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Research article

On a coupled system under coupled integral boundary conditions involving non-singular differential operator

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Abstract: In this work, a coupled system under coupled integral boundary conditions with Caputo-Fabrizio derivative (CFD) is considered. We intend to derive some necessary and sufficient results for the existence of at least one solution. In addition, we extend our analysis further to develop a monotone iterative scheme coupled with the upper and lower solution method to compute extremal solutions. Therefore, in this regard, Perov's fixed point theorem is applied to study the existing criteria for the solution. Also, results related to at least one solution are derived by using Schauder's fixed point theorem. Finally, we use a monotone iterative procedure together with upper and lower solution methods to study extremal solutions. Graphical presentations of upper and lower solutions are provided for some examples to illustrate our results.

Keywords: Caputo-Fabrizio derivative; coupled integral boundary conditions; Perov's theorem; iterative procedure
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1. Introduction

Fractional calculus has been given much attention in the last three decades. Because fractional differential equations (FDEs) and fractional differential systems are important tools to describe physical phenomena arising in engineering, physics, economics, and other fields. It is remarkable that arbitrary order derivative is global in nature as compared to classical integer order. This is a great merit of the said area. Because, ordinary differential equations with respect to time cannot describe processes with dynamic long memory. Due to this reason, to describe dynamical memory, it is possible to use the theory of fractional calculus with derivatives and integrals of fractional orders. Also, the concerned field has a variety of significant applications in many other scientific and engineering fields. For instance, author [1] has employed parameter estimation for fractional dynamical models arising in biology, modeling multiple electrochemical processes with fractional differential equations [2]. The physical understanding of the fractional derivative, the most famous and known one is the continuous time random walk [3], the reader should see the detailed theory about the area and applications in [4]. Keeping in mind the above mentioned details, it was proved that fractional order derivative in comparison with the classical derivative, has an advantage that the initial values take the same form as that for classical integer order differential equations which is more applicable for mathematical modeling. Mathematicians can create mathematical equations called partial and ordinary differential equations using the theory of the rate of change. In recent decades, certain kinds of differential calculus have received a lot of attention. In order to solve those equations, numerical and analytic methods have been suggested. These mathematical equations have proven to be extremely effective in simulating real-world phenomena. However, these mathematical domains of integro differential calculus have consistently failed to recreate the physical phenomena multiple times due to the complexity of several real-world problems. In addition, fractional derivatives include the memory and genetic effects that play a crucial part in investigations of many real world dynamical problems.Due to these significant applications, researchers have given much attention for investigate various real-world problems/phenomena under the concept of the fractional calculus. Recently some interesting work has been done in the engineering side [5, 6], rheology [7], epidemiology [8], physical sciences [9], signal and image processing [10], etc.

Due to the importance of the said area, scientists and research people have given their attention to studying FDEs from different scenarios including qualitative theory, numerical, and stability analysis. In this regards plenty of work has been published. We can refer only few here like the existence theory of FDEs by fixed point theory [11], qualitative results of FDEs via degree theory [12], numerical analysis of FDEs via wavelet [13], spectral analysis of FDEs [14], decomposition technique for FDEs [15], etc. Here we remark that derivative with non-integer order has not a unique definition. The concerned operators have been defined in numbers of ways. The two major groups of operators are called the derivatives with powerlaw kernel and the operators with non-singular kernel. It should be kept in mind that from the said definitions the field of fractional calculus has gotten much more popularity from researchers. Because, researchers use freely different operators for the investigations of their models with different methodology and analysis. The first notable definition was given by Reimann-Liuoville which has given proper attention and still is using very well. In 1967, Caputo gave another definition which has gotten much more popularity and in plenty of research work the said operator used. Besides from the two mentioned operators, other definitions have also been given. Caputo and Fabrizio introduced a new definition in 2016 with exponential kernel [16]. The said definition has also attracted the attention very well. Atangana and his co-author [17] extended the definition of Caputo and Fabrizio further by using Mittag-Leffler type kernel instead of exponential. The said operator has also used very well. The two mentioned operators have used in many papers just for mathematical models or simple boundary value problems.

The qualitative theory is an important aspect to be investigated for FDEs by using different operators. For instance authors [18] have studied the existence mild solutions of coupled hybrid fractional order system with Caputo-Hadamard derivatives. In the same way, authors have derived existence theory for fractional order Volterra integro-differential equation with Mittag-Leffler kernel in [19]. Authors [20], established a detailed analysis for fractional-order nonlinear dynamical systems with general analytic kernels. Also, authors [21, 22] studied stability analysis for fractional order problems using different analysis. In addition, assessment of the performance of the hyperbolic-NILT method has been used to solve FDEs in [23]. In all these mentioned references, authors have applied fixed point theory and different concept of fractional order derivatives to study the qualitative results. Moreover, some recent work we refer also as [24, 25].

Here it is remarkable that boundary value problems (BVPs) play a significant role in modeling various phenomena in mass heat transfer and mechanical engineering. Therefore, the area of BVPs under the concept of fractional calculus has been explored very well see [26]. BVPs of FDEs containing integrals in their conditions have also been studied very well. Because, integral BVPs have numerous applications in applied fields including chemical engineering, blood flow problems, dynamics due to population, underground water flow, and so forth see [27]. It should be kept in mind that mentioned studies have been considered under the concept of the usual Caputo or Reimann-Liouville fractional derivative. As we know that fractional order differential and integral operators have numerous definitions from singular to non-singular. Researchers have significantly used the said operators in different studies. Here we refer to some published work as Caputo-Power law operator in [28], Caputo-Fabrizio in [29] and Mittag-Leffler type in [30], and other operators in [31].

So far we know the area devoted to a coupled system of FDEs under coupled boundary conditions has very rarely been considered for existence and iterative analysis. Although ordinary BVPs under CFD have been considered very well. For instance, authors [32] have considered the following Dirichlet boundary conditions under CFD denoted by ${}^{CF}\mathbf{D}$ as

$$\begin{cases} {}^{CF}_{c}\mathbf{D}^{\delta_1}_{\xi}\mathrm{u}(\xi) = \mathrm{f}_1(\xi, \mathrm{u}(\xi), {}^{CF}_{0}\mathbf{D}^{\delta_1}_{\xi}\mathrm{u}(\xi)), \ \delta_1 \in (1, 2], \quad \xi \in [c, d], \\ \mathrm{u}(c) = \mathrm{u}(d) = 0, \end{cases}$$

while $\mathrm{f}_1 : [c, d] \times R^2 \to R$ is a continuous function. By using nonlinear analysis tools, authors have developed sufficient conditions for the existence of a solution to the given problem.

Since the coupled system of coupled BVPs under CFD has not been discussed yet properly. Therefore, it is worth mentioning to derive the existence result for BVPs for fractional differential systems. So the present paper studies a class of coupled systems of FDEs under coupled boundary conditions. We will consider the following coupled system under coupled boundary conditions with $\xi \in [0, 1] = \mathscr{J}$ as

$$\begin{cases} {}^{CF}_{0}\mathbf{D}^{\delta_1}_{\xi}\mathrm{u}(\xi) = -\mathrm{f}_1(\xi, \mathrm{u}(\xi), \mathrm{v}(\xi)), \\ {}^{CF}_{0}\mathbf{D}^{\delta_2}_{\xi}\mathrm{v}(\xi) = -\mathrm{f}_2(\xi, \mathrm{u}(\xi), \mathrm{v}(\xi)), \\ \mathrm{u}(0) = 0 = \mathrm{v}(0), \\ \mathrm{u}(1) = \int_0^1 \phi(\eta)\mathrm{v}(\eta)d\eta, \ \mathrm{v}(1) = \int_0^1 \psi(\eta)\mathrm{u}(\eta)d\eta, \end{cases} \tag{1.1}$$

where $\delta_i \in (1, 2]$, and $\mathrm{f}_i : \mathscr{J} \times R^2 \to R$, for $i = 1, 2$ are continuous. Also, the boundary functions $\phi, \psi \in L[0, 1]$. By using Perov's and Schauder theorems, sufficient conditions are developed for the uniqueness and existence of a solution. Furthermore, to compute the extremal solution, we establish a monotonic iterative scheme. Two sequences of upper and lower solutions are established in which one is increasing and converges to the upper bound while the other decreases and converges to the lower bound of the iterative solution. Pertinent examples are given to illustrate the results. Some graphical presentations are given to illustrate the behavior of upper and lower solutions.

2. Basic results

Here we recollect some definitions and results which are needed for our analysis.

Definition 2.1. *[33] Let* u *be a absolutely integrable function on* $[0, 1]$*, then the CFD for order* $\delta_i \in (0, 1]$ *is defined as*

$$ {}_{0}^{CF}\mathbf{D}_{\xi}^{\delta_i}\mathrm{u}(\xi) = \frac{M(\delta_i)}{1-\delta_i}\int_0^{\xi}\mathrm{u}'(\eta)\exp\left[\frac{-\delta_i(\xi-\eta)}{1-\delta_i}\right]d\eta, $$

such that $M(\delta_i)(0) = M(\delta_i)(1) = 1$ *satisfying. Further, the derivative for higher order is defined with* $\delta \in [0, 1],\ n \geq 1$ *for* u *a absolutely integrable function on* $[0, 1]$ *as in [34]*

$$ {}_{0}^{CF}\mathbf{D}_{\xi}^{\delta+n}\mathrm{u}(\xi) = {}_{0}^{CF}\mathbf{D}_{\xi}^{\delta}\left[{}_{0}\mathbf{D}_{\xi}^{n}\mathrm{u}(\xi)\right]. $$

Definition 2.2. *[33] Let* u *be absolutely integrable function on* $[0, 1]$*, then the integral with non-singular kernel for order* $\delta_i \in (0, 1]$ *is defined as*

$$ {}_{0}^{CF}\mathbf{I}_{\xi}^{\delta_i}\mathrm{u}(\xi) = \frac{1-\delta_i}{M(\delta_i)}\mathrm{u}(\xi) + \frac{\delta_i}{M(\delta_i)}\int_0^{\xi}\mathrm{u}(\eta)d\eta. $$

Lemma 2.1. *[35] Let* h *be absolutely integrable function on* $[0, 1]$, *if right side vanishes at* $t = 0$ *of*

$$ {}_{0}^{CF}\mathbf{D}_{\xi}^{\delta_i}\mathrm{u}(\xi) = \mathrm{h}(\xi),\ 1 < \delta_i \leq 2, $$

then solution is given by

$$ \mathrm{u}(t) = \mathrm{c}_0 + \mathrm{c}_1\xi + \frac{2-\delta_i}{M(\delta_i-1)}\int_0^{\xi}\mathrm{h}(\eta)d\eta + \frac{\delta_i-1}{M(\delta_i-1)}\int_0^{\xi}(\xi-\eta)\mathrm{h}(\eta)d\eta. $$

Here, we know that a real nonnegative matrix works as a linear operator which acts on suitable Euclidean space. Such a matrix plays an important role in the theory of vector-valued metric spaces and needed in the proof of Perov's fixed point theorem [36].

Definition 2.3. *If* $\mathscr{X} \neq \emptyset$*, then a real valued mapping* $\rho : \mathscr{X} \times \mathscr{X} \to \mathrm{R}^n$ *is called a matric of vector valued type on* $\mathscr{X}$ *if the given hypothesis hold:*

(1) $\rho(\mathrm{u}, \mathrm{v}) = 0$, *with every* $\mathrm{u}, \mathrm{v} \in \mathscr{X}$, *and* $\rho(\mathrm{u}, \mathrm{v}) = 0$ *if and only if* $\mathrm{u} = \mathrm{v}$.
(2) $\rho(\mathrm{u}, \mathrm{v}) = \rho(\mathrm{v}, \mathrm{u})$, *with every* $\mathrm{u}, \mathrm{v} \in \mathscr{X}$.
(3) $\rho(\mathrm{u}, \mathrm{v}) \leq \rho(\mathrm{u}, \mathrm{w}) + \rho(\mathrm{w}, \mathrm{v})$, *with every* $\mathrm{u}, \mathrm{v}, \mathrm{w} \in \mathscr{X}$.

Theorem 2.4. *[37] Let* $(\mathscr{X}, \rho)$ *be a complete generalized metric space with metric* ρ *and let* $\mathscr{Q} : \mathscr{X} \to \mathscr{X}$ *be the operator such that*

$$ \rho(\mathscr{Q}(\mathrm{u}), \mathscr{Q}(\mathrm{v})) \leq \mathscr{M}\rho(\mathrm{u}, \mathrm{v}),\ \textit{for all}\ \mathrm{u}, \mathrm{v} \in \mathscr{X}, $$

where $\mathscr{M}$ is a square matrix of nonnegative entries such that the spectral radius $\varrho(\mathscr{M}) < 1$, then $\mathscr{Q}$ has a unique fixed point u^* *and*

$$\rho(\mathscr{Q}^k(u), u*) \leq \mathscr{M}^k(I - \mathscr{M})^{-1}\rho(\mathscr{Q}u, u),$$

for every $u \in \mathscr{X},\ k \geq 1$.

Remark 1. *[38] Consider a square matrix $\mathscr{M}$ with four nonnegative entries as*

$$\mathscr{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

Now if $m_{11} < 1,\ m_{22} < 1$, *and using famous Gelfand formula of numerical functional which state that*

$$\varrho(\mathscr{M}) = \lim_{n\to\infty} \sqrt[n]{\|\mathscr{M}\|^n}.$$

Then obviously, one has $\varrho(\mathscr{M}) < 1$.

Let $\mathscr{X} = C[0,1] \times C[0,1]$ be a Banach space with norm $\|(u, v)\| = \|u\| + \|v\|$, where $\|u\| = \max_{\xi\in\mathscr{J}} |u(\xi)|$.

Theorem 2.5. *[39] If $\mathscr{X}$ be a Banach space and* $\mathbf{S} \neq \emptyset$ *be closed, bounded, and convex subset of $\mathscr{X}$, then any operator $\mathscr{Q} : \mathbf{S} \to \mathscr{S}$ has at least one fixed point.*

3. Main results

Here we present one of our main result.

Lemma 3.1. *Consider* $x_1, x_2 \in L[0,1]$, *then the solution of the following linear coupled system under coupled boundary condition*

$$\begin{cases} {}^{CF}_{0}\mathbf{D}^{\delta_1}_{\xi} u(\xi) = -x_1(\xi),\ \delta_1 \in (1,2],\ \xi \in \mathscr{J}, \\ {}^{CF}_{0}\mathbf{D}^{\delta_2}_{\xi} v(\xi) = -x_2(\xi),\ \delta_2 \in (1,2],\ \xi \in \mathscr{J}, \\ u(0) = 0 = v(0), \\ u(1) = \displaystyle\int_0^1 \phi(\eta)v(\eta)d\eta, \\ v(1) = \displaystyle\int_0^1 \psi(\eta)u(\eta)d\eta, \end{cases} \tag{3.1}$$

is computed as

$$\begin{cases} u(\xi) = \xi \displaystyle\int_0^1 v(\eta)\phi(\eta)d\eta - \frac{(2-\delta_1)}{M(\delta_1 - 1)}\Big[\int_0^\xi x_1(\eta)d\eta + \xi\int_0^1 x_1(\eta)d\eta\Big] \\ \quad - \dfrac{(\delta_1 - 1)}{M(\delta_1 - 1)}\Big[\displaystyle\int_0^\xi (\xi-\eta)x_1(\eta)d\eta + \xi\int_0^1 (1-\eta)x_1(\eta)d\eta\Big], \\ v(\xi) = \xi \displaystyle\int_0^1 u(\eta)\psi(\eta)d\eta - \frac{(2-\delta_2)}{M(\delta_2 - 1)}\Big[\int_0^\xi x_2(\eta)d\eta + \xi\int_0^1 x_2(\eta)d\eta\Big] \\ \quad - \dfrac{(\delta_2 - 1)}{M(\delta_2 - 1)}\Big[\displaystyle\int_0^\xi (\xi-\eta)x_2(\eta)d\eta + \xi\int_0^1 (1-\eta)x_2(\eta)d\eta\Big]. \end{cases} \tag{3.2}$$

Proof. On applying ${}_{0}^{CF}\mathbf{I}_{\xi}^{\delta_1}$, ${}_{0}^{CF}\mathbf{I}_{\xi}^{\delta_2}$ to (4.1), we have

$$\begin{aligned} \mathrm{u}(\xi) &= \mathrm{c}_0 + \mathrm{c}_1\xi - \frac{(2-\delta_1)}{M(\delta_1-1)}\int_0^{\xi}\mathrm{x}_1(\eta)d\eta - \frac{\delta_1-1}{M(\delta_1-1)}\int_0^{\xi}(\xi-\eta)\mathrm{x}_1(\eta)d\eta, \\ \mathrm{v}(\xi) &= \mathrm{d}_0 + \mathrm{d}_1\xi - \frac{(2-\delta_2)}{M(\delta_2-1)}\int_0^{\xi}\mathrm{x}_2(\eta)d\eta - \frac{\delta_2-1}{M(\delta_2-1)}\int_0^{\xi}(\xi-\eta)\mathrm{x}_2(\eta)d\eta. \end{aligned} \tag{3.3}$$

Using $\mathrm{u}(0) = \mathrm{v}(0) = 0$,, we have from (3.3) that $\mathrm{c}_0 = \mathrm{d}_0 = 0$., Hence (3.3) yields

$$\begin{aligned} \mathrm{u}(\xi) &= \mathrm{c}_1\xi - \frac{(2-\delta_1)}{M(\delta_1-1)}\int_0^{\xi}\mathrm{x}_1(\eta)d\eta - \frac{\delta_1-1}{M(\delta_1-1)}\int_0^{\xi}(\xi-\eta)\mathrm{x}_1(\eta)d\eta, \\ \mathrm{v}(\xi) &= \mathrm{d}_1\xi - \frac{(2-\delta_2)}{M(\delta_2-1)}\int_0^{\xi}\mathrm{x}_2(\eta)d\eta - \frac{\delta_2-1}{M(\delta_2-1)}\int_0^{\xi}(\xi-\eta)\mathrm{x}_2(\eta)d\eta. \end{aligned} \tag{3.4}$$

Now, inview of boundary conditions $\mathrm{u}(1) = \int_0^1 \phi(\eta)\mathrm{v}(\eta)d\eta,\ \mathrm{v}(1) = \int_0^1 \psi(\eta)\mathrm{u}(\eta)d\eta$, one has from (3.4)

$$\begin{aligned} \mathrm{c}_1 &= \int_0^1 \mathrm{v}(\eta)\phi(\eta)d\eta + \frac{(2-\delta_1)}{M(\delta_1-1)}\int_0^1\mathrm{x}_1(\eta)d\eta + \frac{\delta_1-1}{M(\delta_1-1)}\int_0^1(1-\eta)\mathrm{x}_1(\eta)d\eta, \\ \mathrm{d}_1 &= \int_0^1 \mathrm{u}(\eta)\psi(\eta)d\eta + \frac{(2-\delta_2)}{M(\delta_2-1)}\int_0^{\xi}\mathrm{x}_2(\eta)d\eta + \frac{\delta_2-1}{M(\delta_2-1)}\int_0^1(1-\eta)\mathrm{x}_2(\eta)d\eta. \end{aligned} \tag{3.5}$$

Hence using values from (3.5) in (3.6), we get the required solution. □

Corollary 1. *Reference to Lemma 3.1, the solution of* (1.1) *is given by*

$$\begin{cases} \mathrm{u}(\xi) = \xi\displaystyle\int_0^1 \mathrm{v}(\eta)\phi(\eta)d\eta - \frac{(2-\delta_1)}{M(\delta_1-1)}\Big[\int_0^{\xi}\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta + \xi\int_0^1\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta\Big] \\ -\dfrac{(\delta_1-1)}{M(\delta_1-1)}\Big[\displaystyle\int_0^{\xi}(\xi-\eta)\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta + \xi\int_0^1(1-\eta)\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta\Big], \\ \mathrm{v}(\xi) = \xi\displaystyle\int_0^1 \mathrm{u}(\eta)\psi(\eta)d\eta - \frac{(2-\delta_2)}{M(\delta_2-1)}\Big[\int_0^{\xi}\mathrm{f}_2(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta + \xi\int_0^1\mathrm{f}_2(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta\Big] \\ -\dfrac{(\delta_2-1)}{M(\delta_2-1)}\Big[\displaystyle\int_0^{\xi}(\xi-\eta)\mathrm{f}_2(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta + \xi\int_0^1(1-\eta)\mathrm{f}_2(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta\Big]. \end{cases} \tag{3.6}$$

The given hypothesis need to be hold:

(H_1) For any $(\mathrm{u},\mathrm{v}), (\bar{\mathrm{u}},\bar{\mathrm{v}}) \in \mathscr{X}$, and constant $\mathrm{L}_i > 0,\ i = 1,2$, one has

$$|\mathrm{f}_1(\xi,\mathrm{u},\mathrm{v}) - \mathrm{f}_1(\xi,\bar{\mathrm{u}},\bar{\mathrm{v}})| \le \mathrm{L}_1[|\mathrm{u}-\bar{\mathrm{u}}| + |\mathrm{v}-\bar{\mathrm{v}}|],$$

$$|\mathrm{f}_2(\xi,\mathrm{u},\mathrm{v}) - \mathrm{f}_2(\xi,\bar{\mathrm{u}},\bar{\mathrm{v}})| \le \mathrm{L}_2[|\mathrm{u}-\bar{\mathrm{u}}| + |\mathrm{v}-\bar{\mathrm{v}}|].$$

(H_2) For $(\mathrm{u},\mathrm{v}) \in \mathscr{X}$, and constant $\mathrm{a}_i, \mathrm{b}_i, \mathrm{c}_i > 0, i = 1,2$, one has

$$|\mathrm{f}_i(\xi,\mathrm{u},\mathrm{v})| \le \mathrm{a}_i + \mathrm{b}_i|\mathrm{u}(\xi)| + \mathrm{c}_i|\mathrm{v}(\xi)|.$$

Note: For convince, we use $\Omega_1 = \int_0^1 |\phi(\eta)|d\eta,\ \Omega_2 = \int_0^1 |\psi(\eta)|d\eta$.

Theorem 3.1. *Under the hypothesis* (H_1)*, the proposed system* (1.1) *has at most one solution if there exists a* $\mathscr{M}\mathrm{R}^2$ *with nonnegative entries given by*

$$\mathscr{M} = \begin{pmatrix} \mathrm{m}_{11} & \mathrm{m}_{12} \\ \mathrm{m}_{21} & \mathrm{m}_{22} \end{pmatrix},$$

such that $\varrho(\mathscr{M}) < 1$.

Proof. Consider a metric ρ on $\mathscr{X}$ as

$$\rho((\mathrm{u},\mathrm{v}),(\bar{\mathrm{u}},\bar{\mathrm{v}})) = \begin{pmatrix} \|\mathrm{u}-\bar{\mathrm{u}}\| \\ \|\mathrm{v}-\bar{\mathrm{v}}\| \end{pmatrix},$$

clearly, ρ is a vector-valued metric on $\mathscr{X}$. Further, let define the operator $\mathscr{Q} : \mathscr{X} \to \mathscr{X}$ as

$$\mathscr{Q}(\mathrm{u},\mathrm{v}) = (\mathscr{Q}_1, \mathscr{Q}_2)(\mathrm{u},\mathrm{v})$$

by

$$\begin{cases} \mathscr{Q}_1(\mathrm{u},\mathrm{v}) = \xi \int_0^1 \mathrm{v}(\eta)\phi(\eta)d\eta - \frac{(2-\delta_1)}{M(\delta_1-1)}\Big[\int_0^\xi \mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta + \xi\int_0^1 \mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta\Big] \\ -\frac{(\delta_1-1)}{M(\delta_1-1)}\Big[\int_0^\xi (\xi-\eta)\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta + \xi\int_0^1 (1-\eta)\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta\Big], \\ \mathscr{Q}_2(\mathrm{u},\mathrm{v}) = \xi \int_0^1 \mathrm{u}(\eta)\psi(\eta)d\eta - \frac{(2-\delta_2)}{M(\delta_2-1)}\Big[\int_0^\xi \mathrm{f}_2(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta + \xi\int_0^1 \mathrm{f}_2(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta\Big] \\ -\frac{(\delta_2-1)}{M(\delta_2-1)}\Big[\int_0^\xi (\xi-\eta)\mathrm{f}_2(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta + \xi\int_0^1 (1-\eta)\mathrm{f}_2(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))d\eta\Big]. \end{cases} \tag{3.7}$$

Then one has from (3.7) for $(\mathrm{u},\mathrm{v}),(\bar{\mathrm{u}},\bar{\mathrm{v}}) \in \mathscr{X}$

$$\begin{aligned} \|\mathscr{Q}_1(\mathrm{u},\mathrm{v}) - \rho_1(\bar{\mathrm{u}},\bar{\mathrm{v}})\| \leq\ & \max_{\xi\in \mathrm{J}}\Big[|\xi|\int_0^1 |\phi(\eta)||(\mathrm{v}(\eta)-\bar{\mathrm{v}}(\eta))|d\eta\Big| \\ &+ \frac{(2-\delta_1)}{M(\delta_1-1)}\Big(\int_0^\xi |\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta)) - \mathrm{f}_1(\eta,\bar{\mathrm{u}}(\eta),\bar{\mathrm{v}}(\eta))|d\eta \\ &+ |\xi|\int_0^1 |\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta)) - \mathrm{f}_1(\eta,\bar{\mathrm{u}}(\eta),\bar{\mathrm{v}}(\eta))|d\eta\Big) \\ &+ \frac{(\delta_1-1)}{M(\delta_1-1)}\Big(\int_0^\xi |\xi-\eta|\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta)) - \mathrm{f}_1(\eta,\bar{\mathrm{u}}(\eta),\bar{\mathrm{v}}(\eta))|d\eta \\ &+ |\xi|\int_0^1 |1-\xi|\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta)) - \mathrm{f}_1(\eta,\bar{\mathrm{u}}(\eta),\bar{\mathrm{v}}(\eta))|d\eta\Big)\Big] \\ \leq\ & \Omega_1\|\mathrm{v}-\bar{\mathrm{v}}\| + \frac{2(2-\delta_1)\mathrm{L}_1}{M(\delta_1-1)}\Big[\|\mathrm{u}-\bar{\mathrm{u}}\| + \|\mathrm{v}-\bar{\mathrm{v}}\|\Big] \\ &+ \frac{2(\delta_1-1)\mathrm{L}_1}{M(\delta_1-1)}\Big[\|\mathrm{u}-\bar{\mathrm{u}}\| + \|\mathrm{v}-\bar{\mathrm{v}}\|\Big], \end{aligned} \tag{3.8}$$

on rearrangement, from (3.8), one has

$$\begin{aligned}\|\mathscr{Q}_1(\mathrm{u},\mathrm{v})-\rho_1(\bar{\mathrm{u}},\bar{\mathrm{v}})\| &\le 2\mathrm{L}_1\Big(\frac{(2-\delta_1)\mathrm{L}_1}{M(\delta_1-1)}+\frac{2(\delta_1-1)}{M(\delta_1-1)}\Big)\|\mathrm{u}-\bar{\mathrm{u}}\| \\ &+ \Big[2\mathrm{L}_1\Big(\frac{(2-\delta_1)}{M(\delta_1-1)}+\frac{2(\delta_1-1)}{M(\delta_1-1)}\Big)+\Omega_1\Big]\|\mathrm{v}-\bar{\mathrm{v}}\| \end{aligned} \tag{3.9}$$

and in same fashion, one has

$$\begin{aligned}\|\mathscr{Q}_2(\mathrm{u},\mathrm{v})-\mathscr{Q}_1(\bar{\mathrm{u}},\bar{\mathrm{v}})\| &\le \Big[2\mathrm{L}_2\Big(\frac{(2-\delta_1)}{M(\delta_1-1)}+\frac{2(\delta_1-1)}{M(\delta_1-1)}\Big)+\Omega_2\Big]\|\mathrm{u}-\bar{\mathrm{u}}\| \\ &+ 2\mathrm{L}_2\Big(\frac{(2-\delta_1)}{M(\delta_1-1)}+\frac{2(\delta_1-1)}{M(\delta_1-1)}\Big)\|\mathrm{v}-\bar{\mathrm{v}}\|. \end{aligned} \tag{3.10}$$

Let use

$$\begin{aligned}\mathrm{m}_{11} &= 2\mathrm{L}_1\Big(\frac{(2-\delta_1)\mathrm{L}_1}{M(\delta_1-1)}+\frac{2(\delta_1-1)}{M(\delta_1-1)}\Big),\\ \mathrm{m}_{12} &= 2\mathrm{L}_1\Big(\frac{(2-\delta_1)}{M(\delta_1-1)}+\frac{2(\delta_1-1)}{M(\delta_1-1)}\Big)+\Omega_1,\\ \mathrm{m}_{21} &= 2\mathrm{L}_2\Big(\frac{(2-\delta_1)}{M(\delta_1-1)}+\frac{2(\delta_1-1)}{M(\delta_1-1)}\Big)+\Omega_2,\\ \mathrm{m}_{22} &= 2\mathrm{L}_2\Big(\frac{(2-\delta_1)}{M(\delta_1-1)}+\frac{2(\delta_1-1)}{M(\delta_1-1)}\Big). \end{aligned} \tag{3.11}$$

Therefore in view of (3.11), from (3.9), (3.10), one has

$$\begin{pmatrix} \|\mathscr{Q}_1(\mathrm{u},\mathrm{v})-\mathscr{Q}_1(\bar{\mathrm{u}},\bar{\mathrm{v}})\| \\ \|\mathscr{Q}_2(\mathrm{u},\mathrm{v})-\mathscr{Q}_2(\bar{\mathrm{u}},\bar{\mathrm{v}})\| \end{pmatrix} \le \begin{pmatrix} \mathrm{m}_{11} & \mathrm{m}_{12} \\ \mathrm{m}_{21} & \mathrm{m}_{22} \end{pmatrix} \begin{pmatrix} \|\mathrm{u}-\bar{\mathrm{u}}\| \\ \|\mathrm{v}-\bar{\mathrm{v}}\| \end{pmatrix}. \tag{3.12}$$

Hence one has from (3.12)

$$\rho\left(\mathscr{Q}(\mathrm{u},\mathrm{v}),\mathscr{Q}(\bar{\mathrm{u}},\bar{\mathrm{v}})\right) \le \mathscr{M}\rho\left((\mathrm{u},\mathrm{v}),(\bar{\mathrm{u}},\bar{\mathrm{v}})\right). \tag{3.13}$$

Since in (3.13), $\varrho(\mathscr{M}) < 1$, therefore, all conditions of Theorem 2.4 are received. So the considered problem (1.1) has at most one solution. □

Theorem 3.2. *Under the hypothesis* $(\mathrm{H}_1, \mathrm{H}_2)$*, if the operator* $\mathscr{Q} = (\mathscr{Q}_1, \mathscr{Q}_2)$ *is compact and completely continuous, then the considered problem* (1.1) *has at least one solution.*

Proof. Since the functions $\mathrm{f}_i (i = 1, 2)$ are continuous, therefore the operators $\mathscr{Q}_1, \mathscr{Q}_2$ are also continuous. Hence the operator $\mathscr{Q}$ is also continuous. Assume $(\mathrm{u}_n, \mathrm{v}_n) \in \mathscr{X}$, with $(\mathrm{u}_n, \mathrm{v}_n) \to (\mathrm{u}, \mathrm{v})$, as $n \to \infty$. Hence, for every $\xi \in \mathscr{J}$, one has

$$\begin{aligned}|\mathscr{Q}_1(\mathrm{u}_n,\mathrm{v}_n)-\mathscr{Q}_1(\mathrm{u},\mathrm{v})| &= \Big|\xi\int_0^1(\mathrm{v}_n(\eta)-\mathrm{v}(\eta))\phi(\eta)d\eta \\ &+ \frac{(2-\delta_1)}{M(\delta_1-1)}\Big[\int_0^\xi[\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))-\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))]d\eta\end{aligned}$$

$$
\begin{aligned}
&+\ \xi\int_0^1[\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))-\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))]d\eta\Big]\\
&+\ \frac{(\delta_1-1)}{M(\delta_1-1)}\Big[\int_0^\xi(\xi-\eta)[\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))-\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))]d\eta\\
&+\ \xi\int_0^1(1-\eta)[\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))-\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))]d\eta\Big]\Big|.
\end{aligned}
$$

Which further implies that

$$
\begin{aligned}
&|\mathscr{Q}_1(\mathrm{u}_n,\mathrm{v}_n)-\mathscr{Q}_1(\mathrm{u},\mathrm{v})|\\
\leq\ &|\xi|\int_0^1|\mathrm{v}_n(\eta)-\mathrm{v}(\eta)|\phi(\eta)d\eta+\frac{(2-\delta_1)}{M(\delta_1-1)}\Big[\int_0^\xi|\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))-\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))|d\eta\\
+\ &|\xi|\int_0^1|\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))-\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))|d\eta\Big]\\
+\ &\frac{(\delta_1-1)}{M(\delta_1-1)}\Big[\int_0^\xi(\xi-\eta)|\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))-\mathrm{f}_1(\eta,\mathrm{u}(\eta),\mathrm{v}(\eta))|d\eta\\
+\ &|\xi|\int_0^1(1-\eta)|\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))-\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))|d\eta\Big]\\
\leq\ &|\xi|\int_0^1|\mathrm{v}_n(\eta)-\mathrm{v}(\eta)|\phi(\eta)d\eta+\frac{(2-\delta_1)\mathrm{L}_1}{M(\delta_1-1)}\Big[\int_0^\xi[|\mathrm{u}_n(\eta)-\mathrm{u}(\eta)|+|\mathrm{v}_n(\eta)-\mathrm{v}(\eta)|]d\eta\\
+\ &|\xi|\int_0^1[|\mathrm{u}_n(\eta)-\mathrm{u}(\eta)|+|\mathrm{v}_n(\eta)-\mathrm{v}(\eta)|]d\eta\Big]\\
+\ &\frac{\mathrm{L}_1(\delta_1-1)}{M(\delta_1-1)}\Big[\int_0^\xi|(\xi-\eta)|[|\mathrm{u}_n(\eta)-\mathrm{u}(\eta)|+|\mathrm{v}_n(\eta)-\mathrm{v}(\eta)|]d\eta\\
+\ &|\xi|\int_0^1|(1-\eta)|[|\mathrm{u}_n(\eta)-\mathrm{u}(\eta)|+|\mathrm{v}_n(\eta)-\mathrm{v}(\eta)|]d\eta. \qquad (3.14)
\end{aligned}
$$

Since $|\mathrm{u}_n-\mathrm{u}|\to 0$, and $|\mathrm{v}_n-\mathrm{v}|\to 0$, as $n\to\infty$. Therefore, by using Lebesgue dominated convergence theorem [40], right side of (3.14) goes to zero as $n\to\infty$, hence, one has

$$|\mathscr{Q}_1(\mathrm{u}_n,\mathrm{v}_n)-\mathscr{Q}_1(\mathrm{u},\mathrm{v})|\to 0,\ \text{with } n\to\infty. \qquad (3.15)$$

Also with the same process one has

$$|\mathscr{Q}_2(\mathrm{u}_n,\mathrm{v}_n)-\mathscr{Q}_1(\mathrm{u},\mathrm{v})|\to 0,\ \text{with } n\to\infty. \qquad (3.16)$$

Therefore, inview of (3.15) and (3.16), one has

$$|\mathscr{Q}(\mathrm{u}_n,\mathrm{v}_n)-\mathscr{Q}(\mathrm{u},\mathrm{v})|\to 0,\ \text{with } n\to\infty. \qquad (3.17)$$

Thus $\mathscr{Q}$ is continuous. Now to show that $\mathscr{Q}$ is bounded, let there exists a bounded set

$$\mathbf{S}=\{(\mathrm{u},\mathrm{v})\in\mathscr{X}:\ \|(\mathrm{u},\mathrm{v})\|\leq \mathrm{r}\}.$$

Now using (H_2), and consider $(u, v) \in \mathbf{S}$, we have

$$
\begin{aligned}
& \|\mathscr{Q}_1(u,v)\| \\
\leq\ & \max_{\xi\in\mathscr{J}}\Big[|\xi|\int_0^1 |v(\eta)||\phi(\eta)|d\eta + \frac{(2-\delta_1)}{M(\delta_1-1)}\Big[\int_0^\xi |f_1(\eta,u(\eta),v(\eta))|d\eta + |\xi|\int_0^1 |f_1(\eta,u(\eta),v(\eta))|d\eta\Big] \\
+\ & \frac{(\delta_1-1)}{M(\delta_1-1)}\Big[\int_0^\xi |(\xi-\eta)||f_1(\eta,u(\eta),v(\eta))|d\eta + |\xi|\int_0^1 |(1-\eta)||f_1(\eta,u(\eta),v(\eta))|d\eta\Big]\Big] \\
\leq\ & r\int_0^1 |\phi(\eta)|d\eta + \frac{2(2-\delta_1)}{M(\delta_1-1)}\int_0^1 [a_1 + b_1|u| + c_1|u|]d\eta \\
+\ & \frac{2(\delta_1-1)}{M(\delta_1-1)}\int_0^1 (1-\eta)[a_1 + b_1|u| + c_1|u|]d\eta \\
\leq\ & \Omega_1 r + \frac{2(2-\delta_1)}{M(\delta_1-1)}[a_1 + (b_1+c_1)r] + \frac{(\delta_1-1)}{M(\delta_1-1)}[a_1 + (b_1+c_1)r] \\
\leq\ & \Big(\Omega_1 + \frac{2(b_1+c_1)}{M(\delta_1-1)}\Big)r + \frac{2a_1}{M(\delta_1-1)},
\end{aligned} \tag{3.18}
$$

where we have used $1 < \delta_1 \leq 2$, we have $1 \leq 3-\delta_1 < 2$. Hence one has from (3.18), that

$$
\|\mathscr{Q}_1(u,v)\| \leq \frac{r}{2}, \text{ where, } r \geq \frac{4a_1}{M(\delta_1-1) - 2(\Omega_1 M(\delta_1-1) + 2[b_1+c_1])}. \tag{3.19}
$$

Similarly repeating the same process, one has

$$
\|\mathscr{Q}_2(u,v)\| \leq \frac{r}{2}, \text{ where, } r \geq \frac{4a_2}{M(\delta_2-1) - 2(\Omega_2 M(\delta_2-1) + 2[b_2+c_2])}. \tag{3.20}
$$

Thus, from (3.19), (3.20), one has

$$
\|\mathscr{Q}_1(u,v)\| \leq r. \tag{3.21}
$$

Thus (3.21) yields that $\mathscr{Q}$ is bounded. Now the said operator is bounded and continuous. Hence it is uniformly continuous. Now to derive equi-continuity, we take $\xi_1 < \xi_2 \in \mathscr{J}$, and consider

$$
\begin{aligned}
& |\mathscr{Q}_1(u,v)(\xi_2) - \mathscr{Q}_1(u,v)(\xi_1)| \\
\leq\ & (\xi_2-\xi_1)\int_0^1 |v(\eta)||\phi(\eta)|d\eta \\
+\ & \frac{(2-\delta_1)}{M(\delta_1-1)}\Big[\int_{\xi_1}^{\xi_2} |f_1(\eta,u(\eta),v(\eta))|d\eta + (\xi_2-\xi_1)\int_0^1 |f_1(\eta,u(\eta),v(\eta))|d\eta\Big] \\
+\ & \frac{(\delta_1-1)}{M(\delta_1-1)}\Big[\int_0^{\xi_1} [(\xi_1-\eta)-(\xi_2-\eta)]|f_1(\eta,u(\eta),v(\eta))|d\eta \\
+\ & \int_{\xi_1}^{\xi_2} (\xi_2-\eta)|f_1(\eta,u(\eta),v(\eta))|d\eta \\
+\ & |\xi_2-\xi_1|\int_0^1 |(1-\eta)||f_1(\eta,u(\eta),v(\eta))|d\eta\Big]
\end{aligned}
$$

$$\leq \ (\xi_2-\xi_1)\Omega_1 \mathrm{r} + \frac{(2-\delta_1)}{M(\delta_1-1)}\Big[(\mathrm{a}_1+(\mathrm{b}_1+\mathrm{c}_1)\mathrm{r})(\xi_2-\xi_1)+(\xi_2-\xi_1)(\mathrm{a}_1+(\mathrm{b}_1+\mathrm{c}_1)\mathrm{r})\Big]$$
$$+ \ \frac{(\delta_1-1)}{M(\delta_1-1)}\Big[\xi_1^2-\xi_2^2-(\xi_2-\xi_1)^2+(\xi_2-\xi_1)^2\Big](\mathrm{a}_1+(\mathrm{b}_1+\mathrm{c}_1)\mathrm{r}). \tag{3.22}$$

We see in (3.22) that as $\xi_2 \to \xi_1$, the right side goes to zero. Hence, one has

$$|\mathscr{Q}_1(\mathrm{u},\mathrm{v})(\xi_2) - \mathscr{Q}_1(\mathrm{u},\mathrm{v})(\xi_1)| \to 0, \text{ as } \xi_2 \to \xi_1.$$

Since $\mathscr{Q}_1$ is continuous bounded so also uniformly continuous, that is

$$\|\mathscr{Q}_1(\mathrm{u},\mathrm{v})(\xi_2) - \mathscr{Q}_1(\mathrm{u},\mathrm{v})(\xi_1)\| \to 0, \text{ as } \xi_2 \to \xi_1.$$

Also, one has

$$\|\mathscr{Q}_2(\mathrm{u},\mathrm{v})(\xi_2) - \mathscr{Q}_2(\mathrm{u},\mathrm{v})(\xi_1)\| \to 0, \text{ as } \xi_2 \to \xi_1.$$

Therefore, we have

$$\|\mathscr{Q}(\mathrm{u},\mathrm{v})(\xi_2) - \mathscr{Q}(\mathrm{u},\mathrm{v})(\xi_1)\| \to 0, \text{ as } \xi_2 \to \xi_1.$$

Thus $\mathscr{Q}$ fulfills all conditions of Arzelá-Ascoli theorem, thus $\mathscr{Q}$ is compact. Hence, we conclude that problem (1.1) has at least one solution. □

4. Monotone iterative procedure

This part is related to develop the monotone iterative procedure for the problem (1.1).

Definition 4.1. $(\mathrm{u}_0,\mathrm{v}_0) \in \mathscr{X}$ *is the lower solution of* (1.1)*, such that*

$$\begin{cases} -{}_0^{CF}\mathbf{D}_\xi^{\delta_1}\mathrm{u}_0(\xi) \leq \mathrm{f}_1(\xi,\mathrm{u}_0(\xi),\mathrm{v}_0(\xi)), \ -{}_0^{CF}\mathbf{D}_\xi^{\delta_1}\mathrm{v}_0(\xi) \leq \mathrm{f}_1(\xi,\mathrm{u}_0(\xi),\mathrm{v}_0(\xi)),\\ \mathrm{u}_0(0) \leq 0, \ \mathrm{v}_0(0) \leq 0,\\ \mathrm{u}_0(1) \leq \displaystyle\int_0^1 \mathrm{v}_0(\eta)\phi(\eta)d\eta, \ \mathrm{v}_0(1) \leq \int_0^1 \mathrm{u}_0(\xi)\psi(\xi)d\xi. \end{cases}$$

In the same way $(\mathrm{u}_0^*,\mathrm{v}_0^*) \in \mathscr{X}$ *is the upper solutions of* (1.1)*, if*

$$\begin{cases} -{}_0^{CF}\mathbf{D}_\xi^{\delta_1}\mathrm{u}_0^*(\xi) \geq \mathrm{f}_1(\xi,\mathrm{u}_0^*(\xi),\mathrm{v}_0^*(\xi)), \ -{}_0^{CF}\mathbf{D}_\xi^{\delta_1}\mathrm{v}_0^*(\xi) \geq \mathrm{f}_1(\xi,\mathrm{u}_0^*(\xi),\mathrm{v}_0^*(\xi)),\\ \mathrm{u}_0^*(0) \leq 0, \ \mathrm{v}_0^*(0) \leq 0,\\ \mathrm{u}_0^*(1) \leq \displaystyle\int_0^1 \mathrm{v}_0^*(\eta)\phi(\eta)d\eta, \ \mathrm{v}_0^*(1) \leq \int_0^1 \mathrm{u}_0^*(\xi)\psi(\xi)d\xi. \end{cases}$$

Let

$$\mathrm{u}_0 \leq \mathrm{u}_0^*, \ \mathrm{v}_0 \leq \mathrm{v}_0^*, \ \xi \in \mathscr{J}. \tag{4.1}$$

Here we define sector as

$$\mathbf{G} = \{(\mathbf{u},\mathbf{v}) \in \mathscr{X} : (\mathrm{u}_0,\mathrm{v}_0) \leq (\mathbf{u},\mathbf{v}) \leq (\mathrm{u}_0^*,\mathrm{v}_0^*), \ \xi \in \mathscr{J}\}. \tag{4.2}$$

Recalling, the following comparison theorem:

Lemma 4.1. *[41] If $\sigma \in C[0,1]$, $1 < \delta_i \leq 2$ and there exist $\varphi \in C([0,1],\mathrm{R})$, such that*

$$\begin{cases} -{}_0^{CF}\mathbf{D}_\xi^{\delta_1}\mathrm{u}(\xi) \leq -\varphi(\xi)\mathrm{u}(\xi),\ \xi \in \mathscr{J}, \\ \mathrm{u}(0) \leq 0,\ \mathrm{u}(1) \leq 0, \end{cases}$$

then $\mathrm{u}(\xi) \leq 0$, *for all* $\xi \in [0,1]$.

Inview of Lemma 4.1, we state the following theorem following [42].

Theorem 4.2. *If $\varphi_1, \varphi_2 \in C([0,1],\mathrm{R})$. Assume that $(\mathbf{u},\mathbf{v}) \in \mathscr{X}$, such that*

$$\begin{cases} -{}_0^{CF}\mathbf{D}_\xi^{\delta_1}\mathrm{u}(\xi) \leq -\varphi_1(\xi)\mathrm{u}(\xi),\ -{}_0^{CF}\mathbf{D}_\xi^{\delta_2}\mathrm{v}(\xi) \leq -\varphi_2(\xi)\mathrm{v}(\xi), \\ \mathrm{u}(0) \leq 0, \mathrm{v}(0) \leq 0, \\ \mathrm{u}(1) \leq \displaystyle\int_0^1 \mathrm{v}(\xi)\phi(\xi)d\xi,\ \mathrm{v}(1) \leq \displaystyle\int_0^1 \mathrm{u}(\xi)\psi(\xi)d\xi, \end{cases} \tag{4.3}$$

then $\mathrm{u}(\xi) \leq 0$, $\mathrm{v}(\xi) \leq 0$, *for every* $\xi \in \mathscr{J}$.

We need the following assumptions:

(H_3) If $\mathrm{f}_1(\xi,\mathrm{u},\mathrm{v})$ is nondecreasing in v and there exist $\varphi_1 \in C(\mathscr{J},\mathrm{R})$, such that

$$\mathrm{f}_1(\xi,\mathrm{u}_1,\mathrm{v}) - \mathrm{f}_1(\xi,\mathrm{u}_1,\mathrm{v}) \geq -\varphi_1(\xi)(\mathrm{u}_1-\mathrm{u}_2),\ \text{where } \mathrm{u}_0 \leq \mathrm{u}_2 \leq \mathrm{u}_1 \leq \mathbf{u}_0, \mathrm{v}_0 \leq \mathrm{v} \leq \mathbf{v}_0.$$

(H_4) If $\mathrm{f}_2(\xi,\mathrm{u},\mathrm{v})$ is nondecreasing in u, such that $\varphi_2 \in C(\mathscr{J},\mathrm{R})$, that is

$$\mathrm{f}_2(\xi,\mathrm{u}_1,\mathrm{v}) - \mathrm{f}_2(\xi,\mathrm{u}_1,\mathrm{v}) \geq -\varphi_2(\xi)(\mathrm{v}_1-\mathrm{v}_2),\ \text{where } \mathrm{v}_0 \leq \mathrm{v}_2 \leq \mathrm{v}_1 \leq \mathbf{v}_0, \mathrm{u}_0 \leq \mathrm{u} \leq \mathbf{u}_0.$$

Theorem 4.3. *Reference to the hypothesis* $(\mathrm{H}_1,\mathrm{H}_2,\mathrm{H}_3)$ *together with the initial approximation* $(\mathrm{u}_0,\mathrm{v}_0)$ *and* $(\mathrm{u}_0^*,\mathrm{v}_0^*)$ *are lower and upper solutions for* (1.1) *in the sector* $\mathbf{G}$*, then one has two sequences of monotonic type as* $\{(\mathrm{u}_n,\mathrm{v}_n)\}$ *and* $\{(\mathrm{u}_n^*,\ \mathrm{v}_n^*)\}$*, such that* $(\mathrm{u}_n,\mathrm{v}_n) \to (\mathrm{u}_0,\mathrm{v}_0)$, $(\mathrm{u}_n^*,\mathrm{v}_n^*) \to (\mathrm{u}_0^*,\mathrm{v}_0^*)$ *as* $n \to \infty$.

Proof. Consider a sequence describe by

$$\begin{cases} -{}_0^{CF}\mathbf{D}_\xi^{\delta_1}\mathrm{u}_{n+1}(\xi) = \mathrm{f}_1(\xi,\mathrm{w}_n(\xi),\mathrm{z}_n(\xi)) - \varphi_1(\xi)[\mathrm{u}_{n+1}(\xi)-\mathrm{v}_n(\xi)];\ \xi \in \mathscr{J}, \\ -{}_0^{CF}\mathbf{D}_\xi^{\delta_2}\mathrm{u}_{n+1}(\xi) = \mathrm{f}_2(\xi,\mathrm{w}_n(\xi),\mathrm{z}_n(\xi)) - \varphi_1(\xi)[\mathrm{v}_{n+1}(\xi)-\mathrm{v}_n(\xi)];\ \xi \in \mathscr{J}, \\ \mathrm{u}_{n+1}(0) = 0,\ \mathrm{v}_{n+1}(0) = 0, \\ \mathrm{u}_{n+1}(1) = \displaystyle\int_0^1 \phi(\eta)\mathrm{v}_{n+1}(\eta)d\eta,\ \mathrm{v}_{n+1}(1) = \displaystyle\int_0^1 \psi(\eta)\mathrm{u}_{n+1}(\eta)d\eta. \end{cases} \tag{4.4}$$

As from Theorem 3.2, we know that system (4.4) has a unique solutions $(\mathrm{u}_{n+1},\mathrm{v}_{n+1})$. On induction let $n = 0$ in (4.4), one has

$$\begin{cases} -{}_0^{CF}\mathbf{D}_\xi^{\delta_1}\mathrm{u}_1(\xi) = \mathrm{f}_1(\xi,\mathrm{w}_0(\xi),\mathrm{z}_0(\xi)) - \varphi_1(\xi)[\mathrm{u}_1(\xi)-\mathrm{u}_0(\xi)];\ \xi \in \mathscr{J}, \\ -{}_0^{CF}\mathbf{D}_\xi^{\delta_2}\mathrm{u}_{n+1}(\xi) = \mathrm{f}_2(\xi,\mathrm{u}_0(\xi),\mathrm{z}_0(\xi)) - \varphi_1(\xi)[\mathrm{v}_1(\xi)-\mathrm{v}_0(\xi)];\ \xi \in \mathscr{J}, \\ \mathrm{u}_1(0) = 0,\ \mathrm{v}_1(0) = 0, \\ \mathrm{u}_1(1) = \displaystyle\int_0^1 \phi(\eta)\mathrm{v}_1(\eta)d\eta,\ \mathrm{v}_1(1) = \displaystyle\int_0^1 \psi(\eta)\mathrm{u}_1(\eta)d\eta. \end{cases} \tag{4.5}$$

For proving $(\mathrm{u}_0,\mathrm{v}_0)\leq(\mathrm{u}_1,\mathrm{v}_1)\leq(\mathrm{u}_0^*,\mathrm{v}_0^*)$ setting $p(\xi)=\mathrm{u}_0(\xi)-\mathrm{u}_1(\xi),\ q(\xi)=\mathrm{v}_0(\xi)-\mathrm{v}_1(\xi)$ in (4.5), one can get

$$\begin{cases}-{}_0^{CF}\mathbf{D}_\xi^{\delta_1}p(\xi)=-{}_0^{CF}\mathbf{D}_\xi^{\delta_1}[\mathrm{u}_0(\xi)-\mathrm{u}_1(\xi)]\leq-\varphi_1(\xi)p(\xi);\ \xi\in\mathscr{J},\\ -{}_0^{CF}\mathbf{D}_\xi^{\delta_2}q(\xi)=-{}_0^{CF}\mathbf{D}_\xi^{\delta_2}[\mathrm{v}_0(\xi)-\mathrm{v}_1(\xi)]\leq-\varphi_2(\xi)q(\xi);\ \xi\in\mathscr{J},\\ p(0)\leq0,\ q(0)\leq0,\ p(1)\leq\int_0^1q(\eta)\phi(\eta)d\eta,\ q(1)\leq\int_0^1\psi(\eta)p(\eta)d\eta.\end{cases}\tag{4.6}$$

Inview of Theorem 4.2,

$$-{}_0^{CF}\mathbf{D}_\xi^{\delta_1}p(\xi)\leq-\varphi_1(\xi)p(\xi),\ -{}_0^{CF}\mathbf{D}_\xi^{\delta_2}q(\xi)\leq-\varphi_2(\xi)q(\xi)$$

which yields that $p(\xi)\leq0,\ q(\xi)\leq0$. Therefore, $\mathrm{u}_0(\xi)\leq\mathrm{u}_1(\xi),\ \mathrm{v}_0(\xi)\leq\mathrm{v}_1(\xi)$. So $(\mathrm{u}_0,\mathrm{v}_0)\leq(\mathrm{u}_1,\mathrm{v}_1),\ \xi\in\mathscr{J}$. On the same manner, $p(\xi)=\mathrm{u}_1(\xi)-\mathrm{u}_1^*(\xi),\ q(\xi)=\mathrm{v}_1(\xi)-\mathrm{v}_1^*(\xi)$ in (4.5), one can get $(\mathrm{u}_1,\mathrm{v}_1)\leq(\mathrm{u}_1^*,\mathrm{v}_1^*),\ \xi\in\mathscr{J}$. Moreover, to prove that $(\mathrm{u}_1^*,\mathrm{v}_1^*)\leq(\mathrm{u}_0^*,\mathrm{v}_0^*)$. Let $p(\xi)=\mathrm{u}_1^*(\xi)-\mathrm{u}_0^*(\xi),\ q(\xi)=\mathrm{v}_1^*(\xi)-\mathrm{v}_0^*(\xi)$, obviously one has $\mathrm{u}_1^*(\xi)\leq\mathrm{u}_0^*(\xi),\ \mathrm{v}_1^*(\xi)\leq\mathrm{v}_0^*(\xi)$, which yields $(\mathrm{u}_1^*,\mathrm{v}_1^*)\leq(\mathrm{u}_0^*,\mathrm{v}_0^*)$. Therefore,

$$(\mathrm{u}_0,\mathrm{v}_0)\leq(\mathrm{u}_1,\mathrm{v}_1)\leq(\mathrm{u}_1^*,\mathrm{v}_1^*)\leq(\mathrm{u}_0^*,\mathrm{v}_0^*).$$

Also, inview of induction, for $k>1$ and set $p(\xi)=\mathrm{u}_k-\mathrm{u}_{k+1},\ q(\xi)=\mathrm{v}_k-\mathrm{v}_{k+1}$ in the system (4.4), one has

$$\begin{cases}-{}_0^{CF}\mathbf{D}_\xi^{\delta_1}p(\xi)=-{}_0^{CF}\mathbf{D}_\xi^{\delta_1}[\mathrm{u}_k(\xi)-\mathrm{u}_{k+1}(\xi)]\leq-\varphi_1(\xi)\phi(\xi),\\ -{}_0^{CF}\mathbf{D}_\xi^{\delta_2}q(t)=-{}_0^{CF}\mathbf{D}_\xi^{\delta_1}[\mathrm{v}_k(\xi)-\mathrm{v}_{k+1}(\xi)]\leq-\varphi_2(\xi)\psi(\xi),\ \xi\in\mathscr{J},\\ p(0)\leq0,\ q(0)\leq0,\\ p(1)\leq\int_0^1q(\eta)\phi(\eta)d\eta,\ q(1)\leq\int_0^1\psi(\eta)p(\eta)d\eta.\end{cases}\tag{4.7}$$

Using Theorem 4.2, one has $p(\xi)\leq0,\ q(\xi)\leq0$. Hence, $\mathrm{u}_k\leq\mathrm{u}_{k+1},\ \mathrm{v}_k\leq\mathrm{v}_{k+1}$ which yields that $(\mathrm{u}_k,\mathrm{v}_k)\leq(\mathrm{u}_{k+1},\mathrm{v}_{k+1})$, on the same fashion one can prove that $(\mathrm{u}_{k+1},\mathrm{v}_{k+1})\leq(\mathrm{u}_{k+1}^*,\mathrm{v}_{k+1}^*)$. The solution of (1.1) is given as system of integral equations as

$$\begin{cases}\mathrm{u}_{n+1}(\xi)=\xi\int_0^1\mathrm{v}_n(\eta)\phi(\eta)d\eta-\frac{(2-\delta_1)}{M(\delta_1-1)}\Big[\int_0^\xi\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))d\eta+\xi\int_0^1\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))d\eta\Big]\\ -\frac{(\delta_1-1)}{M(\delta_1-1)}\Big[\int_0^\xi(\xi-\eta)\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))d\eta+\xi\int_0^1(1-\eta)\mathrm{f}_1(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))d\eta\Big],\\ \mathrm{v}_{n+1}(\xi)=\xi\int_0^1\mathrm{u}(\eta)\psi(\eta)d\eta-\frac{(2-\delta_2)}{M(\delta_2-1)}\Big[\int_0^\xi\mathrm{f}_2(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))d\eta+\xi\int_0^1\mathrm{f}_2(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))d\eta\Big]\\ -\frac{(\delta_2-1)}{M(\delta_2-1)}\Big[\int_0^\xi(\xi-\eta)\mathrm{f}_2(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))d\eta+\xi\int_0^1(1-\eta)\mathrm{f}_2(\eta,\mathrm{u}_n(\eta),\mathrm{v}_n(\eta))d\eta\Big].\end{cases}\tag{4.8}$$

As (u,v) is the system of solutions of (4.8). We prove that $(\mathrm{u}_0,\mathrm{v}_0),(\mathrm{u}_0^*,\mathrm{v}_0^*)$ are upper/lower solutions of (1.1). So, let $(\mathbf{u},\mathbf{v})$ be result different from $(\mathrm{u}_0,\mathrm{v}_0),(\mathrm{u}_0^*,\mathrm{v}_0^*)$, then for some positive integer $n=k$, we take $(\mathrm{u}_k,\mathrm{v}_k)\leq(\mathbf{u}_k,\mathbf{v}_k)\leq(\mathrm{u}_k^*,\mathrm{v}_k^*),\ \xi\in\mathscr{J}$. Sitting $p(\xi)=\mathrm{u}_{k+1}-\mathbf{u}_k,\ q(\xi)=\mathrm{v}_{k+1}-\mathbf{v}_k$, then due to Theorem 4.2, one has $p(\xi)\leq0,q(\xi)\leq0$, so $\mathrm{u}_{k+1}\leq\mathbf{u}_k,\ \mathrm{v}_{k+1}\leq\mathbf{v}_k$, on $\mathscr{J}$. Analogously, one can $\mathbf{u}_k\leq\mathrm{u}_{k+1}^*,\ \mathbf{v}_k\leq\mathrm{v}_{k+1}^*$, for all positive integer. Thus, $(\mathrm{u}_k,\mathrm{v}_k)\leq(\mathbf{u}_k,\mathbf{v}_k)\leq(\mathrm{u}_k^*,\mathrm{v}_k^*)$, on the application of limit $k\to\infty$ yields that $(\mathrm{u}_0,\mathrm{v}_0)\leq(\mathbf{u}_0,\mathbf{v}_0)\leq(\mathrm{u}_0^*,\mathrm{v}_0^*)$, which implies that extremal solutions lies in the sector $\mathbf{G}$. □

5. Illustrative examples

Here to testify our results, we give some test problems.

Example 1. *Taking the given system under CFD as*

$$\begin{cases} -{}_0^{CF}\mathbf{D}_\xi^{1.5}\mathrm{u}(\xi) = 32\left(\frac{-\xi}{2}\right)^5 \mathrm{u}(\xi) + \mathrm{v}^5(\xi);\ \xi \in [0,1], \\ -{}_0^{CF}\mathbf{D}_\xi^{1.5}\mathrm{v}(\xi) = \mathrm{u}^3(\xi) - \xi^4\mathrm{v}(\xi);\ \xi \in [0,1],, \\ \mathrm{w}(0) = \mathrm{z}(0) = 0, \\ \mathrm{w}(1) = \int_0^1 \eta\mathrm{v}(\eta)d\eta,\ \mathrm{v}(1) = \int_0^1 \eta\mathrm{u}(\eta)d\eta. \end{cases} \tag{5.1}$$

From the above system(5.1)*, we see*

$$\mathrm{f}_1(\xi, \mathrm{w}(\xi), \mathrm{z}(\xi)) \;=\; 32\left(\frac{-\xi}{2}\right)^5 \mathrm{u}(\xi) + \mathrm{v}^5(\xi),\ \mathrm{f}_1(\xi, \mathrm{w}(\xi), \mathrm{z}(\xi)) = \mathrm{u}^3(\xi) - \xi^4\mathrm{v}(\xi).$$

Also, $\phi(\xi) = \psi(\xi) = \xi$ *are non-negative on* $\mathscr{J}$*. Let* $(-1,-1) = (\mathrm{u}_0, \mathrm{v}_0),\ (1,1) = (\mathrm{u}_0^*, \mathrm{v}_0^*)$ *be initial approximation of system of lower and upper solutions respectively. For* $(-1,-1) = (\mathrm{u}_0, \mathrm{v}_0)$*, we have*

$$\begin{cases} {}_0^{CF}\mathbf{D}_\xi^{1.5}\mathrm{u}(\xi) + 32\left(\frac{-\xi}{2}\right)^5 \mathrm{u}_0(\xi) + \mathrm{v}_0^5(\xi) = \xi^5 - 1 \leq 0;\ \xi \in \mathscr{J}, \\ {}_0^{CF}\mathbf{D}_\xi^{1.5}\mathrm{v}(\xi) + \mathrm{u}_0^3(\xi) - \xi^4\mathrm{v}_0(\xi) = \xi^4 - 1 \leq 0;\ \xi \in \mathscr{J}, \\ \mathrm{u}_0(0) = \mathrm{v}_0(0) = 0,\ \mathrm{u}_0(1) = -1 \leq \int_0^1 \eta d\eta,\ \mathrm{v}_0(1) = -1 \leq \int_0^1 \eta d\eta. \end{cases}$$

Similarly by taking $(1,1) = (\mathrm{u}_0^*, \mathrm{v}_0^*)$*, one can get*

$$\begin{cases} {}_0^{CF}\mathbf{D}_\xi^{1.5}\mathrm{u}_0^*(\xi) + 32\left(\frac{-\xi}{2}\right)^5 \mathrm{u}_0^*(\xi) + \mathrm{v}_0^{*5}(\xi) = 1 - \xi^5 \geq 0;\ \xi \in \mathscr{J}, \\ {}_0^{CF}\mathbf{D}_\xi^{1.5\frac{3}{2}}\mathrm{v}_0^*(\xi) + \mathrm{u^*}_0^3(\xi)) - \xi^4\mathrm{v}_0^*(\xi) \geq 1 - \xi^4 \geq 0;\ \xi \in \mathscr{J}, \\ \mathrm{u}_0^*(0) = \mathrm{v}_0^*(0) = 0, \\ \mathrm{u}_0^*(1) = 1 \geq \int_0^1 \eta\mathrm{v}_0^*(\eta)d\eta,\ \mathrm{v}_0^*(1) = 1 \geq \int_0^1 \eta\mathrm{u}_0^*(\eta)d\eta. \end{cases}$$

Thus it relies that $(-1,-1),\ (1,1)$ *are lower and upper solutions of problem* (5.1) *respectively. Also,*

$$\begin{aligned} \mathrm{f}_1(\xi, \mathrm{u}_2(\xi), \mathrm{v}(\xi)) - \mathrm{f}_1(\xi, \mathrm{u}_1(\xi), \mathrm{v}(\xi)) \;&\geq\; -\xi^5\left(\mathrm{u}_2(\xi) - \mathrm{u}_1(\xi)\right), \\ \mathrm{f}_2(\xi, \mathrm{u}(\xi), \mathrm{v}_2(\xi) - \mathrm{f}_2(\xi, \mathrm{u}(\xi), \mathrm{v}_2(\xi)) \;&\geq\; -\xi^4\left(\mathrm{v}_2(\xi) - \mathrm{v}_1(\xi)\right), \end{aligned}$$

where $\mathrm{u}_0(\xi) \leq \mathrm{u}_1(\xi) \leq \mathrm{u}_2(\xi) \leq \mathrm{u}_0^*(\xi),\ \mathrm{v}_0(\xi) \leq \mathrm{v}_1(\xi) \leq \mathrm{v}_2(\xi) \leq \mathrm{v}_0^*(\xi)$*, thus all the hypothesis of Theorem 4.3 are satisfied. Thus system* (5.1) *has an extremal system of solutions. Further, in Figure 1, the behavior of extremal solutions has been displayed using the initial iterations.*

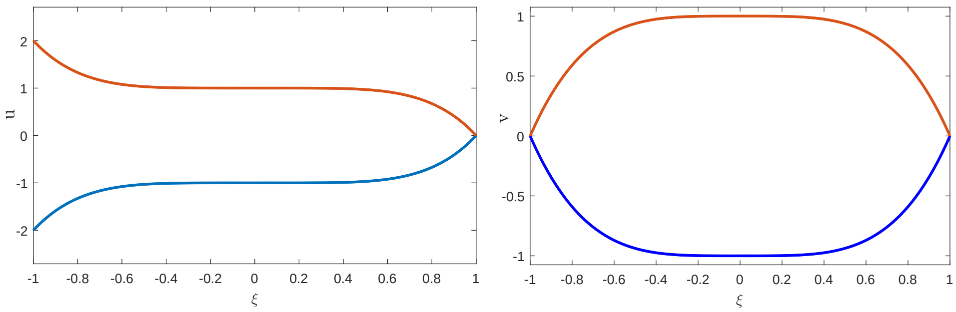

Figure 1. Behavior of Extremal solutions for Example 1.

Moreover, it is easy to verify that $(0,0)$ *is a unique solution of the system* (5.1).

Example 2. *Consider another test problem as*

$$\begin{cases} {}^{CF}_{0}\mathbf{D}^{1.5}_{\xi}\mathrm{u}^{*}_{0}(\xi) = -\sin(\xi)\mathrm{u}(\xi) + \dfrac{1}{4}\mathrm{v}^{3}(\xi);\ \xi \in \mathscr{J}, \\ {}^{CF}_{0}\mathbf{D}^{1.5}_{\xi}\mathrm{v}^{*}_{0}(\xi) = \dfrac{1}{8}\mathrm{u}^{3}(\xi) - \sin(\xi)\mathrm{v}(\xi);\ \xi \in \mathscr{J}, \\ \mathrm{u}(0) = \mathrm{v}(0) = 0,\ \mathrm{u}(1) = \displaystyle\int_0^1 \eta^2 \mathrm{v}(\eta)d\eta,\ \mathrm{v}(1) = \int_0^1 \eta^2 \mathrm{u}(\eta)d\eta. \end{cases} \tag{5.2}$$

From the above system (5.2)*, we see*

$$\mathrm{f}_1(\xi, \mathrm{u}(\xi), \mathrm{v}(\xi)) \;=\; -\sin(\xi)\mathrm{u}(\xi) + \frac{1}{4}\mathrm{v}^{3}(\xi),\ \mathrm{f}_2(\xi, \mathrm{u}(\xi), \mathrm{v}(\xi)) = \frac{1}{8}\mathrm{u}^{5}(\xi) - \sin(\xi)\mathrm{v}(\xi).$$

Note that $\phi(\xi) = \psi(\xi) = \xi^2$*, are non-negative on* $\mathscr{J}$.

Taking $(-2,-2) = (\mathrm{u}_0, \mathrm{v}_0)$ *and* $(2,2) = (\mathrm{u}^{*}_0, \mathrm{v}^{*}_0)$ *be initial approximation of lower and upper solutions. Let* $(-2,-2) = (\mathrm{u}_0, \mathrm{v}_0)$*, then one has*

$$\begin{cases} {}^{CF}_{0}\mathbf{D}^{1.5}_{\xi}\mathrm{u}^{*}_{0}(\xi) + \sin(\xi)\mathrm{u}_0(\xi) + \dfrac{1}{4}\mathrm{v}^{3}_{0}(\xi) = 2\sin(\xi) - 2 \le 0;\ \xi \in \mathscr{J}, \\ {}^{CF}_{0}\mathbf{D}^{1.5}_{\xi}\mathrm{v}^{*}_{0}(\xi) + \dfrac{1}{8}\mathrm{u}^{3}_{0}(\xi) - \sin(\xi)\mathrm{v}_0(\xi)) = 2\sin(\xi) - 1 \le 0;\ \xi \in \mathscr{J}, \\ \mathrm{u}_0(0) = \mathrm{v}_0(0) = 0, \\ \mathrm{u}_0(1) = -2 \le \displaystyle\int_0^1 \eta^2 \mathrm{v}_0(\eta)d\eta,\ \mathrm{v}_0(1) = -2 \le \int_0^1 \eta^2 \mathrm{u}_0(\eta)d\eta. \end{cases}$$

In the same way, from $(2,2) = (\mathrm{u}^{*}_0, \mathrm{v}^{*}_0)$*, we get*

$$\begin{cases} {}^{CF}_{0}\mathbf{D}^{1.5}_{\xi}\mathrm{u}^{*}_{0}(\xi) + \sin(\xi)\mathrm{u}^{*}_{0}(\xi) + \dfrac{1}{4}\mathrm{v}^{*3}_{0}(\xi) = 2 - 2\sin(\xi) \ge 0;\ \xi \in \mathscr{J}, \\ {}^{CF}_{0}\mathbf{D}^{1.5}_{\xi}\mathrm{v}^{*}_{0}(\xi) + \dfrac{1}{8}\mathrm{u}^{*3}{}_{0}(\xi)) - \sin(\xi)\mathrm{v}^{*}_{0}(\xi) = 1 - 2\sin(\xi) \ge 0;\ \xi \in \mathscr{J}, \\ w *_0 (0) = \mathrm{v}^{*}_{0}(0) = 0, \\ \mathrm{u}^{*}_{0}(1) = 2 \ge \displaystyle\int_0^1 \eta^2 \mathrm{v}^{*}_{0}(\eta)d\eta,\ \mathrm{v}^{*}_{0}(1) = 2 \ge \int_0^1 \eta^2 \mathrm{u}^{*}_{0}(\eta)d\eta. \end{cases}$$

Hence, $(-2,-2)$, $(2,2)$ *justify the conditions for lower and upper solutions of* (5.2)*. Moreover,*

$$\begin{aligned} f_1(\xi, u_2(\xi), v(\xi)) - f_1(\xi, u_1(\xi), v(\xi)) &\geq -\sin(\xi)\,(u_2(\xi) - u_1(\xi))\,, \\ f_2(\xi, u(\xi), v_2) - f_2(\xi, u, v_1) &\geq -\sin(\xi)\,(v_2(\xi) - v_1(\xi))\,, \end{aligned}$$

where

$$u_0(\xi) \leq u_1(\xi) \leq u_2(\xi) \leq u_0^*(\xi), \;\; v_0(\xi) \leq v_1(\xi) \leq v_2(\xi) \leq v_0^*(\xi). \tag{5.3}$$

If we take $(u_0, v_0) = (-2,-2), (u_1, v_1) = (-1,-1), (u_2, v_2) = (1,1), (u_0^*, v_0^*) = (2,2)$*, then clearly condition* (5.5) *holds. Thus all the conditions of Theorem 4.3 are satisfied. Thus system* (5.2) *has an extremal system of solutions. Also, it is simple to verify that* $(0,0)$ *is a unique exact solution of* (5.2)*. Further, in Figure 2, the behavior of extremal solutions has been displayed using the initial iteration.*

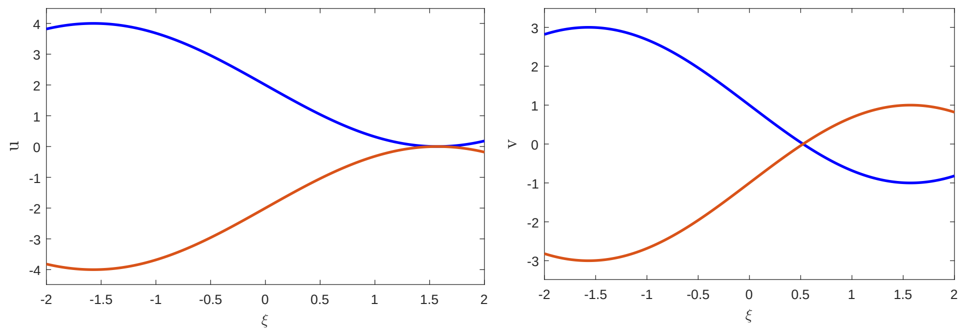

Figure 2. Behavior of Extremal solutions for Example 2.

Example 3. *Consider another test example as*

$$\begin{cases} {}_0^{CF}\mathbf{D}_\xi^{1.7} u_0^*(\xi) = -\xi^3 u(\xi) + \xi v^4(\xi);\; \xi \in \mathscr{J}, \\ {}_0^{CF}\mathbf{D}_\xi^{1.7} v_0^*(\xi) = \xi^2 u^4(\xi) - \xi^5 v(\xi);\; \xi \in \mathscr{J}, \\ u(0) = v(0) = 0, \;\; u(1) = \displaystyle\int_0^1 \sqrt{\eta}\, v(\eta) d\eta, \;\; v(1) = \int_0^1 \sqrt{\eta}\, u(\eta) d\eta. \end{cases} \tag{5.4}$$

From the above system (5.4)*, one has*

$$f_1(\xi, u(\xi), v(\xi)) \;=\; -\xi^3 u(\xi) + \xi v^4(\xi), \;\; f_2(\xi, u(\xi), v(\xi)) = \xi^2 u^4(\xi) - \xi^5 v(\xi).$$

Note that $\phi(\xi) = \psi(\xi) = \sqrt{\xi}$*, are non-negative on* $\mathscr{J}$.

Taking $(-1,-1) = (u_0, v_0)$ *and* $(1,1) = (u_0^*, v_0^*)$ *be initial approximations of lower and upper solutions. Let* $(-1,-1) = (u_0, v_0)$*, then one has*

$$\begin{cases} {}_0^{CF}\mathbf{D}_\xi^{1.7} u_0^*(\xi) + \xi^4 u_0(\xi) - v_0^4(\xi) = \xi^3 - \xi \leq 0;\; \xi \in \mathscr{J}, \\ {}_0^{CF}\mathbf{D}_\xi^{1.5} v_0^*(\xi) - u_0^4(\xi) + \xi^4 v_0(\xi)) = \xi^5 - \xi^2 \leq 0;\; \xi \in \mathscr{J}, \\ u_0(0) = v_0(0) = 0, \\ u_0(1) = -1 \leq \displaystyle\int_0^1 \sqrt{\eta}\, v_0(\eta) d\eta, \;\; v_0(1) = -1 \leq \int_0^1 \sqrt{\eta}\, u_0(\eta) d\eta. \end{cases}$$

In the same way, from $(1,1) = (u_0^*, v_0^*)$, *we get*

$$\begin{cases} {}_0^{CF}\mathbf{D}_\xi^{1.7} u_0^*(\xi) + \xi^4 u_0^*(\xi) + (v_0^*)^4(\xi) = \xi - \xi^3 \geq 0;\ \xi \in \mathscr{J}, \\ {}_0^{CF}\mathbf{D}_\xi^{1.7} v_0^*(\xi) + (u_0^*)^4(\xi)) - \xi^4 v_0^*(\xi) = \xi^2 - \xi^5 \geq 0;\ \xi \in \mathscr{J}, \\ w *_0 (0) = v_0^*(0) = 0, \\ u_0^*(1) = 1 \geq \int_0^1 \sqrt{\eta} v_0^*(\eta) d\eta,\ v_0^*(1) = 1 \geq \int_0^1 \sqrt{\eta} u_0^*(\eta) d\eta. \end{cases}$$

Hence, $(-1,-1)$, $(1,1)$ *satisfy the conditions for lower and upper solutions of* (5.4). *Moreover,*

$$\begin{aligned} f_1(\xi, u_2(\xi), v(\xi)) - f_1(\xi, u_1(\xi), v(\xi)) &\geq -\xi^3 (u_2(\xi) - u_1(\xi)), \\ f_2(\xi, u(\xi), v_2) - f_2(\xi, u, v_1) &\geq -\xi^5 (v_2(\xi) - v_1(\xi)), \end{aligned}$$

where

$$u_0(\xi) \leq u_1(\xi) \leq u_2(\xi) \leq u_0^*(\xi),\ v_0(\xi) \leq v_1(\xi) \leq v_2(\xi) \leq v_0^*(\xi). \tag{5.5}$$

If we take $(u_0, v_0) = (-1,-1)$, $(u_2, v_2) = (1,1)$, *then clearly condition* (5.5) *holds. Thus all the conditions of Theorem 4.3 are satisfied. Thus system* (5.4) *has an extremal system of solutions. Also, it is simple to verify that* $(0,0)$ *is a unique solution of* (5.4). *Further, in Figure 3, the behavior of extremal solutions has been displayed using the initial iteration.*

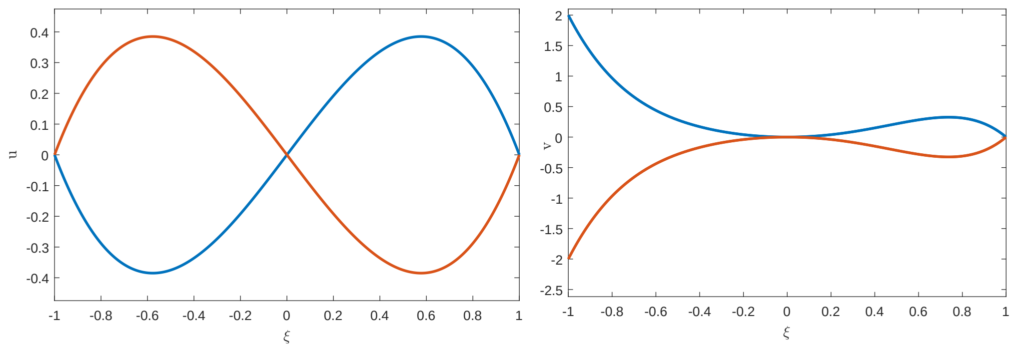

Figure 3. Behavior of Extremal solutions for Example 3.

6. Conclusions

This manuscript is related to some mathematical analysis of the coupled system of FDEs under coupled boundary conditions involving integration. The presented system has been studied by using CFD of non-singular nature. We have constructed adequate conditions addressing the existence and uniqueness of at least one solution by using Perov's and Schauder's fixed point theorems. Since exact solutions for nonlinear problems with fractional derivatives are difficult tasks. Up till now, no such a powerful procedure has yet been established to compute exact solutions to every problem of FDEs. Therefore, sophisticated tools are needed use to compute the best approximate solutions to nonlinear problems. Hence, on the application of the monotone iterative technique coupled with the upper and lower solution method, two sequences of upper and lower solutions have been constructed. In this way, extremal solutions have been developed. Also, the procedure has been testified by two pertinent examples. It should be kept in mind that the system under coupled integral BVPs with CFD has not been studied before this using the aforesaid analysis. Also, the behavior of upper and lower solutions has been presented graphically. In the future, such huge analysis can be extended to more BVPs involving fractal-fractional order derivatives and Mittag-Leffler type operators with integral boundary conditions.

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Conflict of interest

There does not exist any kind of conflicts of interest.

References

1. F. Liu, K. Burrag, Novel techniques in parameter estimation for fractional dynamical models arising from biological systems, *Comput. Math. Appl.*, **62** (2011), 822–833. https://doi.org/10.1016/j.camwa.2011.03.002
2. K. B. Oldham, Fractional differential equations in electrochemistry, *Adv. Eng. Softw.*, **41** (2010), 9–12. https://doi.org/10.1016/j.advengsoft.2008.12.012
3. M. Magdziarz, A. Weron, K. Burnecki, J. Klafter, Fractional Brownian motion versus the continuous-time random walk: a simple test for subdiffusive dynamics, *Phys. Rev. Lett.*, **103** (2009), 180602. https://doi.org/10.1103/PhysRevLett.103.180602
4. F. Mainardi, Fractional calculus: theory and applications, *Mathematics*, **6** (2021), 145. https://doi.org/10.3390/math609014
5. J. A. Tenreiro Machado, M. F. Silva, R. S. Barbosa, I. S. Jesus, C. M. Reis, M. G. Marcos, et al., Some applications of fractional calculus in engineering, *Math. Probl. Eng.*, **2010** (2010), 639801. https://doi.org/10.1155/2010/639801
6. H. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Chen, A new collection of real world applications of fractional calculus in science and engineering, *Commun. Nonlinear Sci. Numer. Simul.*, **64** (2018), 213–231. https://doi.org/10.1016/j.cnsns.2018.04.019
7. S. Majumdar, S. Hazra, M. Dutta Choudhury, S. D. Sinha, S. Das, T. R. Middya, et al., A study of the rheological properties of visco-elastic materials using fractional calculus, *Colloids Surf. A: Phys. Eng. Aspects*, **516** (2017), 181–189. https://doi.org/10.1016/j.colsurfa.2016.12.019
8. A. Atangana, Application of fractional calculus to epidemiology, *Fractional Dyn.*, **2015** (2015), 174–190. https://doi.org/10.1515/9783110472097-011
9. R. Hilfer, *Applications of fractional calculus in physics*, Singapore: World Scientific, 2000. https://doi.org/10.1142/3779

10. Q. Yang, D. Chen, T. Zhao, Y. Chen, Fractional calculus in image processing: a review, *FCAA*, **19** (2016), 1222–1249. https://doi.org/10.1515/fca-2016-0063

11. X. Zhang, L. Liu, Y. Zou, Fixed-point theorems for systems of operator equations and their applications to the fractional differential equations, *J. Funct. Spaces*, **2018** (2018), 7469868. https://doi.org/10.1155/2018/7469868

12. K. Shah, R. A. Khan, Existence and uniqueness results to a coupled system of fractional order boundary value problems by topological degree theory, *Numer. Func. Anal. Opt.*, **37** (2016), 887–899. https://doi.org/10.1080/01630563.2016.1177547

13. Y. Li, W. Zhao, Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations, *Appl. Math. Comput.*, **216** (2010), 2276–2285. https://doi.org/10.1016/j.amc.2010.03.063

14. M. Hamid, M. Usman, R. U. Haq, Z. Tian, A spectral approach to analyze the nonlinear oscillatory fractional-order differential equations, *Chaos Solitons Fract.*, **146** (2021), 110921. https://doi.org/10.1016/j.chaos.2021.110921

15. J. S. Duan, R. Rach, D. Baleanu, A. M. Wazwaz, A review of the Adomian decomposition method and its applications to fractional differential equations, *Commun. Frac. Calc.*, **3** (2012), 73–99.

16. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 73–85. http://dx.doi.org/10.12785/pfda/010201

17. A. Atangana, I. Koca, Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order, *Chaos Solitons Fract.*, **89** (2016), 447–454. https://doi.org/10.1016/j.chaos.2016.02.012

18. P. Bedi, A. Kumar, T. Abdeljawad, A. Khan, J. F. Gómez-Aguilar, Mild solutions of coupled hybrid fractional order system with Caputo-Hadamard derivatives, *Fractals*, **29** (2021), 2150158. https://doi.org/10.1142/S0218348X21501589

19. H. Khan, T. Abdeljawad, J. F. Gómez-Aguilar, H. Tajadodi, A. Khan, Fractional order Volterra integro-differential equation with Mittag-Leffler kernel, *Fractals*, **29** (2021), 2150154. https://doi.org/10.1142/S0218348X21501541

20. O. Martínez-Fuentes, F. Meléndez-Vàzquez, G. Fernàndez-Anaya, J. F. Gómez-Aguilar, Analysis of fractional-order nonlinear dynamic systems with general analytic kernels: lyapunov stability and inequalities, *Mathematics*, **9** (2021), 2084. https://doi.org/10.3390/math9172084

21. J. F. Gómez-Aguilar, G. Rahman, M. Javed, Stability analysis for fractional order implicit φ-Hilfer differential equations, *Math. Methods Appl. Sci.*, **45** (2022), 2701–2712.

22. A. I. K. Butt, W. Ahmad, M. Rafiq, D. Baleanu, Numerical analysis of Atangana-Baleanu fractional model to understand the propagation of a novel corona virus pandemic, *Alexandria Eng. J.*, **61** (2022), 7007–7027. https://doi.org/10.1016/j.aej.2021.12.042

23. A. Gonzàlez-Calderón, L. X. Vivas-Cruz, M. A. Taneco-Hernández, J. F. Gómez-Aguilar, Assessment of the performance of the hyperbolic-NILT method to solve fractional differential equations, *Math. Comput. Simul.*, *206* (2023), 375–390. https://doi.org/10.1016/j.matcom.2022.11.022

24. H. Khan, J. Alzabut, O. Tunç, M. K. Kaabar, A fractal-fractional COVID-19 model with a negative impact of quarantine on the diabetic patients, *Results Control Optim.*, **10** (2023), 100199. https://doi.org/10.1016/j.rico.2023.100199

25. A. Shah, H. Khan, M. De la Sen, J. Alzabut, S. Etemad, C. T. Deressa, et al., On non-symmetric fractal-fractional modeling for ice smoking: mathematical analysis of solutions, *Symmetry*, **15** (2022), 87. https://doi.org/10.3390/sym15010087

26. Y. Li, K. Shah, R. A. Khan, Iterative technique for coupled integral boundary value problem of non-integer order differential equations, *Adv. Differ. Equ.*, **2017** (2017), 251. https://doi.org/10.1186/s13662-017-1308-x

27. L. Liu, X. Hao, Y. Wu, Positive solutions for singular second order differential equations with integral boundary conditions, *Math. Comput. Model.*, **57** (2013), 836–847. https://doi.org/10.1016/j.mcm.2012.09.012

28. M. Arfan, K. Shah, T. Abdeljawad, N. Mlaiki, A. Ullah, A Caputo power law model predicting the spread of the COVID-19 outbreak in Pakistan, *Alexandria Eng. J.*, **60** (2021), 447–456. https://doi.org/10.1016/j.aej.2020.09.011

29. M. Al-Refai, K. Pal, New aspects of Caputo-Fabrizio fractional derivative, *Progr. Fract. Differ. Appl.*, **5** (2019), 157–166. http://dx.doi.org/10.18576/pfda/050206

30. J. F. Gómez, L. Torres, R. F. Escobar, *Fractional derivatives with Mittag-Leffler Kernel*, Fractional Derivatives with Mittag-Leffler Kernel, Springer Cham, 2019. https://doi.org/10.1007/978-3-030-11662-0

31. Y. Nawaz, M. S. Arif, W. Shatanawi, A new numerical scheme for time fractional diffusive seair model with non-linear incidence rate: an application to computational biology, *Fractal Fract.*, **6** (2022), 78. https://doi.org/10.3390/fractalfract6020078

32. R. Gul, M. Sarwar, K. Shah, T. Abdeljawad, F. Jarad, Qualitative analysis of implicit Dirichlet boundary value problem for Caputo-Fabrizio fractional differential equations, *J. Funct. Spaces*, **2020** (2020), 4714032. https://doi.org/10.1155/2020/4714032

33. T. M. Atanacković, S. Pilipović, D. Zorica, Properties of the Caputo-Fabrizio fractional derivative and its distributional settings, *FCAA*, **21** (2018), 29–44. https://doi.org/10.1515/fca-2018-0003

34. J. Losada, J. J. Nieto, Properties of a new fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 87–92. http://dx.doi.org/10.12785/pfda/010202

35. T. Abdeljawad, Fractional operators with exponential kernels and a Lyapunov type inequality, *Adv. Differ. Equ.*, **2017** (2017), 313. https://doi.org/10.1186/s13662-017-1285-0

36. C. Urs, Coupled fixed point theorems and applications to periodic boundary value problems, *Miskolc Math. Notes*, **14** (2013), 323–333. https://doi.org/10.18514/MMN.2013.598

37. A. D. Filip, A. Petruşel, Fixed point theorems on spaces endowed with vector-valued metrics, *Fixed Point Theory Appl.*, **2010** (2010), 281381. https://doi.org/10.1155/2010/281381

38. R. Precup, The role of matrices that are convergent to zero in the study of semilinear operator systems, *Math. Comput. Model.*, **49** (2009), 703–708. https://doi.org/10.1016/j.mcm.2008.04.006

39. K. Latrach, M. A. Taoudi, A. Zeghal, Some fixed point theorems of the Schauder and the Krasnosel'skii type and application to nonlinear transport equations, *J. Differ. Equ.*, **221** (2006), 256–271. https://doi.org/10.1016/j.jde.2005.04.010
40. A. Browder, *Mathematical analysis*, New York: Springer-Verlag, 1996. https://doi.org/10.1007/978-1-4612-0715-3
41. Y. Cui, Y. Zou, Existence results and the monotone iterative technique for nonlinear fractional differential systems with coupled four-point boundary value problems, *Abstr. Appl. Anal.*, **2014** (2014), 242591. https://doi.org/10.1155/2014/242591
42. K. Shah, H. Khalil, R. A. Khan, Upper and lower solutions to a coupled system of nonlinear fractional differential equations, *Progr. Fract. Differ. Appl.*, **2** (2016), 31–39. http://dx.doi.org/10.18576/pfda/020104

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