4(0,0,3)$ is the only configuration that can arise from the complete graph K_4.

For every configuration C, let $p(C)$ denote the probability that the configuration occurs, and $\beta_C(\lambda)$ be the conditional probability that v is occupied in given configuration C. The dissociation occupancy fraction of G can be written as:

$$\begin{aligned}\beta_G(\lambda) &= \frac{1}{|V(G)|}\sum_{v\in G}\Pr[v \in D] \\ &= \sum_{C\in\mathcal{C}}\Pr[v \in D \mid C]\cdot p(C) \\ &= \sum_{C\in\mathcal{C}}\beta_C(\lambda)\cdot p(C).\end{aligned}$$

We select a vertex u uniformly from the neighbors of v at random, and consider the following conditional probabilities:

$$\begin{aligned}&\beta_t^v(C) = \Pr[v \in D \text{ and } d_{G[D]}(v) = t \mid C] \quad \text{and} \\ &\beta_t^u(C) = \Pr[u \in D \text{ and } d_{G[D]}(u) = t \mid C],\end{aligned}$$

where $t \in \{0, 1\}$.

The expressions for $\beta_C(\lambda), \beta_0^v(C), \beta_1^v(C), \beta_0^u(C)$ and $\beta_1^u(C)$ and all configurations $C \in \mathcal{C}$ are evaluated and listed in Appendix A.

By consistency conditions, we use the fact that, for any $t \in \{0, 1\}$, the probability that v is in D and has degree t in the induced subgraph $G[D]$ equals the probability that a random neighbor u of v is in D and has degree t in $G[D]$, that is,

$$\sum_{C\in\mathcal{C}}\beta_t^v(C)\cdot p(C) = \sum_{C\in\mathcal{C}}\beta_t^u(C)\cdot p(C), \text{ for } t = 0, 1.$$

Hence, we have two constraints on the probability distribution on configurations.

Now, we write the following linear programming with decision variables $p(C)$ and three constraints:

$$\begin{aligned}(LP) \qquad \beta_{\max}(\lambda) = \max \quad & \sum_{C\in\mathcal{C}}\beta_C(\lambda)p(C) \\ s.t. \quad & \sum_{C\in\mathcal{C}} p(C) = 1 \\ & \sum_{C\in\mathcal{C}} p(C)\cdot(\beta_t^v(C) - \beta_t^u(C)) = 0 \quad \text{for } t = 0, 1 \\ & p(C) \geq 0 \quad \forall C \in \mathcal{C}.\end{aligned}$$

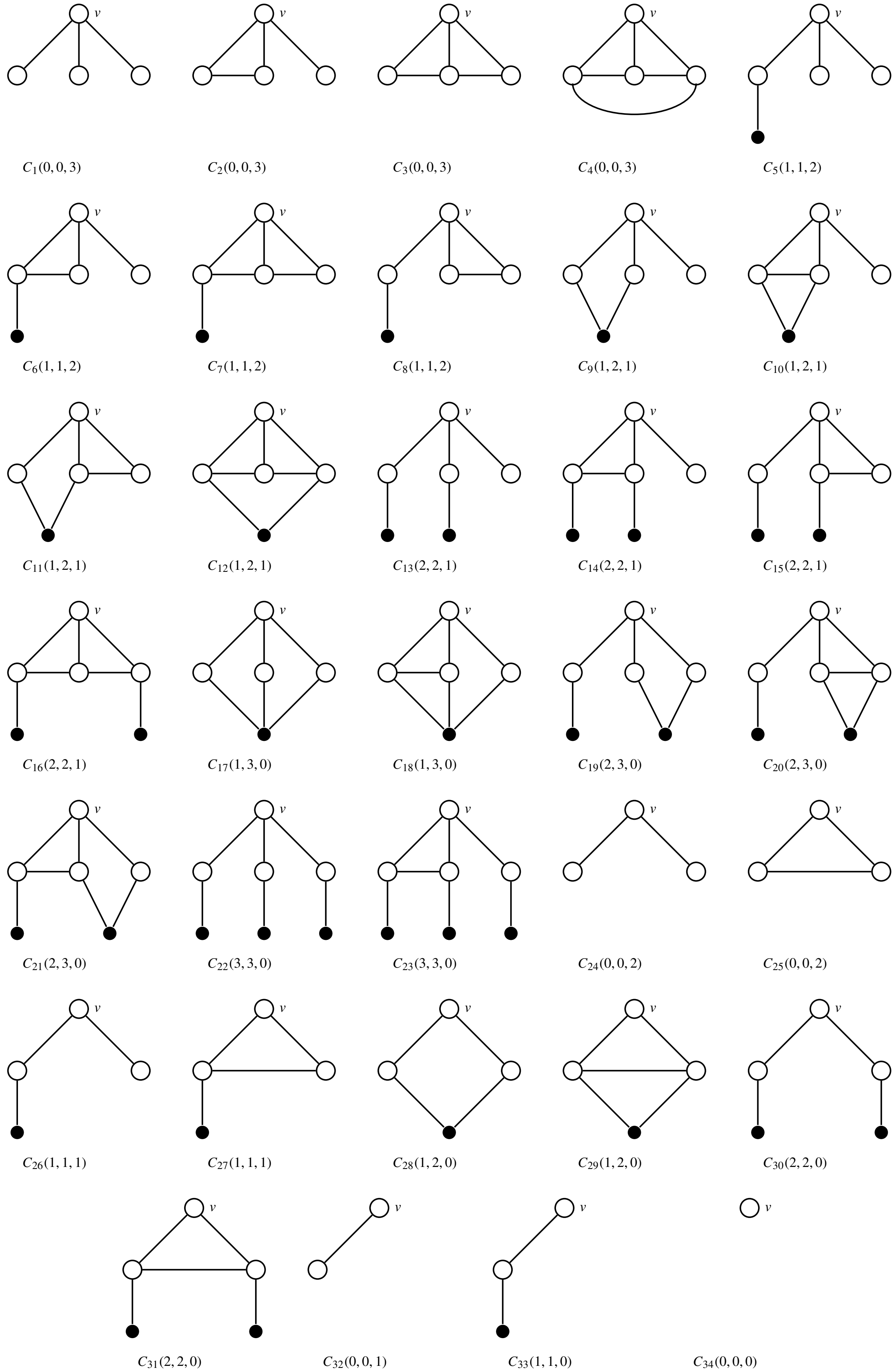

Figure 2. All possible configurations that can arise from a cubic graph.

The dual linear programming of LP is as follows:

$$(DP) \qquad \beta_{\max}(\lambda) = \min \ \Lambda_p$$
$$s.t. \quad \Lambda_p + \sum_{t=0}^{1} \Lambda_t[\beta_t^v(C) - \beta_t^u(C)] \geq \beta_C(\lambda) \quad \forall C \in \mathcal{C},$$

where $\Lambda_p, \Lambda_0, \Lambda_1$ are the decision variables of DP.

Our goal is to show that, when $\lambda \in (0, 1]$, the optimal value of LP is $\beta_{\max}(\lambda) = \beta_{K_4}(\lambda)$. The solution that $p(C_4(0, 0, 3)) = 1$ and $p(C) = 0$ for all other configurations is clearly feasible to LP. It suffices to find a feasible solution to DP with $\Lambda_p^* = \beta_{K_4}(\lambda)$ for $\lambda \in (0, 1]$. Define the slack function of every configuration C as:

$$S_C(\lambda, \Lambda_0, \Lambda_1) = \beta_{K_4}(\lambda) - \beta_C(\lambda) + \sum_{t=0}^{1} \Lambda_t[\beta_t^v(C) - \beta_t^u(C)].$$

Claim 2.1. *Let*

$$\Lambda_0^*(\lambda) = \frac{3\lambda^2}{1 + 4\lambda + 6\lambda^2},$$
$$\Lambda_1^*(\lambda) = \frac{3\lambda + 9\lambda^2}{2 + 8\lambda + 12\lambda^2}.$$

Then, for every configuration $C \in \mathcal{C}$ and any $\lambda \in (0, 1]$,

$$S_C(\lambda, \Lambda_0^*(\lambda), \Lambda_1^*(\lambda)) \geq 0.$$

Proof. Proof of Claim 2.1. The values of $S_C(\lambda, \Lambda_0^*(\lambda), \Lambda_1^*(\lambda))$ for all configurations $C \in \mathcal{C}$ are calculated and listed in Table 1. Let $\mathcal{C}_1 := \{C_4, C_{25}, C_{32}, C_{34}\}$ and $\mathcal{C}_2 := \{C_2, C_5, C_8, C_{13}, C_{22}\}$.

For every configuration $C \in \mathcal{C}_1$, as can be seen from Table 1, we have

$$S_C(\lambda, \Lambda_0^*(\lambda), \Lambda_1^*(\lambda)) = 0,$$

for all $\lambda > 0$.

For every configuration $C \in \mathcal{C}_2$, we use an auxiliary function $\lambda(t) = \frac{t}{1+t}$ which maps $[0, +\infty)$ to $[0, 1)$. Also shown in Table 1 is that $S_C(\lambda(t), \Lambda_0^*(\lambda(t)), \Lambda_1^*(\lambda(t)))$ is the ratio of two polynomials in t with positive coefficients. Thus,

$$S_C(\lambda, \Lambda_0^*(\lambda), \Lambda_1^*(\lambda)) > 0$$

for all $\lambda \in (0, 1)$. It is easy to check that when $\lambda = 1$, $S_C(\lambda, \Lambda_0^*(\lambda), \Lambda_1^*(\lambda)) > 0$. Thus, we have

$$S_C(\lambda, \Lambda_0^*(\lambda), \Lambda_1^*(\lambda)) > 0$$

for all $\lambda \in (0, 1]$.

For every configuration $C \in \mathcal{C} \setminus (\mathcal{C}_1 \cup \mathcal{C}_2)$, $S_C(\lambda, \Lambda_0^*(\lambda), \Lambda_1^*(\lambda))$ is the ratio of two polynomials in λ with positive coefficients, it follows that

$$S_C(\lambda, \Lambda_0^*(\lambda), \Lambda_1^*(\lambda)) > 0,$$

for all $\lambda > 0$. ∎

Table 1. The values of $S_C(\lambda, \Lambda_0^*(\lambda), \Lambda_1^*(\lambda))$ for all configurations $C \in \mathcal{C}$.

	$S_C(\lambda, \Lambda_0^*, \Lambda_1^*)$	$\lambda(t) = \frac{t}{1+t}$
$C_1(0,0,3)$	$\frac{3\lambda^3+4\lambda^4}{1+8\lambda+28\lambda^2+49\lambda^3+40\lambda^4+6\lambda^5}$	
$C_2(0,0,3)$	$\frac{2\lambda^3+2\lambda^4-\lambda^5}{1+8\lambda+28\lambda^2+49\lambda^3+40\lambda^4+6\lambda^5}$	$\frac{2t^3+6t^4+3t^5}{1+13t+70t^2+191t^3+259t^4+132t^5}$
$C_3(0,0,3)$	$\frac{\lambda^3+\lambda^4}{1+8\lambda+28\lambda^2+48\lambda^3+36\lambda^4}$	
$C_4(0,0,3)$	0	
$C_5(1,1,2)$	$\frac{\lambda^2+7\lambda^3+5\lambda^4-\lambda^5}{2+16\lambda+54\lambda^2+90\lambda^3+68\lambda^4+12\lambda^5}$	$\frac{t^2+10t^3+22t^4+12t^5}{2+26t+138t^2+368t^3+484t^4+242t^5}$
$C_6(1,1,2)$	$\frac{\lambda^2+6\lambda^3+3\lambda^4}{2+16\lambda+52\lambda^2+80\lambda^3+48\lambda^4}$	
$C_7(1,1,2)$	$\frac{\lambda^2+4\lambda^3+\lambda^4}{2+16\lambda+52\lambda^2+80\lambda^3+48\lambda^4}$	
$C_8(1,1,2)$	$\frac{\lambda^2+5\lambda^3+\lambda^4-3\lambda^5}{2+16\lambda+54\lambda^2+90\lambda^3+68\lambda^4+12\lambda^5}$	$\frac{t^2+8t^3+14t^4+4t^5}{2+26t+138t^2+368t^3+484t^4+242t^5}$
$C_9(1,2,1)$	$\frac{\lambda^4+4\lambda^3+\lambda^2}{1+8\lambda+25\lambda^2+36\lambda^3+18\lambda^4}$	
$C_{10}(1,2,1)$	$\frac{\lambda^4+4\lambda^3+\lambda^2}{1+8\lambda+25\lambda^2+36\lambda^3+18\lambda^4}$	
$C_{11}(1,2,1)$	$\frac{2\lambda^2+7\lambda^3+\lambda^4}{2+16\lambda+48\lambda^2+64\lambda^3+24\lambda^4}$	
$C_{12}(1,2,1)$	$\frac{\lambda^2+3\lambda^3}{1+8\lambda+23\lambda^2+28\lambda^3+6\lambda^4}$	
$C_{13}(2,2,1)$	$\frac{\lambda^2+4\lambda^3+\lambda^4-\lambda^5}{1+8\lambda+26\lambda^2+41\lambda^3+28\lambda^4+6\lambda^5}$	$\frac{t^2+7t^3+12t^4+5t^5}{1+13t+68t^2+177t^3+225t^4+110t^5}$
$C_{14}(2,2,1)$	$\frac{\lambda^2+4\lambda^3+\lambda^4}{1+8\lambda+25\lambda^2+36\lambda^3+18\lambda^4}$	
$C_{15}(2,2,1)$	$\frac{2\lambda^2+7\lambda^3+\lambda^4}{2+16\lambda+50\lambda^2+72\lambda^3+36\lambda^4}$	
$C_{16}(2,2,1)$	$\frac{\lambda^2+3\lambda^3}{1+8\lambda+24\lambda^2+32\lambda^3+12\lambda^4}$	
$C_{17}(1,3,0)$	$\frac{3\lambda^2+9\lambda^3}{2+16\lambda+44\lambda^2+48\lambda^3}$	
$C_{18}(1,3,0)$	$\frac{3\lambda^2+9\lambda^3}{2+16\lambda+44\lambda^2+48\lambda^3}$	
$C_{19}(2,3,0)$	$\frac{3\lambda^2+9\lambda^3}{2+16\lambda+48\lambda^2+64\lambda^3+24\lambda^4}$	
$C_{20}(2,3,0)$	$\frac{3\lambda^2+9\lambda^3}{2+16\lambda+48\lambda^2+64\lambda^3+24\lambda^4}$	
$C_{21}(2,3,0)$	$\frac{3\lambda^2+9\lambda^3}{2+16\lambda+46\lambda^2+56\lambda^3+12\lambda^4}$	
$C_{22}(3,3,0)$	$\frac{3\lambda^2+9\lambda^3-\lambda^4-3\lambda^5}{2+16\lambda+50\lambda^2+74\lambda^3+44\lambda^4+12\lambda^5}$	$\frac{3t^2+18t^3+26t^4+8t^5}{2+26t+134t^2+340t^3+416t^4+198t^5}$
$C_{23}(3,3,0)$	$\frac{3\lambda^2+9\lambda^3}{2+16\lambda+48\lambda^2+64\lambda^3+24\lambda^4}$	
$C_{24}(0,0,2)$	$\frac{\lambda^3+\lambda^4}{1+7\lambda+21\lambda^2+30\lambda^3+18\lambda^4}$	
$C_{25}(0,0,2)$	0	
$C_{26}(1,1,1)$	$\frac{\lambda^2+4\lambda^3+\lambda^4}{2+14\lambda+40\lambda^2+52\lambda^3+24\lambda^4}$	
$C_{27}(1,1,1)$	$\frac{\lambda^2+3\lambda^3}{2+14\lambda+38\lambda^2+44\lambda^3+12\lambda^4}$	
$C_{28}(1,2,0)$	$\frac{\lambda^2}{1+4\lambda+6\lambda^2}$	
$C_{29}(1,2,0)$	$\frac{\lambda^2}{1+4\lambda+6\lambda^2}$	
$C_{30}(2,2,0)$	$\frac{\lambda^2+3\lambda^3}{1+7\lambda+19\lambda^2+22\lambda^3+6\lambda^4}$	
$C_{31}(2,2,0)$	$\frac{\lambda^2}{1+4\lambda+6\lambda^2}$	
$C_{32}(0,0,1)$	0	
$C_{33}(1,1,0)$	$\frac{\lambda^2+3\lambda^3}{2+12\lambda+28\lambda^2+24\lambda^3}$	
$C_{34}(0,0,0)$	0	

Now, we have obtained a feasible solution to DP with $\Lambda_p^* = \beta_{K_4}(\lambda)$ for $\lambda \in (0,1]$ and proved that $\beta_G(\lambda) \leq \beta_{K_4}(\lambda)$ for all cubic graphs G and all $\lambda \in (0,1]$. Next, we will prove that unions of copies of the complete graph K_4 are the only graphs that maximize $\beta_G(\lambda)$ among all cubic graphs.

Claim 2.2. *Let G be a cubic graph with $\beta_G(\lambda) = \beta_{K_4}(\lambda)$, only the configuration $C_4(0,0,3)$ appears with positive probability.*

Proof. Proof of Claim 2.2. It can be seen from the proof of Claim 2.1 that for every configuration $C \in \mathcal{C} \setminus \mathcal{C}_1$ and any $\lambda \in (0,1]$,

$$S_C(\lambda, \Lambda_0^*(\lambda), \Lambda_1^*(\lambda)) > 0.$$

It follows from complementary slackness that $p(C) = 0$ for every configuration $C \in \mathcal{C} \setminus \mathcal{C}_1$. It suffices to prove that $p(C_{25}) = p(C_{32}) = p(C_{34}) = 0$.

Suppose that the random dissociation set chosen is the empty dissociation set. If $p(C_{25}) > 0$, then either $p(C_2) > 0$ or $p(C_3) > 0$. If $p(C_{32}) > 0$, then either $p(C_1) > 0$, or $p(C_2) > 0$, or $p(C_3) > 0$. If $p(C_{34}) > 0$, then either $p(C_1) > 0$ or $p(C_2) > 0$. In each case, we have a contradiction. ∎

Therefore, the configuration $C_4(0,0,3)$ is the unique maximizer of LP, which implies that unions of copies of the complete graph K_4 are the only extremal graphs. We complete the proof of Theorem 1.1.

3. Conclusions

In this paper, we show that for $\lambda \in (0,1]$, unions of copies of the complete graph K_4 are optimal on the level of dissociation occupancy fraction among all cubic graphs, which implies that a union of copies of the complete graph K_4 maximizes the number of dissociation sets and the dissociation polynomial for $\lambda \in (0,1]$ of a cubic graph on the same number of vertices.

Acknowledgments

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Conflict of interest

We declare that there are no conflicts of interest.

Appendix A. The expressions for $\beta_C(\lambda)$, $\beta_0^v(C)$, $\beta_1^v(C)$, $\beta_0^u(C)$ and $\beta_1^u(C)$, and all configurations $C \in \mathcal{C}$.

We write the expressions for $\beta_C(\lambda)$, $\beta_0^v(C)$, $\beta_1^v(C)$, $\beta_0^u(C)$ and $\beta_1^u(C)$:

$$D_C(\lambda) = \lambda^{a_1{}'} \cdot (\lambda + a_2\lambda^2 + D_H(\lambda)),$$
$$\beta_C(\lambda) = \frac{\lambda^{a_1{}'}}{D_C(\lambda)} \cdot (\lambda + a_2\lambda^2),$$
$$\beta_0^v(C) = \frac{\lambda^{a_1{}'}}{D_C(\lambda)} \cdot \lambda,$$
$$\beta_1^v(C) = \frac{\lambda^{a_1{}'}}{D_C(\lambda)} \cdot a_2\lambda^2,$$
$$\beta_0^u(C) = \frac{1}{3} \cdot \frac{\lambda^{a_1{}'}}{D_C(\lambda)} \cdot \sum_{u\in A_2} \sum_{D\in\mathcal{D}(H-N(u))} \lambda^{1+|D|},$$
$$\beta_1^u(C) = \frac{1}{3} \cdot \frac{\lambda^{a_1{}'}}{D_C(\lambda)} \cdot (\sum_{u\in A_1} \sum_{D\in\mathcal{D}(H-N(u))} \lambda^{1+|D|} + \sum_{u\in A_2} \sum_{x\in N(u)} \mathbf{1}_{x\notin A_1} \sum_{D\in\mathcal{D}(A_2\setminus(N(u)\cup N(x)))} \lambda^{2+|D|}).$$

For all configurations $C \in \mathcal{C}$, their accurate expressions are computed and listed as follows.

$$\mathbf{C_1(0,0,3)}: \beta_C(\lambda) = \frac{\lambda+3\lambda^2}{1+4\lambda+6\lambda^2+\lambda^3}, \quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+6\lambda^2+\lambda^3}, \quad \beta_1^v(C) = \frac{3\lambda^2}{1+4\lambda+6\lambda^2+\lambda^3},$$
$$\beta_0^u(C) = \frac{1}{3} \cdot \frac{3\lambda+6\lambda^2+3\lambda^3}{1+4\lambda+6\lambda^2+\lambda^3}, \quad \beta_1^u(C) = \frac{1}{3} \cdot \frac{3\lambda^2}{1+4\lambda+6\lambda^2+\lambda^3}.$$
$$\mathbf{C_2(0,0,3)}: \beta_C(\lambda) = \frac{\lambda+3\lambda^2}{1+4\lambda+6\lambda^2+\lambda^3}, \quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+6\lambda^2+\lambda^3}, \quad \beta_1^v(C) = \frac{3\lambda^2}{1+4\lambda+6\lambda^2+\lambda^3},$$
$$\beta_0^u(C) = \frac{1}{3} \cdot \frac{3\lambda+4\lambda^2+\lambda^3}{1+4\lambda+6\lambda^2+\lambda^3}, \quad \beta_1^u(C) = \frac{1}{3} \cdot \frac{5\lambda^2+2\lambda^3}{1+4\lambda+6\lambda^2+\lambda^3}.$$
$$\mathbf{C_3(0,0,3)}: \beta_C(\lambda) = \frac{\lambda+3\lambda^2}{1+4\lambda+6\lambda^2}, \quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+6\lambda^2}, \quad \beta_1^v(C) = \frac{3\lambda^2}{1+4\lambda+6\lambda^2},$$
$$\beta_0^u(C) = \frac{1}{3} \cdot \frac{3\lambda+2\lambda^2}{1+4\lambda+6\lambda^2}, \quad \beta_1^u(C) = \frac{1}{3} \cdot \frac{7\lambda^2}{1+4\lambda+6\lambda^2}.$$
$$\mathbf{C_4(0,0,3)}: \beta_C(\lambda) = \frac{\lambda+3\lambda^2}{1+4\lambda+6\lambda^2}, \quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+6\lambda^2}, \quad \beta_1^v(C) = \frac{3\lambda^2}{1+4\lambda+6\lambda^2},$$
$$\beta_0^u(C) = \frac{1}{3} \cdot \frac{3\lambda}{1+4\lambda+6\lambda^2}, \quad \beta_1^u(C) = \frac{1}{3} \cdot \frac{9\lambda^2}{1+4\lambda+6\lambda^2}.$$
$$\mathbf{C_5(1,1,2)}: \beta_C(\lambda) = \frac{\lambda+2\lambda^2}{1+4\lambda+5\lambda^2+\lambda^3}, \quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+5\lambda^2+\lambda^3}, \quad \beta_1^v(C) = \frac{2\lambda^2}{1+4\lambda+5\lambda^2+\lambda^3},$$
$$\beta_0^u(C) = \frac{1}{3} \cdot \frac{2\lambda+4\lambda^2+2\lambda^3}{1+4\lambda+5\lambda^2+\lambda^3}, \quad \beta_1^u(C) = \frac{1}{3} \cdot \frac{\lambda+4\lambda^2+\lambda^3}{1+4\lambda+5\lambda^2+\lambda^3}.$$
$$\mathbf{C_6(1,1,2)}: \beta_C(\lambda) = \frac{\lambda+2\lambda^2}{1+4\lambda+4\lambda^2}, \quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+4\lambda^2}, \quad \beta_1^v(C) = \frac{2\lambda^2}{1+4\lambda+4\lambda^2},$$
$$\beta_0^u(C) = \frac{1}{3} \cdot \frac{2\lambda+3\lambda^2}{1+4\lambda+4\lambda^2}, \quad \beta_1^u(C) = \frac{1}{3} \cdot \frac{\lambda+3\lambda^2}{1+4\lambda+4\lambda^2}.$$
$$\mathbf{C_7(1,1,2)}: \beta_C(\lambda) = \frac{\lambda+2\lambda^2}{1+4\lambda+4\lambda^2}, \quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+4\lambda^2}, \quad \beta_1^v(C) = \frac{2\lambda^2}{1+4\lambda+4\lambda^2},$$
$$\beta_0^u(C) = \frac{1}{3} \cdot \frac{2\lambda+\lambda^2}{1+4\lambda+4\lambda^2}, \quad \beta_1^u(C) = \frac{1}{3} \cdot \frac{\lambda+5\lambda^2}{1+4\lambda+4\lambda^2}.$$
$$\mathbf{C_8(1,1,2)}: \beta_C(\lambda) = \frac{\lambda+2\lambda^2}{1+4\lambda+5\lambda^2+\lambda^3}, \quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+5\lambda^2+\lambda^3}, \quad \beta_1^v(C) = \frac{2\lambda^2}{1+4\lambda+5\lambda^2+\lambda^3},$$

$$\beta_0^u(C) = \frac{1}{3}\cdot\frac{2\lambda+2\lambda^2}{1+4\lambda+5\lambda^2+\lambda^3},\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{\lambda+6\lambda^2+3\lambda^3}{1+4\lambda+5\lambda^2+\lambda^3}.$$

$$\mathbf{C_9(1,2,1)}:\beta_C(\lambda) = \frac{\lambda+\lambda^2}{1+4\lambda+3\lambda^2},\quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+3\lambda^2},\quad \beta_1^v(C) = \frac{\lambda^2}{1+4\lambda+3\lambda^2},$$
$$\beta_0^u(C) = \frac{1}{3}\cdot\frac{\lambda+2\lambda^2}{1+4\lambda+3\lambda^2},\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{2\lambda+3\lambda^2}{1+4\lambda+3\lambda^2}.$$

$$\mathbf{C_{10}(1,2,1)}:\beta_C(\lambda) = \frac{\lambda+\lambda^2}{1+4\lambda+3\lambda^2},\quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+3\lambda^2},\quad \beta_1^v(C) = \frac{\lambda^2}{1+4\lambda+3\lambda^2},$$
$$\beta_0^u(C) = \frac{1}{3}\cdot\frac{\lambda+2\lambda^2}{1+4\lambda+3\lambda^2},\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{2\lambda+3\lambda^2}{1+4\lambda+3\lambda^2}.$$

$$\mathbf{C_{11}(1,2,1)}:\beta_C(\lambda) = \frac{\lambda+\lambda^2}{1+4\lambda+2\lambda^2},\quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+2\lambda^2},\quad \beta_1^v(C) = \frac{\lambda^2}{1+4\lambda+2\lambda^2},$$
$$\beta_0^u(C) = \frac{1}{3}\cdot\frac{\lambda+\lambda^2}{1+4\lambda+2\lambda^2},\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{2\lambda+2\lambda^2}{1+4\lambda+2\lambda^2}.$$

$$\mathbf{C_{12}(1,2,1)}:\beta_C(\lambda) = \frac{\lambda+\lambda^2}{1+4\lambda+\lambda^2},\quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+\lambda^2},\quad \beta_1^v(C) = \frac{\lambda^2}{1+4\lambda+\lambda^2},$$
$$\beta_0^u(C) = \frac{1}{3}\cdot\frac{\lambda}{1+4\lambda+\lambda^2},\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{2\lambda+\lambda^2}{1+4\lambda+\lambda^2}.$$

$$\mathbf{C_{13}(2,2,1)}:\beta_C(\lambda) = \frac{\lambda+\lambda^2}{1+4\lambda+4\lambda^2+\lambda^3},\quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+4\lambda^2+\lambda^3},\quad \beta_1^v(C) = \frac{\lambda^2}{1+4\lambda+4\lambda^2+\lambda^3},$$
$$\beta_0^u(C) = \frac{1}{3}\cdot\frac{\lambda+2\lambda^2+\lambda^3}{1+4\lambda+4\lambda^2+\lambda^3},\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{2\lambda+5\lambda^2+2\lambda^3}{1+4\lambda+4\lambda^2+\lambda^3}.$$

$$\mathbf{C_{14}(2,2,1)}:\beta_C(\lambda) = \frac{\lambda+\lambda^2}{1+4\lambda+3\lambda^2},\quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+3\lambda^2},\quad \beta_1^v(C) = \frac{\lambda^2}{1+4\lambda+3\lambda^2},$$
$$\beta_0^u(C) = \frac{1}{3}\cdot\frac{\lambda+2\lambda^2}{1+4\lambda+3\lambda^2},\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{2\lambda+3\lambda^2}{1+4\lambda+3\lambda^2}.$$

$$\mathbf{C_{15}(2,2,1)}:\beta_C(\lambda) = \frac{\lambda+\lambda^2}{1+4\lambda+3\lambda^2},\quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+3\lambda^2},\quad \beta_1^v(C) = \frac{\lambda^2}{1+4\lambda+3\lambda^2},$$
$$\beta_0^u(C) = \frac{1}{3}\cdot\frac{\lambda+\lambda^2}{1+4\lambda+3\lambda^2},\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{2\lambda+4\lambda^2}{1+4\lambda+3\lambda^2}.$$

$$\mathbf{C_{16}(2,2,1)}:\beta_C(\lambda) = \frac{\lambda+\lambda^2}{1+4\lambda+2\lambda^2},\quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+2\lambda^2},\quad \beta_1^v(C) = \frac{\lambda^2}{1+4\lambda+2\lambda^2},$$
$$\beta_0^u(C) = \frac{1}{3}\cdot\frac{\lambda}{1+4\lambda+2\lambda^2},\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{2\lambda+3\lambda^2}{1+4\lambda+2\lambda^2}.$$

$$\mathbf{C_{17}(1,3,0)}:\beta_C(\lambda) = \frac{\lambda}{1+4\lambda},\quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda},\quad \beta_1^v(C) = 0,\quad \beta_0^u(C) = 0,\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{3\lambda}{1+4\lambda}.$$

$$\mathbf{C_{18}(1,3,0)}:\beta_C(\lambda) = \frac{\lambda}{1+4\lambda},\quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda},\quad \beta_1^v(C) = 0,\quad \beta_0^u(C) = 0,\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{3\lambda}{1+4\lambda}.$$

$$\mathbf{C_{19}(2,3,0)}:\beta_C(\lambda) = \frac{\lambda}{1+4\lambda+2\lambda^2},\ \beta_0^v(C) = \frac{\lambda}{1+4\lambda+2\lambda^2},\quad \beta_1^v(C) = 0,$$
$$\beta_0^u(C) = 0,\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{3\lambda+4\lambda^2}{1+4\lambda+2\lambda^2}.$$

$$\mathbf{C_{20}(2,3,0)}:\beta_C(\lambda) = \frac{\lambda}{1+4\lambda+2\lambda^2},\quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+2\lambda^2},\quad \beta_1^v(C) = 0,$$
$$\beta_0^u(C) = 0,\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{3\lambda+4\lambda^2}{1+4\lambda+2\lambda^2}.$$

$$\mathbf{C_{21}(2,3,0)}:\beta_C(\lambda) = \frac{\lambda}{1+4\lambda+\lambda^2},\quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+\lambda^2},\quad \beta_1^v(C) = 0,$$
$$\beta_0^u(C) = 0,\quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{3\lambda+2\lambda^2}{1+4\lambda+\lambda^2}.$$

$\mathbf{C_{22}(3,3,0)}: \beta_C(\lambda) = \frac{\lambda}{1+4\lambda+3\lambda^2+\lambda^3}, \quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+3\lambda^2+\lambda^3}, \quad \beta_1^v(C) = 0,$

$\beta_0^u(C) = 0, \quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{3\lambda+6\lambda^2+3\lambda^3}{1+4\lambda+3\lambda^2+\lambda^3}.$

$\mathbf{C_{23}(3,3,0)}: \beta_C(\lambda) = \frac{\lambda}{1+4\lambda+2\lambda^2}, \quad \beta_0^v(C) = \frac{\lambda}{1+4\lambda+2\lambda^2}, \quad \beta_1^v(C) = 0,$

$\beta_0^u(C) = 0, \quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{3\lambda+4\lambda^2}{1+4\lambda+2\lambda^2}.$

$\mathbf{C_{24}(0,0,2)}: \beta_C(\lambda) = \frac{\lambda+2\lambda^2}{1+3\lambda+3\lambda^2}, \quad \beta_0^v(C) = \frac{\lambda}{1+3\lambda+3\lambda^2}, \quad \beta_1^v(C) = \frac{2\lambda^2}{1+3\lambda+3\lambda^2},$

$\beta_0^u(C) = \frac{1}{3}\cdot\frac{2\lambda+2\lambda^2}{1+3\lambda+3\lambda^2}, \quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{2\lambda^2}{1+3\lambda+3\lambda^2}.$

$\mathbf{C_{25}(0,0,2)}: \beta_C(\lambda) = \frac{\lambda+2\lambda^2}{1+3\lambda+3\lambda^2}, \quad \beta_0^v(C) = \frac{\lambda}{1+3\lambda+3\lambda^2}, \quad \beta_1^v(C) = \frac{2\lambda^2}{1+3\lambda+3\lambda^2},$

$\beta_0^u(C) = \frac{1}{3}\cdot\frac{2\lambda}{1+3\lambda+3\lambda^2}, \quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{4\lambda^2}{1+3\lambda+3\lambda^2}.$

$\mathbf{C_{26}(1,1,1)}: \beta_C(\lambda) = \frac{\lambda+\lambda^2}{1+3\lambda+2\lambda^2}, \quad \beta_0^v(C) = \frac{\lambda}{1+3\lambda+2\lambda^2}, \quad \beta_1^v(C) = \frac{\lambda^2}{1+3\lambda+2\lambda^2},$

$\beta_0^u(C) = \frac{1}{3}\cdot\frac{\lambda+\lambda^2}{1+3\lambda+2\lambda^2}, \quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{\lambda+2\lambda^2}{1+3\lambda+2\lambda^2}.$

$\mathbf{C_{27}(1,1,1)}: \beta_C(\lambda) = \frac{\lambda+\lambda^2}{1+3\lambda+3\lambda^2}, \quad \beta_0^v(C) = \frac{\lambda}{1+3\lambda+3\lambda^2}, \quad \beta_1^v(C) = \frac{\lambda^2}{1+3\lambda+3\lambda^2},$

$\beta_0^u(C) = \frac{1}{3}\cdot\frac{\lambda}{1+3\lambda+3\lambda^2}, \quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{\lambda+\lambda^2}{1+3\lambda+3\lambda^2}.$

$\mathbf{C_{28}(1,2,0)}: \beta_C(\lambda) = \frac{\lambda}{1+3\lambda}, \quad \beta_0^v(C) = \frac{\lambda}{1+3\lambda}, \quad \beta_1^v(C) = 0, \quad \beta_0^u(C) = 0, \quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{2\lambda}{1+3\lambda}.$

$\mathbf{C_{29}(1,2,0)}: \beta_C(\lambda) = \frac{\lambda}{1+3\lambda}, \quad \beta_0^v(C) = \frac{\lambda}{1+3\lambda}, \quad \beta_1^v(C) = 0, \quad \beta_0^u(C) = 0, \quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{2\lambda}{1+3\lambda}.$

$\mathbf{C_{30}(2,2,0)}: \beta_C(\lambda) = \frac{\lambda}{1+3\lambda+\lambda^2}, \quad \beta_0^v(C) = \frac{\lambda}{1+3\lambda+\lambda^2}, \quad \beta_1^v(C) = 0,$

$\beta_0^u(C) = 0, \quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{2\lambda+2\lambda^2}{1+3\lambda+\lambda^2}.$

$\mathbf{C_{31}(2,2,0)}: \beta_C(\lambda) = \frac{\lambda}{1+3\lambda}, \quad \beta_0^v(C) = \frac{\lambda}{1+3\lambda}, \quad \beta_1^v(C) = 0, \quad \beta_0^u(C) = 0, \quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{2\lambda}{1+3\lambda}.$

$\mathbf{C_{32}(0,0,1)}: \beta_C(\lambda) = \frac{\lambda+\lambda^2}{1+2\lambda+\lambda^2}, \quad \beta_0^v(C) = \frac{\lambda}{1+2\lambda+\lambda^2}, \quad \beta_1^v(C) = \frac{\lambda^2}{1+2\lambda+\lambda^2},$

$\beta_0^u(C) = \frac{1}{3}\cdot\frac{\lambda}{1+2\lambda+\lambda^2}, \quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{\lambda^2}{1+2\lambda+\lambda^2}.$

$\mathbf{C_{33}(1,1,0)}: \beta_C(\lambda) = \frac{\lambda}{1+2\lambda}, \quad \beta_0^v(C) = \frac{\lambda}{1+2\lambda}, \quad \beta_1^v(C) = 0, \quad \beta_0^u(C) = 0, \quad \beta_1^u(C) = \frac{1}{3}\cdot\frac{\lambda}{1+2\lambda}.$

$\mathbf{C_{34}(0,0,0)}: \beta_C(\lambda) = \frac{\lambda}{1+\lambda}, \quad \beta_0^v(C) = \frac{\lambda}{1+\lambda}, \quad \beta_1^v(C) = 0, \quad \beta_0^u(C) = 0, \quad \beta_1^u(C) = 0.$

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