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Research article

Common fixed points and convergence results for α-Krasnosel'skii mappings

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Abstract: We present convergence and common fixed point conclusions of the Krasnosel'skii iteration which is one of the iterative methods associated with α-Krasnosel'skii mappings satisfying condition (E). Our conclusions extend, generalize and improve numerous conclusions existing in the literature. Examples are given to support our results.

Keywords: α-Krasnosel'skii iteration; approximation; asymptotically regular; iterative methods; metric projection; Schauder theorem; uniformly convex Banach space
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1. Introduction

A self-mapping $\mathcal{F}$ on a convex, closed, and bounded subset K of a Banach space U is known as nonexpansive if $\|\mathcal{F}u - \mathcal{F}v\| \leq \|u - v\|$, $u, v \in U$ and need not essentially possess a fixed point. It is widely known that a point $u \in U$ is a fixed point or an invariant point if $\mathcal{F}u = u$. However, some researchers ensured the survival of a fixed point of nonexpansive mapping in Banach spaces utilizing suitable geometric postulates. Numerous mathematicians have extended and generalized these conclusions to consider several nonlinear mappings. One such special class of mapping is Suzuki generalized nonexpansive mapping (SGNM). Many extensions, improvements and generalizations of nonexpansive mappings are given by eminent researchers (see [8–10, 13, 15, 17, 19, 21, 22, 25], and so on). On the other hand, Krasnosel'skii [16] investigated a novel iteration of approximating fixed points of nonexpansive mapping. A sequence $\{u_i\}$ utilizing the Krasnosel'skii iteration is defined as: $u_1 = u, u_{i+1} = (1 - \alpha)u_i + \alpha\mathcal{F}u_i$, where $\alpha \in (0, 1)$ is a real constant. This iteration is one of the iterative methods which is the extension of the celebrated Picard iteration [24], $u_{i+1} = \mathcal{F}u_i$. The convergence rate of the Picard iteration [24] is better than the Krasnosel'skii iteration although the Picard iterative scheme is not essentially convergent for nonexpansive self-mappings. It is interesting to see that the fixed point of a self-mapping $\mathcal{F}$ is also a fixed point of the iteration $\mathcal{F}^n$ ($n \in \mathbb{N}$), of the self-mapping $\mathcal{F}$ but the reverse implication is not feasible. Recently several authors presented extended and generalized results for better approximation of fixed points (see [1, 3, 11, 23, 26, 27]).

We present convergence and common fixed point conclusions for the associated α-Krasnosel'skii mappings satisfying condition (E) in the current work. Also, we support these with nontrivial illustrative examples to demonstrate that our conclusions improve, generalize and extend comparable conclusions of the literature.

2. Preliminaries

We symbolize $F(\mathcal{F})$, to be the collection of fixed points of a self-mapping $\mathcal{F}$, that is, $F(\mathcal{F}) = \{u \in U : \mathcal{F}u = u\}$. We begin with the discussion of convex Banach spaces, α-Krasnosel'skii mappings and the condition (E) (see [12, 18, 20, 23]).

Definition 2.1. *[14] A Banach space U is uniformly convex if, for $\epsilon \in (0,2]$ $\exists\, \delta > 0$ satisfying, $\|\frac{u+v}{2}\| \leq 1-\delta$ so that $\|u-v\| > \epsilon$ and $\|u\| = \|v\| = 1$, $u, v \in U$.*

Definition 2.2. *[14] A Banach space U is strictly convex if, $\|\frac{u+v}{2}\| < 1$ so that $u \neq v, \|u\| = \|v\| = 1$, $u, v \in U$.*

Theorem 2.1. *[5] Suppose U is a uniformly convex Banach space. Then $\exists$ a $\gamma > 0$, satisfying $\|\frac{1}{2}(u+v)\| \leq [1-\gamma\frac{\epsilon}{\delta}]\,\delta$ for every ϵ, $\delta > 0$ so that $\|u-v\| \geq \epsilon$, $\|u\| \leq \delta$ and $\|v\| \leq \delta$, for $u, v \in U$.*

Theorem 2.2. *[14] The subsequent postulates are equivalent in a Banach space U:*
(i) U is strictly convex.
(ii) $u = 0$ or $v = 0$ or $v = cu$ for $c > 0$, whenever $\|u+v\| = \|u\| + \|v\|$, $u, v \in U$.

Definition 2.3. *Suppose $\mathcal{F}$ is a self-mapping on a non-void subset V of a Banach space U.*
(i) Suppose for $u \in U$, $\exists\, v \in V$ so that for all $w \in V$, $\|v-u\| \leq \|w-u\|$. Then v is a metric projection [6] of U onto V, and is symbolized by $P_V(.)$. The mapping $P_V(u) : U \to V$ is the metric projection if $P_V(x)$ exists and is determined uniquely for each $x \in U$.
(ii) $\mathcal{F}$ satisfies condition (E_μ) [23] on V if $\exists$ $\mu \geq 1$, satisfying $\|u - \mathcal{F}v\| \leq \mu\|u - \mathcal{F}u\| + \|u-v\|$, $u, v \in V$. Moreover, $\mathcal{F}$ satisfies condition (E) on V, if $\mathcal{F}$ satisfies (E_μ).
(iii) $\mathcal{F}$ satisfies condition (E) [23] and $F(\mathcal{F}) \neq 0$, then $\mathcal{F}$ is quasi-nonexpansive.
(iv) $\mathcal{F}$ is a generalized α-Reich-Suzuki nonexpansive [21] if for an $\alpha \in [0,1)$, $\frac{1}{2}\|u-\mathcal{F}u\| \leq \|u-v\| \implies \|\mathcal{F}u - \mathcal{F}v\| \leq \max\{\alpha\|\mathcal{F}u-u\| + \alpha\|\mathcal{F}v - v\| + (1-2\alpha)\|u-v\|,\ \alpha\|\mathcal{F}u - v\| + \alpha\|\mathcal{F}v-u\| + (1-2\alpha)\|u-v\|\}$, $\forall\, u, v \in V$.
(v) A self-mapping $\mathcal{F}_\alpha : V \to V$ is an α-Krasnosel'skii associated with $\mathcal{F}$ [2] if, $\mathcal{F}_\alpha u = (1-\alpha)u + \alpha\mathcal{F}u$, for $\alpha \in (0,1)$, $u \in V$.
(vi) $\mathcal{F}$ is asymptotically regular [4] if $\lim_{n\to\infty}\|\mathcal{F}^n u - \mathcal{F}^{n+1}u\| = 0$.
(vii) $\mathcal{F}$ is a generalized contraction of Suzuki type [2], if $\exists\, \beta \in (0,1)$ and $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$, where $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$, satisfying $\beta\|u - \mathcal{F}u\| \leq \|u-v\|$ implies

$$\|\mathcal{F}u - \mathcal{F}v\| \leq \alpha_1\|u-v\| + \alpha_2(\|u-\mathcal{F}u\| + \|v - \mathcal{F}v\|) + \alpha_3(\|u - \mathcal{F}v\| + \|v - \mathcal{F}u\|), \quad u, v \in U.$$

(viii) $\mathcal{F}$ is α-nonexpansive [7] if $\exists$ an $\alpha < 1$ satisfying

$$\|\mathcal{F}u - \mathcal{F}v\| \leq \alpha\|\mathcal{F}u - v\| + \alpha\|\mathcal{F}v - u\| + (1 - 2\alpha)\|u - v\|, \quad u, v \in U.$$

Theorem 2.3. *[5] A continuous mapping on a non-void, convex and compact subset V of a Banach space U has a fixed point in V.*

Pant et al. [23] derived a proposition that if $\beta = \frac{1}{2}$, then a generalized contraction of Suzuki type is a generalized α-Reich-Suzuki nonexpansive. Moreover, the reverse implication may not necessarily hold.

Lemma 2.1. *[2] Let $\mathcal{F}$ be a generalized contraction of the Suzuki type on a non-void subset V of a Banach space U. Let $\beta \in [\frac{1}{2}, 1)$, then*

$$\|u - \mathcal{F}v\| \leq \left(\frac{2 + \alpha_1 + \alpha_2 + 3\alpha_3}{1 - \alpha_2 - \alpha_3}\right)\|u - \mathcal{F}u\| + \|u - v\|.$$

Proposition 2.1. *[23] Let $\mathcal{F}$ be a generalized contraction of the Suzuki type on a non-void subset V of a Banach space U, then $\mathcal{F}$ satisfies condition (E).*

The converse of this proposition is not true, which can be verified by the following example.

Example 2.1. Suppose $U = (\mathbb{R}^2, \|.\|)$ with the Euclidean norm and $V = [-1, 1] \times [-1, 1]$ be a subset of U. Let $\mathcal{F} : V \to V$ be defined as

$$\mathcal{F}(u_1, u_2) = \begin{cases} (\frac{u_1}{2}, u_2), & \text{if } |u_1| \leq \frac{1}{2} \\ (-u_1, u_2), & \text{if } |u_1| > \frac{1}{2}. \end{cases}$$

Case I. Let $x = (u_1, u_2), y = (v_1, v_2)$ with $|u_1| \leq \frac{1}{2}$, $|v_1| \leq \frac{1}{2}$. Then,

$$\begin{aligned} \|\mathcal{F}x - \mathcal{F}y\| &= \left\|\left(\frac{u_1}{2}, u_2\right) - \left(\frac{v_1}{2}, v_2\right)\right\| \\ &= \sqrt{\frac{(u_1 - v_1)^2}{4} + (u_2 - v_2)^2} \\ &\leq \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2} \\ &= \|x - y\|, \end{aligned}$$

which implies

$$\|x - \mathcal{F}y\| \leq \|x - \mathcal{F}x\| + \|\mathcal{F}x - \mathcal{F}y\| \leq \|x - \mathcal{F}x\| + \|x - y\|.$$

Case II. If $|u_1| \leq \frac{1}{2}$, $|v_1| > \frac{1}{2}$

$$\begin{aligned} \|x - \mathcal{F}y\| &= \sqrt{(u_1 + v_1)^2 + (u_2 - v_2)^2} \\ \|x - y\| &= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2} \\ \|x - \mathcal{F}x\| &= \frac{|u_1|}{2}. \end{aligned}$$

Consider

$$\begin{aligned}\|x-\mathcal{F}y\| &= \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+4u_1v_1}\\ &\leq \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+4|u_1|}\\ &\leq \sqrt{(u_1-v_1)^2+(u_2-v_2)^2}+4|u_1|.\end{aligned}$$

Hence,

$$\|x-\mathcal{F}y\| \leq 8\,\|x-\mathcal{F}x\| + \|x-y\|.$$

Here $\mu = 8$ satisfies the inequality.
Case III. If $|u_1| > \frac{1}{2}$, $|v_1| \leq \frac{1}{2}$

$$\begin{aligned}\|x-\mathcal{F}y\| &= \sqrt{\left(u_1-\frac{v_1}{2}\right)^2+(u_2-v_2)^2}\\ \|x-y\| &= \sqrt{(u_1+v_1)^2+(u_2-v_2)^2}\\ \|x-\mathcal{F}x\| &= 2|u_1|.\end{aligned}$$

Consider

$$\begin{aligned}\|x-\mathcal{F}y\| &= \sqrt{\left(u_1-\frac{v_1}{2}\right)^2+(u_2-v_2)^2}\\ &\leq \sqrt{(u_1-v_1)^2+(u_2-v_2)^2}\\ &\leq \sqrt{(u_1-v_1)^2+(u_2-v_2)^2}+|u_1|\\ &\leq \sqrt{(u_1-v_1)^2+(u_2-v_2)^2}+2|u_1|.\end{aligned}$$

So,

$$\|x-\mathcal{F}y\| \leq \|x-\mathcal{F}y\| + \|x-y\|.$$

Case IV. If $|u_1| > \frac{1}{2}$ and $|v_1| > \frac{1}{2}$, then

$$\begin{aligned}\|x-\mathcal{F}y\| &= \sqrt{(u_1+v_1)^2+(u_2-v_2)^2}\\ \|x-y\| &= \sqrt{(u_1-v_1)^2+(u_2-v_2)^2}\\ \|x-\mathcal{F}x\| &= 2|u_1|.\end{aligned}$$

Since $|u_1| > \frac{1}{2}$ and $|v_1| > \frac{1}{2}$, by simple calculation as above, we attain

$$\|x-\mathcal{F}y\| \leq \mu\,\|x-\mathcal{F}x\| + \|x-y\|.$$

Thus, $\mathcal{F}$ satisfies condition (E) for $\mu = 4$.
Now, suppose $x = (\frac{1}{2}, 1)$ and $y = (1, 1)$, so

$$\beta\,\|x-\mathcal{F}x\| = \beta\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\beta}{4}$$

$$\leq \|x-y\| = \frac{1}{2}.$$

Clearly, $\|\mathcal{F}x-\mathcal{F}y\| = \sqrt{(\frac{5}{4})^2+(1-1)^2} = \frac{5}{4}$.
Consider

$$\begin{aligned}
&\alpha_1 \|x-y\| + \alpha_2(\|x-\mathcal{F}x\|+\|y-\mathcal{F}y\|) + \alpha_3(\|x-\mathcal{F}y\|+\|y-\mathcal{F}x\|) \\
&= \alpha_1 \left\|\left(\frac{1}{2},1\right)-(1,1)\right\| + \alpha_2\left(\left\|\left(\frac{1}{2},1\right)-\left(\frac{1}{4},1\right)\right\| + \|(1,1)-(-1,1)\|\right) \\
&\quad + \alpha_3\left(\left\|\left(\frac{1}{2},1\right)-(-1,1)\right\| + \left\|(1,1)-\left(\frac{1}{4},1\right)\right\|\right) \\
&= \frac{\alpha_1}{2} + \frac{\alpha_2}{4} + 2\alpha_2 + \frac{3\alpha_3}{2} + \frac{3\alpha_3}{4} \\
&= \frac{\alpha_1}{2} + \frac{9}{4}(\alpha_2+\alpha_3) \\
&= \frac{\alpha_1}{2} + \frac{9}{4}\left(\frac{1-\alpha_1}{2}\right) \qquad \text{(by Definition 2.3 (vii))} \\
&= \frac{\alpha_1}{2} + \frac{9}{8} - \frac{9\alpha_1}{8} \\
&= \frac{9}{8} - \frac{5\alpha_1}{8}.
\end{aligned}$$

Since $\alpha_1, \alpha_2, \alpha_3 \geq 0$, therefore

$$\|\mathcal{F}x-\mathcal{F}y\| > \alpha_1 \|x-y\| + \alpha_2(\|x-\mathcal{F}y\|+\|y-\mathcal{F}y\|) + \alpha_3(\|x-\mathcal{F}y\|+\|y-\mathcal{F}x\|),$$

which is a contradiction.
Thus, $\mathcal{F}$ is not a generalized contraction of the Suzuki type.

3. Results

Now, we establish results for a pair of α-Krasnosel'skii mappings using condition (E).

Theorem 3.1. *Let $\mathcal{F}_i$, for $i \in \{1,2\}$, be self-mappings on a non-void convex subset V of a uniformly convex Banach space U and satisfy condition (E) so that $F(\mathcal{F}_1 \cap \mathcal{F}_2) \neq \phi$. Then the α-Krasnosel'skii mappings $\mathcal{F}_{i_\alpha}$, $\alpha \in (0,1)$ and $i \in \{1,2\}$ are asymptotically regular.*

Proof. Let $v_0 \in V$. Define $v_{n+1} = \mathcal{F}_{i_\alpha} v_n$ for $i \in \{1,2\}$ and $n \in N \cup \{0\}$. Thus,

$$\mathcal{F}_{i_\alpha} v_n = y_{n+1} = (1-\alpha)v_n + \alpha\mathcal{F}_i v_n \quad \text{for } i \in \{1,2\},$$

and

$$\mathcal{F}_{i_\alpha} v_n - v_n = \mathcal{F}_{i_\alpha} v_n - \mathcal{F}_{i_\alpha} v_{n-1} = \alpha(\mathcal{F}_i v_n - v_n) \quad \text{for } i \in \{1,2\}.$$

It is sufficient to show that $\lim_{n\to\infty}\|\mathcal{F}_i v_n - v_n\| = 0$ to prove $\mathcal{F}_{i_\alpha}$ is asymptotically regular.
By definition, for $u_0 \in F(\mathcal{F}_1 \cap \mathcal{F}_2)$, we have

$$\|u_0 - \mathcal{F}_i v_n\| \leq \|u_0 - v_n\| \quad \text{for } i \in \{1,2\} \tag{3.1}$$

and for $i \in \{1,2\}$,

$$\begin{aligned}
\|u_0 - v_{n+1}\| &= \|u_0 - \mathcal{F}_{i_\alpha} v_n\| \\
&= \|u_0 - (1-\alpha)v_n - \alpha \mathcal{F}_i v_n\| \\
&\leq (1-\alpha)\|u_0 - v_n\| + \alpha\|u_0 - \mathcal{F}_i v_n\| \\
&= (1-\alpha)\|u_0 - v_n\| + \alpha\|u_0 - v_n\| \\
&= \|u_0 - v_n\|. \qquad (3.2)
\end{aligned}$$

Thus, the sequence $\{\|u_0 - v_n\|\}$ is bounded by $s_0 = \|u_0 - v_0\|$. From inequality (3.2), $v_n \to u_0$ as $n \to \infty$, if $v_{n_0} = u_0$, for some $n_0 \in \mathbb{N}$. So, assume $v_n \neq u_0$, for $n \in \mathbb{N}$, and

$$w_n = \frac{u_0 - v_n}{\|u_0 - v_n\|} \quad \text{and} \quad e_n = \frac{u_0 - \mathcal{F}_i v_n}{\|u_0 - v_n\|}, \quad \text{for } i \in \{1,2\}. \qquad (3.3)$$

If $\alpha \leq \frac{1}{2}$ and using Eq (3.3), we obtain

$$\begin{aligned}
\|u_0 - v_{n+1}\| &= \|u_0 - \mathcal{F}_{i_\alpha} v_n\|, \text{ for } i \in \{1,2\} \\
&= \|u_0 - (1-\alpha)v_n - \alpha\mathcal{F}_i v_n\|, \text{ for } i \in \{1,2\} \\
&= \|u_0 - v_n + \alpha v_n - \alpha\mathcal{F}_i v_n - 2\alpha u_0 + 2\alpha u_0 + \alpha v_n - \alpha v_n\|, \text{ for } i \in \{1,2\} \\
&= \|(1-2\alpha)u_0 - (1-2\alpha)v_n + (2\alpha u_0 - \alpha v_n - \alpha\mathcal{F}_i v_n)\|, \text{ for } i \in \{1,2\} \\
&\leq (1-2\alpha)\|u_0 - v_n\| + \alpha\|2u_0 - v_n - \mathcal{F}_i v_n\| \\
&= 2\alpha\|u_0 - v_n\|\|\frac{w_n + e_n}{2}\| + (1-2\alpha)\|u_0 - v_n\|. \qquad (3.4)
\end{aligned}$$

As the space U is uniformly convex with $\|w_n\| \leq 1$, $\|e_n\| \leq 1$ and $\|w_n - e_n\| = \frac{\|v_n - \mathcal{F}_i v_n\|}{\|u_0 - v_n\|} \geq \frac{\|v_n - \mathcal{F}_i v_n\|}{s_0} = \epsilon$ (say) for $i \in \{1,2\}$, we obtain

$$\frac{\|w_n + e_n\|}{2} \leq 1 - \delta\frac{\|v_n - \mathcal{F}_i v_n\|}{s_o} \quad \text{for } i \in \{1,2\}. \qquad (3.5)$$

From inequalities (3.4) and (3.5),

$$\begin{aligned}
\|u_0 - v_{n+1}\| &\leq \left(2\alpha\left(1 - \delta\frac{\|v_n - \mathcal{F}_i v_n\|}{s_o}\right) + (1-2\alpha)\right)\|u_0 - v_n\| \\
&= \left(1 - 2\alpha\delta\left(\frac{\|v_n - \mathcal{F}_i v_n\|}{s_0}\right)\right)\|u_0 - v_n\|. \qquad (3.6)
\end{aligned}$$

By induction, it follows that

$$\|u_0 - v_{n+1}\| \leq \prod_{j=1}^{n}\left(1 - 2\alpha\delta\left(\frac{\|v_n - \mathcal{F}_i v_n\|}{s_0}\right)\right)s_0. \qquad (3.7)$$

We shall prove that $\lim_{n\to\infty}\|\mathcal{F}_i v_n - v_n\| = 0$ for $i \in \{1,2\}$. On the contrary, consider that $\{\|\mathcal{F}_i v_n - v_n\|\}$ for $i \in \{1,2\}$ is not converging to zero, and we have a subsequence $\{v_{n_k}\}$, of $\{v_n\}$, satisfying $\|\mathcal{F}_i v_{n_k} - v_{n_k}\|$ converges to $\zeta > 1$. As $\delta \in [\,0,1]$ is increasing and $\alpha \leq \frac{1}{2}$, $1 - 2\alpha\delta\frac{\|v_k - \mathcal{F}_i v_k\|}{s_0} \in [\,0,1]\,$, $i \in \{1,2\}$, for all

$k \in \mathbb{N}$. Since $\|\mathcal{F}_i v_{n_k} - v_{n_k}\| \to \zeta$ so, for sufficiently large k, $\|\mathcal{F}_i v_{n_k} - v_{n_k}\| \geq \frac{\zeta}{2}$, from inequality (3.7), we have

$$\|u_0 - v_{n_k+1}\| \leq s_0 \left(1 - 2\alpha\delta\left(\frac{\zeta}{2 - s_0}\right)\right)^{(n_k+1)}. \tag{3.8}$$

Making $k \to \infty$, it follows that $v_{n_k} \to u_0$. By inequality (3.1), we get $\mathcal{F}_i v_{n_k} \to u_0$ and $\|v_{n_k} - \mathcal{F}_i v_{n_k}\| \to 0$ as $k \to \infty$, which is a contradiction. If $\alpha > \frac{1}{2}$, then $1 - \alpha < \frac{1}{2}$, because $\alpha \in (0, 1)$. Now, for $i \in \{1, 2\}$

$$\begin{aligned}
\|u_0 - v_{n+1}\| &= \|u_0 - (1-\alpha)v_n - \alpha\mathcal{F}_i v_n\| \\
&= \|u_0 - v_n + \alpha v_n - \alpha\mathcal{F}_i v_n + (2-2\alpha)u_0 - (2-2\alpha)u_0 + \mathcal{F}_i v_n - \mathcal{F}_i v_n + \alpha\mathcal{F}_i v_n - \alpha\mathcal{F}_i v_n\| \\
&= \|(2u_0 - v_n - \mathcal{F}_i v_n) - \alpha(2u_0 - v_n - \mathcal{F}_i v_n) + 2\alpha(u_0 - \mathcal{F}_i v_n) - (u_0 - \mathcal{F}_i v_n)\| \\
&\leq (1-\alpha)\|2u_0 - v_n - \mathcal{F}_i v_n\| + (2\alpha - 1)\|u_0 - v_n\| \\
&\leq 2(1-2\alpha)\ \|u_0 - v_n\| \ \frac{\|w_n + e_n\|}{2} + (2\alpha - 1)\ \|u_0 - v_n\|.
\end{aligned}$$

By the uniform convexity of U, we attain, for $i \in \{1, 2\}$,

$$\|x_0 - y_{n+1}\| \leq \left(2(1-\alpha) - 2(1-\alpha)\ \delta\ \frac{\|y_n - \mathcal{F}_i y_n\|}{s_o} + (1-2\alpha)\right)\|x_0 - y_n\|. \tag{3.9}$$

By induction, we get

$$\|u_0 - v_{n+1}\| \leq \prod_{j=1}^{n}\left(1 - 2(1-\alpha)\ \delta\ \left(\frac{\|v_j - \mathcal{F}_i v_j\|}{s_0}\right)\right)s_0.$$

Similarly, it can be easily proved that $\|\mathcal{F}_i v_n - v_n\| \to 0$ as $n \to \infty$, which implies that $\mathcal{F}_{i_\alpha}$ for $i \in \{1, 2\}$, is asymptotically regular. □

Next, we demonstrate by a numerical experiment that a pair of α-Krasnosel'skii mappings are asymptotically regular for fix $\alpha \in (0, 1)$.

Example 3.1. *Assume $U = (R^2, \|.\|)$ with Euclidean norm and $V = \{u \in R^2 : \|u\| \leq 1\}$, to be a convex subset of U. $\mathcal{F}_i$ for $i \in \{1, 2\}$ be self-mappings on V, satisfying*

$$\begin{aligned}
\mathcal{F}_1(u_1, u_2) &= (u_1, u_2) \\
\mathcal{F}_2(u_1, u_2) &= (\frac{u_1}{2}, 0)
\end{aligned}$$

Then, clearly both $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy the condition (E) and $F(\mathcal{F}_1 \cap \mathcal{F}_2) = (0, 0)$. Now, we will show that the α-Krasnosel'skii mappings $\mathcal{F}_{i_\alpha}$ for $\alpha \in (0, 1)$ and $i \in \{1, 2\}$ are asymptotically regular.
Since $\mathcal{F}_1$ is the identity map, α- Krasnosel'skii mapping $\mathcal{F}_{1\alpha}$ is also identity and hence asymptotically regular.
Now, we show $\mathcal{F}_{2\alpha}$ is asymptotically regular, let $u = (u_1, u_2) \in V$

$$\begin{aligned}
\mathcal{F}_{2\alpha}(u_1, u_2) &= (1-\alpha)(u_1, u_2) + \alpha\mathcal{F}_2(u_1, u_2) \\
&= ((1-\alpha)u_1, (1-\alpha)u_2) + \alpha(\frac{u_1}{2}, 0)
\end{aligned}$$

$$= (u_1 - \frac{\alpha u_1}{2}, (1-\alpha)u_2),$$

$$\begin{aligned}
\mathcal{F}_{2\alpha}^2(u_1,u_2) &= (1-\alpha)(u_1 - \frac{\alpha u_1}{2}, (1-\alpha)u_2) + \alpha\mathcal{F}_2(u_1 - \frac{\alpha u_1}{2}, (1-\alpha)u_2) \\
&= (x_1 + \frac{\alpha^2 u_1}{2} - 3\frac{\alpha u_1}{2}, (1-\alpha)^2 u_2) + (\frac{\alpha u}{2} - \frac{\alpha^2 u_1}{4}, 0) \\
&= (u_1 - \alpha u_1 + \frac{\alpha^2 u_1}{4}, (1-\alpha)^2 x_2).
\end{aligned}$$

Continuing in this manner, we get

$$f_{2\alpha}^n(u_1,u_2) = ((u_1 - \frac{\alpha}{2})^n, (1-\alpha)^n u_2).$$

Since $(u_1,u_2) \in V$ *and* $\alpha \in (0,1)$*, we get that* $\lim_{n\to\infty}(u_1 - \frac{\alpha}{2})^n = 0$ *and* $\lim_{n\to\infty}(1-\alpha)^n = 0$*. Now, consider*

$$\begin{aligned}
&\lim_{n\to\infty}\|\mathcal{F}_{2\alpha}^n(u_1,u_2) - \mathcal{F}_{2\alpha}^{n+1}(u_1,u_2)\| \\
&\qquad = \sup_{u\in M}\lim_{n\to\infty}\|(u_1 - \frac{\alpha}{2})^n - (u_1 - \frac{\alpha}{2})^{n+1}, ((1-\alpha)^n - (1-\alpha)^{n+1})x_2\| \\
&\qquad = 0.
\end{aligned}$$

Hence, $\mathcal{F}_{2\alpha}$ *is also asymptotically regular.*

Theorem 3.2. *Let* $\mathcal{F}_i$ *be quasi-nonexpansive self-mappings on a non-void and closed subset* V *of a Banach space* U *for* $i \in \{1,2\}$*, and satisfy condition (E) so that* $F(\mathcal{F}_1 \cap \mathcal{F}_2) \neq 0$*. Then,* $F(\mathcal{F}_1 \cap \mathcal{F}_2)$ *is closed in* V*. Also, if* U *is strictly convex, then* $F(\mathcal{F}_1 \cap \mathcal{F}_2)$ *is convex. Furthermore, if* U *is strictly convex,* V *is compact, and* $\mathcal{F}$ *is continuous, then for any* $s_0 \in V, \alpha \in (0,1)$*, the* α*-Krasnosel'skii sequence* $\{\mathcal{F}_{i_\alpha}^n(s_0)\}$*, converges to* $s \in F(\mathcal{F}_1 \cap \mathcal{F}_2)$ *.*

Proof. (i) We assume $\{s_n\} \in F(\mathcal{F}_1 \cap \mathcal{F}_2)$ so that $s_n \to s \in F(\mathcal{F}_1 \cap \mathcal{F}_2)$ as $n \to \infty$. Hence, $\mathcal{F}_i s_n = s_n$ for $i \in \{1,2\}$. Next, we show that $\mathcal{F}_i s = s$ for $i \in \{1,2\}$. Since $\mathcal{F}_i$ are quasi-nonexpansive, we get

$$\|s_n - \mathcal{F}_i s\| \leq \|s_n - s\| \text{ for } i \in \{1,2\},$$

that is, $\mathcal{F}_i s = s$ for $i = 1,2$, hence $F(\mathcal{F}_2 \cap \mathcal{F}_2)$ is closed.
(ii) V is convex since U is strictly convex. Also fix $\gamma \in (0,1)$ and $u,v \in F(\mathcal{F}_1 \cap \mathcal{F}_2)$ so that $u \neq v$. Take $s = \gamma u + (1-\gamma)v \in V$. Since mapping $\mathcal{F}_i$ satisfy condition (E),

$$\|u - \mathcal{F}_i s\| \leq \|u - \mathcal{F}_i u\| + \|u - s\| = \|u - s\| \text{ for } i \in \{1,2\}.$$

Similarly,

$$\|v - \mathcal{F}_i s\| \leq \|v - s\| \text{ for } i \in \{1,2\}.$$

Using strict convexity of U, there is a $\theta \in [0,1]$ so that $\mathcal{F}_i s = \theta u + (1-\theta)v$ for $i = 1,2$

$$(1-\theta)\|u-v\| = \|\mathcal{F}_i u - \mathcal{F}_i s\| \leq \|u - s\| = (1-\gamma)\|u-v\|, \text{ for } i \in \{1,2\}, \tag{3.10}$$

and

$$\theta\|u - v\| = \|\mathcal{F}_i v - \mathcal{F}_i s\| \le \|v - s\| = \gamma\|u - v\|, \text{ for } i \in \{1, 2\}. \tag{3.11}$$

From inequalities (3.10) and (3.11), we obtain

$$1 - \theta \le 1 - \gamma \text{ and } \theta \le \gamma \text{ implies that } \theta = \gamma.$$

Hence, $\mathcal{F}_i s = s$ for i = 1,2, implies $s \in F(\mathcal{F}_1 \cap \mathcal{F}_2)$.
(iii) Let us define $\{s_n\}$ by $s_n = \mathcal{F}_{i_\alpha}^n s_0, s_0 \in V$, where $\mathcal{F}_{i_\alpha} s_0 = (1-\alpha)s_0 + \alpha\mathcal{F}_i s_0, \alpha \in (0, 1)$. We have a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ converging to some $s \in V$, since V is compact. Using the Schauder theorem and the continuity of $\mathcal{F}_i$, we have $F(\mathcal{F}_1 \cap \mathcal{F}_2) \neq \phi$. We shall demonstrate that $s \in F(\mathcal{F}_1 \cap \mathcal{F}_2)$. Let $w_0 \in F(\mathcal{F}_1 \cap \mathcal{F}_2)$, consider

$$\begin{aligned}\|s_n - w_0\| &= \|\mathcal{F}_{i_\alpha}^n s_0 - w_0\| \\ &\le \|\mathcal{F}_{i_\alpha}^{n-1} s_0 - w_0| \\ &= \|s_{n-1} - w_0\|.\end{aligned}$$

Therefore, $\{\|s_n - w_0\|\}$ converges as it is a decreasing sequence that is bounded below by 0. Moreover, since $\mathcal{F}_{i_\alpha}$ for $i = 1, 2$ is continuous, we have

$$\begin{aligned}\|w_0 - s_0\| &= \lim_{k\to\infty}\|s_{n_k+1} - s_o\| \\ &= \lim_{k\to\infty}\|\mathcal{F}_{i_\alpha} s_{n_k} - s_0\| \\ &= \|\mathcal{F}_{i\alpha} s - s_0\| \\ &= \|(1-\alpha)s + \alpha\mathcal{F}_i s - s_0\| \\ &\le (1-\alpha)\|s - s_0\| + \alpha\|\mathcal{F}_i s - s_0\| \quad \text{for } i \in \{1, 2\}.\end{aligned} \tag{3.12}$$

Since $\alpha > 0$, we get

$$\|s - s_0\| \le \|\mathcal{F}_i s - s_0\|, \text{ for } i \in \{1, 2\}. \tag{3.13}$$

Since $\mathcal{F}_i$ are quasi-nonexpansive maps, we get

$$\|\mathcal{F}_i s - s_0\| \le \|s - s_0\|, \text{ for } i \in \{1, 2\}, \tag{3.14}$$

and from inequalities (3.13) and (3.14), we get

$$\|\mathcal{F}_i s - s_0\| = \|s - s_0\|, \text{ for } i \in \{1, 2\}. \tag{3.15}$$

Now, from inequality (3.12), we have

$$\begin{aligned}\|s - s_0\| &\le \|(1-\alpha)s + \alpha\mathcal{F}_i s - s_0\|, \text{ for } i \in \{1, 2\} \\ &\le (1-\alpha)\|s - s_0\| + \alpha\|\mathcal{F}_i s - s_0\|, \text{ for } i \in \{1, 2\} \\ &= \|s - s_0\|,\end{aligned}$$

which implies that

$$\|(1-\alpha)s + \alpha\mathcal{F}_i s - s_0\| = (1-\alpha)\|s - s_0\| + \alpha\|\mathcal{F}_i s - s_0\|, \text{ for } i \in \{1, 2\}.$$

Since U is strictly convex, either $\mathcal{F}_i s - s_0 = a(s - s_0)$ for some $a \gtrsim 0$ or $s = s_0$. From Eq (15), it follows that $a = 1$, then, $\mathcal{F}_i s = s$ for $i = 1, 2$ and $s \in F(\mathcal{F}_1 \cap \mathcal{F}_2)$. Since $\lim_{n\to\infty}\|s_n - s_0\|$ exists and $\{s_{nk}\}$ converges strongly to s. Hence, $\{s_n\}$ converges strongly to $s \in F(\mathcal{F}_1 \cap \mathcal{F}_2)$. □

The next conclusion for metric projection is slightly more fascinating.

Theorem 3.3. *Let $\mathcal{F}_i$ be quasi-nonexpansive self-mappings on a non-void, closed, and convex subset V of a uniformly convex Banach space U for i $\in \{1,2\}$, and satisfies condition (E) so that $F(\mathcal{F}_1 \cap \mathcal{F}_2) \neq \phi$. Let $P : U \to F(\mathcal{F}_1 \cap \mathcal{F}_2)$ be the metric projection. Then, for every $u \in U$, the sequence $\{P\mathcal{F}_i^n u\}$ for $i = \{1,2\}$, converges to $s \in F(\mathcal{F}_1 \cap \mathcal{F}_2)$.*

Proof. Let $u \in V$. For $n, m \in N$

$$\|P\mathcal{F}_i^n u - \mathcal{F}_i^n u\| \leq \|P\mathcal{F}_i^m u - \mathcal{F}_i^n u\|, \text{ for } n \geq m,\ i \in \{1,2\}. \tag{3.16}$$

Since $u \in F(\mathcal{F}_1 \cap \mathcal{F}_2)$, $n \in N$ and $\mathcal{F}_i$ are quasi-nonexpansive maps, for $i \in \{1,2\}$ we have

$$\begin{aligned}\|P\mathcal{F}_i^m u - \mathcal{F}_i^n u\| &= \|P\mathcal{F}_i^m u - \mathcal{F}_i\mathcal{F}_i^{n-1} u\| \\ &\leq \|P\mathcal{F}_i^m u - \mathcal{F}_i^{n-1} u\|.\end{aligned}$$

Therefore, for $n \geq m$, it follows that

$$\|P\mathcal{F}_i^m u - \mathcal{F}_i^n u\| \leq \|P\mathcal{F}_i^m u - \mathcal{F}_i^m u\|, \text{ for } i \in \{1,2\}. \tag{3.17}$$

From inequalities (3.16) and (3.17), we have

$$\|P\mathcal{F}_i^n u - \mathcal{F}_i^n u\| \leq \|P\mathcal{F}_i^m u - \mathcal{F}_i^m u\|, \text{ for } i \in \{1,2\},$$

which implies that $\lim_{n\to\infty}\|P\mathcal{F}_i^n u - \mathcal{F}_i^n u\|$ exists. Taking $\lim_{n\to\infty}\|P\mathcal{F}_i^n u - \mathcal{F}_i^n u\| = l$.
If $l = 0$, then we have an integer $n_0(\epsilon)$ for $\epsilon > 0$, satisfying

$$\|P\mathcal{F}_i^n u - \mathcal{F}_i^n u\| > \frac{\epsilon}{4}, \text{ for } i \in \{1,2\}, \tag{3.18}$$

for $n \geq n_0$. Therefore, if $n \geq m \geq n_0$ and using inequalities (3.17) and (3.18), we have, for $i \in \{1,2\}$,

$$\begin{aligned}\|P\mathcal{F}_i^n u - P\mathcal{F}_i^m u\| &\leq \|P\mathcal{F}_i^n u - P\mathcal{F}_i^{n_0} u\| + \|P\mathcal{F}_i^{n_0} u - \mathcal{F}_i^m u\| \\ &\leq \|P\mathcal{F}_i^n u - \mathcal{F}_i^n u\| + \|\mathcal{F}_i^n u - P\mathcal{F}_i^{n_0} u\| + \|P\mathcal{F}_i^m u - \mathcal{F}_i^m u\| + \|\mathcal{F}_i^m u - P\mathcal{F}_i^{n_0} u\| \\ &\leq \|P\mathcal{F}_i^n u - \mathcal{F}_i^n u\| + \|\mathcal{F}_i^{n_0} u - P\mathcal{F}_i^{n_0} u\| + \|P\mathcal{F}_i^m u - \mathcal{F}_i^m u\| + \|\mathcal{F}_i^{n_0} u - P\mathcal{F}_i^{n_0} u\| \\ &\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \epsilon.\end{aligned}$$

That is, $\{P\mathcal{F}_i^n u\}$ for $i = \{1,2\}$ is a Cauchy sequence in $F(\mathcal{F}_1 \cap \mathcal{F}_2)$. Using the completeness of U and the closedness of $F(\mathcal{F}_1 \cap \mathcal{F}_2)$ from the above theorem, $\{P\mathcal{F}_i^n x\}$ for $i = 1,2$, converges in $F(\mathcal{F}_1 \cap \mathcal{F}_2)$. Taking $l > 0$, we claim that the sequence $\{P\mathcal{F}_i^n u\}$ for $i = 1,2$, is a Cauchy sequence in U . Also we have, an $\epsilon_0 > 0$ so that, for each $n_0 \in N$, we have some $r_0, s_0 \geq n_0$ satisfying

$$\|P\mathcal{F}_i^{r_0} u - P\mathcal{F}_i^{s_0} u\| \geq \epsilon_0, \text{ for } i \in \{1,2\}.$$

Now, we choose a $\theta > 0$

$$(l + \theta)\left(1 - \delta\frac{\epsilon_0}{l + \theta}\right) < \theta.$$

Let m_0 be as large as possible such that for $q \geq m_0$

$$l \leq \|P\mathcal{F}_i^q u - \mathcal{F}_i^q u\| \leq l + \theta.$$

For this m_0, there exist q_1, q_2 such that $q_1, q_2 > m_0$ and

$$\|P\mathcal{F}_i^{q_1} u - P\mathcal{F}_i^{q_2} u\| \geq \epsilon_0 \text{ for } i \in \{1, 2\}.$$

Thus, for $q_0 \geq max\{q_1, q_2\}$, we attain

$$\|P\mathcal{F}_i^{q_1} x - \mathcal{F}_i^{q_0} x\| \geq \|P\mathcal{F}_i^{q_1} x - \mathcal{F}_i^{q_1} x\| < l + \theta,$$

and

$$\|P\mathcal{F}_i^{q_2} x - \mathcal{F}_i^{q_0} x\| \geq \|P\mathcal{F}_i^{q_1} x - \mathcal{F}_i^{q_1} x\| < l + \theta \text{ for } i \in \{1, 2\}.$$

Now, using the uniform convexity of U, we attain

$$\begin{aligned} l \leq \|P\mathcal{F}_i^{q_0} x - \mathcal{F}_i^{q_0} x\| &\leq \left\|\frac{P\mathcal{F}_i^{q_1} x + P\mathcal{F}_i^{q_2} x}{2} - \mathcal{F}_i^{q_0} x\right\|, \text{ for } i \in \{1, 2\} \\ &\leq (\ l + \theta) \left(1 - \delta \frac{\epsilon_0}{l + \theta}\right) \\ &< \theta, \end{aligned}$$

a contradiction. Hence for every $u \in V$, the sequence $\{P\mathcal{F}_i^n u\}$ for $i = 1, 2$, converges to some $s \in F(\mathcal{F}_1 \cap \mathcal{F}_2)$. □

4. Conclusions

We have proved some properties of common fixed points and also showed that if two mappings have common fixed points, then their α-Krasnosel'skii mappings are asymptotically regular. To show the superiority of our results, we have provided an example. Further, we have proved that the α-Krasnosel'skii sequence and its projection converge to a common fixed whose collection is closed.

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Conflict of interest

The authors declare no conflict of interest.

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