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Research article

Local well-posedness to the thermal boundary layer equations in Sobolev space

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Abstract: In this paper, we study the local well-posedness of the thermal boundary layer equations for the two-dimensional incompressible heat conducting flow with nonslip boundary condition for the velocity and Neumann boundary condition for the temperature. Under Oleinik's monotonicity assumption, we establish the local-in-time existence and uniqueness of solutions in Sobolev space for the boundary layer equations by a new weighted energy method developed by Masmoudi and Wong.

Keywords: thermal boundary layer equations; Oleinik's monotonicity condition; local well-posedness
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1. Introduction

The concept of the boundary layer was first proposed by Ludwig Prandtl in 1904 ([19]). For the incompressible viscous fluid satisfying the non-slip boundary condition, Prandtl obtained a degenerate parabolic equation coupled with the elliptic equation, namely the famous Prandtl equation, to describe the fluid motion in the boundary layer.

Since Prandtl boundary layer theory was put forward, many mathematicians have devoted themselves to establishing its mathematical theory (cf. [2, 3, 5, 7–9, 17, 18, 23, 25, 27, 29–31, 34–36]). Oleinik [17] performed the first rigorous mathematical systematic work by showing that under the monotonic condition of the boundary normal tangential velocity field, local well-posedness of the Prandtl system can be proved in two-dimensional by using the Crocco transformation. This well-posedness result was also obtained in the Sobolev spaces by using energy method (cf. [1, 16]). The key ingredient in the proof is a nonlinear cancellation mechanism that can be used to eliminate the problematic terms in the equations.

Without the monotonicity condition, Caflisch and Sammartino [21, 22] established the local well-posedness in the framework of analytic functions. If the initial data is neither monotonic in the normal variable nor analytic, E and Engquist [4] constructed a finite time blowup solution to the Prandtl equations. See also the instability results of Gérard-Varet and Dormy [6]. Recently, there are also many important studies on the boundary layer problem for some more complex fluids, such as the MHD system and viscoelastic equations. Interested readers can refer to [11–15, 20, 24, 32, 33] for more details.

In reality, most fluids have thermal conductivity, so the study of heat-conducting viscous fluid has important theoretical significance and application background. The main object of this paper is to establish the local well-posedness of the thermal boundary layer equations for two-dimensional incompressible heat conducting flow with non-slip boundary condition. Namely, we will consider the following system in the two-dimensional half space $\Omega := \mathbb{T} \times \mathbb{R}^+ = \{(x, y) \mid x \in \mathbb{R}/\mathbb{Z}, 0 < y < \infty\}$

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u = \partial_y^2 u - \partial_x P - (\theta - \theta_\infty)\,, \\ \partial_t \theta + u\partial_x \theta + v\partial_y \theta = \partial_y^2 \theta + \left(\partial_y u\right)^2, \\ \partial_x u + \partial_y v = 0, \\ u(t, x, y) \mid_{t=0} = u_0(x, y), \qquad \theta(t, x, y) \mid_{t=0} = \theta_0(x, y), \\ u(t, x, y) \mid_{y=0} = 0, \quad v(t, x, y) \mid_{y=0} = 0, \quad \partial_y \theta(t, x, y) \mid_{y=0} = \theta_b(t, x), \\ \lim\limits_{y \to +\infty} u(t, x, y) = U(t, x), \quad \lim\limits_{y \to +\infty} \theta(t, x, y) = \Theta(t, x), \end{cases} \tag{1.1}$$

where (u, v) is the velocity field, and θ is the absolute temperature. The $U(t, x), \Theta(t, x)$ and $P(t, x)$ are the traces at the boundary $\{y = 0\}$ of the tangential velocity, temperature, and pressure of the outer inviscid flow with heat conduction, respectively. The reference temperature θ_∞ is assumed to be a positive constant in this paper. The states U, Θ, and P are interrelated through

$$\begin{cases} \partial_t U + U\partial_x U = -\partial_x P - (\Theta - \theta_\infty)\,, \\ \partial_t \Theta + U\partial_x \Theta = 0. \end{cases} \tag{1.2}$$

The mathematical theory of the thermal boundary layer equations was first studied by Wang and Zhu in [26], where they proved the local existence and uniqueness of solutions under the assumption of analyticity. In [28], they also proved finite time blowup of the solutions if the monotonic condition is violated. On back flow of boundary layers in two-dimensional unsteady incompressible heat conducting flow be studied in [29]. Recently, Liu, Wang and Yang [10] developed energy method to prove the well-posedness of a viscous layer problem when the tangential velocity is monotonically increasing in the normal variable. In this paper, we are going to show the local well-posedness of the system (1.1) in Sobolev space under the monotonic condition. This extends the Oleinik local well-posedness theory to the thermal boundary layer equations.

To state the main result, we first introduce some notations and the function spaces in which the initial-boundary value problem (1.1) will be solved under the strictly monotonic assumption on the tangential velocity in the normal variable

$$\omega := \partial_y u > 0.$$

First, C is a genetic constant which may change from line to line throughout this paper. We denote the tangential derivative operator by

$$\partial_\chi^\beta = \partial_t^{\beta_1} \partial_x^{\beta_2}, \quad \beta = (\beta_1, \beta_2) \in \mathbb{N}^2,$$

and the full derivative operator is given by

$$D^\alpha = \partial_\chi^\beta \partial_y^m, \quad \alpha = (\beta, m) = (\beta_1, \beta_2, m) \in \mathbb{N}^3.$$

We also use the following notations

$$e_1 = (1, 0), \quad e_2 = (0, 1),$$

and

$$E_1 = (1, 0, 0), \quad E_2 = (0, 1, 0), \quad E_3 = (0, 0, 1).$$

Second, $H^s(\Omega)$ and $H^s(\mathbb{T})$ is the usual Sobolev space on spatial domain Ω and $\mathbb{T}$ respectively. We also define the weighted Sobolev space $H^{s,\gamma}(\Omega)$ by

$$\|u\|_{H^{s,\gamma}}^2 := \sum_{|\alpha|\le s} \left\| \langle y\rangle^{\gamma+m} D^\alpha u \right\|_{L^2(\Omega)}^2,$$

with $u(t, x, y) : [0, T] \times \Omega \to \mathbb{R}$ and

$$\langle y\rangle = 1 + y.$$

Finally, we set $\tilde{\theta} = \theta - \Theta$ and define the space $H^{s,\gamma}_{\mu,\delta}$ for $(\omega, \tilde{\theta}) : [0, T] \times \Omega \to \mathbb{R}$ by

$$H^{s,\gamma}_{\mu,\delta} := \left\{ \omega \middle| \|(\omega, \tilde{\theta})\|_{H^{s,\gamma}} < \infty, \langle y\rangle^\mu \omega \ge \delta, \sum_{|\alpha|\le 2} \left| \langle y\rangle^{\mu+m} D^\alpha (\omega, \tilde{\theta}) \right| \le \frac{1}{\delta} \right\}, \tag{1.3}$$

with $s \ge 6, \gamma \ge 1, \mu > \gamma + \frac{1}{2}$ and $\delta \in (0, 1)$.

Remark 1.1. *The condition $\mu > \gamma + \frac{1}{2}$ is indispensable for the definition of the space $H^{s,\gamma}_{\mu,\delta}$. Actually, if $\mu \le \gamma + \frac{1}{2}$, one can check that $H^{s,\gamma}_{\mu,\delta}$ is an empty set. For example, taking $\mu = \gamma = 1, \alpha = 0$, we find that $\|\langle y\rangle\omega\|_{L^2} < \infty, \langle y\rangle\omega \ge \delta, |\langle y\rangle\omega| \le \frac{1}{\delta}$ can not hold at the same time. The same hypothesis is also explained by Masmoudi and Wong (see Remark 2.1 in [16]). The reason for introducing the weighted space $H^{s,\gamma}_{\mu,\delta}$ is to give the control of terms like $\frac{\partial_x\omega}{\omega}, \frac{\partial_x\theta}{\omega}, \frac{\partial_x^2\omega}{\omega}, \frac{\partial_x^2\theta}{\omega}$.*

Before state the main result, we assume $\theta_b = 0$ throughout this paper for the sake of simplicity. We claim that the result still holds if we have a non-trivial θ_b. Moreover, it is easy to find that the vorticity ω satisfies

$$\partial_t\omega + u\partial_x\omega + v\partial_y\omega = \partial_y^2\omega - \partial_y\theta. \tag{1.4}$$

Now, we are ready to state the main results of this paper in the following theorem.

Theorem 1.2. *Given any even integer $s \ge 6$, real numbers $\gamma \ge 1, \mu > \gamma + \frac{1}{2}, \delta \in (0, 1)$, assume the following conditions on the initial data and the outer flow U and Θ:*
i) The initial data $u_0 - U \in H^{s,\gamma-1}(\Omega)$ and $(\omega_0, \tilde{\theta}_0) \in H^{s,\gamma}_{\mu,2\delta}$. Here, the time derivatives of the initial data is expressed by solving equation (1.1) *and* (1.4) *repeatly for $\partial_t^k(\omega, \tilde{\theta})$ and substituting the initial data into the result, for example:*

$$\partial_t\omega_0 := -[u_0\partial_x\omega_0 + v_0\partial_y\omega_0 - \partial_y^2\omega_0 + \partial_y\theta_0], \tag{1.5}$$

with $v_0 = -\int_0^y \partial_x u_0 dy$.
ii)The outer flow U *and* Θ *is supposed to satisfy*

$$\sup_t \sum_{l=0}^{\frac{s}{2}+1} \|\partial_t^l (U,\Theta)\|_{H^{s-2l+2}(\mathbb{T})} < +\infty.$$

Then there exists a time $T := T\left(s,\gamma,\delta,\|(w_0,\tilde{\theta}_0)\|_{H^{s,\gamma}},U,\Theta\right) > 0$ *such that the initial-boundary value problem* (1.1) *has a unique classical solution* (u,v,θ) *satisfying*

$$u - U \in L^\infty([0,T];H^{s,\gamma-1}) \cap C\left([0,T];H^s - w\right),$$

and

$$(\partial_y u,\tilde{\theta}) \in L^\infty([0,T];H^{s,\gamma}_{\mu,\delta}) \cap C\left([0,T];H^s - w\right),$$

where $H^s - w$ *is the space* H^s *endowed with its weak topology.*

Remark 1.3. *If the Dirichlet boundary condition for temperature is given, some boundary terms cannot be handled in the proof. In this paper, we give the Neumann boundary condition to the temperature* θ *and it is interesting to investigate the Dirichlet boundary condition case. Another interesting question is how to extend the results to fractional problems.*

Remark 1.4. *We assume* $s \geq 6$ *in Theorem 1.2 mainly because we need to derive the uniform upper bound and lower bound of the solutions, and* s *needs to be an even number. Moreover, we didn't get the result similar to (5.3) in [16], so we need to assume* $s \geq 6$ *to get our results.*

Remark 1.5. *From the definition of* $H^{s,\gamma}_{\mu,\delta}$*, we can see that both the vorticity and the temperature enjoy some decay properties with respect to* $\langle y\rangle$ *at the far field* $y = +\infty$*. We refer to Appendix C of [16] for more details about the far-fields behavior of the vorticity, and the decay rates of the temperature can be obtained similarly.*

Let us briefly describe the strategy of the proof of our main theorem. As mentioned earlier, we will use the energy method developed by Masmoudi and Wong [16] to prove the local well-posedness of the thermal boundary layer equations. To do this, we first need to construct a regularized system by adding the viscous terms $\varepsilon\partial_x^2 u$ and $\varepsilon\partial_x^2\theta$ to the original equations. This will make the system no longer degenerate and the local existence of the regularized system can be established by using the classical local well-posedness theory of the hyperbolic-parabolic system. Next, to construct local solutions of the original system, we will derive the uniform-in-ε estimates of the solutions to the regularized system, which is the main part of this paper. The uniform estimates are divided into two parts. The first part is the weighted L^2 estimates on $D^\alpha(\omega,\tilde{\theta})$ with $|\alpha| \leq s, |\beta| \leq s-1$ and the second part is to get the estimate of $\partial_\chi^\beta(\omega,\tilde{\theta})$ with $|\beta| = s$.

Different from classical Prandtl equations [16] where only spatial derivatives of the solutions need to be estimated, here we give the control of both spatial and time derivatives of the solutions, because the time derivatives of θ will be involved in estimating the boundary integral $\int_{\mathbb{T}} \partial_y D^\alpha\omega D^\alpha\omega dx$ (see Lemma 3.3 for example). Similarly, the control of $\partial_t^k\omega$ is also needed when we encountered with the boundary term $\int_{\mathbb{T}} \partial_y D^\alpha\tilde{\theta} D^\alpha\tilde{\theta} dx$. However, estimating $D^\alpha(\omega,\tilde{\theta})$ will bring us new difficulties when we have the term $\int_{\mathbb{T}} \partial_y D^\alpha\omega D^\alpha\omega dx$ with $|\alpha| = |(\beta_1,0,m)| = s$ and $m = 2k+1$ is odd. Since there are no

x-derivatives here, we can not use integrating by parts to reduce the $s+1$ order derivatives which will prevent us from using trace estimate to estimate the boundary term. The remedy is to replace $\partial_y^{m+1}\omega$ and $\partial_y^m\omega$ by using the equations repeatedly to get

$$\int_{\mathbb{T}} \partial_y D^\alpha \omega D^\alpha \omega|_{y=0} = \int_{\mathbb{T}} \partial_t^{\beta_1}[\left(\partial_t - \varepsilon^2\partial_x^2\right)^{k+1} \omega + Q_k]\partial_t^{\beta_1}[\left(\partial_t - \varepsilon^2\partial_x^2\right)^k (\partial_x P - \theta) + \mathcal{P}_k],$$

where $\mathcal{P}_k$ and Q_k are low order terms. Now, as we already have x-derivatives in the above boundary integral, we can use integrating by parts and the trace lemma to control it.

The estimate of $\partial_\chi^\beta(\omega, \tilde{\theta})$ mainly based on the nonlinear cancellation method invented in [16]. Here we note that we only need the monotonic assumption on the tangential velocity u in the normal variable but have no restrictions on the absolute temperature θ. With the uniform estimates of the solutions, we show that the solutions of the regularized system actually exists in a time interval $[0, T]$ independent of ε. Moreover, we can use the Aubin-Lions lemma to extract a solution sequence and prove that the limit of this sequence is the solution of the original thermal boundary layer equations. Thus the existence of the local solution is constructed. Finally, the uniqueness of the solution is also proved by the energy estimate.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we first introduce the regularized system (3.1) and the regularized vorticity system (3.3) in order to construct the approximate solutions. Then we give the uniform estimates of the solutions. The local-in-time existence and uniqueness of the solution to the initial-boundary value problem (1.1) will be proved in Sections 4 and Section 5 respectively based on the uniform weighted estimates derived in Section 3. The proof of some useful inequalities and the derivation of some equations will be given in the Appendix.

2. Preliminaries

In this section, we introduce some notations and collect some preliminary results which will be used in the rest part of this paper.

As the Prandtl system, the key point for obtaining the energy estimates of solutions is to eliminate the terms $v\partial_y u$ and $v\partial_y\theta$ appeared in the first and the second equations of (1.1) respectively. Recalling that in [10] (see also [1, 16] for a similar transformation), the authors introduce $\omega = \partial_y(\frac{u}{\partial_y u})$ and $\bar{\theta} = \theta - \frac{\partial_y\theta}{\partial_y u}u$. Here a little different from [10], we define

$$g_\beta := \partial_\chi^\beta\omega - \frac{\partial_y\omega}{\omega}\partial_\chi^\beta(u - U), \qquad h_\beta := \partial_\chi^\beta\tilde{\theta} - \frac{\partial_y\tilde{\theta}}{\omega}\partial_\chi^\beta(u - U),$$

then we can introduce a weighted norm for the vorticity

$$\|\omega(t)\|^2_{H_g^{s,\gamma}(\Omega)} := \sum_{|\beta|=s} \left\|\langle y\rangle^\gamma g_\beta\right\|^2_{L^2(\Omega)} + \sum_{\substack{|\alpha|\le s \\ |\beta|\le s-1}} \left\|\langle y\rangle^{\gamma+m} D^\alpha\omega(t)\right\|^2_{L^2(\Omega)}, \tag{2.1}$$

and a weighted norm for the absolute temperature

$$\|\tilde{\theta}(t)\|^2_{H_h^{s,\gamma}(\Omega)} := \sum_{|\beta|=s} \left\|\langle y\rangle^\gamma h_\beta\right\|^2_{L^2(\Omega)} + \sum_{\substack{|\alpha|\le s \\ |\beta|\le s-1}} \|\langle y\rangle^{\gamma+m} D^\alpha\tilde{\theta}(t)\|^2_{L^2(\Omega)}. \tag{2.2}$$

Obviously, the main difference between norms $\|\cdot\|_{H^{s,\gamma}(\Omega)}$ and $\|\cdot\|_{H_g^{s,\gamma}(\Omega)}, \|\cdot\|_{H_h^{s,\gamma}(\Omega)}$ is that the weighted L^2 norm of $\partial_\chi^\beta \omega$ and $\partial_\chi^\beta \tilde{\theta}$ with $|\beta| = s$ is replaced by that of g_β, h_β. As we will see later, by estimating the weighted norms (2.1) and (2.2) of the solutions, we can avoid the loss of x-derivative through a delicate nonlinear cancellation.

Moreover, similar to those in [16], one can show that $\|\omega\|_{H^{s,\gamma}(\Omega)}$ and $\|\omega\|_{H_g^{s,\gamma}(\Omega)}$ are almost equivalent. That is, for any $(\omega, \tilde{\theta}) \in H_{\mu,\delta}^{s,\gamma}(\Omega)$, there exists a positive constant C such that

$$C^{-1}\|\omega\|_{H_g^{s,\gamma}} \leq \|\omega\|_{H^{s,\gamma}} + \|(u-U)\|_{H^{s,\gamma-1}} \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2}), \tag{2.3}$$

and

$$\|\tilde{\theta}\|_{H^{s,\gamma}} \leq C(\|\tilde{\theta}\|_{H_h^{s,\gamma}} + \|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2}). \tag{2.4}$$

Remark 2.1. *Although h_β is similar to g_β in form, the weighted norm $\|\tilde{\theta}\|_{H_h^{s,\gamma}}$ does not share the almost equivalent relationship with the norm $\|\tilde{\theta}\|_{H^{s,\gamma}}$. However, the above inequality* (2.4) *is enough to solve the problem.*

Next, let us introduce several useful inequalities which will be frequently used in this paper. We omit the proofs of these inequalities for the sake of simplicity and interested readers may refer to [11] and [16] and the references therein for more details.

Lemma 2.2 (Hardy type inequality)**.** *Let $u : \Omega \to \mathbb{R}$. Then*
i) if $\gamma > -\frac{1}{2}$ and $\lim_{y\to+\infty} u(x,y) = 0$, we have

$$\|\langle y\rangle^\gamma u\|_{L^2(\Omega)} \leq \frac{2}{2\gamma+1}\left\|\langle y\rangle^{\gamma+1}\partial_y u\right\|_{L^2(\Omega)},$$

ii) if $\gamma < -\frac{1}{2}$, we have

$$\|\langle y\rangle^\gamma u\|_{L^2(\Omega)} \leq \sqrt{-\frac{1}{2\gamma+1}}\|u|_{y=0}\|_{L^2(\mathbb{T})} - \frac{2}{2\gamma+1}\left\|\langle y\rangle^{\gamma+1}\partial_y u\right\|_{L^2(\Omega)}.$$

Lemma 2.3 (Sobolev type inequality)**.** *Let $u : \Omega \to \mathbb{R}$. Then there exists a positive constant C such that*

$$\|u\|_{L^\infty(\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|\partial_x u\|_{L^2(\Omega)} + \left\|\partial_y^2 u\right\|_{L^2(\Omega)}\right).$$

Lemma 2.4 (Trace estimate)**.** *Let $u, v : \Omega \to \mathbb{R}$. If $\lim_{y\to+\infty}(uv)(x,y) = 0$, then*

$$\left|\int_{\mathbb{T}} (uv)|_{y=0} dx\right| \leq \left\|\partial_y u\right\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\left\|\partial_y v\right\|_{L^2(\Omega)}.$$

Lemma 2.5 (Aubin-Lions)**.** *Let $\Omega_0 \subset \Omega \subset \Omega_1$ be three Banach spaces, with compact embedding $\Omega_0 \subset \Omega$ and continuous embedding $\Omega \subset \Omega_1$. Let $p, q \geq 1$, then*

$$L^p([0,T];\Omega_0) \cap H^{1,q}([0,T];\Omega_1),$$

is compactly embedded into $L^p([0,T];\Omega)$.

Finally, we also need the following lemma which will be used to control certain L^2 and L^∞ norms of $u, v, \omega, \tilde{\theta}, g_\beta, h_\beta$ and their derivatives in terms of the weighted norms $\|\omega(t,\cdot)\|_{H_g^{s,\gamma}}$ and $\|\tilde{\theta}(t,\cdot)\|_{H_h^{s,\gamma}}$. The proof of this lemma is given in the Appendix A.

Lemma 2.6. *Let the vector field (u, v) defined on Ω satisfy the condition $\partial_x u + \partial_y v = 0$, the Dirichlet boundary condition $u|_{y=0} = v|_{y=0} = 0$ and $\lim\limits_{y\to+\infty} u = U$. If $(\omega, \tilde{\theta}) \in H_{\mu,\delta}^{s,\gamma}$ for some constants $s \geq 6, \gamma \geq 1, \mu > \gamma + \frac{1}{2}$ and $\delta \in (0, 1)$, then we have the following estimates:*

A) Weighted L^2 estimates.

(i) *For all $|\beta| = 0, 1, \cdots, s$,*

$$\|\langle y\rangle^{\gamma-1}\partial_\chi^\beta(u - U)\|_{L^2} \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2}). \tag{2.5}$$

(ii) *For all $|\beta| = 0, 1, \cdots, s-1$,*

$$\|\langle y\rangle^{-1}(\partial_\chi^\beta v + y\partial_\chi^{\beta+e_2} U)\|_{L^2} \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2}). \tag{2.6}$$

(iii) *For all $|\beta| \leq s$,*

$$\|\langle y\rangle^{\gamma+m} D^\beta \omega\|_{L^2} \leq \begin{cases} C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2}) & \text{if } |\beta| = s, \\ C\|\omega\|_{H_g^{s,\gamma}} & \text{if } |\beta| \neq s, \end{cases} \tag{2.7}$$

and

$$\|\langle y\rangle^{\gamma+m} D^\beta \tilde{\theta}\|_{L^2} \leq \begin{cases} C(\|\tilde{\theta}\|_{H_g^{s,\gamma}} + \|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2}) & \text{if } |\beta| = s, \\ C\|\tilde{\theta}\|_{H_h^{s,\gamma}} & \text{if } |\beta| \neq s. \end{cases} \tag{2.8}$$

(iv) *For all $|\beta| = 1, 2, \cdots, s$,*

$$\left\|\langle y\rangle^{\gamma} g_\beta\right\|_{L^2} \leq \begin{cases} C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2}) & \text{if } |\beta| \leq s-1, \\ C\|\omega\|_{H_g^{s,\gamma}} & \text{if } |\beta| = s, \end{cases} \tag{2.9}$$

and

$$\|\langle y\rangle^{\gamma} h_\beta\|_{L^2} \leq \begin{cases} C(\|\tilde{\theta}\|_{H_h^{s,\gamma}} + \|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2}) & \text{if } |\beta| \leq s-1, \\ C\|\tilde{\theta}\|_{H_h^{s,\gamma}} & \text{if } |\beta| = s. \end{cases} \tag{2.10}$$

B) Weighted L^∞ estimates.

(v) *For all $|\beta| = 0, 1, \cdots, s-1$,*

$$\|\partial_\chi^\beta u\|_{L^\infty} \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2}). \tag{2.11}$$

(vi) *For all For all $|\beta| = 0, 1, \cdots, s-2$,*

$$\|\langle y\rangle^{-1}\partial_\chi^\beta v\|_{L^\infty} \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2} + 1). \tag{2.12}$$

(vii) *For all $|\alpha| \leq s-2$,*

$$\|\langle y\rangle^{\gamma+m} D^\beta \omega\|_{L^\infty} \leq C\|\omega\|_{H_g^{s,\gamma}}, \qquad \|\langle y\rangle^{\gamma+m} D^\beta \tilde{\theta}\|_{L^\infty} \leq C\|\tilde{\theta}\|_{H_h^{s,\gamma}}. \tag{2.13}$$

3. Uniform estimates to the regularized system

In this section, in order to prove the local-in-time existence of the initial-boundary value problem (1.1), we consider the following regularized equations for any $\varepsilon > 0$:

$$\begin{cases}\partial_t u^\varepsilon + u^\varepsilon\partial_x u^\varepsilon + v^\varepsilon\partial_y u^\varepsilon = \varepsilon^2\partial_x^2 u^\varepsilon + \partial_y^2 u^\varepsilon - \partial_x P^\varepsilon - (\theta^\varepsilon - \theta_\infty),\\ \partial_t\theta^\varepsilon + u^\varepsilon\partial_x\theta^\varepsilon + v^\varepsilon\partial_y\theta^\varepsilon = \varepsilon^2\partial_x^2\theta^\varepsilon + \partial_y^2\theta^\varepsilon + (\omega^\varepsilon)^2,\\ \partial_x u^\varepsilon + \partial_y v^\varepsilon = 0,\\ u^\varepsilon(t,x,y)\mid_{t=0} = u_0(x,y), \qquad \theta^\varepsilon(t,x,y)\mid_{t=0} = \theta_0(x,y),\\ u^\varepsilon(t,x,y)\mid_{y=0} = 0, \quad v^\varepsilon(t,x,y)\mid_{y=0} = 0, \quad \partial_y\theta^\varepsilon(t,x,y)\mid_{y=0} = 0,\\ \lim\limits_{y\to+\infty} u^\varepsilon(t,x,y) = U^\varepsilon(t,x), \quad \lim\limits_{y\to+\infty}\theta^\varepsilon(t,x,y) = \Theta^\varepsilon(t,x).\end{cases} \tag{3.1}$$

The states U^ε and P^ε are interrelated through

$$\begin{cases}\partial_t U^\varepsilon + U^\varepsilon\partial_x U^\varepsilon = \varepsilon^2\partial_x^2 U^\varepsilon - \partial_x P^\varepsilon - (\Theta^\varepsilon - \theta_\infty),\\ \partial_t\Theta^\varepsilon + U^\varepsilon\partial_x\Theta^\varepsilon = \varepsilon^2\partial_x^2\Theta^\varepsilon.\end{cases} \tag{3.2}$$

By a direct calculation, we find that the regularized vorticity $\omega^\varepsilon := \partial_y u^\varepsilon$ and $\tilde\theta^\varepsilon = \theta^\varepsilon - \Theta^\varepsilon$ satisfies the following regularized system

$$\begin{cases}\partial_t\omega^\varepsilon + u^\varepsilon\partial_x\omega^\varepsilon + v^\varepsilon\partial_y\omega^\varepsilon = \varepsilon^2\partial_x^2\omega^\varepsilon + \partial_y^2\omega^\varepsilon - \partial_y\theta^\varepsilon,\\ \partial_t\tilde\theta^\varepsilon + u^\varepsilon\partial_x\tilde\theta^\varepsilon + v^\varepsilon\partial_y\tilde\theta^\varepsilon = \varepsilon^2\partial_x^2\tilde\theta^\varepsilon + \partial_y^2\tilde\theta^\varepsilon + (\omega^\varepsilon)^2 - \tilde u^\varepsilon\partial_x\Theta^\varepsilon,\\ \partial_y\omega^\varepsilon\mid_{y=0} = \partial_x P^\varepsilon - (\theta^\varepsilon\mid_{y=0} - \theta_\infty), \quad \partial_y\tilde\theta^\varepsilon\mid_{y=0} = 0.\end{cases} \tag{3.3}$$

Here, the velocity field $(u^\varepsilon, v^\varepsilon)$ is given by

$$u^\varepsilon(t,x,y) := U - \int_y^{+\infty}\omega^\varepsilon(t,x,\tilde y)d\tilde y, \qquad v^\varepsilon(t,x,y) := -\int_0^y \partial_x u^\varepsilon(t,x,\tilde y)d\tilde y.$$

Now, the regularized system (3.3) constitutes a hyperbolic-parabolic equations. For any fixed $\varepsilon > 0$, the well-posedness can be established in a standard way. Actually, we have

Lemma 3.1 (Local Existence of the Regularized Equations)**.** *Let $s \geq 6$ be an even integer, $\gamma \geq 1, \mu > \gamma + \frac{1}{2}$, $\delta \in \left(0, \frac{1}{2}\right)$, and $\varepsilon \in (0,1)$. If $(\omega_0, \theta_0) \in H^{s+12,\gamma}_{\mu,2\delta}$, then there exist a time*

$$T := T(s,\gamma,\delta,\varepsilon,\omega_0,\theta_0,U,\Theta),$$

and a solution

$$(\omega^\varepsilon, \tilde\theta^\varepsilon) \in C([0,T]; H^{s+4,\gamma}_{\mu,\delta}) \cap C^1([0,T]; H^{s+2,\gamma}),$$

to the regularized system (3.3)*. Moreover, the velocity $(u^\varepsilon, v^\varepsilon)$ and the absolute temperature θ^ε satisfy the regularized system* (3.1) *as well.*

By Lemma 3.1, we have obtained the local existence of solution in $[0,T]$ which depends on $(s,\gamma,\delta,\omega_0,\theta_0,U,\Theta)$ as well as the parameter $\varepsilon > 0$. To get a solution in a time interval independent of ε for the original system, we need to derive the uniform-in-ε estimates of the solutions. From now on, we omit the superscript ε of the solution for the sake of simplicity.

The proof of the uniform estimates will be divided into four parts. First, we will give the weighted estimates of $D^\alpha(\omega,\tilde\theta)$ with $|\alpha|\le s$ and $|\beta|\le s-1$ in Subsection 3.1. Then we study the estimates of $D^\beta_\chi(\omega,\tilde\theta)$ with $|\beta|=s$ in Subsection 3.2. In Subsection 3.3, the weighted H^s estimates of the solution are obtained by combining the estimates in the last two parts. Finally, to ensure that our solution belongs to the function space $H^{s,\gamma}_{\mu,\delta}$, we also need to deal with the L^∞ estimates of the solution and this will be given in Subsection 3.4.

3.1. Weighted L^2 estimates on $D^\alpha(\omega,\tilde\theta)$ with $|\alpha|\le s, |\beta|\le s-1$

The main goal of this part is to prove:

Theorem 3.2. *Let $s\ge 6$ be an even integer, $\gamma\ge 1, \mu>\gamma+\frac12, \delta\in(0,1)$, and $\varepsilon\in(0,1]$. If*

$$(\omega,\tilde\theta)\in C([0,T];H^{s+4,\gamma}_{\mu,\delta})\cap C^1([0,T];H^{s+2,\gamma}_{\mu,\delta}),$$

and $(u,v,\omega,\tilde\theta)$ solves (3.1) and (3.3), then we have

$$\begin{aligned}
&\frac12\frac{d}{dt}\sum_{\substack{|\alpha|\le s\\|\beta|\le s-1}}\|\langle y\rangle^{\gamma+m}D^\alpha(\omega,\tilde\theta)\|^2_{L^2}-\varepsilon^2\sum_{|\beta|=s}\|\partial_y\partial^\beta_\chi\tilde\theta\|^2_{L^2}\\
&\le-\varepsilon^2\sum_{\substack{|\alpha|\le s\\|\beta|\le s-1}}\|\langle y\rangle^{\gamma+m}\partial_xD^\alpha(\omega,\tilde\theta)\|^2_{L^2}-\frac12\sum_{\substack{|\alpha|\le s\\|\beta|\le s-1}}\|\langle y\rangle^{\gamma+m}\partial_yD^\alpha(\omega,\tilde\theta)\|^2_{L^2}\\
&\quad+C(1+\|\omega\|_{H^{s,\gamma}_g})^{s-2}\|\omega\|^2_{H^{s,\gamma}_g}+C(\|\omega\|_{H^{s,\gamma}_g}+\|\partial^\beta_\chi U\|_{L^2(\mathbb{T})}+1)\|\omega\|^2_{H^{s,\gamma}_g}\\
&\quad+C\sum_{l=0}^{s/2}\|\partial^l_t(\partial_xP)\|^2_{H^{s-2l}(\mathbb{T})}+C\|\omega\|_{H^{s,\gamma}_g}\|\tilde\theta\|^2_{H^{s,\gamma}_h}+C\|\omega\|_{H^{s,\gamma}_g}\|\tilde\theta\|_{H^{s,\gamma}_h}\\
&\quad+C(1+\|\tilde\theta\|_{H^{s,\gamma}_h})^{s-2}\|\tilde\theta\|^2_{H^{s,\gamma}_h}+C(\|\omega\|_{H^{s,\gamma}_g}+\|\partial^\beta_\chi U\|_{L^2(\mathbb{T})}+1)\|\tilde\theta\|^2_{H^{s,\gamma}_h}\\
&\quad+C\|\partial^{\beta+e_2}_\chi\Theta\|_{L^\infty}(\|\omega\|_{H^{s,\gamma}_g}+\|\partial^\beta_\chi U\|_{L^2(\mathbb{T})})\|\tilde\theta\|_{H^{s,\gamma}_h},
\end{aligned}\tag{3.4}$$

where C is a constant independent of ε and t.

Proof. Applying the operator $D^\alpha=\partial^\beta_\chi\partial^m_y$ for $\alpha=(\beta,m)=(\beta_1,\beta_2,m)$ with $|\alpha|\le s,|\beta|\le s-1$ to the equation $(3.3)_1,(3.3)_2$, multiplying by $\langle y\rangle^{2\gamma+2m}D^\alpha\omega$, $\langle y\rangle^{2\gamma+2m}D^\alpha\tilde\theta$ respectively, then integrating over Ω, we have

$$\begin{aligned}
\frac12\frac{d}{dt}\left\|\langle y\rangle^{\gamma+m}D^\alpha\omega\right\|^2_{L^2}&=\varepsilon^2\int_\Omega\langle y\rangle^{2\gamma+2m}\partial^2_xD^\alpha\omega D^\alpha\omega+\int_\Omega\langle y\rangle^{2\gamma+2m}\partial^2_yD^\alpha\omega D^\alpha\omega\\
&\quad-\int_\Omega\langle y\rangle^{2\gamma+2m}\left(u\partial_xD^\alpha\omega+v\partial_yD^\alpha\omega\right)D^\alpha\omega-\int_\Omega\langle y\rangle^{2\gamma+2m}D^\alpha\omega\partial_yD^\alpha\tilde\theta\\
&\quad-\sum_{0<\sigma\le\alpha}\binom{\alpha}{\sigma}\int_\Omega\langle y\rangle^{2\gamma+2m}\left(D^\sigma u\partial_xD^{\alpha-\sigma}\omega+D^\sigma v\partial_yD^{\alpha-\sigma}\omega\right)D^\alpha\omega\\
&:=\sum_{i=1}^5J_i,
\end{aligned}\tag{3.5}$$

and

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|\langle y\rangle^{\gamma+m}D^\alpha\tilde{\theta}\right\|_{L^2}^2 =& \varepsilon^2\int_\Omega \langle y\rangle^{2\gamma+2m}\partial_x^2 D^\alpha\tilde{\theta}D^\alpha\tilde{\theta} + \int_\Omega \langle y\rangle^{2\gamma+2m}\partial_y^2 D^\alpha\tilde{\theta}D^\alpha\tilde{\theta}\\
&- \int_\Omega \langle y\rangle^{2\gamma+2m}\left(u\partial_x D^\alpha\tilde{\theta} + v\partial_y D^\alpha\tilde{\theta}\right)D^\alpha\tilde{\theta}\\
&- \sum_{0<\sigma\le\alpha}\binom{\alpha}{\sigma}\int_\Omega \langle y\rangle^{2\gamma+2m}\left(D^\sigma u\partial_x D^{\alpha-\sigma}\tilde{\theta} + D^\sigma v\partial_y D^{\alpha-\sigma}\tilde{\theta}\right)D^\alpha\tilde{\theta}\\
&- \sum_{0\le\sigma\le\alpha}\binom{\alpha}{\sigma}\int_\Omega \langle y\rangle^{2\gamma+2m}D^\sigma\omega D^{\alpha-\sigma}\omega D^\alpha\tilde{\theta}\\
&- \sum_{0\le\sigma\le\alpha}\binom{\alpha}{\sigma}\int_\Omega \langle y\rangle^{2\gamma+2m}D^\sigma\tilde{u}\partial_x D^{\alpha-\sigma}\Theta D^\alpha\tilde{\theta}\\
:=& \sum_{i=1}^{6}K_i.
\end{aligned}
\tag{3.6}
$$

Now, we will give the estimates of J_i and K_i as follows. First of all, for J_1, it holds that

$$
J_1 = -\varepsilon^2\left\|\langle y\rangle^{\gamma+m}\partial_x D^\alpha\omega\right\|_{L^2}^2,
$$

where an integration by parts in the x-variable is used. For J_2, utilizing integration by parts in the y-variable, we have

$$
\begin{aligned}
J_2 &= -\left\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha\omega\right\|_{L^2}^2 - (2\gamma+2m)\int_\Omega \langle y\rangle^{2\gamma+2m-1}\partial_y D^\alpha\omega D^\alpha\omega + \int_{\mathbb{T}}\partial_y D^\alpha\omega D^\alpha\omega\Big|_{y=0}\\
&= -\left\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha\omega\right\|_{L^2}^2 + J_2^1 + J_2^2.
\end{aligned}
$$

Clearly, J_2^1 can be controlled by using the Cauchy inequality

$$
J_2^1 \le \frac{1}{4}\left\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha\omega\right\|_{L^2}^2 + C\left\|\langle y\rangle^{\gamma+m}D^\alpha\omega\right\|_{L^2}^2 \le \frac{1}{4}\left\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha\omega\right\|_{L^2}^2 + C\|\omega\|_{H_g^{s,\gamma}}^2.
$$

However, the estimate of the boundary integral J_2^2 is very complicated. To handle it, we need to introduce the following lemma and the proof of this lemma is given in the Appendix B.

Lemma 3.3. *(Reduction of Boundary Data). Under the hypotheses of Theorem 3.2, we have*

$$
\begin{cases}
\partial_y\omega\Big|_{y=0} = \partial_x P - \theta|_{y=0} + \theta_\infty,\\
\partial_y^3\omega\Big|_{y=0} = \left(\partial_t - \varepsilon^2\partial_x^2\right)(\partial_x P - \theta|_{y=0}) + \omega\partial_x\omega|_{y=0} + \partial_y^2\theta|_{y=0}.
\end{cases}
\tag{3.7}
$$

For any $2 \le k \le \frac{s}{2}$, we have

$$
\partial_y^{2k+1}\omega|_{y=0} = \left(\partial_t - \varepsilon^2\partial_x^2\right)^k(\partial_x P - \theta|_{y=0}) + \mathcal{P}_k|_{y=0},
\tag{3.8}
$$

where $\mathcal{P}_k$ denotes a polynomial

$$
\mathcal{P}_k = \mathcal{P}[D^\alpha_{|\alpha|\le 2k-1}\omega, D^\pi_{|\pi|\le 2k}(\partial_x P, \theta)].
$$

Now, we claim that

$$J_2^2 \leq \frac{1}{8}\left\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha \omega\right\|_{L^2}^2 + C(1+\|\omega\|_{H_g^{s,\gamma}})^{s-2}\|\omega\|_{H_g^{s,\gamma}}^2 + C\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2 + \varepsilon^2\sum_{|\beta|=s}\|\partial_y\partial_\chi^\beta\tilde{\theta}\|_{L^2}^2 + C\sum_{l=0}^{s/2}\|\partial_t^l(\partial_x P)\|_{H^{s-2l}(\mathbb{T})}^2. \tag{3.9}$$

We prove the above claim in two cases $|\alpha| \leq s-1$ and $|\alpha| = s$. When $|\alpha| \leq s-1$, we find

$$\begin{aligned}\int_{\mathbb{T}} \partial_y D^\alpha \omega D^\alpha \omega\Big|_{y=0} &\leq \|\partial_y^2 D^\alpha\omega\|_{L^2}\|D^\alpha\omega\|_{L^2} + \|\partial_y D^\alpha\omega\|_{L^2}\|\partial_y D^\alpha\omega\|_{L^2} \\ &\leq \frac{1}{8}\|\partial_y^2 D^\alpha\omega\|_{L^2}^2 + C(\|D^\alpha\omega\|_{L^2}^2 + \|\partial_y D^\alpha\omega\|_{L^2}^2) \\ &\leq \frac{1}{8}\|\partial_y^2 D^\alpha\omega\|_{L^2}^2 + C\|\omega\|_{H_g^{s,\gamma}}^2.\end{aligned}$$

While when $|\alpha| = s$, we must have $m \geq 1$ since $|\beta| \leq s-1$. Two cases need to be considered here.

Case 1: $m = 2k$ **is an even number.**

In this case, we can use Lemma 3.3 to obtain

$$\int_{\mathbb{T}} \partial_y D^\alpha\omega D^\alpha\omega|_{y=0} = \int_{\mathbb{T}} D^\alpha\omega\partial_\chi^\beta\left(\partial_t - \varepsilon^2\partial_x^2\right)^k(\partial_x P - \theta) + \int_{\mathbb{T}} D^\alpha\omega\mathcal{P}_k|_{y=0}.$$

Therefore, applying the trace estimate in Lemma 2.3 and the Cauchy-Schwarz inequality to the above equation, we get

$$J_2^2 \leq \frac{1}{8}\left\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha \omega\right\|_{L^2}^2 + C(1+\|\omega\|_{H_g^{s,\gamma}})^{s-2}\|\omega\|_{H_g^{s,\gamma}}^2 + C\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2 + \varepsilon^2\sum_{|\beta|=s}\|\partial_y\partial_\chi^\beta\tilde{\theta}\|_{L^2}^2 + C\sum_{l=0}^{s/2}\|\partial_t^l(\partial_x P)\|_{H^{s-2l}(\mathbb{T})}^2. \tag{3.10}$$

Case 2: $m = 2k+1$ **is an odd number.**

In this case, since s is even, we have $|\beta| = |\beta_1| + |\beta_2| \geq 1$.

(1) When $\beta_2 \geq 1$. Using integration by parts in x, we have

$$\int_{\mathbb{T}} \partial_y D^\alpha\omega D^\alpha\omega dx|_{y=0} = -\int_{\mathbb{T}} D^{\alpha-E_2+E_3}\omega\partial_x D^\alpha\omega dx|_{y=0}. \tag{3.11}$$

Now, the term $\partial_x D^\alpha\omega|_{y=0} = \partial_\chi^{\beta+e_2}\partial_y^{2m+1}\omega|_{y=0}$ has an odd number of y derivatives. Hence, we can apply Lemma 3.2 to reduce the order of the right hand side of (3.11). Similar to the Case 1, we can further apply Lemma 2.4 to eventually obtain the following estimates:

$$J_2^2 \leq \frac{1}{8}\left\|\langle y\rangle^{\gamma+m+1}D^{\alpha-E_2+E_3}\omega\right\|_{L^2}^2 + C(1+\|\omega\|_{H_g^{s,\gamma}})^{s-2}\|\omega\|_{H_g^{s,\gamma}}^2 + C\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2 + \varepsilon^2\sum_{|\beta|=s}\|\partial_y\partial_\chi^\beta\tilde{\theta}\|_{L^2}^2 + C\sum_{l=0}^{s/2}\|\partial_t^l(\partial_x P)\|_{H^{s-2l}(\mathbb{T})}^2. \tag{3.12}$$

(2) When $\beta_2 = 0$. There is no derivative of x here, so we cannot integrate by parts with respect to x directly. However, we can substitute the $y-$derivative of ω by using Lemma 3.3 and

$$\partial_y^{2k+2}\omega = \left(\partial_t - \varepsilon^2\partial_x^2\right)^{k+1}\omega + Q_k,$$

which is obtained by applying ∂_y^{2k} to $(3.3)_1$ and Q_k is a polynomial of $D^\alpha(\omega, \tilde{\theta})$ with $|\alpha| \leq 2k+1$. This substitution gives us the derivative with respect to x, so we have

$$\begin{aligned}\int_{\mathbb{T}} \partial_y D^\alpha\omega D^\alpha\omega|_{y=0} &= \int_{\mathbb{T}} \partial_t^{\beta_1}\partial_y^{2k+2}\omega\partial_t^{\beta_1}\partial_y^{2k+1}\omega \\ &= \int_{\mathbb{T}} \partial_t^{\beta_1}[\left(\partial_t - \varepsilon^2\partial_x^2\right)^{k+1}\omega + Q_k]\partial_t^{\beta_1}[\left(\partial_t - \varepsilon^2\partial_x^2\right)^{k}(\partial_x P - \theta) + \mathcal{P}_k].\end{aligned}$$

The difficulty lies in the term $\varepsilon^{2k+2}\int_{\mathbb{T}} \partial_t^{\beta_1}\partial_x^{2k+2}\omega\partial_t^{\beta_1}(\partial_t - \varepsilon^2\partial_x^2)^k(\partial_x P - \theta)$, since the others can be estimated directly by the trace estimate. Integrating by parts with respect to x, we have

$$\int_{\mathbb{T}} \partial_t^{\beta_1}\partial_x^{2k+2}\omega\partial_t^{\beta_1}(\partial_t - \varepsilon^2\partial_x^2)^k(\partial_x P - \theta) = -\int_{\mathbb{T}} \partial_t^{\beta_1}\partial_x^{2k+1}\omega\partial_t^{\beta_1}(\partial_t - \varepsilon^2\partial_x^2)^k\partial_x(\partial_x P - \theta). \tag{3.13}$$

Now, the order of the derivatives of (ω, θ) in (3.13) is no larger than $\beta_1 + 2k + 1 = s$, so we can further apply Lemma 2.4 to eventually obtain the following estimates:

$$\begin{aligned}J_2^2 \leq{}& \frac{1}{8}\left\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha\omega\right\|_{L^2}^2 + C(1 + \|\omega\|_{H_g^{s,\gamma}})^{s-2}\|\omega\|_{H_g^{s,\gamma}}^2 \\ &+ C\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2 + \varepsilon^2\sum_{|\beta|=s}\|\partial_y\partial_\chi^\beta\tilde{\theta}\|_{L^2}^2 + C\sum_{l=0}^{s/2}\|\partial_t^l(\partial_x P)\|_{H^{s-2l}(\mathbb{T})}^2.\end{aligned} \tag{3.14}$$

Thus, combining estimates (3.10), (3.12), (3.14), we prove (3.9). As a result, J_2 can be estimated as

$$\begin{aligned}J_2 \leq{}& -\frac{1}{2}\left\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha\omega\right\|_{L^2}^2 + C(1 + \|\omega\|_{H_g^{s,\gamma}})^{s-2}\|\omega\|_{H_g^{s,\gamma}}^2 \\ &+ C\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2 + \varepsilon^2\sum_{|\beta|=s}\|\partial_y\partial_\chi^\beta\tilde{\theta}\|_{L^2}^2 + C\sum_{l=0}^{s/2}\|\partial_t^l(\partial_x P)\|_{H^{s-2l}(\mathbb{T})}^2.\end{aligned}$$

For the term J_3, we can use integration by parts and the equation $\partial_x u + \partial_y v = 0$ to get

$$\begin{aligned}J_3 = -\int_\Omega \langle y\rangle^{2\gamma+2m}\left(u\partial_x D^\alpha\omega + v\partial_y D^\alpha\omega\right)D^\alpha\omega &= (2\gamma+2m)\int_\Omega \langle y\rangle^{2\gamma+2m}(\langle y\rangle^{-1}v)(D^\alpha\omega)^2 \\ &\leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})} + 1)\|\omega\|_{H_g^{s,\gamma}}^2.\end{aligned}$$

Next, we can estimate the term J_4 by using Cauchy-Schwarz inequality directly:

$$J_4 = -\int_\Omega \langle y\rangle^{2\gamma+2m} D^\alpha\omega\partial_y D^\alpha\tilde{\theta} \leq \frac{1}{4}\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha\tilde{\theta}\|_{L^2}^2 + C\|\omega\|_{H_g^{s,\gamma}}^2.$$

Finally, we will show the estimate of

$$J_5 = -\sum_{0<\sigma\leq\alpha}\binom{\alpha}{\sigma}\int_\Omega \langle y\rangle^{2\gamma+2m}\left(D^\sigma u\partial_x D^{\alpha-\sigma}\omega + D^\sigma v\partial_y D^{\alpha-\sigma}\omega\right)D^\alpha\omega := J_5^1 + J_5^2,$$

where $\sigma = (\bar{\sigma}, \bar{m}) = (\sigma_1, \sigma_2, \bar{m})$. For J_5^1, we have two cases:
Case 1: $\bar{m} = 0$, and $|\bar{\sigma}| \leq s-1$. Here we have

$$\begin{aligned}\int_\Omega \langle y\rangle^{2\gamma+2m}\, \partial_\chi^{\bar{\sigma}} u D^{\alpha-\sigma-E_2}\omega D^\alpha\omega &\leq \left\|\partial_\chi^{\bar{\sigma}} u\right\|_{L^\infty} \left\|\langle y\rangle^{\gamma+m} D^{\alpha-\sigma+E_2}\omega\right\|_{L^2} \left\|\langle y\rangle^{\gamma+m} D^\alpha\omega\right\|_{L^2} \\ &\leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\omega\|^2_{H_g^{s,\gamma}}.\end{aligned} \tag{3.15}$$

Case 2: $\bar{m} \geq 1$. In this case, by Lemma 2.4, we find

$$\begin{aligned}\int_\Omega \langle y\rangle^{2\gamma+2m} D^\sigma u\partial_x D^{\alpha-\sigma}\omega D^\alpha\omega &\leq \left\|\langle y\rangle^{\gamma+m} D^{\sigma-E_3}\omega D^{\alpha-\sigma+E_2}\omega\right\|_{L^2} \left\|\langle y\rangle^{\gamma+m} D^\alpha\omega\right\|_{L^2} \\ &\leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\omega\|^2_{H_g^{s,\gamma}}.\end{aligned} \tag{3.16}$$

While for J_5^2, we need to consider the following four cases:
Case 1: $\bar{m} = 0, \sigma_2 = s-1$,

$$\begin{aligned}&\int_\Omega \langle y\rangle^{2\gamma+2m} D^\sigma v\partial_y D^{\alpha-\sigma}\omega D^\alpha\omega \\ &\quad\leq C\|\langle y\rangle^{-1}(\partial_\chi^{\bar{\sigma}} v + y\partial_\chi^{\beta+e_2}U)\|_{L^2}\|\langle y\rangle^{\gamma+m}D^\alpha\omega\|_{L^2}\|\langle y\rangle^{\gamma+m+1}D^{\alpha-\sigma+E_3}\omega\|_{L^\infty} \\ &\quad\leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\omega\|^2_{H_g^{s,\gamma}}.\end{aligned} \tag{3.17}$$

For the other three cases, by using $\partial_x u + \partial_y v = 0$ and Lemma 2.4, one has
Case 2: $m = 0, \sigma_2 \leq s-2$,

$$J_5^2 \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\omega\|^2_{H_g^{s,\gamma}}. \tag{3.18}$$

Case 3: $m = 1$,

$$J_5^2 \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\omega\|^2_{H_g^{s,\gamma}}. \tag{3.19}$$

Case 4: $m \leq 2$,

$$J_5^2 \leq C\|\omega\|^3_{H_g^{s,\gamma}}. \tag{3.20}$$

Combining estimates (3.15)-(3.20), we get

$$J_5 \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})} + 1)\|\omega\|^2_{H_g^{s,\gamma}}. \tag{3.21}$$

In the following part, we will give the estimates of the right-hand side terms of (3.6). First, for K_1, it directly follows from an integration by parts in the x-variable that

$$K_1 = -\varepsilon^2\|\langle y\rangle^{\gamma+m}\partial_x D^\alpha\tilde{\theta}\|^2_{L^2}. \tag{3.22}$$

Integrating by parts in the y-variable in K_2, we have

$$\begin{aligned}K_2 &= -\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha\tilde{\theta}\|^2_{L^2} - (2\gamma+2m)\int_\Omega \langle y\rangle^{2\gamma+2m-1}\partial_y D^\alpha\tilde{\theta}D^\alpha\tilde{\theta} + \int_{\mathbb{T}} \partial_y D^\alpha\tilde{\theta}D^\alpha\tilde{\theta}|_{y=0} \\ &= -\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha\tilde{\theta}\|^2_{L^2} + K_2^1 + K_2^2.\end{aligned} \tag{3.23}$$

Here K_2^1 can be controlled by the Cauchy-Schwarz inequality

$$K_2^1 \le \frac{1}{4}\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha\tilde{\theta}\|_{L^2}^2 + C\|\langle y\rangle^{\gamma+m}D^\alpha\tilde{\theta}\|_{L^2}^2 \le \frac{1}{4}\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha\tilde{\theta}\|_{L^2}^2 + C\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2. \tag{3.24}$$

Similar to the estimate of J_2^2, the boundary integral K_2^2 is controlled in the following two cases: $|\alpha| \le s-1$ and $|\alpha| = s$. When $|\alpha| \le s-1$, we can use the basic trace estimate directly to control K_2^2 since the order of the derivatives in the boundary integral is no larger than s. While for the case $|\alpha| = s$, we have to appeal to the boundary reduction argument as before. Actually, we have the following lemma.

Lemma 3.4. *(Reduction of Boundary Data). Under the hypotheses of Theorem 3.2, for any* $2 \le k \le \frac{s}{2}$*, we have*

$$\partial_y^{2k+1}\tilde{\theta}|_{y=0} = \mathcal{H}_k,$$

$\mathcal{H}_k$ *denotes a polynomial, and*

$$\mathcal{H}_k = \mathcal{H}[D^\alpha_{|\alpha|\le 2k-1}\omega, D^\pi_{|\pi|\le 2k}(\tilde{\theta},\Theta)].$$

The proof of this Lemma is based on an elementary use of the original equation (1.1), so we just omit it. With this Lemma, we can give the estimate of K_2^2 in a similar fashion with J_2^2. By direct calculations, one has

$$K_2^2 \le \frac{1}{8}\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha\tilde{\theta}\|_{L^2}^2 + C(1+\|\tilde{\theta}\|_{H_h^{s,\gamma}})^{s-2}\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2 + C\|\omega\|_{H_g^{s,\gamma}}^2. \tag{3.25}$$

Collecting the estimates (3.23), (3.24) and (3.25), we obtain

$$K_2 \le -\frac{1}{2}\|\langle y\rangle^{\gamma+m}\partial_y D^\alpha\tilde{\theta}\|_{L^2}^2 + C(1+\|\tilde{\theta}\|_{H_h^{s,\gamma}})^{s-2}\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2 + C\|\omega\|_{H_g^{s,\gamma}}^2.$$

For K_3 we have

$$K_3 = -\int_\Omega \langle y\rangle^{2\gamma+2m}\left(u\partial_x D^\alpha\tilde{\theta} + v\partial_y D^\alpha\tilde{\theta}\right)D^\alpha\tilde{\theta} \le C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})} + 1)\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2.$$

The estimate of K_4 is similar to J_5. Namely, we can use a similar strategy to get

$$K_4 \le C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})} + 1)\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2.$$

The term K_5 can be estimated by using Lemma 2.6 directly:

$$K_5 \le C\|\omega\|_{H_g^{s,\gamma}}^2\|\tilde{\theta}\|_{H_h^{s,\gamma}}.$$

Similarly, K_6 satisfies

$$K_6 \le C\|\partial_\chi^{\beta+e_2}\Theta\|_{L^\infty}(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\tilde{\theta}\|_{H_h^{s,\gamma}}.$$

Finally, putting all the above estimates of J_i and K_i into (3.5) and (3.6), we can obtain the desired estimate (3.4). This completes the proof of Theorem **3.2**. □

3.2. Weighted L^2 estimates on g_β, h_β with $|\beta| = s$

In this subsection, we will derive the weighted L^2 estimates on g_β and h_β by using standard energy method. We need to derive the evolution equations of g_β and h_β first.

Let $a = \frac{\partial_y \omega}{\omega}$, $b = \frac{\partial_y \tilde{\theta}}{\omega}$, then after a tedious but straightforward calculation (see Appendix C), we have

$$\begin{aligned}
(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)g_\beta = &\, 2\varepsilon^2(\partial_\chi^{\beta+e_2}\tilde{u} - \frac{\partial_x\omega}{\omega}\partial_\chi^\beta \tilde{u})\partial_x a + 2g_\beta \partial_y a - g_{e_2}\partial_\chi^\beta U \\
&+ \frac{\partial_y^2\tilde{\theta}}{\omega}\partial_\chi^\beta \tilde{u} - ab\partial_\chi^\beta \tilde{u} - \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}} u g_{\bar{\beta}+e_2} \\
&+ a\sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}+e_2}U\partial_\chi^{\bar{\beta}}\tilde{u} \\
&- \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}} v\left(\partial_y\partial_\chi^{\bar{\beta}}\omega - a\partial_\chi^{\bar{\beta}}\omega\right) - \partial_\chi^\beta\partial_y\tilde{\theta} + a\partial_\chi^\beta\tilde{\theta},
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)h_\beta = &\, 2\varepsilon^2\left(\partial_\chi^{\beta+e_2}\tilde{u} - \frac{\partial_x\omega}{\omega}\partial_\chi^\beta\tilde{u}\right)\partial_x b - \partial_y(ab)\partial_\chi^\beta\tilde{u} \\
&- h_{e_2}\partial_\chi^\beta U - b^2\partial_\chi^\beta\tilde{u} - a\frac{\partial_y^2\tilde{\theta}}{\omega}\partial_\chi^\beta\tilde{u} + b\frac{\partial_y^2\omega}{\omega}\partial_\chi^\beta\tilde{u} \\
&- 2\partial_y\omega\partial_\chi^\beta\tilde{u} + 2\partial_\chi^\beta\omega\partial_y b + \partial_x\Theta\partial_\chi^\beta\tilde{u} \\
&- \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}} u h_{\bar{\beta}+e_2} + b\sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}+e_2}U\partial_\chi^{\bar{\beta}}\tilde{u} \\
&- \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}} v\left(\partial_y\partial_\chi^{\bar{\beta}}\tilde{\theta} - b\partial_\chi^{\bar{\beta}}\omega\right) - \partial_\chi^\beta\omega^2 \\
&+ b\partial_\chi^\beta\tilde{\theta} - \sum_{0\le\bar{\beta}\le\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}\tilde{u}\partial_\chi^{\bar{\beta}}\partial_x\Theta.
\end{aligned} \tag{3.27}$$

Theorem 3.5. *Let $s \ge 6$ be an even integer, $\gamma \ge 1, \mu > \gamma + \frac{1}{2}, \delta \in (0,1)$, and $\varepsilon \in (0,1]$. If*

$$(\omega, \tilde{\theta}) \in C([0,T]; H_{\mu,\delta}^{s+4,\gamma}) \cap C^1([0,T]; H_{\mu,\delta}^{s+2,\gamma}),$$

and $(u, v, \omega, \tilde{\theta})$ solves (3.1) and (3.3), then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|\beta|=s}\left\|\langle y\rangle^\gamma(g_\beta, h_\beta)\right\|_{L^2}^2 + \frac{\varepsilon^2}{2}\sum_{|\beta|=s}\left\|\langle y\rangle^\gamma\partial_x(g_\beta, h_\beta)\right\|_{L^2}^2 + \frac{1}{2}\sum_{|\beta|=s}\left\|\langle y\rangle^\gamma\partial_y(g_\beta, h_\beta)\right\|_{L^2}^2 \\
&\le C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^\infty(\mathbb{T})} + 1)(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^{\beta+e_2}U\|_{L^\infty(\mathbb{T})})\|\omega\|_{H_g^{s,\gamma}} + C\|\partial_\chi^\beta\partial_x P\|_{L^2(\mathbb{T})} \\
&+ C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^\infty(\mathbb{T})} + 1)(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^{\beta+e_2}U\|_{L^\infty(\mathbb{T})})\|\tilde{\theta}\|_{H_h^{s,\gamma}} \\
&+ C(\|\partial_\chi^\beta U\|_{L^\infty(\mathbb{T})}^2 + \|\partial_\chi^{\beta+e_2}\Theta\|_{L^\infty(\mathbb{T})}^2)(\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2 + \|\omega\|_{H_g^{s,\gamma}}^2) \\
&+ C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^\infty(\mathbb{T})} + 1)\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2.
\end{aligned} \tag{3.28}$$

where C is a constant independent of ε and t.

Proof. Multiplying the equation (3.26) by $\langle y\rangle^{2\gamma} g_\beta$ and integrating the resulting equation over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\langle y\rangle^{\gamma} g_\beta\right\|_{L^2}^2 =& \varepsilon^2\int_\Omega \langle y\rangle^{2\gamma}\partial_x^2 g_\beta g_\beta + \int_\Omega \langle y\rangle^{2\gamma}\partial_y^2 g_\beta g_\beta - \int_\Omega \langle y\rangle^{2\gamma}\left(u\partial_x g_\beta + v\partial_y g_\beta\right) g_\beta\\ &+ 2\varepsilon^2\int_\Omega \langle y\rangle^{2\gamma} g_\beta\left(\partial_\chi^{\beta+e_2}\tilde{u} - \frac{\partial_x\omega}{\omega}\partial_\chi^\beta\tilde{u}\right)\partial_x a + 2\int_\Omega \langle y\rangle^{2\gamma} g_\beta^2\partial_y a\\ &- \int_\Omega \langle y\rangle^{2\gamma} g_\beta g_{e_2}\partial_\chi^\beta U + \int_\Omega \langle y\rangle^{2\gamma} g_\beta\frac{\partial_y^2\tilde{\theta}}{\omega}\partial_\chi^\beta\tilde{u} - \int_\Omega \langle y\rangle^{2\gamma} g_\beta ab\partial_\chi^\beta\tilde{u}\\ &- \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\int_\Omega \langle y\rangle^{2\gamma} g_\beta\partial_\chi^{\beta-\bar{\beta}} u g_{\bar{\beta}+e_2} - \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\int_\Omega \langle y\rangle^{2\gamma} g_\beta\partial_\chi^{\beta-\bar{\beta}} v\left(\partial_y\partial_\chi^{\bar{\beta}}\omega - a\partial_\chi^{\bar{\beta}}\omega\right)\\ &+ a\sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\int_\Omega \langle y\rangle^{2\gamma} g_\beta\partial_\chi^{\beta-\bar{\beta}+e_2} U\partial_\chi^{\bar{\beta}}\tilde{u} - \int_\Omega \langle y\rangle^{2\gamma} g_\beta\partial_\chi^\beta\partial_y\tilde{\theta} + \int_\Omega \langle y\rangle^{2\gamma} g_\beta a\partial_\chi^\beta\tilde{\theta}\\ :=& \sum_{i=1}^{13} L_i,\end{aligned}$$

where $g_i = \partial_\chi^i\omega - \frac{\partial_y\omega}{\omega}\partial_\chi^i(u-U)$.
Then we shall estimate L_i term by term as follows. It directly follows from an integration by parts in the x-variable that

$$L_1 = -\varepsilon^2\left\|\langle y\rangle^\gamma\partial_x g_\beta\right\|_{L^2}^2.$$

Integrating by parts in the y-variable, we have

$$L_2 = -\left\|\langle y\rangle^\gamma\partial_y g_\beta\right\|_{L^2}^2 - (2\gamma)\int_\Omega \langle y\rangle^{2\gamma-1}\partial_y g_\beta g_\beta + \int_{\mathbb{T}}\partial_y g_\beta g_\beta\Big|_{y=0} := -\left\|\langle y\rangle^\gamma\partial_y g_\beta\right\|_{L^2}^2 + L_2^1 + L_2^2. \quad (3.29)$$

L_2^1 is controlled by the Cauchy inequality

$$L_2^1 \le \frac{1}{8}\left\|\langle y\rangle^\gamma\partial_y g_\beta\right\|_{L^2}^2 + C\left\|\langle y\rangle^\gamma g_\beta\right\|_{L^2}^2 \le \frac{1}{8}\left\|\langle y\rangle^\gamma\partial_y g_\beta\right\|_{L^2}^2 + C\|\omega\|_{H_g^{s,\gamma}}^2. \quad (3.30)$$

For the boundary integral L_2^2, a direct calculation yields

$$\partial_y g_\beta = \partial_y(\partial_\chi^\beta\omega - \frac{\partial_y\omega}{\omega}\partial_\chi^\beta\tilde{u}) = \partial_\chi^\beta\partial_y\omega - \partial_\chi^\beta\tilde{u}\frac{\partial_y^2\omega}{\omega} - a(\partial_\chi^\beta\omega - \frac{\partial_y\omega}{\omega}\partial_\chi^\beta\tilde{u}) = \partial_\chi\partial_y\omega - \partial_\chi^\beta\tilde{u}\frac{\partial_y^2\omega}{\omega} - ag_\beta. \quad (3.31)$$

This combined with $\partial_y\omega|_{y=0} = \partial_x P - \theta|_{y=0} + \theta_\infty$ and $u|_{y=0} = 0$ gives

$$\partial_y g_\beta\Big|_{y=0} = \partial_\chi^\beta\left(\partial_x P - \theta|_{y=0} + \theta_\infty\right) + \partial_\chi^\beta U\frac{\partial_y^2\omega}{\omega}\Big|_{y=0} - \left(ag_\beta\right)\Big|_{y=0}. \quad (3.32)$$

Substituting (3.32) to L_2^2 and using Lemma 2.4, we get

$$\begin{aligned}L_2^2 =& \int_{\mathbb{T}} g_\beta\left(\partial_\chi^\beta(\partial_x P - \theta + \theta_\infty) + \partial_\chi^\beta U\frac{\partial_y^2\omega}{\omega} - ag_\beta\right)\Big|_{y=0}\\ \le& \frac{1}{4}\left\|\langle y\rangle^\gamma\partial_y g_\beta\right\|_{L^2}^2 + \frac{1}{4}\left\|\langle y\rangle^\gamma\partial_y h_\beta\right\|_{L^2}^2 + C\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2 + C\|\omega\|_{H_g^{s,\gamma}}^2(1 + \|\partial_\chi^\beta U\|_{L^\infty(\mathbb{T})}^2) + \|\partial_\chi^\beta(\partial_x P)\|_{L^2(\mathbb{T})}^2,\end{aligned} \quad (3.33)$$

where $|\frac{\partial_y^2\omega}{\omega}| \leq \frac{1}{\delta^2}$ is used. Combining (3.29)-(3.33), we have

$$L_2 \leq -\frac{1}{4}\left\|\langle y\rangle^\gamma \partial_y g_\beta\right\|_{L^2}^2 - \frac{1}{4}\left\|\langle y\rangle^\gamma \partial_y h_\beta\right\|_{L^2}^2 + C\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2 + C\|\omega\|_{H_g^{s,\gamma}}^2(1 + \|\partial_\chi^\beta U\|_{L^\infty(\mathbb{T})}^2) + \|\partial_\chi^\beta(\partial_x P)\|_{L^2(\mathbb{T})}^2.$$

For L_3, by integrating by parts in the x-variable and y-variable, and using the equation $\partial_x u + \partial_y v = 0$, we have

$$L_3 = -\int_\Omega \langle y\rangle^{2\gamma}\left(u\partial_x g_\beta + v\partial_y g_\beta\right) g_\beta = 2\gamma\int_\Omega \langle y\rangle^{2\gamma}(\langle y\rangle^{-1}v)(g_\beta)^2 \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})} + 1)\|\omega\|_{H_g^{s,\gamma}}^2.$$

The term L_4 is estimated as follows:

$$\begin{aligned} L_4 =& 2\varepsilon^2\int_\Omega \langle y\rangle^{2\gamma} g_\beta\left(\partial_\chi^{\beta+e_2}\tilde{u} - \frac{\partial_x\omega}{\omega}\partial_\chi^\beta\tilde{u}\right)\partial_x a \\ \leq& 2\varepsilon^2\Big[\|\langle y\rangle^{\gamma-1}\partial_\chi^{\beta+e_2}\tilde{u}\|_{L^2} + \|\langle y\rangle^{\gamma-1}\frac{\partial_x\omega}{\omega}\partial_\chi^\beta\tilde{u}\|_{L^2}\Big]\left\|\langle y\rangle^\gamma g_\beta\right\|_{L^2}\|\langle y\rangle\partial_x a\|_{L^\infty}. \end{aligned}$$

Since $\delta \leq \langle y\rangle^\mu\omega \leq \delta^{-1}$, $\quad \sum_{|\alpha|\leq 2}|\langle y\rangle^{\mu+m}D^\alpha\omega|^2 \leq \delta^{-2}$, we have $\|\langle y\rangle\partial_x a\|_{L^\infty} \leq \delta^{-2}$. Note that $\omega\partial_y\left(\frac{\partial_\chi^{\beta+e_2}\tilde{u}}{\omega}\right) = \partial_x g_\beta + \partial_x a\partial_\chi^\beta\tilde{u}$, and since $\mu > \gamma + \frac{1}{2}$, we can use Lemma 2.2 to get

$$\begin{aligned} \left\|\langle y\rangle^{\gamma-1}\partial_\chi^{\beta+e_2}\tilde{u}\right\|_{L^2} =& \left\|\langle y\rangle^{\gamma-1}\omega\frac{\partial_\chi^{\beta+e_2}\tilde{u}}{\omega}\right\|_{L^2} \\ \leq& \frac{1}{\delta}\left\|\langle y\rangle^{\gamma-\mu-1}\frac{\partial_\chi^{\beta+e_2}\tilde{u}}{\omega}\right\|_{L^2} \\ \leq& C\|\partial_\chi^{\beta+e_2}U\|_{L^2} + C\left\|\langle y\rangle^\gamma\omega\partial_y\left(\frac{\partial_\chi^{\beta+e_2}\tilde{u}}{\omega}\right)\right\|_{L^2} \\ \leq& C\left(\|\partial_\chi^{\beta+e_2}U\|_{L^2} + \left\|\langle y\rangle^\gamma\partial_x g_\beta\right\|_{L^2} + \|\langle y\rangle^\gamma\partial_x a\partial_\chi^\beta\tilde{u}\|_{L^2}\right). \end{aligned}$$

Thus, we get

$$L_4 \leq \frac{\varepsilon^2}{2}\left\|\langle y\rangle^\gamma\partial_x g_\beta\right\|_{L^2}^2 + C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^{\beta+e_2}U\|_{L^2})\|\omega\|_{H_g^{s,\gamma}}.$$

Since $\|\partial_y a\|_{L^\infty} \leq C$, we have

$$L_5 \leq C\|\omega\|_{H_g^{s,\gamma}}^2.$$

The estimates of $L_6 - L_{13}$ are straightforward, so we omit the details for simplicity. Actually, we have

$$\begin{aligned} L_6 \leq& C\|\partial_\chi^{\beta+e_2}U\|_{L^\infty}(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^{\beta+e_2}U\|_{L^2})\|\omega\|_{H_g^{s,\gamma}}, \\ L_7 + L_8 + L_9 + L_{10} \leq& C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})} + 1)\|\omega\|_{H_g^{s,\gamma}}^2, \\ L_{11} \leq& C\|\partial_\chi^{\beta+e_2}U\|_{L^\infty}(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^{\beta+e_2}U\|_{L^2})\|\omega\|_{H_g^{s,\gamma}}, \\ L_{12} \leq& \frac{1}{4}\left\|\langle y\rangle^\gamma\partial_y h_\beta\right\|_{L^2}^2 + C\|\omega\|_{H_g^{s,\gamma}}^2, \\ L_{13} \leq& C(\|\tilde{\theta}\|_{H_h^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2})\|\omega\|_{H_g^{s,\gamma}}. \end{aligned}$$

Now let us estimate h_β. Multiplying the equation (3.27) by $\langle y\rangle^{2\gamma} h_\beta$ and integrating the resulting equation over Ω, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|\langle y\rangle^\gamma h_\beta\right\|_{L^2}^2 = {} & \varepsilon^2\int_\Omega \langle y\rangle^{2\gamma}\partial_x^2 h_\beta h_\beta + \int_\Omega \langle y\rangle^{2\gamma}\partial_y^2 h_\beta h_\beta - \int_\Omega \langle y\rangle^{2\gamma}\left(u\partial_x h_\beta + v\partial_y h_\beta\right)h_\beta \\
& + 2\varepsilon^2\int_\Omega \langle y\rangle^{2\gamma} h_\beta\left(\partial_\chi^{\beta+e_2}\tilde{u} - \frac{\partial_x\omega}{\omega}\partial_\chi^\beta\tilde{u}\right)\partial_x b - \int_\Omega \langle y\rangle^{2\gamma} h_\beta\partial_y(ab)\partial_\chi^\beta\tilde{u} \\
& - \int_\Omega \langle y\rangle^{2\gamma} h_\beta h_{e_2}\partial_\chi^\beta U - \int_\Omega \langle y\rangle^{2\gamma} h_\beta b^2\partial_\chi^\beta\tilde{u} - \int_\Omega \langle y\rangle^{2\gamma} h_\beta a\frac{\partial_y^2\tilde{\theta}}{\omega}\partial_\chi^\beta\tilde{u} \\
& + \int_\Omega \langle y\rangle^{2\gamma} h_\beta b\frac{\partial_y^2\omega}{\omega}\partial_\chi^\beta\tilde{u} + 2\int_\Omega \langle y\rangle^{2\gamma} h_\beta\partial_y b\partial_\chi^\beta\omega - 2\int_\Omega \langle y\rangle^{2\gamma} h_\beta\partial_y\omega\partial_\chi^\beta\tilde{u} \\
& - \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\int_\Omega \langle y\rangle^{2\gamma} h_\beta\partial_\chi^{\beta-\bar{\beta}} u h_{\bar{\beta}+e_2} - \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\int_\Omega \langle y\rangle^{2\gamma} h_\beta\partial_\chi^{\beta-\bar{\beta}} v\left(\partial_y\partial_\chi^{\bar{\beta}}\tilde{\theta} - b\partial_\chi^{\bar{\beta}}\omega\right) \\
& + b\sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\int_\Omega \langle y\rangle^{2\gamma} h_\beta\partial_\chi^{\beta-\bar{\beta}+e_2} U\partial_\chi^{\bar{\beta}}\tilde{u} - \sum_{0\le\bar{\beta}\le\beta}\binom{\beta}{\bar{\beta}}\int_\Omega \langle y\rangle^{2\gamma} h_\beta\partial_\chi^{\beta-\bar{\beta}}\omega\partial_\chi^{\bar{\beta}}\omega \\
& + \int_\Omega \langle y\rangle^{2\gamma} h_\beta b\partial_\chi^\beta\tilde{\theta} - \sum_{0\le\bar{\beta}\le\beta}\binom{\beta}{\bar{\beta}}\langle y\rangle^{2\gamma} h_\beta\partial_\chi^{\beta-\bar{\beta}}\tilde{u}\partial_\chi^{\bar{\beta}+e_2}\Theta + \int_\Omega \langle y\rangle^{2\gamma} h_\beta\partial_x\Theta\partial_\chi^\beta\tilde{u} \\
:= {} & \sum_{i=1}^{18} M_i,
\end{aligned}$$

where $h_i = \partial_\chi^i\tilde{\theta} - \frac{\partial_y\tilde{\theta}}{\omega}\partial_\chi^i(u-U)$.
Each M_i need to be estimated now. However, since the estimate of M_i is similar to L_i, we only give a skeleton of the proof. From integration by parts in the x-variable, we have

$$M_1 = -\varepsilon^2\left\|\langle y\rangle^\gamma\partial_x h_\beta\right\|_{L^2}^2.$$

Similar to the estimate of L_2, we have

$$M_2 = -\left\|\langle y\rangle^\gamma\partial_y h_\beta\right\|_{L^2}^2 - (2\gamma)\int_\Omega \langle y\rangle^{2\gamma-1}\partial_y h_\beta h_\beta + \int_{\mathbb{T}} \partial_y h_\beta h_\beta\Big|_{y=0} := -\left\|\langle y\rangle^\gamma\partial_y h_\beta\right\|_{L^2}^2 + M_2^1 + M_2^2, \quad (3.34)$$

where M_2^1 is controlled by the Cauchy inequality

$$M_2^1 \le \frac{1}{8}\left\|\langle y\rangle^\gamma\partial_y h_\beta\right\|_{L^2}^2 + C\left\|\langle y\rangle^\gamma h_\beta\right\|_{L^2}^2 \le \frac{1}{8}\left\|\langle y\rangle^\gamma\partial_y h_\beta\right\|_{L^2}^2 + C\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2. \quad (3.35)$$

Using the boundary conditions $\partial_y\tilde{\theta}|_{y=0} = 0$ and $u|_{y=0} = 0$, we have

$$\partial_y h_\beta\Big|_{y=0} = \partial_\chi^\beta U\frac{\partial_y^2\tilde{\theta}}{\omega}\Bigg|_{y=0} - \left(bg_\beta\right)\Big|_{y=0}. \quad (3.36)$$

Substituting (3.36) to M_2^2 and using Lemma 2.4, we get

$$M_2^2 = \int_{\mathbb{T}} h_\beta\left(\partial_\chi^\beta U\frac{\partial_y^2\tilde{\theta}}{\omega} - bg_\beta\right)\Bigg|_{y=0} \le \frac{1}{4}\left\|\langle y\rangle^\gamma(\partial_y g_\beta, \partial_y h_\beta)\right\|_{L^2}^2 + C\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2(1+\|\partial_\chi^\beta U\|_{L^\infty(\mathbb{T})}^2). \quad (3.37)$$

Combining (3.34)–(3.37), we have

$$M_2 \leq -\frac{1}{4}\left\|\langle y\rangle^{\gamma}\partial_y g_\beta\right\|_{L^2}^2 - \frac{1}{4}\left\|\langle y\rangle^{\gamma}\partial_y h_\beta\right\|_{L^2}^2 + C\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2(1+\|\partial_\chi^\beta U\|_{L^\infty(\mathbb{T})}^2).$$

We can give the control of M_3 as

$$M_3 = -\int_\Omega \langle y\rangle^{2\gamma}\left(u\partial_x h_\beta + v\partial_y h_\beta\right)h_\beta = (2\gamma)\int_\Omega \langle y\rangle^{2\gamma}(\langle y\rangle^{-1}v)(h_\beta)^2 \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})}+1)\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2.$$

The term M_4 can be controlled in a similar fashion as L_4, we just skip the details to get

$$\begin{aligned} M_4 &= 2\varepsilon^2\int_\Omega \langle y\rangle^{2\gamma}h_\beta\left(\partial_\chi^{\beta+e_2}\tilde{u} - \frac{\partial_x\omega}{\omega}\partial_\chi^\beta\tilde{u}\right)\partial_x b \\ &\leq 2\varepsilon^2\left\|\langle y\rangle^{\gamma-1}\left(\partial_\chi^{\beta+e_2}\tilde{u} - \frac{\partial_x\omega}{\omega}\partial_\chi^\beta\tilde{u}\right)\right\|_{L^2}\left\|\langle y\rangle^{\gamma}h_\beta\right\|_{L^2}\|\langle y\rangle\partial_x b\|_{L^\infty} \\ &\leq \frac{\varepsilon^2}{2}\left\|\langle y\rangle^{\gamma}\partial_x g_\beta\right\|_{L^2}^2 + C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^{\beta+e_2}U\|_{L^2})\|\tilde{\theta}\|_{H_h^{s,\gamma}}. \end{aligned}$$

By tedious but straightforward calculations, we find

$$\begin{aligned} M_5 &\leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\tilde{\theta}\|_{H_h^{s,\gamma}}, \\ M_6 &\leq C\|\partial_\chi^\beta U\|_{L^\infty(\mathbb{T})}\|\tilde{\theta}\|_{H_h^{s,\gamma}}(\|\tilde{\theta}\|_{H_h^{s,\gamma}} + \|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})}), \\ M_7+M_8+M_9+M_{10} &\leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\tilde{\theta}\|_{H_h^{s,\gamma}}, \\ M_{11} &\leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\tilde{\theta}\|_{H_h^{s,\gamma}}\|\omega\|_{H_g^{s,\gamma}}. \end{aligned}$$

Here we have used $\|\langle y\rangle a\|_{L^\infty} \leq C, \|\langle y\rangle b\|_{L^\infty} \leq C, \|\langle y\rangle\frac{\partial_y^2\tilde{\theta}}{\omega}\|_{L^\infty} \leq C, \|\langle y\rangle\frac{\partial_y^2\omega}{\omega}\|_{L^\infty} \leq C, \|\langle y\rangle^2 ab\|_{L^\infty} \leq C, \|\langle y\rangle^2 b^2\|_{L^\infty} \leq C, \|\langle y\rangle^2\partial_y(ab)\|_{L^\infty} \leq C$, since $(\omega,\tilde{\theta}) \in H_{\mu,\delta}^{s,\gamma}$. For M_{12}, we have

$$\begin{aligned} M_{12} = -\sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\int_\Omega \langle y\rangle^{2\gamma}h_\beta\partial_\chi^{\beta-\bar{\beta}}u h_{\bar{\beta}+e_2} &\leq \|\partial_\chi^{\beta-\bar{\beta}}\tilde{u}\|_{L^\infty}\|\langle y\rangle^{\gamma}h_{\bar{\beta}+e_2}\|_{L^2}\|\langle y\rangle^{\gamma}h_\beta\|_{L^2} \\ &\leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})(\|\tilde{\theta}\|_{H_h^{s,\gamma}} + \|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\tilde{\theta}\|_{H_h^{s,\gamma}}. \end{aligned}$$

By direct calculations, one can show that

$$\begin{aligned} M_{13} &\leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})}+1)(\|\tilde{\theta}\|_{H_h^{s,\gamma}}^2 + \|\tilde{\theta}\|_{H_h^{s,\gamma}}\|\omega\|_{H_g^{s,\gamma}}), \\ M_{14} &\leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\tilde{\theta}\|_{H_h^{s,\gamma}}\|\partial_\chi^\beta U\|_{L^\infty(\mathbb{T})}, \\ M_{15} &\leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\tilde{\theta}\|_{H_h^{s,\gamma}}\|\omega\|_{H_g^{s,\gamma}}, \\ M_{16} &\leq C(\|\tilde{\theta}\|_{H_h^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\tilde{\theta}\|_{H_h^{s,\gamma}}. \end{aligned}$$

The term M_{17} can be estimated by using Lemma 2.6 directly:

$$\begin{aligned} M_{17} = \sum_{0\leq\bar{\beta}\leq\beta}\binom{\beta}{\bar{\beta}}\langle y\rangle^{2\gamma}h_\beta\partial_\chi^{\beta-\bar{\beta}}\tilde{u}\partial_\chi^{\bar{\beta}+e_2}\Theta &\leq C\|\partial_\chi^{\bar{\beta}+e_2}\Theta\|_{L^\infty}\|\langle y\rangle^{\gamma}\partial_\chi^{\beta-\bar{\beta}}\tilde{u}\|_{L^2}\|\langle y\rangle^{\gamma}h_\beta\|_{L^2} \\ &\leq C\|\partial_\chi^{\bar{\beta}+e_2}\Theta\|_{L^\infty}(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})})\|\tilde{\theta}\|_{H_h^{s,\gamma}}. \end{aligned}$$

The last term M_{18} can also be estimated by using Lemma 2.6:

$$M_{18} = \int_\Omega \langle y\rangle^{2\gamma} h_\beta \partial_x \Theta \partial_\chi^\beta \tilde{u} \leq \|\partial_x \Theta\|_{L^\infty} \|\langle y\rangle^\gamma h_\beta\|_{L^2} \|\langle y\rangle^\gamma \partial_\chi^\beta \tilde{u}\|_{L^2} \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})}) \|\partial_x \Theta\|_{L^\infty} \|\tilde{\theta}\|_{H_h^{s,\gamma}}.$$

Finally combining all the above estimates, we obtain the estimate (3.28), and this completes the proof of Theorem 3.5. □

3.3. Weighted H^s estimates on ω and $\tilde{\theta}$

In this subsection, we can derive the weighted H^s estimates on $\omega, \tilde{\theta}$ by employing Theorem 3.2 and Theorem 3.5. The aim of this subsection is to derive the growth rate control (3.38) on the weighted H^s energy of $\omega, \tilde{\theta}$.

Theorem 3.6. *Let $s \geq 6$ be an even integer, $\gamma \geq 1, \mu > \gamma + \frac{1}{2}, \delta \in (0,1)$, and $\varepsilon \in (0,1]$. If*

$$(\omega, \tilde{\theta}) \in C([0,T]; H_{\mu,\delta}^{s+4,\gamma}) \cap C^1([0,T]; H_{\mu,\delta}^{s+2,\gamma}),$$

and $(u, v, \omega, \tilde{\theta})$ solves (3.1) and (3.3), then we have

$$\begin{aligned} \|\omega\|_{H_g^{s,\gamma}}^2 + \|\tilde{\theta}\|_{H_h^{s,\gamma}}^2 \leq & \left\{ \|\omega_0\|_{H_g^{s,\gamma}}^2 + \|\tilde{\theta}_0\|_{H_h^{s,\gamma}}^2 + \int_0^t Y(\tau) d\tau \right\} \\ & \cdot \left\{ 1 - C\left(\frac{s}{2} - 1\right) \left(\|\omega_0\|_{H_g^{s,\gamma}}^2 + \|\tilde{\theta}_0\|_{H_h^{s,\gamma}}^2 + \int_0^t Y(\tau) d\tau \right)^{\frac{s-2}{2}} t \right\}^{-\frac{2}{s-2}} := \mathcal{G}, \end{aligned} \tag{3.38}$$

where C is a constant independent of ε and t. The function $Y(t)$ is expressed by

$$Y(t) = C(\|\partial_\chi^{\beta+e_2} U(t)\|_{L^\infty(\mathbb{T})} + \|\partial_\chi^{\beta+e_2} \Theta(t)\|_{L^\infty(\mathbb{T})} + 1)^2 + C \sum_{l=0}^{s/2} \|\partial_t^l (\partial_x P)(t)\|_{H^{s-2l}(\mathbb{T})}^2. \tag{3.39}$$

Proof. Combining estimates (3.4) and (3.28), we find

$$\begin{aligned} \frac{d}{dt} \{\|\omega(t)\|_{H_g^{s,\gamma}}^2 + \|\tilde{\theta}(t)\|_{H_h^{s,\gamma}}^2\} \leq & C\|\omega(t)\|_{H_g^{s,\gamma}}^s + C\|\tilde{\theta}(t)\|_{H_h^{s,\gamma}}^s + C \sum_{l=0}^{s/2} \|\partial_t^l (\partial_x P)(t)\|_{H^{s-2l}(\mathbb{T})}^2 \\ & + C(\|\partial_\chi^{\beta+e_2} U(t)\|_{L^\infty(\mathbb{T})} + \|\partial_\chi^{\beta+e_2} \Theta(t)\|_{L^\infty(\mathbb{T})} + 1)^2. \end{aligned} \tag{3.40}$$

Then it follows from the comparison principle of ordinary differential equations that

$$\begin{aligned} \|\omega\|_{H_g^{s,\gamma}}^2 + \|\tilde{\theta}\|_{H_h^{s,\gamma}}^2 \leq & \left\{ \|\omega_0\|_{H_g^{s,\gamma}}^2 + \|\tilde{\theta}_0\|_{H_h^{s,\gamma}}^2 + \int_0^t Y(\tau) d\tau \right\} \\ & \cdot \left\{ 1 - C\left(\frac{s}{2} - 1\right) \left(\|\omega_0\|_{H_g^{s,\gamma}}^2 + \|\tilde{\theta}_0\|_{H_h^{s,\gamma}}^2 + \int_0^t Y(\tau) d\tau \right)^{\frac{s-2}{2}} t \right\}^{-\frac{2}{s-2}}, \end{aligned}$$

provided

$$1 - C\left(\frac{s}{2} - 1\right) \left(\|\omega_0\|_{H_g^{s,\gamma}}^2 + \|\tilde{\theta}_0\|_{H_h^{s,\gamma}}^2 + \int_0^t Y(\tau) d\tau \right)^{\frac{s-2}{2}} t \quad \geq 0,$$

which truly holds if $t > 0$ is chosen small enough. This completes the proof of Theorem 3.6. □

3.4. Uniform existence of the regularized system

In this subsection, we are going to prove the uniform existence of the solutions to the regularized system. To this end, we need to derive the uniform upper bound and lower bound of the solutions.

For $|\alpha| \le 2$, applying the operator $D^\alpha = \partial_\chi^\beta \partial_y^m$ to the equation $(3.3)_1$ and multiplying $\langle y\rangle^{\mu+m}$, we have

$$\begin{aligned}\partial_t \langle y\rangle^{\mu+m} D^\alpha \omega = &- \sum_{0\le\sigma\le\alpha} \binom{\alpha}{\sigma} \langle y\rangle^{\mu+m} D^\sigma u \partial_x D^{\alpha-\sigma}\omega - \sum_{0\le\sigma\le\alpha} \binom{\alpha}{\sigma} \langle y\rangle^{\mu+m} D^\sigma v \partial_y D^{\alpha-\sigma}\omega \\ &+ \varepsilon^2 \langle y\rangle^{\mu+m} \partial_x^2 D^\alpha \omega + \langle y\rangle^{\mu+m} \partial_y^2 D^\alpha \omega - \langle y\rangle^{\mu+m} \partial_y D^\alpha \tilde{\theta}.\end{aligned} \tag{3.41}$$

From Lemma 2.6, when $|\sigma| \le 2$ with $\sigma = (\sigma_1, \sigma_2, \bar{m})$, we have

$$\left\|\langle y\rangle^{\bar m} D^\sigma u\right\|_{L^\infty} \le \begin{cases} C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})}) & \bar m = 0, \\ C\|\omega\|_{H_g^{s,\gamma}} & \bar m \ge 1, \end{cases}$$

$$\left\|\langle y\rangle^{\bar m -1} D^\sigma v\right\|_{L^\infty} \le C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2(\mathbb{T})} + 1), \quad \|\langle y\rangle^{\bar m} D^\sigma \tilde\theta\|_{L^\infty} \le C\|\tilde\theta\|_{H_h^{s,\gamma}}, \quad \|\langle y\rangle^{\bar m+1} \partial_y D^\sigma \tilde\theta\|_{L^\infty} \le C\|\tilde\theta\|_{H_h^{s,\gamma}}.$$

Then, by direct calculation and using the above inequalities, we get

$$\|\partial_t \langle y\rangle^{\mu+m} D^\alpha \omega\|_{L^\infty} \le C\|\omega\|^2_{H_g^{s,\gamma}} + C\|\tilde\theta\|^2_{H_h^{s,\gamma}} + C(\|\partial_\chi^{\beta+e_2} U\|_{L^2} + 1)^2 \le C\mathcal{G}(t) + CY(t). \tag{3.42}$$

Integrating (3.41) with respect to t, we have

$$\begin{aligned}\sum_{|\alpha|\le 2} \left\|\langle y\rangle^{\mu+m} D^\alpha \omega\right\|_{L^\infty} &\le \sum_{|\alpha|\le 2} \left\|\langle y\rangle^{\mu+m} D^\alpha \omega_0\right\|_{L^\infty} + \sum_{|\alpha|\le 2} \int_0^t \left\|\partial_t \langle y\rangle^{\mu+m} D^\alpha \omega\right\|_{L^\infty} d\tau \\ &\le \sum_{|\alpha|\le 2} \left\|\langle y\rangle^{\mu+m} D^\alpha \omega_0\right\|_{L^\infty} + \sum_{|\alpha|\le 2} \sup_{0\le\tau\le t} \left\|\partial_t \langle y\rangle^{\mu+m} D^\alpha \omega\right\|_{L^\infty} t \\ &\le \sum_{|\alpha|\le 2} \left\|\langle y\rangle^{\mu+m} D^\alpha \omega_0\right\|_{L^\infty} + C\mathcal{G}t + CYt,\end{aligned} \tag{3.43}$$

where (3.42) is used. Specifically, when $\alpha = (0,0,0)$ and $m = 0$, we have

$$\langle y\rangle^\mu \omega \ge \langle y\rangle^\mu \omega_0 - \int_0^t \|\partial_t \langle y\rangle^\mu \omega\|_{L^\infty} d\tau \ge \langle y\rangle^\mu \omega_0 - \sup_{0\le\tau\le t} \|\partial_t \langle y\rangle^\mu \omega\|_{L^\infty} t \ge \langle y\rangle^\mu \omega_0 - C\mathcal{G}t - CYt. \tag{3.44}$$

Applying the operator $D^\alpha = \partial_\chi^\beta \partial_y^m$ to the equation $(3.3)_2$ and multiplying $\langle y\rangle^{\mu+m}$, we have

$$\begin{aligned}\partial_t \langle y\rangle^{\mu+m} D^\alpha \tilde\theta = &- \sum_{0\le\sigma\le\alpha} \binom{\alpha}{\sigma} \langle y\rangle^{\mu+m} D^\sigma u \partial_x D^{\alpha-\sigma}\tilde\theta - \sum_{0\le\sigma\le\alpha} \binom{\alpha}{\sigma} \langle y\rangle^{\mu+m} D^\sigma v \partial_y D^{\alpha-\sigma}\tilde\theta \\ &+ \varepsilon^2 \langle y\rangle^{\mu+m} \partial_x^2 D^\alpha \tilde\theta + \langle y\rangle^{\mu+m} \partial_y^2 D^\alpha \tilde\theta + \langle y\rangle^{\mu+m} D^\alpha \omega^2 \\ &- \sum_{0\le\sigma\le\alpha} \binom{\alpha}{\sigma} \langle y\rangle^{\mu+m} D^\sigma \tilde u D^{\alpha-\sigma} \partial_x \Theta.\end{aligned}$$

Similar to (3.42), we get from the above equation that

$$\|\partial_t \langle y\rangle^{\mu+m} D^\alpha \tilde\theta\|_{L^\infty} \le C\|\omega\|^2_{H_g^{s,\gamma}} + C\|\tilde\theta\|^2_{H_h^{s,\gamma}} + (\|\partial_\chi^{\beta+e_2} U(t)\|_{L^2} + \|\partial_\chi^{\beta+e_2} \Theta(t)\|_{L^2} + 1)^2 \le C\mathcal{G}(t) + CY(t),$$

which further gives

$$
\begin{aligned}
\sum_{|\alpha|\le 2}\|\langle y\rangle^{\mu+m}D^{\alpha}\tilde{\theta}\|_{L^\infty} &\le \sum_{|\alpha|\le 2}\|\langle y\rangle^{\mu+m}D^{\alpha}\tilde{\theta}_0\|_{L^\infty}+\sum_{|\alpha|\le 2}\int_0^t\|\partial_t\langle y\rangle^{\mu+m}D^{\alpha}\tilde{\theta}\|_{L^\infty}d\tau\\
&\le \sum_{|\alpha|\le 2}\|\langle y\rangle^{\mu+m}D^{\alpha}\tilde{\theta}_0\|_{L^\infty}+\sum_{|\alpha|\le 2}\sup_{0\le\tau\le t}\|\partial_t\langle y\rangle^{\mu+m}D^{\alpha}\tilde{\theta}\|_{L^\infty}t\\
&\le \sum_{|\alpha|\le 2}\|\langle y\rangle^{\mu+m}D^{\alpha}\tilde{\theta}_0\|_{L^\infty}+C\mathcal{G}t+CYt.
\end{aligned}
\tag{3.45}
$$

Now the uniform existence of the regularized system (3.3) can be stated as follows.

Theorem 3.7. *Let $s\ge 6$ be an even integer, $\gamma\ge 1,\mu>\gamma+\frac{1}{2},\delta\in(0,1)$ and $\varepsilon\in(0,1)$. If $(\omega_0,\tilde{\theta}_0)\in H^{s,\gamma}_{\mu,2\delta}$ and $|Y(t)|\le M$ for any $t\ge 0$, then there exists a positive T independent of ε, such that the regularized system* (3.3) *has solutions*

$$
(\omega,\tilde{\theta})\in C\left([0,T];H^{s,\gamma}_{\mu,\delta}\right)\cap C^1\left([0,T];H^{s-2,\gamma}\right).
$$

Moreover, for any $t\in[0,T]$, the solutions satisfy the following uniform estimates.
i) Uniform weighted H^s estimates

$$
\|\omega(t)\|^2_{H^{s,\gamma}_g}+\|\tilde{\theta}(t)\|^2_{H^{s,\gamma}_h}\le 4\left(\|\omega_0\|^2_{H^{s,\gamma}_g}+\|\tilde{\theta}_0\|^2_{H^{s,\gamma}_h}\right).
\tag{3.46}
$$

ii) Uniform weighted L^∞ upper bound

$$
\sum_{|\alpha|\le 2}\|\langle y\rangle^{\mu+m}D^{\alpha}(\omega,\tilde{\theta})(t)\|_{L^\infty}\le\frac{1}{\delta}.
$$

iii) Uniform weighted L^∞ lower bound

$$
\langle y\rangle^{\mu}\omega\ge\delta.
$$

Proof. Since $Y(t)\le M$, if we take

$$
T_1=\min\left\{\frac{3(\|\omega_0\|^2_{H^{s,\gamma}_g}+\|\tilde{\theta}_0\|^2_{H^{s,\gamma}_h})}{M},\frac{1-2^{-s+1}}{C2^{s-2}\left(\|\omega_0\|^2_{H^{s,\gamma}_g}+\|\tilde{\theta}_0\|^2_{H^{s,\gamma}_h}\right)^{s-2}}\right\},
$$

then by inequality (3.38), we have

$$
\|\omega\|^2_{H^{s,\gamma}_g}+\|\tilde{\theta}\|^2_{H^{s,\gamma}_h}\le\mathcal{G}(t)\le 4(\|\omega_0\|^2_{H^{s,\gamma}_g}+\|\tilde{\theta}_0\|^2_{H^{s,\gamma}_h}),
$$

for $t\in[0,T_1]$. This gives (3.46). Note that $(\omega_0,\tilde{\theta}_0)\in H^{s,\gamma}_{\mu,2\delta}$, we have

$$
\sum_{|\alpha|\le 2}\left\|\langle y\rangle^{\mu+m}D^{\alpha}\omega_0\right\|_{L^\infty}\le\frac{1}{2\delta}.
$$

Choosing T_2 as

$$T_2 = \min\left\{T_1, \frac{\delta^{-1}}{16C\left(\|\omega_0\|^2_{H^{s,\gamma}_g} + \|\tilde{\theta}_0\|^2_{H^{s,\gamma}_h}\right)}, \frac{\delta^{-1}}{4CM}\right\},$$

then from (3.43), we have for all $t \in [0, T_2]$,

$$\sum_{|\alpha|\le 2} \left\|\langle y\rangle^{\mu+m} D^\alpha \omega\right\|_{L^\infty} \le \frac{1}{\delta}.$$

Next, due to $\langle y\rangle^\mu \omega_0 \ge 2\delta$, if T_3 is chosen as

$$T_3 = \min\left\{T_1, \frac{\delta}{8C\left(\|\omega_0\|^2_{H^{s,\gamma}_g} + \|\tilde{\theta}_0\|^2_{H^{s,\gamma}_h}\right)}, \frac{\delta}{2CM}\right\},$$

then from (3.44), we have for all $t \in [0, T_3]$, $\langle y\rangle^\mu \omega \ge \delta$. Similarly, if we choose

$$T_4 = \min\left\{T_1, \frac{\delta^{-1}}{16C\left(\|\omega_0\|^2_{H^{s,\gamma}_g} + \|\tilde{\theta}_0\|^2_{H^{s,\gamma}_h}\right)}, \frac{\delta^{-1}}{4CM}\right\},$$

then we can get from (3.45) that for all $t \in [0, T_4]$,

$$\sum_{|\alpha|\le 2} \|\langle y\rangle^{\mu+m} D^\alpha \tilde{\theta}\|_{L^\infty} \le \frac{1}{\delta}.$$

Finally, letting $T := \min\{T_1, T_2, T_3, T_4\}$, we complete the proof of the theorem. □

4. Local-in-time existence

In this section, we will establish the local existence of solution to the original system (1.1) by compactness argument. Using the almost equivalence relation (2.3)-(2.4) and the uniform weighted H^s estimate (3.46), we have

$$\begin{aligned}
&\sup_{0\le t\le T} (\|\omega^\varepsilon\|^2_{H^{s,\gamma}} + \|u^\varepsilon - U\|^2_{H^{s,\gamma-1}} + \|\tilde{\theta}^\varepsilon\|^2_{H^{s,\gamma}}) \\
&\qquad \le C \sup_{0\le t\le T} (\|\omega^\varepsilon\|^2_{H^{s,\gamma}_g} + \|\tilde{\theta}^\varepsilon\|^2_{H^{s,\gamma}_h} + \|\partial^\beta_\chi U\|^2_{L^2}) \\
&\qquad \le 4C(\|\omega^\varepsilon_0\|^2_{H^{s,\gamma}_g} + \|\tilde{\theta}^\varepsilon_0\|^2_{H^{s,\gamma}_h} + \|\partial^\beta_\chi U\|^2_{L^2}).
\end{aligned}$$

Furthermore, we also know that $\partial_t(\omega^\varepsilon, \tilde{\theta}^\varepsilon)$ and $\partial_t \tilde{u}^\varepsilon$ are uniformly bounded in $L^\infty([0,T]; H^{s-2,\gamma})$ and $L^\infty([0,T]; H^{s-2,\gamma-1})$, respectively. By Aubin-Lions Lemma 2.5 and compact embedding of $H^{s,\gamma}$ in $H^{s'}_{loc}(s' < s)$, we can find limit function

$$\omega = \partial_y u \in L^\infty([0,T]; H^{s,\gamma}) \cap \bigcap_{s'<s} C\left([0,T]; H^{s'}_{loc}\right), \tag{4.1}$$

$$\tilde{u} \in L^{\infty}([0,T]; H^{s,\gamma-1}) \cap \bigcap_{s'<s} C\left([0,T]; H^{s'}_{loc}\right), \tag{4.2}$$

$$\tilde{\theta} \in L^{\infty}([0,T]; H^{s,\gamma}) \cap \bigcap_{s'<s} C\left([0,T]; H^{s'}_{loc}\right), \tag{4.3}$$

such that, after taking a subsequence, as $\varepsilon_k \to 0^+$:

$$\begin{cases} \omega^{\varepsilon_k} \overset{*}{\rightharpoonup} \omega & \text{in } L^{\infty}([0,T]; H^{s,\gamma}), \\ \omega^{\varepsilon_k} \to \omega & \text{in } C\left([0,T]; H^{s'}_{loc}\right), \\ u^{\varepsilon_k} - U \overset{*}{\rightharpoonup} u - U & \text{in } L^{\infty}\left([0,T]; H^{s,\gamma-1}\right), \\ u^{\varepsilon_k} \to u & \text{in } C\left([0,T]; H^{s'}_{loc}\right), \\ \theta^{\varepsilon_k} - \Theta \overset{*}{\rightharpoonup} \theta - \Theta & \text{in } L^{\infty}([0,T]; H^{s,\gamma}), \\ \theta^{\varepsilon_k} \to \theta & \text{in } C\left([0,T]; H^{s'}_{loc}\right). \end{cases} \tag{4.4}$$

Using the local uniform convergence of $\partial_x u^{\varepsilon_k}$, we also have the pointwise convergence of v^{ε_k}, as $\varepsilon_k \to 0^+$

$$v^{\varepsilon_k} = -\int_0^y \partial_x u^{\varepsilon_k} dy \to v = -\int_0^y \partial_x u dy. \tag{4.5}$$

Combining (4.4)-(4.5), one may justify the pointwise convergence of all terms in the regularized equation (3.1). Passing to the limit $\varepsilon_k \to 0^+$ in (3.1), we know that the limit function $(u, v, \tilde{\theta})$ solves the problem in the classical sense. Thus the local existence of solutions is obtained and we complete the proof of our main Theorem 1.2.

5. Uniqueness

The purpose of this section is to prove the uniqueness of $H^{s,\gamma}_{\mu,\delta}$ solutions constructed in Section 4. Assume (u_1, v_1, θ_1) and (u_2, v_2, θ_2) are two solutions to the initial-boundary value problem (1.1) and $\omega_i = \partial_y u_i (i = 1, 2)$. Setting $\bar{u} = u_1 - u_2, \bar{v} = v_1 - v_2, \bar{\theta} = \theta_1 - \theta_2, \bar{\omega} = \omega_1 - \omega_2$, we obtain the following equations

$$\begin{cases} (\partial_t + u_1\partial_x + v_1\partial_y - \partial_y^2)\bar{\omega} + \bar{u}\partial_x\omega_2 + \bar{v}\partial_y\omega_2 = -\partial_y\bar{\theta}, \\ (\partial_t + u_1\partial_x + v_1\partial_y - \partial_y^2)\bar{\theta} + \bar{u}\partial_x\theta_2 + \bar{v}\partial_y\theta_2 = (\omega_1 + \omega_2)\bar{\omega}, \\ (\partial_t + u_1\partial_x + v_1\partial_y - \partial_y^2)\bar{u} + \bar{u}\partial_x u_2 + \bar{v}\partial_y u_2 = -\bar{\theta}, \\ \partial_x\bar{u} + \partial_y\bar{v} = 0, \\ \bar{\omega}|_{t=0} = \omega_{10} - \omega_{20}, \quad \bar{\theta}|_{t=0} = \theta_{10} - \theta_{20}, \quad \bar{u}|_{t=0} = u_{10} - u_{20}, \\ (\bar{u}, \bar{v}, \partial_y\bar{\omega}, \partial_y\bar{\theta})|_{y=0} = 0. \end{cases} \tag{5.1}$$

Furthermore, set $\varpi = \bar{\omega} - a_2\bar{u}$ and $\vartheta = \bar{\theta} - b_2\bar{u}$ with $a_2 = \frac{\partial_y \omega_2}{\omega_2}$ and $b_2 = \frac{\partial_y \tilde{\theta}_2}{\omega_2}$. By direct calculations, we get

$$\begin{cases} (\partial_t + u_1\partial_x + v_1\partial_y - \partial_y^2)\varpi \\ \qquad = -\left\{(\partial_t + u_1\partial_x + v_1\partial_y - \partial_y^2)a_2\right\}\bar{u} + 2\partial_y a_2\bar{\omega} + a_2\bar{u}\partial_x u_2 \\ \qquad +a_2\bar{v}\partial_y u_2 + a_2\bar{\theta} - \bar{u}\partial_x\omega_2 - \bar{v}\partial_y\omega_2 - \partial_y\bar{\theta}, \\ (\partial_t + u_1\partial_x + v_1\partial_y - \partial_y^2)\vartheta \\ \qquad = -\left\{(\partial_t + u_1\partial_x + v_1\partial_y - \partial_y^2)b_2\right\}\bar{u} + 2\partial_y b_2\bar{\omega} + b_2\bar{u}\partial_x u_2 \\ \qquad +b_2\bar{v}\partial_y u_2 + b_2\bar{\theta} - \bar{u}\partial_x\theta_2 - \bar{v}\partial_y\theta_2 + (\omega_1 + \omega_2)\bar{\omega}, \\ \varpi|_{t=0} = (\omega_{10} - \omega_{20}) - a_{20}(u_{10} - u_{20}), \\ \vartheta|_{t=0} = (\theta_{10} - \theta_{20}) - b_{20}(u_{10} - u_{20}), \\ (\partial_y\varpi, \partial_y\vartheta)|_{y=0} = 0. \end{cases} \tag{5.2}$$

Since $(\omega_2, \tilde{\theta}_2) \in H^{s,\gamma}_{\mu,\delta}$, it follows from the weighted L^∞ bounds on ω_2, θ_2 of Theorem 3.7 that

$$\|\langle y\rangle a_2\|_{L^\infty} \le \delta^{-2}, \qquad \|\langle y\rangle\partial_x a_2\|_{L^\infty} + \left\|\langle y\rangle^2\partial_y a_2\right\|_{L^\infty} \le \delta^{-2} + \delta^{-4},$$

and

$$\|\langle y\rangle b_2\|_{L^\infty} \le \delta^{-2}, \qquad \|\langle y\rangle\partial_x b_2\|_{L^\infty} + \left\|\langle y\rangle^2\partial_y b_2\right\|_{L^\infty} \le \delta^{-2} + \delta^{-4}.$$

Multiplying the equations $(5.2)_1$ and $(5.2)_2$ by 2ϖ and 2ϑ, respectively, then integrating the resulting equations over Ω, and using Lemma 2.6 we obtain

$$\frac{d}{dt}\left(\|\varpi\|_{L^2}^2 + \|\vartheta\|_{L^2}^2\right) \le C\left(\|\varpi\|_{L^2}^2 + \|\vartheta\|_{L^2}^2 + \left\|\langle y\rangle^{-1}\bar{u}\right\|_{L^2}^2\right). \tag{5.3}$$

Since $\delta \le \langle y\rangle^\mu\omega_2 \le \delta^{-1}$, $\bar{u}|_{y=0} = 0$ and $\varpi = \omega_2\partial_y\left(\frac{\bar{u}}{\omega_2}\right)$, we can use the Hardy inequality of Lemma 2.2 to obtain

$$\left\|\langle y\rangle^{-1}\bar{u}\right\|_{L^2} \le \frac{1}{\delta}\left\|\langle y\rangle^{-\mu-1}\frac{\bar{u}}{\omega_2}\right\|_{L^2} \le C\left\|\langle y\rangle^{-\mu}\partial_y\left(\frac{\bar{u}}{\omega_2}\right)\right\|_{L^2} \le C\,\|\varpi\|_{L^2}. \tag{5.4}$$

Substituting (5.4) into (5.3), we get

$$\frac{d}{dt}\left(\|\varpi\|_{L^2}^2 + \|\vartheta\|_{L^2}^2\right) \le C\left(\|\varpi\|_{L^2}^2 + \|\vartheta\|_{L^2}^2\right). \tag{5.5}$$

Applying Gronwall's inequality to (5.5), we obtain

$$\|\varpi(t)\|_{L^2}^2 + \|\vartheta(t)\|_{L^2}^2 \le \left(\|\varpi(0)\|_{L^2}^2 + \|\vartheta(0)\|_{L^2}^2\right)e^{Ct},$$

which further gives

$$\|\varpi\|_{L^2}^2 + \|\vartheta\|_{L^2}^2 \equiv 0,$$

provided that $u_1|_{t=0} = u_2|_{t=0}$ and $\theta_1|_{t=0} = \theta_2|_{t=0}$. As a result, we have $\varpi \equiv 0$ and $\vartheta \equiv 0$. Note that $\bar{u}$ can be expressed by $\bar{u} = \omega_2\int_0^y \frac{\varpi}{\omega_2}dy$, we get $\bar{u} \equiv 0$. It is easy to see $\theta_1 = \theta_2$ due to $\bar{\theta} = \vartheta + b_2\bar{u} = 0$. Finally we get $v_1 = v_2$ from $(1.1)_3$ and $\bar{u} = 0$. This completes the proof of the uniqueness.

6. Conclusions

In this paper, we study the local well-posedness of the thermal boundary layer equations for the two-dimensional incompressible heat conducting flow by using a new weighted energy method. Our results show that we only need the monotonic assumption on the tangential velocity u in the normal variable but have no restrictions on the absolute temperature θ. Furthermore, this analytical approach can be applied to the boundary layer problems involving more complex fluids.

Appendix A: Some inequalities

In this appendix, we will prove the inequalities given in Lemma 2.6. Here we give a proof for the reader's convenience.

Proof. Only need to prove $\beta_2 = 0$. In other words, we only prove when it's all derivatives with respect to t , other cases can be found in [16].

(i)It follows from the definition of $\|u\|_{H^{s,\gamma-1}}$ that $\|\langle y\rangle^{\gamma-1}\partial_t(u-U)\| \leq \|(u-U)\|_{H^{s,\gamma-1}}$, so it is a direct consequence of the almost equivalence inequality (2.3).

(ii) Using Lemma 2.2 and (2.5), we have

$$\|\langle y\rangle^{-1}\left(\partial_t^{s-1}v + y\partial_t^{s-1}\partial_x U)\right\|_{L^2} \leq 2\|\partial_t^{s-1}\partial_x(u-U)\|_{L^2} \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U|_{L^2}),$$

which is inequality (2.6).

(iii)Inequality (2.7) and (2.8) follows directly from the defnition of $\|\omega\|_{H_g^{s,\gamma}}, \|\tilde\theta\|_{H_h^{s,\gamma}}$ and inequality (2.3)-(2.4).

(iv) Since $(\omega,\tilde\theta) \in H_{\mu,\delta}^{s,\gamma}$, so we know that $\|\langle y\rangle\frac{\partial_y\tilde\theta}{\omega}\|_{L^\infty} \leq \delta^{-2}, \|\langle y\rangle\frac{\partial_y\omega}{\omega}\|_{L^\infty} \leq \delta^{-2}$. Thus

$$\left\|\langle y\rangle^\gamma g_\beta\right\|_{L^2} \leq \|\langle y\rangle^\gamma\partial_\chi^\beta\omega\|_{L^2} + \delta^{-2}\|\langle y\rangle^{\gamma-1}\partial_\chi^\beta(u-U)\|_{L^2} \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2}),$$

$$\left\|\langle y\rangle^\gamma h_\beta\right\|_{L^2} \leq \|\langle y\rangle^\gamma\partial_\chi^\beta\tilde\theta\|_{L^2} + \delta^{-2}\|\langle y\rangle^{\gamma-1}\partial_\chi^\beta(u-U)\|_{L^2} \leq C(\|\tilde\theta\|_{H_h^{s,\gamma}} + \|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2}),$$

which is inequality for $|\beta| \leq s-1$. When $|\beta| = s$, the better upper bound in (2.9) and (2.10) follows directly from the definition of $\|\omega\|_{H_g^{s,\gamma}}, \|\tilde\theta\|_{H_h^{s,\gamma}}$.

(v)Using Lemma 2.3, (2.5) and (2.7), we have

$$\|\partial_t^{s-1}(u-U)\|_{L^\infty} \leq C\{\|\partial_t^{s-1}(u-U)\|_{L^2} + \|\partial_t^{s-1}\partial_x(u-U)\|_{L^2} + \|\partial_t^{s-1}\partial_y\omega\|_{L^2}\} \leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2}),$$

which implies inequality (2.11).

(vi) Applying triangle inequality, Lemma 2.3 , $\partial_x u + \partial_y v = 0$ and $\omega = \partial_y u$, we have

$$\begin{aligned}
\|\langle y\rangle^{-1}\partial_t^{s-2}v\|_{L^\infty} &\leq \|\langle y\rangle^{-1}y\partial_t^{s-2}\partial_x U\|_{L^\infty} + \|\langle y\rangle^{-1}(\partial_t^{s-2}v + y\partial_t^{s-2}\partial_x U)\|_{L^\infty}\\
&\leq C\{\|\partial_t^{s-2}\partial_x U\|_{L^2} + \|\partial_t^{s-2}\partial_x^2 U\|_{L^2} + \|\langle y\rangle^{-1}(\partial_t^{s-2}v + y\partial_t^{s-2}\partial_x U)\|_{L^2}\\
&+ \|\langle y\rangle^{-1}(\partial_t^{s-2}\partial_x v + y\partial_t^{s-2}\partial_x^2 U)\|_{L^2} + \|\langle y\rangle^{-3}(\partial_t^{s-2}v + y\partial_t^{s-2}\partial_x U)\|_{L^2}\\
&+ \|\langle y\rangle^{-2}\partial_t^{s-2}\partial_x(u-U)\|_{L^2} + \|\langle y\rangle^{-1}\partial_t^{s-1}\partial_x\omega\|_{L^2}\}\\
&\leq C(\|\omega\|_{H_g^{s,\gamma}} + \|\partial_\chi^\beta U\|_{L^2} + 1),
\end{aligned}$$

which implies inequality (2.12), because of (2.5)-(2.8).

(vii) The inequality follows directly from Lemma 2.3 and inequality (2.7)-(2.8).

□

Appendix B: Proof of Lemma 3.3

Proof. In order to illustrate the idea, let us derive the formula (3.8) for the case $k = 2$ as follows. Applying ∂_y^3 to the vorticity equation $(3.3)_1$ and evaluating at $y = 0$, we obtain, by using $(3.7)_2$ and $u|_{y=0} = v|_{y=0} = \partial_y\theta|_{y=0} = 0$, that

$$\begin{aligned}\partial_y^5\omega\Big|_{y=0} =& \left(\partial_t - \varepsilon^2\partial_x^2\right)^2(\partial_x P - \theta) + \left(\partial_t - \varepsilon^2\partial_x^2\right)(\omega\partial_x\omega) + \left(\partial_t - \varepsilon^2\partial_x^2\right)\left(\partial_y^2\theta\right)\\ &+ 3\omega\partial_x\partial_y^2\omega + 2\partial_y\omega\partial_x\partial_y\omega - 2\partial_x\omega\partial_y^2\omega + \partial_y^4\theta\Big|_{y=0}.\end{aligned}$$

Since the last four terms on the right-hand side are our desired forms, we only need to deal with the terms $(\partial_t - \varepsilon^2\partial_x^2)(\omega\partial_x\omega)|_{y=0}$ and $(\partial_t - \varepsilon^2\partial_x^2)(\partial_y^2\theta)|_{y=0}$. Using the evolution equations for $\omega, \partial_x\omega$ and $\partial_y^2\theta$ as well as $u|_{y=0} = v|_{y=0} = \partial_y\theta|_{y=0} = 0$, one may check that

$$\begin{aligned}\left(\partial_t - \varepsilon^2\partial_x^2\right)(\omega\partial_x\omega)\Big|_{y=0} &= \omega\partial_x\partial_y^2\omega + \partial_x\omega\partial_y^2\omega - 2\varepsilon^2\partial_x\omega\partial_x^2\omega + \partial_y^3\theta\Big|_{y=0},\\ \left(\partial_t - \varepsilon^2\partial_x^2\right)\left(\partial_y^2\theta\right)|_{y=0} &= \partial_y^4\theta + 2\omega\partial_y^2\omega + 2(\partial_y\omega)^2 - \partial_y\omega\partial_x\theta|_{y=0}.\end{aligned}$$

Assuming that the lemma holds for $k = n$, we will show that it also holds for $k = n + 1$. Applying ∂_y^{2n+1} to the vorticity equation and evaluating the resulting equation at $y = 0$ yields

$$\begin{aligned}\partial_y^{2n+3}\omega|_{y=0} =& (\partial_t - \varepsilon^2\partial_x^2)\partial_y^{2n+1}\omega + \sum_{i=1}^{2n+1}\binom{2n+1}{i}\partial_y^{i-1}\omega\partial_x\partial_y^{2n-i+1}\omega + \partial_y^{2n+1}\theta\\ &- \sum_{i=1}^{2n+1}\binom{2n+1}{i}\partial_y^{i}\omega\partial_x\partial_y^{2n-i+1}\theta + \sum_{i=2}^{2n+1}\binom{2n+1}{i}\partial_x\partial_y^{i-2}\omega\partial_y^{2n-i+2}\omega + 2\partial_y^{2n+2}\theta|_{y=0}.\end{aligned}$$

Thanks to the induction hypothesis, we have

$$\partial_y^{2n+1}\omega\Big|_{y=0} = \left(\partial_t - \varepsilon^2\partial_x^2\right)^n(\partial_x P - \theta) + \mathcal{P}_k|_{y=0}.$$

This completes the proof of Lemma 3.3. □

Appendix C: Equations for a, b, g_β, and h_β

In this appendix, we will derive the evolution equations for a, b, g_β, and h_β. The equations satisfied by $(\tilde{u}, \omega) = (u - U, \partial_y u)$ is

$$\begin{cases}\partial_t\tilde{u} + u\partial_x\tilde{u} + v\partial_y\tilde{u} = \varepsilon^2\partial_x^2\tilde{u} + \partial_y^2\tilde{u} - \tilde{\theta} - \tilde{u}\partial_x U,\\ \partial_t\omega + u\partial_x\omega + v\partial_y\omega = \varepsilon^2\partial_x^2\omega + \partial_y^2\omega - \partial_y\tilde{\theta}.\end{cases} \tag{6.1}$$

Equation for a:
Differentiating the equation $(6.1)_2$ with respect to y, we have

$$(\partial_t + u\partial_x + v\partial_y)\partial_y\omega = \varepsilon^2\partial_x^2\partial_y\omega + \partial_y^3\omega - \partial_y^2\tilde{\theta} - \omega\partial_x\omega + \partial_x u\partial_y\omega,$$

which implies

$$
\begin{aligned}
&(\partial_t + u\partial_x + v\partial_y)a \\
&= \frac{(\partial_t + u\partial_x + v\partial_y)\partial_y\omega}{\omega} - \frac{\partial_y\omega(\partial_t + u\partial_x + v\partial_y)\omega}{\omega^2} \\
&= \varepsilon^2\left(\frac{\partial_x^2\partial_y\omega}{\omega} - a\frac{\partial_x^2\omega}{\omega}\right) + \left(\frac{\partial_y^3\omega}{\omega} - a\frac{\partial_y^2\omega}{\omega}\right) - \frac{\partial_y^2\tilde{\theta}}{\omega} - \partial_x\omega + a\partial_x u + ab.
\end{aligned}
\tag{6.2}
$$

Note that

$$
\partial_x^2 a = \frac{\partial_x^2\partial_y\omega}{\omega} - a\frac{\partial_x^2\omega}{\omega} - 2\frac{\partial_x\omega}{\omega}\partial_x a, \quad \partial_y^2 a = \frac{\partial_y^3\omega}{\omega} - a\frac{\partial_y^2\omega}{\omega} - 2a\partial_y a. \tag{6.3}
$$

Substituting (6.3) into (6.2), we have (where $g_{e_2} = \partial_x\omega - a\partial_x\tilde{u}$):

$$
(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)a = 2\varepsilon^2\frac{\partial_x\omega}{\omega}\partial_x a + 2a\partial_y a + ab - \frac{\partial_y\tilde{\theta}}{\omega} - g_{e_2} + a\partial_x U.
$$

Equation for g_β:
Differentiating the equations (6.1) with ∂_χ^β respectively, one has

$$
\begin{aligned}
&(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)\partial_\chi^\beta\tilde{u} + \partial_\chi^\beta v\omega \\
&= -\sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}u\partial_\chi^{\bar{\beta}+e_2}\tilde{u} - \sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}+e_2}U\partial_\chi^{\bar{\beta}}\tilde{u} - \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}v\partial_\chi^{\bar{\beta}}\omega - \partial_\chi^\beta\tilde{\theta},
\end{aligned}
\tag{6.4}
$$

and

$$
\begin{aligned}
&(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)\partial_\chi^\beta\omega + \partial_\chi^\beta v\partial_y\omega \\
&= -\sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}u\partial_\chi^{\bar{\beta}+e_2}\omega - \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}v\partial_\chi^{\bar{\beta}}\partial_y\omega - \partial_\chi^\beta\partial_y\theta.
\end{aligned}
\tag{6.5}
$$

Subtracting (6.4) $\times a$ from (6.5), we have

$$
\begin{aligned}
&(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)g_\beta + \left\{(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)a\right\}\partial_\chi^\beta\tilde{u} \\
&= 2\varepsilon^2\partial_\chi^{\beta+e_2}\tilde{u}\partial_x a + 2\partial_y a\partial_\chi^\beta\omega - \sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}ug_{\bar{\beta}+e_2} - \partial_\chi^\beta\partial_y\tilde{\theta} + a\partial_\chi^\beta\tilde{\theta} \\
&+ a\sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}+e_2}U\partial_\chi^{\bar{\beta}}\tilde{u} - \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}v\left(\partial_y\partial_\chi^{\bar{\beta}}\omega - a\partial_\chi^{\bar{\beta}}\omega\right),
\end{aligned}
$$

and then we get the equation satisfied by g_β

$$
\begin{aligned}
&(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)g_\beta \\
&= 2\varepsilon^2(\partial_\chi^{\beta+e_2}\tilde{u} - \frac{\partial_x\omega}{\omega}\partial_\chi^\beta\tilde{u})\partial_x a + 2g_\beta\partial_y a - g_{e_2}\partial_\chi^\beta U + \frac{\partial_y^2\tilde{\theta}}{\omega}\partial_\chi^\beta\tilde{u} - ab\partial_\chi^\beta\tilde{u} - \partial_\chi^\beta\partial_y\tilde{\theta} + a\partial_\chi^\beta\tilde{\theta} \\
&- \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}ug_{\bar{\beta}+e_2} + a\sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}+e_2}U\partial_\chi^{\bar{\beta}}\tilde{u} - \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}v\left(\partial_y\partial_\chi^{\bar{\beta}}\omega - a\partial_\chi^{\bar{\beta}}\omega\right).
\end{aligned}
$$

Similarly, we can write down the evolution equation of $h_\beta := \partial_\chi^\beta\tilde{\theta} - \frac{\partial_y\tilde{\theta}}{\omega}\partial_\chi^\beta(u-U)$ as follows.
Equation for b:
The equations satisfied by $(\tilde{u},\tilde{\theta}) = (u-U, \theta-\Theta)$ is

$$\begin{cases} \partial_t\tilde{u} + u\partial_x\tilde{u} + v\partial_y\tilde{u} = \varepsilon^2\partial_x^2\tilde{u} + \partial_y^2\tilde{u} - \tilde{\theta} - \tilde{u}\partial_x U, \\ \partial_t\tilde{\theta} + u\partial_x\tilde{\theta} + v\partial_y\tilde{\theta} = \varepsilon^2\partial_x^2\tilde{\theta} + \partial_y^2\tilde{\theta} + \omega^2 - \tilde{u}\partial_x\Theta. \end{cases} \tag{6.6}$$

Differentiating the equation $(6.6)_2$ with respect to y, we have

$$(\partial_t + u\partial_x + v\partial_y)\partial_y\tilde{\theta} = \varepsilon^2\partial_x^2\partial_y\tilde{\theta} + \partial_y^3\tilde{\theta} - \omega\partial_x\tilde{\theta} + \partial_x u\partial_y\tilde{\theta} + 2\omega\partial_y\omega - \omega\partial_x\Theta,$$

and

$$(\partial_t + u\partial_x + v\partial_y)b = \varepsilon^2\left(\frac{\partial_x^2\partial_y\tilde{\theta}}{\omega} - b\frac{\partial_x^2\omega}{\omega}\right) + \left(\frac{\partial_y^3\omega}{\omega} - b\frac{\partial_y^2\omega}{\omega}\right) - \partial_x\tilde{\theta} + b\partial_x u + 2\partial_y\omega - \partial_x\Theta + b^2. \tag{6.7}$$

Since

$$\partial_x^2 b = \frac{\partial_x^2\partial_y\tilde{\theta}}{\omega} - b\frac{\partial_x^2\omega}{\omega} - 2\frac{\partial_x\omega}{\omega}\partial_x b, \quad \partial_y^2 b = \frac{\partial_y^3\tilde{\theta}}{\omega} - a\frac{\partial_y^2\tilde{\theta}}{\omega} - \partial_y(ab), \tag{6.8}$$

we get

$$\begin{aligned}&(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)b \\ &= 2\varepsilon^2\frac{\partial_x\omega}{\omega}\partial_x b + \partial_y(ab) + b^2 + a\frac{\partial_y^2\tilde{\theta}}{\omega} - b\frac{\partial_y^2\omega}{\omega} - h_{e_2} + b\partial_x U + 2\partial_y\omega - \partial_x\Theta,\end{aligned}$$

Equation for h_β:
Differentiating the equations (6.6) with ∂_χ^β respectively, one has

$$\begin{aligned}&(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)\partial_\chi^\beta\tilde{u} + \partial_\chi^\beta v\omega \\ &= -\sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}u\partial_\chi^{\bar{\beta}+e_2}\tilde{u} - \sum_{0\le\bar{\beta}\le\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}+e_2}U\partial_\chi^{\bar{\beta}}\tilde{u} - \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}v\partial_\chi^{\bar{\beta}}\omega - \partial_\chi^\beta\tilde{\theta},\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}&(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)\partial_\chi^\beta\tilde{\theta} + \partial_\chi^\beta v\partial_y\tilde{\theta} \\ &= -\sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}u\partial_\chi^{\bar{\beta}+e_2}\tilde{\theta} - \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}v\partial_\chi^{\bar{\beta}}\partial_y\tilde{\theta} - \sum_{0\le\bar{\beta}\le\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}\tilde{u}\partial_\chi^{\bar{\beta}+e_2}\Theta - \partial_\chi^\beta\omega^2.\end{aligned} \tag{6.10}$$

Subtracting (6.9) $\times b$ from (6.10), we have

$$\begin{aligned}&(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)h_\beta + \left\{(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)b\right\}\partial_\chi^\beta\tilde{u} \\ &= 2\varepsilon^2\partial_\chi^{\beta+e_2}\tilde{u}\partial_x b + 2\partial_\chi^\beta\omega\partial_y b - \sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}uh_{\bar{\beta}+e_2} + b\sum_{0\le\bar{\beta}\le\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}+e_2}U\partial_\chi^{\bar{\beta}}\tilde{u} \\ &\quad - \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}v\left(\partial_y\partial_\chi^{\bar{\beta}}\tilde{\theta} - b\partial_\chi^{\bar{\beta}}\omega\right) - \partial_\chi^\beta\omega^2 + b\partial_\chi^\beta\tilde{\theta} - \sum_{0\le\bar{\beta}\le\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}\tilde{u}\partial_\chi^{\bar{\beta}}\partial_x\Theta,\end{aligned}$$

and then we get the equation satisfied by h_β

$$
\begin{aligned}
&(\partial_t + u\partial_x + v\partial_y - \varepsilon^2\partial_x^2 - \partial_y^2)h_\beta \\
&= 2\varepsilon^2\left(\partial_\chi^{\beta+e_2}\tilde{u} - \frac{\partial_x\omega}{\omega}\partial_\chi^\beta\tilde{u}\right)\partial_x b - \partial_y(ab)\partial_\chi^\beta\tilde{u} - h_{e_2}\partial_\chi^\beta U - b^2\partial_\chi^\beta\tilde{u} - a\frac{\partial_y^2\tilde{\theta}}{\omega}\partial_\chi^\beta\tilde{u} + b\frac{\partial_y^2\omega}{\omega}\partial_\chi^\beta\tilde{u} \\
&- 2\partial_y\omega\partial_\chi^\beta\tilde{u} + 2\partial_\chi^\beta\omega\partial_y b + \partial_x\Theta\partial_\chi^\beta\tilde{u} - \sum_{0\le<\beta<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}uh_{\bar{\beta}+e_2} + b\sum_{0\le\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}+e_2}U\partial_\chi^{\bar{\beta}}\tilde{u} \\
&- \sum_{0<\bar{\beta}<\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}v\left(\partial_y\partial_\chi^{\bar{\beta}}\tilde{\theta} - b\partial_\chi^{\bar{\beta}}\omega\right) - \partial_\chi^\beta\omega^2 + b\partial_\chi^\beta\tilde{\theta} - \sum_{0\le\bar{\beta}\le\beta}\binom{\beta}{\bar{\beta}}\partial_\chi^{\beta-\bar{\beta}}\tilde{u}\partial_\chi^{\bar{\beta}}\partial_x\Theta.
\end{aligned}
$$

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Conflict of interest

The authors declare no conflicts of interest.

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