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Research article

A complete comparison for the number of palindromes in different bases

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Abstract: Let b and b_1 be distinct positive integers larger than 1, and let $A_b(n)$ and $A_{b_1}(n)$ be the number of palindromes in bases b and b_1 that are less than or equal to n, respectively. In this article, we finish the comparative study of the functions $A_b(n)$ and $A_{b_1}(n)$. As a result, we present the full picture of the asymptotic behavior of their difference.

Keywords: palindrome; palindromic number; b-adic expansion; extremal order; sign change
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1. Introduction

Let $b \geq 2$ and $n \geq 1$ be integers. We call n a palindrome in base b (or b-adic palindrome) if the b-adic expansion of $n = (a_k a_{k-1} \cdots a_0)_b$ with $a_k \neq 0$ has the symmetric property $a_{k-i} = a_i$ for $0 \leq i \leq k$. As usual, if we write a number without specifying the base, then it is always in base 10, and if we write $n = (a_k a_{k-1} \cdots a_0)_b$, then it means that $n = \sum_{i=0}^{k} a_i b^i$, $a_k \neq 0$, and $0 \leq a_i < b$ for all $i = 0, 1, \ldots, k$. Throughout this article, we let $A_b(n)$ be the number of b-adic palindromes not exceeding n.

Previously, we [1] obtained an extremal order of $A_b(n)$ and proved that if $b > b_1 \geq 2$ are integers, then

$$\limsup_{n\to\infty} (A_b(n) - A_{b_1}(n)) = +\infty \quad \text{and} \quad \liminf_{n\to\infty} (A_b(n) - A_{b_1}(n)) < 0.$$

In addition, if $\frac{\log b}{\log b_1}$ is irrational, then

$$\liminf_{n\to\infty} (A_b(n) - A_{b_1}(n)) = -\infty. \tag{1.1}$$

Therefore, it is interesting to determine the value of the left-hand side of (1.1) when b is a rational power of b_1. In this article, we show in Theorem 3.1 that if $\log b / \log b_1$ is an integer, then the left-hand side of (1.1) is -1, and we obtain in Theorem 3.2 that if $\log b/\log b_1$ is rational but not integral, then the left-hand side of (1.1) is $-\infty$. We also propose some possible research problems at the end of this article. We remark that the study on the ratio $A_b(n)/A_{b_1}(n)$ may be interesting too, but we were previously interested in the sign changes of $A_b(n) - A_{b_1}(n)$, and so we focus only on the difference not the ratio. Nevertheless, since $A_b(n) - A_{b_1}(n)$ has an infinite number of sign changes, if the limit of $A_b(n)/A_{b_1}(n)$ as $n \to \infty$ exists, then it must be one.

Perhaps, one of the popular problems in palindromes is to determine whether or not there are infinitely many palindromic primes. Although this problem is still open, Banks, Hart and Sakata [2] showed that almost all palindromes in any fixed base $b \geq 2$ are composite. Banks and Shparlinski [3] also obtained results on prime divisors of palindromes, and there are many other interesting articles on palindromes too. We refer the reader to Banks [4], Cilleruelo, Luca and Baxter [5], and Rajasekaran, Shallit and Smith [6] for additive properties of palindromes, Bašić [7, 8], Di Scala and Sombra [9], Goins [10], Luca and Togbé [11] for the study of palindromes in different bases, Cilleruelo, Luca and Tesoro [12] for palindromes in linear recurrence sequences, Harminc and Soták [13] for b-adic palindromes in arithmetic progressions, and Pongsriiam [14] for the longest arithmetic progressions of palindromes.

2. Preliminaries and lemmas

In this section, we provide some results which are needed in the proof of the main theorems. Recall that for a real number x, $\lfloor x \rfloor$ is the largest integer less than or equal to x, $\lceil x \rceil$ is the smallest integer greater than or equal to x, and $\{x\}$ is the fractional part of x given by $\{x\} = x - \lfloor x \rfloor$. It is also convenient to define $C_b(n)$ as follows.

Definition 2.1. *Let $b \geq 2$ and $n = (a_k a_{k-1} \cdots a_1 a_0)_b$ be positive integers. We define $C_b(n) = (c_k c_{k-1} \cdots c_1 c_0)_b$ to be the b-adic palindrome satisfying $c_i = a_i$ for $k - \lfloor k/2 \rfloor \leq i \leq k$. In other words, $C_b(n)$ is the b-adic palindrome having $k+1$ digits whose first half digits are the same as those of n in its b-adic expansion, that is, $C_b(n) = (a_k a_{k-1} \cdots a_{k-\lfloor \frac{k}{2} \rfloor} \cdots a_{k-1} a_k)_b$.*

In the following lemma, if P is a mathematical statement, then the Iverson notation $[P]$ is defined by $[P] = 1$ if P holds, and $[P] = 0$ otherwise. Then the formula for $A_b(n)$ is as follows.

Lemma 2.1. [15] *Let $b \geq 2$, $n \geq 1$, and $n = (a_k a_{k-1} \cdots a_1 a_0)_b$ be integers. Then the number of b-adic palindromes less than or equal to n is given by*

$$A_b(n) = b^{\lceil \frac{k}{2} \rceil} + \sum_{0 \leq i \leq \lfloor \frac{k}{2} \rfloor} a_{k-i} b^{\lfloor \frac{k}{2} \rfloor - i} + [n \geq C_b(n)] - 2.$$

Lemma 2.2. *Let $a, r, s \geq 2$ be integers and $(r, s) = 1$. If $a^{\frac{r}{s}}$ is an integer, then there exists an integer $c \geq 2$ such that $a = c^s$.*

Proof. Suppose $a^{\frac{r}{s}} = m$ is an integer. Then $a^r = m^s$, and so a and m have the same set of prime divisors. Let $a = \prod_{i=1}^{k} p_i^{a_i}$ and $m = \prod_{i=1}^{k} p_i^{m_i}$. Then $a_i r = m_i s$ for all i. Since $s \mid a_i r$ and $(s, r) = 1$, $s \mid a_i$ for all i. Let $c = \prod_{i=1}^{k} p_i^{a_i/s}$. Then c is an integer, $c \geq 2$, and $a = c^s$. So the proof is complete. □

3. Main results

Theorem 3.1. *Let $b > b_1 \geq 2$ and $\ell \geq 2$ be integers. Suppose that $b = b_1^\ell$. Then, the following statements hold.*

(i) $A_b(n) - A_{b_1}(n) \geq -1$ *for all* $n \in \mathbb{N}$.
(ii) $A_b(n) - A_{b_1}(n) = -1$ *for infinitely many* $n \in \mathbb{N}$.
(iii) $\liminf_{n\to\infty} (A_b(n) - A_{b_1}(n)) = -1$.

Proof. We first prove (i). Let $n \geq 1$ and write

$$n = (a_k a_{k-1} \cdots a_0)_{b_1} = (c_r c_{r-1} \cdots c_0)_b \,.$$

Since $b^r \leq c_r b^r \leq n < b^{r+1}$, we see that $r = \left\lfloor \frac{\log n}{\log b} \right\rfloor$. Similarly, we have $k = \left\lfloor \frac{\log n}{\log b_1} \right\rfloor$, and so $r = \lfloor k/\ell \rfloor$. By the uniqueness of the b-adic and b_1-adic representations, we can write $c_0, c_1, c_2, \ldots, c_r$ in terms of b_1 and the a_j as follows:

Considering n modulo b, we obtain

$$c_0 \equiv a_0 + a_1 b_1 + a_2 b_1^2 + \cdots + a_{\ell-1} b_1^{\ell-1} \pmod{b},$$

and both sides of the congruence are nonnegative integers less than b, and so they are equal. Similarly, after reducing n modulo $b^2, b^3, \ldots, b^{r+1}$, respectively, we obtain $c_1, c_2, \ldots c_r$. Thus

$$c_j = \sum_{i=0}^{\ell-1} a_{j\ell+i} b_1^i \text{ for every } j = 0, 1, 2, \ldots, r,$$

where $a_m = 0$ if $m > k$. By Lemma 2.1, we have

$$A_{b_1}(n) = b_1^{\left\lceil \frac{k}{2} \right\rceil} + \sum_{0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor} a_{k-i} b_1^{\left\lfloor \frac{k}{2} \right\rfloor - i} + [n \geq C_{b_1}(n)] - 2, \tag{3.1}$$

$$\begin{aligned} A_b(n) &= b^{\left\lceil \frac{r}{2} \right\rceil} + \sum_{0 \leq j \leq \left\lfloor \frac{r}{2} \right\rfloor} c_{r-j} b^{\left\lfloor \frac{r}{2} \right\rfloor - j} + [n \geq C_b(n)] - 2 \\ &= b_1^{\ell \left\lceil \frac{\left\lfloor \frac{k}{\ell} \right\rfloor}{2} \right\rceil} + \sum_{0 \leq j \leq \left\lfloor \frac{k}{2\ell} \right\rfloor} \left(\sum_{i=0}^{\ell-1} a_{(r-j)\ell+i} b_1^i \right) b_1^{\ell(\left\lfloor \frac{k}{2\ell} \right\rfloor - j)} + [n \geq C_b(n)] - 2. \end{aligned} \tag{3.2}$$

It is useful to recall that if $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, then $\lfloor k + x \rfloor = k + \lfloor x \rfloor$, and if $k \in \mathbb{N}$ and $x \in \mathbb{R}$, then $\left\lfloor \frac{\lfloor x \rfloor}{k} \right\rfloor = \left\lfloor \frac{x}{k} \right\rfloor$. We also let $s = k \bmod \ell$ be the least nonnegative residue of k modulo ℓ, that is, $k \equiv s \pmod{\ell}$ and $0 \leq s < \ell$. Then, from (3.1) and (3.2), we obtain

$$A_{b_1}(n) = \begin{cases} b_1^{\frac{k}{2}} + \sum_{0 \leq i \leq \frac{k}{2}} a_{k-i} b_1^{\frac{k}{2} - i} + [n \geq C_{b_1}(n)] - 2, & \text{if } k \text{ is even;} \\ b_1^{\frac{k+1}{2}} + \sum_{0 \leq i \leq \frac{k-1}{2}} a_{k-i} b_1^{\frac{k-1}{2} - i} + [n \geq C_{b_1}(n)] - 2, & \text{if } k \text{ is odd,} \end{cases} \tag{3.3}$$

$$A_b(n) = \begin{cases} b_1^{\frac{k-s}{2}} + \sum_{0\le j\le \frac{k-s}{2\ell}} \sum_{i=0}^{\ell-1} a_{k-s-\ell j+i} b_1^{\frac{k-s}{2}-\ell j+i} + [n \ge C_b(n)] - 2, & \text{if } \left\lfloor \frac{k}{\ell} \right\rfloor \text{ is even;} \\ b_1^{\frac{k-s+\ell}{2}} + \sum_{0\le j\le \frac{k-s-\ell}{2\ell}} \sum_{i=0}^{\ell-1} a_{k-s-\ell j+i} b_1^{\frac{k-s-\ell}{2}-\ell j+i} + [n \ge C_b(n)] - 2, & \text{if } \left\lfloor \frac{k}{\ell} \right\rfloor \text{ is odd.} \end{cases} \tag{3.4}$$

Next, we will reduce the double sum in (3.4) into a sum. We see that if $\left\lfloor \frac{k}{\ell} \right\rfloor$ is even, then $-\ell j + i$ runs through the integers from $-\frac{k-s}{2}$ to $\ell - 1$ exactly once as j runs through $0, 1, 2, \ldots, \frac{k-s}{2\ell}$ and i runs through 0 to $\ell - 1$. Similarly, if $\left\lfloor \frac{k}{\ell} \right\rfloor$ is odd, then $-\ell j + i$ ranges over the integers from $-\frac{k-s-\ell}{2}$ to $\ell - 1$ exactly once as j ranges over $0, 1, 2, \ldots, \frac{k-s-\ell}{2\ell}$ and i ranges over 0 to $\ell - 1$. So if $\left\lfloor \frac{k}{\ell} \right\rfloor$ is even, the first double sum in (3.4) reduces to

$$\sum_{-\frac{k-s}{2}\le i\le \ell-1} a_{k-s+i} b_1^{\frac{k-s}{2}+i}.$$

We replace the index i by $i - \frac{k-s}{2}$ and recall that $s \le \ell - 1$, $a_{\frac{k-s}{2}+i} = 0$ if $i > \frac{k+s}{2}$, and $\ell - 1 + \frac{k-s}{2} \ge \frac{k+s}{2}$. So the first double sum in (3.4) further reduces to

$$\sum_{0\le i\le \frac{k+s}{2}} a_{\frac{k-s}{2}+i} b_1^{i}.$$

Similarly, if $\left\lfloor \frac{k}{\ell} \right\rfloor$ is odd, then the second double sum in (3.4) reduces to

$$\sum_{-\frac{k-s-\ell}{2}\le i\le \ell-1} a_{k-s+i} b_1^{\frac{k-s-\ell}{2}+i} = \sum_{0\le i\le \frac{k+s-\ell}{2}} a_{\frac{k-s+\ell}{2}+i} b_1^{i}.$$

From (3.4) and the above calculation, we obtain

$$A_b(n) = \begin{cases} b_1^{\frac{k-s}{2}} + \sum_{0\le i\le \frac{k+s}{2}} a_{\frac{k-s}{2}+i} b_1^{i} + [n \ge C_b(n)] - 2, & \text{if } \left\lfloor \frac{k}{\ell} \right\rfloor \text{ is even;} \\ b_1^{\frac{k-s+\ell}{2}} + \sum_{0\le i\le \frac{k+s-\ell}{2}} a_{\frac{k-s+\ell}{2}+i} b_1^{i} + [n \ge C_b(n)] - 2, & \text{if } \left\lfloor \frac{k}{\ell} \right\rfloor \text{ is odd.} \end{cases} \tag{3.5}$$

Next, we divide the proof into four cases according to the parity of k and $\left\lfloor \frac{k}{\ell} \right\rfloor$.

Case 1. Assume that k and $\left\lfloor \frac{k}{\ell} \right\rfloor$ are even. Then, s is even and

$$\sum_{0\le i\le \frac{k+s}{2}} a_{\frac{k-s}{2}+i} b_1^{i} \ge b_1^{\frac{s}{2}} \sum_{\frac{s}{2}\le i\le \frac{k+s}{2}} a_{\frac{k-s}{2}+i} b_1^{i-\frac{s}{2}} = b_1^{\frac{s}{2}} \sum_{0\le i\le \frac{k}{2}} a_{k-i} b_1^{\frac{k}{2}-i}.$$

By (3.3), (3.5) and the above inequality, we obtain that $A_b(n) - A_{b_1}(n)$ is at least

$$b_1^{\frac{k-s}{2}} - b_1^{\frac{k}{2}} + \left(b_1^{\frac{s}{2}} - 1\right) \sum_{0\le i\le \frac{k}{2}} a_{k-i} b_1^{\frac{k}{2}-i} - 1 \ge b_1^{\frac{k-s}{2}} - b_1^{\frac{k}{2}} + a_k b_1^{\frac{k}{2}} \left(b_1^{\frac{s}{2}} - 1\right) - 1 = b_1^{\frac{k}{2}} \left(a_k b_1^{\frac{s}{2}} + b_1^{-\frac{s}{2}} - a_k - 1\right) - 1.$$

Since the function $x \mapsto a_k x + x^{-1}$ is increasing on $[1, \infty)$, the number in the above parenthesis is nonnegative, and so $A_b(n) - A_{b_1}(n) \geq -1$.

Case 2. Assume that k is odd and $\left\lfloor \frac{k}{\ell} \right\rfloor$ is even. Then s is odd. Similar to Case 1, we obtain

$$\sum_{0 \leq i \leq \frac{k+s}{2}} a_{\frac{k-s}{2}+i} b_1^i \geq b_1^{\frac{s+1}{2}} \sum_{\frac{s+1}{2} \leq i \leq \frac{k+s}{2}} a_{\frac{k-s}{2}+i} b_1^{i-\frac{s+1}{2}} \geq b_1^{\frac{s+1}{2}} \sum_{0 \leq i \leq \frac{k-1}{2}} a_{k-i} b_1^{\frac{k-1}{2}-i}.$$

By (3.3), (3.5) and the above inequality, we see that $A_b(n) - A_{b_1}(n)$ is at least

$$b_1^{\frac{k-s}{2}} - b_1^{\frac{k+1}{2}} + a_k b_1^{\frac{k-1}{2}} \left(b_1^{\frac{s+1}{2}} - 1 \right) - 1 = b_1^{\frac{k}{2}} \left(a_k b_1^{\frac{s}{2}} + b_1^{-\frac{s}{2}} - a_k b_1^{-\frac{1}{2}} - b_1^{\frac{1}{2}} \right) - 1.$$

Since $x \mapsto a_k x + x^{-1}$ is increasing on $[1, \infty)$ and $b_1^{\frac{s}{2}} \geq b_1^{\frac{1}{2}} \geq 1$, we have $a_k b_1^{\frac{s}{2}} + b_1^{-\frac{s}{2}} \geq a_k b_1^{\frac{1}{2}} + b_1^{-\frac{1}{2}}$, and

$$A_b(n) - A_{b_1}(n) \geq b_1^{\frac{k}{2}} \left(a_k b_1^{\frac{1}{2}} + b_1^{-\frac{1}{2}} - a_k b_1^{-\frac{1}{2}} - b_1^{\frac{1}{2}} \right) - 1 = b_1^{\frac{k}{2}} (a_k - 1) \left(b_1^{\frac{1}{2}} - b_1^{-\frac{1}{2}} \right) - 1 \geq -1.$$

Case 3. Assume that k is even and $\left\lfloor \frac{k}{\ell} \right\rfloor$ is odd. Then $\ell \equiv k - s \equiv s \pmod 2$. So $\frac{\ell-s}{2}$ is an integer and $\frac{\ell-s}{2} > 0$. So $\frac{\ell-s}{2} \geq 1$. Considering the first sum in (3.3) and changing the index from i to $\frac{k+s-\ell}{2} - i$, we see that

$$\begin{aligned}\sum_{0 \leq i \leq \frac{k}{2}} a_{k-i} b_1^{\frac{k}{2}-i} &= \sum_{-\frac{\ell-s}{2} \leq i \leq \frac{k+s-\ell}{2}} a_{\frac{k-s+\ell}{2}+i} b_1^{i+\frac{\ell-s}{2}} = b_1^{\frac{\ell-s}{2}} \sum_{0 \leq i \leq \frac{k+s-\ell}{2}} a_{\frac{k-s+\ell}{2}+i} b_1^i + \sum_{-\frac{\ell-s}{2} \leq i < 0} a_{\frac{k-s+\ell}{2}+i} b_1^{i+\frac{\ell-s}{2}} \\ &= b_1^{\frac{\ell-s}{2}} \sum_{0 \leq i \leq \frac{k+s-\ell}{2}} a_{\frac{k-s+\ell}{2}+i} b_1^i + \sum_{0 \leq i < \frac{\ell-s}{2}} a_{\frac{k}{2}+i} b_1^i \leq b_1^{\frac{\ell-s}{2}} \sum_{0 \leq i \leq \frac{k+s-\ell}{2}} a_{\frac{k-s+\ell}{2}+i} b_1^i + \left(b_1^{\frac{\ell-s}{2}} - 1 \right).\end{aligned}$$

By (3.3), (3.5) and the above inequality, we obtain

$$A_b(n) - A_{b_1}(n) \geq b_1^{\frac{k-s+\ell}{2}} - b_1^{\frac{k}{2}} + \left(1 - b_1^{\frac{\ell-s}{2}} \right) \sum_{0 \leq i \leq \frac{k+s-\ell}{2}} a_{\frac{k-s+\ell}{2}+i} b_1^i - \left(b_1^{\frac{\ell-s}{2}} - 1 \right) - 1. \tag{3.6}$$

The sum on the right-hand side of (3.6) is less than or equal to $b_1^{\frac{k+s-\ell}{2}+1} - 1$ and $1 - b_1^{\frac{\ell-s}{2}}$ is negative. Therefore, $A_b(n) - A_{b_1}(n)$ is at least

$$b_1^{\frac{k-s+\ell}{2}} - b_1^{\frac{k}{2}} + \left(1 - b_1^{\frac{\ell-s}{2}} \right) \left(b_1^{\frac{k+s-\ell}{2}+1} - 1 \right) + \left(1 - b_1^{\frac{\ell-s}{2}} \right) - 1 = b_1^{\frac{k}{2}} \left(b_1^{\frac{\ell-s}{2}} + b_1^{1-\frac{\ell-s}{2}} - b_1 - 1 \right) - 1.$$

Since the function $x \mapsto b_1^x + b_1^{1-x}$ is increasing on $[1, \infty)$ and $\frac{\ell-s}{2} \geq 1$, the number in the above parenthesis is nonnegative, and so $A_b(n) - A_{b_1}(n) \geq -1$.

Case 4. Assume that k and $\left\lfloor \frac{k}{\ell} \right\rfloor$ are odd. Changing the index from i to $\frac{k-1}{2} - \frac{\ell-s-1}{2} - i$, the second sum in (3.3) is

$$\begin{aligned}\sum_{0 \leq i \leq \frac{k-1}{2}} a_{k-i} b_1^{\frac{k-1}{2}-i} &= \sum_{-\frac{\ell-s-1}{2} \leq i \leq \frac{k+s-\ell}{2}} a_{\frac{k-s+\ell}{2}+i} b_1^{i+\frac{\ell-s-1}{2}} = b_1^{\frac{\ell-s-1}{2}} \sum_{0 \leq i \leq \frac{k+s-\ell}{2}} a_{\frac{k-s+\ell}{2}+i} b_1^i + \sum_{-\frac{\ell-s-1}{2} \leq i < 0} a_{\frac{k-s+\ell}{2}+i} b_1^{i+\frac{\ell-s-1}{2}} \\ &\leq b_1^{\frac{\ell-s-1}{2}} \sum_{0 \leq i \leq \frac{k+s-\ell}{2}} a_{\frac{k-s+\ell}{2}+i} b_1^i + \left(b_1^{\frac{\ell-s-1}{2}} - 1 \right).\end{aligned}$$

By (3.3), (3.5) and the above inequality, we obtain that $A_b(n) - A_{b_1}(n)$ is at least

$$\begin{aligned}
&b_1^{\frac{k-s+\ell}{2}} - b_1^{\frac{k+1}{2}} + \left(1 - b_1^{\frac{\ell-s-1}{2}}\right) \sum_{0 \le i \le \frac{k+s-\ell}{2}} a_{\frac{k-s+\ell}{2}+i} b_1^i - \left(b_1^{\frac{\ell-s-1}{2}} - 1\right) - 1 \\
\ge & b_1^{\frac{k-s+\ell}{2}} - b_1^{\frac{k+1}{2}} + \left(1 - b_1^{\frac{\ell-s-1}{2}}\right)\left(b_1^{\frac{k+s-\ell}{2}+1} - 1\right) + \left(1 - b_1^{\frac{\ell-s-1}{2}}\right) - 1 \\
= & b_1^{\frac{k+1}{2}} \left(b_1^{\frac{\ell-s-1}{2}} + b_1^{-\frac{\ell-s-1}{2}} - 2\right) - 1.
\end{aligned}$$

Since $x + x^{-1} \ge 2$ for all $x > 0$, we see that $A_b(n) - A_{b_1}(n) \ge -1$.

In any case, we obtain $A_b(n) - A_{b_1}(n) \ge -1$, as desired. This proves (i).

Next, we prove (ii). For each $k \in \mathbb{N}$, let $n = n_k = b_1^{2\ell k-1} + 1$. Then $n = b_1^{\ell-1} b^{2k-1} + 1$. By Lemma 2.1, we obtain

$$A_{b_1}(n) = b_1^{\left\lceil \frac{2\ell k-1}{2} \right\rceil} + b_1^{\left\lfloor \frac{2\ell k-1}{2} \right\rfloor} - 1 = b_1^{k\ell} + b_1^{k\ell-1} - 1,$$

$$A_b(n) = b^{\left\lceil \frac{2k-1}{2} \right\rceil} + b_1^{\ell-1} b^{\left\lfloor \frac{2k-1}{2} \right\rfloor} - 2 = b^k + b_1^{\ell-1} b^{k-1} - 2 = b_1^{k\ell} + b_1^{k\ell-1} - 2.$$

Therefore $A_b(n) - A_{b_1}(n) = -1$. Since $n = n_k$ and k is arbitrary, this shows that there are infinitely many $n \in \mathbb{N}$ such that $A_b(n) - A_{b_1}(n) = -1$. This proves (ii). Then (iii) follows immediately from (i) and (ii). So the proof is complete. □

Since we have already got the answers to the cases when $\log b / \log b_1$ is irrational and when it is integral, it remains to consider the case when $\log b / \log b_1$ is a rational number and is not an integer.

Theorem 3.2. *Let $b > b_1 \ge 2$ be integers and $b = b_1^{\frac{r}{s}}$ where $r, s \in \mathbb{N}$, $r > s > 1$, and $(r, s) = 1$. Then*

$$\liminf_{n \to \infty} (A_b(n) - A_{b_1}(n)) = -\infty.$$

Proof. To prove this theorem, it is enough to find a sequence $(n_k)_{k \ge 1}$ of positive integers such that $n_k \to +\infty$ and $A_{b_1}(n_k) - A_b(n_k) \to +\infty$ as $k \to +\infty$. We divide the calculation into three cases according to parity of r and s, and adjust the exponents so that $A_{b_1}(n)$ is large and $A_b(n)$ is small.

Case 1. Assume that s is even. Let k be a positive integer and $n = n_k = b^{s(2k+1)} + 1$. Since $(r, s) = 1$, s is even, and $b_1^r = b^s$, we see that r is odd and $n = b_1^{r(2k+1)} + 1$. By Lemma 2.1, we obtain $A_b(n) = 2b^{sk+\frac{s}{2}} - 1$ and

$$A_{b_1(n)} = b_1^{\left\lceil \frac{r(2k+1)}{2} \right\rceil} + b_1^{\left\lfloor \frac{r(2k+1)}{2} \right\rfloor} - 1 = b_1^{kr+\frac{r+1}{2}} + b_1^{kr+\frac{r-1}{2}} - 1 = b_1^{\frac{1}{2}} b^{sk+\frac{s}{2}} + b_1^{-\frac{1}{2}} b^{sk+\frac{s}{2}} - 1.$$

Since $x + x^{-1} > 2$ for all $x > 1$, we see that $b_1^{\frac{1}{2}} + b_1^{-\frac{1}{2}} - 2$ is a positive constant. Therefore,

$$A_{b_1}(n) - A_b(n) = b^{sk+\frac{s}{2}} \left(b_1^{\frac{1}{2}} + b_1^{-\frac{1}{2}} - 2\right) \to +\infty \text{ as } k \to +\infty.$$

Case 2. Assume that s and r are odd. Since $b_1^{\frac{r}{s}} = b$ is an integer, we obtain by Lemma 2.2 an integer $b_2 \ge 2$ such that $b_1 = b_2^s$, and so $b = b_2^r$. Since $(2s, r) = 1$, there exists a negative integer x such that $2sx \equiv 1 \pmod r$. Since the proof of Case 1 is finished, we will define a new sequence $(n_k)_{k \ge 1}$ using the same notation. Let k be a positive integer and let $n = n_k = b_1^{2x(1-s)+2rk+1} + 1$. For convenience, let

$\ell = x(1-s) + rk$. So $\ell \in \mathbb{N}$, $\ell \geq 5$, and $n = b_1^{2\ell+1} + 1$. By Lemma 2.1 and the fact that $b_1 = b_2^s$, we obtain

$$A_{b_1}(n) = b_1^{\ell+1} + b_1^{\ell} - 1 = b_2^{s\ell+s} + b_2^{s\ell} - 1. \tag{3.7}$$

Next, let $y = \frac{s(2\ell+1)-1}{r}$. By the definition of ℓ and x, we have

$$s(2\ell+1) = 2sx(1-s) + 2srk + s \equiv 1 \pmod{r}.$$

Therefore y is a positive integer. Since $s(2\ell+1) - 1$ is even and r is odd, we see that y is even. To calculate $A_b(n)$, we write

$$n = b_1^{2\ell+1} + 1 = b_2^{s(2\ell+1)} + 1 = b_2^{ry+1} + 1 = b_2 \cdot b^y + 1.$$

So b_2 is the leading digit in the b-adic representation of n. Since y is even and $b = b_2^r$, we obtain by Lemma 2.1 that

$$A_b(n) = b_2^{\frac{ry}{2}}(1 + b_2) - 2 = b_2^{s\ell + \frac{s-1}{2}}(1 + b_2) - 2. \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$A_{b_1}(n) - A_b(n) = b_2^{s\ell+\frac{s}{2}} B + 1, \quad \text{where } B = b_2^{\frac{s}{2}} + b_2^{-\frac{s}{2}} - b_2^{-\frac{1}{2}} - b_2^{\frac{1}{2}}.$$

Since the function $x \mapsto x + x^{-1}$ is strictly increasing on $(1, \infty)$ and $b_2^{\frac{s}{2}} > b_2^{\frac{1}{2}} > 1$, we see that B is a positive constant. Therefore, $A_{b_1}(n) - A_b(n) = b_2^{s\ell+\frac{s}{2}} B + 1 \to +\infty$ as $\ell \to +\infty$. Since $\ell = x(1-s) + rk \geq k$, we see that $A_{b_1}(n) - A_b(n) \to +\infty$ as $k \to +\infty$. So the proof of Case 2 is complete.

Case 3. Assume that s is odd and r is even. This case is similar to Case 2 and we only need to modify some calculations. Again, we have $b_1 = b_2^s$ and $b = b_2^r$ for some integer $b_2 \geq 2$. Since $(2r, 2s) = 2$ and $2 \mid 1 - s$, there exists a positive integer k such that

$$2sk \equiv 1 - s \pmod{2r}. \tag{3.9}$$

In fact, there are infinitely many positive integers k satisfying (3.9), so we can choose k to be arbitrarily large. Let $n = n_k = b_1^{2k+1} + 1$. By Lemma 2.1 and the fact that $b_1 = b_2^s$, we obtain

$$A_{b_1}(n) = b_1^{k+1} + b_1^{k} - 1 = b_2^{sk+s} + b_2^{sk} - 1. \tag{3.10}$$

Next, let $y = \frac{s(2k+1)-1}{r}$. By (3.9), we have $s(2k+1) = 2sk + s \equiv 1 \pmod{2r}$. Therefore $2r$ divides $s(2k+1) - 1$, and thus $y = 2\left(\frac{s(2k+1)-1}{2r}\right)$ is an even positive integer. To calculate $A_b(n)$, we write

$$n = b_1^{2k+1} + 1 = b_2^{s(2k+1)} + 1 = b_2^{ry+1} + 1 = b_2 \cdot b^y + 1.$$

By Lemma 2.1, we obtain

$$A_b(n) = b_2^{\frac{ry}{2}}(1 + b_2) - 2 = b_2^{ks+\frac{s-1}{2}}(1 + b_2) - 2. \tag{3.11}$$

From (3.10), (3.11) and a similar reason as in Case 2, we obtain

$$A_{b_1}(n) - A_b(n) = b_2^{ks+\frac{s}{2}}\left(b_2^{\frac{s}{2}} + b_2^{-\frac{s}{2}} - b_2^{-\frac{1}{2}} - b_2^{\frac{1}{2}}\right) + 1 \to +\infty \ \text{ as } k \to +\infty.$$

This completes the proof. □

4. Conclusions

Let us record all related results that we obtained as follows.

Summary of the results:

Let $b > b_1 \geq 2$ be integers. Then, the following statements hold.

(i) If $\frac{\log b}{\log b_1}$ is not an integer, then,

$$\limsup_{n\to\infty} (A_b(n) - A_{b_1}(n)) = +\infty \text{ and } \liminf_{n\to\infty} (A_b(n) - A_{b_1}(n)) = -\infty.$$

(ii) If $\frac{\log b}{\log b_1}$ is an integer, then,

$$\limsup_{n\to\infty} (A_b(n) - A_{b_1}(n)) = +\infty \text{ and } \liminf_{n\to\infty} (A_b(n) - A_{b_1}(n)) = -1.$$

(iii) $A_b(n) - A_{b_1}(n)$ has an infinite number of sign changes as $n \to \infty$.

(iv) If s_b and s_{b_1} are the sums of the reciprocal of palindromes in bases b and b_1, respectively, then s_b and s_{b_1} are finite and $s_b > s_{b_1}$.

The statements (i)–(iii) come directly from [1, Theorems 11 and 12], and Theorems 3.1 and 3.2 of this article. The finiteness of s_b and s_{b_1} in (iv) is given by Shallit and Klauser [16] and the inequality $s_b > s_{b_1}$ is obtained from [17, Theorem 3].

Now that for all cases we have obtained results for the comparison between the number of palindromes in two bases, it is natural to extend this to more than two bases. So let $k \geq 2$ and $b_1 > b_2 > \cdots > b_k \geq 2$ be integers. Let $c_1, c_2, \ldots, c_k$ be any permutation of $b_1, b_2, \ldots, b_k$.

Question 4.1. Does the inequality $A_{c_1}(n) < A_{c_2}(n) < \cdots < A_{c_k}(n)$ hold for infinitely many $n \in \mathbb{N}$? We conjecture that if $\log b_i / \log b_j$ is irrational for every distinct $i, j = 1, 2, \ldots, k$, then the inequality holds for infinitely many $n \in \mathbb{N}$. What are the answers when one of the following situations occur?

(i) $\frac{\log b_i}{\log b_{i+1}}$ is integral for every $i = 1, 2, \ldots, k-1$;

(ii) $\frac{\log b_i}{\log b_j}$ is rational but not integral for each distinct i, j;

(iii) The set $\left\{ \frac{\log b_i}{\log b_j} \;\middle|\; 1 \leq i < j \leq k \right\}$ contains both an irrational number and a rational number.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this article.

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