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Research article

Structure connectivity and substructure connectivity of data center network

Bo Zhu[1], **Shumin Zhang**[2,4,*], **Jinyu Zou**[3] **and Chengfu Ye**[2,4]

[1] Department of Computer, Qinghai Normal University, Xining, Qinghai 810008, China
[2] School of Mathematics and Statistics, Qinghai Normal University, Xining, Qinghai 810008, China
[3] Department of Basic Research, Qinghai University, Xining, Qinghai 810016, China
[4] Academy of Plateau Science and Sustainability, People's Government of Qinghai Province and Beijing Normal University

* **Correspondence:** Email: zhangshumin@qhnu.edu.cn.

Abstract: The structure connectivity $\kappa(G; H)$ and substructure connectivity $\kappa^s(G; H)$ are important indicators to measure interconnection network's fault tolerance and reliability. The data center network, denoted by $D_{k,n}$, have been proposed for data centers as a server-centric interconnection network structure, which can support millions of servers with high network capacity by only using commodity switches. In this paper, we obtain $\kappa(D_{k,n}; S_m)$ and $\kappa^s(D_{k,n}; S_m)$ when $k \geq 2$, $n \geq 4$ and $1 \leq m \leq n+k-2$. Furthermore, we obtain both $\kappa(D_{k,n}; S_{23})$ and $\kappa^s(D_{k,n}; S_{23})$ for $k \geq 8$ and $n \geq 8$.

Keywords: structure connectivity; substructure connectivity; data center network
Mathematics Subject Classification: 05C40, 05C07

1. Introduction

The topological structure of a computer interconnection network can be represented by a graph, where the vertices represent processors and the edges represent communication links between processors. The connectivity of a graph is an important parameter reflecting the strength between two nodes in an interconnection network. The connectivity of a graph G, denoted by $\kappa(G)$, is to delete the minimum number of vertices such that the remaining part is disconnected. The classical connectivity has certain limitations to measure the fault tolerance of the network, then Harary [6] proposed the concept of the conditional connectivity. Later, Fàbrega et al. [3] proposed the concept of the g-extra connectivity. The g-extra connectivity of G, denoted by $\kappa_g(G)$, is the minimum cardinality of vertices in G whose deletion would disconnect G, and each remaining component has more than g vertices. It has triggered extensive research by scholars, and some results can be found in [2, 5, 7, 13, 18, 21].

With the development of large-scale integration technology, a multi-processor system can contain thousands of processors. When one of the processors fails, the processors around it may all be affected. Therefore, it is necessary to consider deleting a certain structure in a network to measure the reliability of the network. Considering the fault status of a certain structure, rather than individual vertices, Lin et al. [10] have given the concepts of the structure connectivity and substructure connectivity. Recently, the results on the structure connectivity and substructure connectivity have come out focusing on networks. For example: hypercube network, folded hypercube network, star network, alternating group network and so on. Many results of networks can be found in the literature [8–12,14–16,19,20].

A network may have thousands of substructures, so it is an important topic to study which substructures are more valuable for the network reliability. A star as a substructure of a network is very important. Because when the central node fails, all of its neighbors are affected. It is reasonable to assume that a node in a network has different degrees of influence on its surrounding nodes. Therefore, we can assign an impactability to each node v, denoted by $imp(v)$. When $imp(v) = 0$, it means that v has no effect on its neighbors; $imp(v) = 1$ means that v affects all its direct neighbors; $imp(v) = 2$ means that v affects not only all of its direct neighbors, but also its immediate neighbor's neighbors. In a network, the structure corresponding to the node v with $imp(v) = 1$ is an m-leaves star with v as the center, denoted by S_m. For v with $imp(v) = 2$, its corresponding structure is called a 2-step star with m-leaves, denoted by S_{2m}, centered on v. (See Figure 1.)

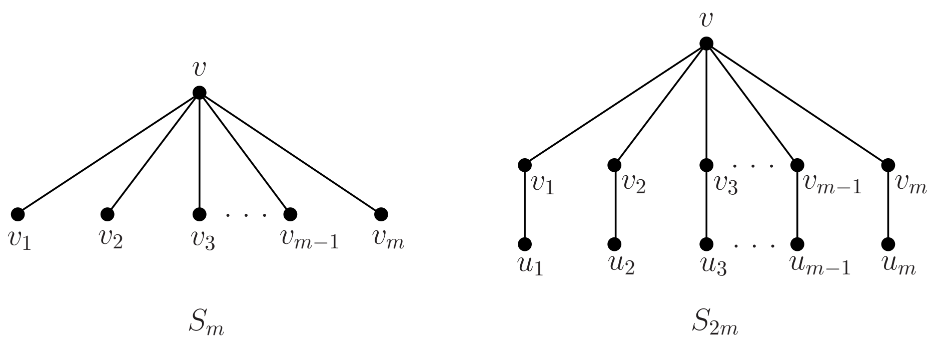

Figure 1. A star and a 2-step star with center vertices v.

2. Preliminaries

2.1. Basic notations and definitions

Given a graph G, let $V(G)$, $E(G)$ and (u, v) denote the set of vertices, the set of edges, and the edge whose end vertices are u and v. The degree of the vertex u in graph G is the number of neighbors of u, denoted by $d(u)$. The neighbors of a vertex u in G is denoted by $N_G(u)$. For a set $U \subseteq V(G)$, the neighbors in $V(G) - U$ of vertices in U are called the neighbors of U, denoted by $N_G(U)$. We denote a complete graph with n vertices by K_n. A graph G is said to be k-regular if every vertex of it has k neighbors. If G_1 is a subgraph of G, denoted by $G_1 \subseteq G$, then $V(G_1) \subseteq V(G)$ and $E(G_1) \subseteq E(G)$. If $G \cong H$, then G is isomorphic to H. Let $G_1 \preceq H$ denote G_1 to be isomorphic to a connected subgraph of H. We use $G[H]$ to represent the subgraph induced by H, which consists of the vertex set of H and the edge set $\{(u, v)|u, v \in V(H), (u, v) \in E(G)\}$. Terminologies not given here can be referred to [1].

Here is the definitions of the structure connectivity and substructure connectivity:

Definition 2.1. *Let H be a connected subgraph of G and F be a set of subgraphs of G such that every element in F is isomorphic to H. If $G - V(F)$ is disconnected, then F is called an H-structure cut. The minimum cardinality of H-structure cuts is called H-structure connectivity of G, denoted by $\kappa(G; H)$.*

Definition 2.2. *Let H be a connected subgraph of G and F^s be a set of subgraphs of G such that every element in F^s is isomorphic to connected subgraph of H. If $G - V(F)$ is disconnected, then F^s is called an H-substructure cut. The minimum cardinality of H-substructure cuts is called H-substructure connectivity of G, denoted by $\kappa^s(G; H)$.*

Obviously, $\kappa^s(G; H) \leq \kappa(G; H)$.

2.2. The data center networks

For a positive integer n, we use $[n]$ and $\langle n\rangle$ to denote the sets $\{1, 2, \ldots, n\}$ and $\{0, 1, 2 \ldots, n\}$, respectively. For any positive integers $k \geq 0$ and $n \geq 2$, we use $D_{k,n}$ to denote a k-dimensional *DCell* with n-port switches. We use $t_{k,n}$ to denote the number of vertices in $D_{k,n}$ with $t_{0,n} = n$ and $t_{k,n} = t_{k-1,n} \times (t_{k-1,n} + 1)$, where $i \in [k]$. Let $I_{0,n} = \langle n - 1\rangle$ and $I_{i,n} = \langle t_{i-1,n}\rangle$ for any $i \in [k]$. Let $V_{k,n} = \{u_k u_{k-1} \ldots u_0 | u_i \in \langle t_{i-1,n}\rangle$ and $i \in \langle k\rangle\}$, and $V^l_{k,n} = \{u_k u_{k-1} \ldots u_l | u_i \in \langle t_{i-1,n}\rangle$ and $i \in \{l, l + 1, \ldots, k\}$ for any $l \in [k]\}$. Clearly, $|V_{k,n}| = t_{k,n}$ and $|V^l_{k,n}| = t_{k,n}/t_{l-1,n}$. The $D_{k,n}$ is defined as follows.

Definition 2.3. *The data center network $D_{k,n}$ is a graph with the vertex set $V_{k,n}$, where a vertex $u = u_k u_{k-1} \ldots u_i \ldots u_0$ is adjacent to a vertex $v = v_k v_{k-1} \ldots v_i \ldots v_0$ if and only if there is an positive integer l with*

(1) $u_k u_{k-1} \ldots u_l = v_k v_{k-1} \ldots v_l$,

(2) $u_{l-1} \neq v_{l-1}$,

(3) $u_{l-1} = v_0 + \sum_{j=1}^{l-2}(v_j \times t_{j-1,n})$ and $v_{l-1} = u_0 + \sum_{j=1}^{l-2}(u_j \times t_{j-1,n}) + 1$ with $l \geq 1$.

Lemma 2.4. [4] *Let $D_{k,n}$ be the data center network with $k \geq 0$ and $n \geq 2$.*

(1) $D_{0,n}$ is a complete graph with n vertices labeled as $0, 1, 2, \ldots, n - 1$.

(2) For $k \geq 1$, $D_{k,n}$ consists of $t_{k-1,n} + 1$ copies of $D_{k-1,n}$ denoted by $D^i_{k-1,n}$, for each $i \in \langle t_{k-1,n}\rangle$. There is one edge between $D^i_{k-1,n}$ and $D^j_{k-1,n}$ for any $i, j \in I_{k,n}$ and $i \neq j$. This implies that the outside neighbors of vertices in $D^i_{k-1,n}$ belong to different copies of $D^j_{k-1,n}$ for $j \neq i$ and $i, j \in I_{k,n}$.

(3) For any two distinct vertices u, v in $D^i_{k-1,n}$, $N_{D^{I_{k,n}\setminus\{i\}}_{k-1,n}}(u) \cap N_{D^{I_{k,n}\setminus\{i\}}_{k-1,n}}(v) = \emptyset$ and $|N_{D^{I_{k,n}\setminus\{i\}}_{k-1,n}}(u)| = 1$.

Lemma 2.5. [4] *For any positive integers $n \geq 2$ and $k \geq 0$, $D_{k,n}$ has the following combinatorial properties.*

(1) $D_{k,n}$ is $(n + k - 1)$-regular with $t_{k,n}$ vertices and $\frac{(n+k-1)t_{k,n}}{2}$ edges.

(2) $\kappa(D_{k,n}) = \lambda(D_{k,n}) = n + k - 1$.

(3) For any integer $k \geq 0$, there is no cycle of length 3 in $D_{k,2}$ and for any integer $n \geq 3$ and $k \geq 0$, there exist cycles of length 3 in $D_{k,n}$.

(4) The number of vertices in $D_{k,n}$ satisfies $t_{k,n} \geq (n + \frac{1}{2})^{2^k} - \frac{1}{2}$.

Lemma 2.6. [17] *There exist $t_{k-1,n}$ disjoint paths (in which any two paths have no common vertices) joining $D^i_{k-1,n}$ and $D^j_{k-1,n}$ for $i, j \in I_{k,n}$, denoted by $P(D^i_{k-1,n}, D^j_{k-1,n})$.*

Lemma 2.7. [13] *For any positive integers $n \geq 2$, $k \geq 2$, and $0 \leq g \leq n - 1$, the g-extra connectivity of $D_{k,n}$ is $\kappa_g(D_{k,n}) = (g + 1)(k - 1) + n$.*

The graph $D_{0,n}$ generates $D_{k,n}$ after k iterations. For any vertex u in $D_{0,n}$, an out neighbor is added every iteration. The graph $D_{i,n}$ consists of $t_{i-1,n} + 1$ copies of $D_{i-1,n}$. Let u^i be the out neighbor of u in $D_{i,n}$, and (u, u^i) be denoted by i edge for $1 \leq i \leq k$. So each vertex in some $D_{0,n}$ has k neighbors and k edges outside of $D_{0,n}$ in $D_{k,n}$. Several data center networks with small parameters k and n, see Figure 2.

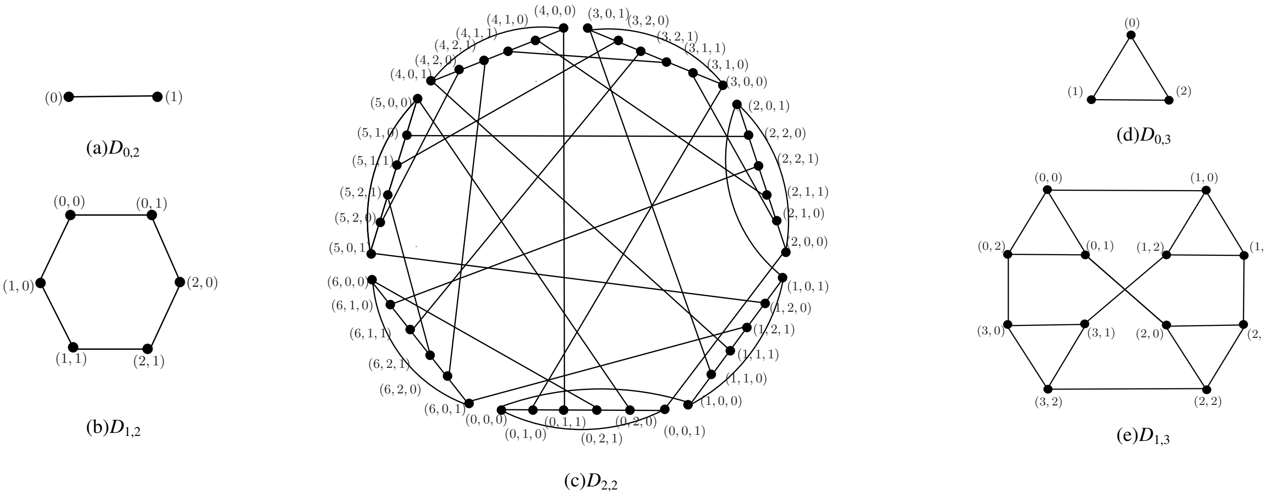

Figure 2. Several data center networks with small parameters k and n.

3. Results of S_m-structure and substructure connectivity of $D_{k,n}$

Lemma 3.1. *$\kappa(D_{k,n}; S_m) \leq \lceil \frac{n-1}{m+1} \rceil + k$ for $n \geq 4$, $k \geq 2$ and $1 \leq m \leq n + k - 2$.*

Proof. For any $v \in V(D^i_{k-1,n})$ for $i \in I_{k,n}$. By the structure of $D_{k,n}$, we know that v belongs to some $D_{0,n}$. Let the $D_{0,n}$ which v is in it be $D'_{0,n}$. Since v has $n-1$ neighbors in $D'_{0,n}$ and has k neighbors $v^1, v^2, \ldots, v^k$ outside of the $D'_{0,n}$, $d(v) = n + k - 1$ in $D_{k,n}$. By the construction of $D_{k,n}$, we know that v^j is the out neighbor of v in $D_{j,n}$ and v^j in a $D_{0,n}$, denoted by $D'^j_{0,n}$ and let v^j be the center vertex of an S_m in $D'^j_{0,n}$ for $1 \leq j \leq k$. Since there is only one edge between different copies in the same dimension, the S_m in $D'^j_{0,n}$ and the S_m in $D'^i_{0,n}$ have no common vertices for $1 \leq i, j \leq k$ and $i \neq j$. Thus, there are k S_m's outside of $D'_{0,n}$ connecting to v. (See Figure3.)

When $1 \leq m \leq n - 3$. Let $p \geq 0$, $q \geq 0$ be two positive integers such that $n - 1 = (m + 1)p + q$, where $0 \leq q \leq m$. If $q = 0$, then there are p S_m's connecting to v in $D'_{0,n}$ and k S_m's connecting to v outside of $D'_{0,n}$. If $1 \leq q \leq m$, then it means that after deleting p S_m's in $D'_{0,n}$ there are q vertices left, except for v. Suppose that w is one of the remaining q vertices and w is the center vertex of an S_m. Then these $q - 1$ neighbors of w in $D'_{0,n}$ and the k neighbors outside of $D'_{0,n}$ can construct an S_m. Thus, there are $(\lceil \frac{n-1}{m+1} \rceil + k)$ S_m's connecting to v. The graph $D_{k,n}$ will be disconnected by deleting $(\lceil \frac{n-1}{m+1} \rceil + k)$ S_m's. Hence, the lemma holds. □

When $n - 2 \leq m \leq n + k - 2$, we have $\lceil \frac{n-1}{m+1} \rceil + k = 1 + k$. Let u be the center vertex of an S_m in $D'_{0,n}$. Then u has $n - 2$ neighbors in $D'_{0,n}$ and k neighbors outside of $D'_{0,n}$ which can construct an S_m connecting to v. It is clearly that there are $(k + 1)$ S_m's connecting to v. Thus, $D_{k,n}$ will be disconnected by deleting $(k + 1)$ S_m's.

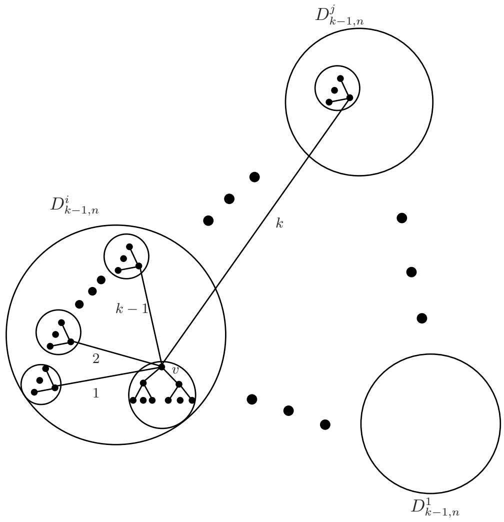

Figure 3. Graph Explanation of Lemma 3.1.

Lemma 3.2. *Let* $F = \{T | T \cong K_1$ *or* $T \cong S_m, n-2 \leq m \leq n\}$*. Then* $D_{2,n} - F$ *is connected for* $n \geq 4$ *and* $|F| \leq 2$*.*

Proof. To prove this lemma by induction n. Clearly, $D_{2,4} - F$ is connected when $|F| \leq 2$. Suppose that $D_{2,n-1} - F$ is connected when $|F| \leq 2$ for $F = \{T | T \cong K_1$ or $T \cong S_m, n-3 \leq m \leq n-1\}$. When $F = \{T | T \cong K_1\}$ and $|F| \leq 2$, it is obviously that $D_{2,n} - F$ is connected. When $F = \{T | T \cong S_m, n-2 \leq m \leq n\}$, it means that the center vertex of each S_m in $D_{2,n}$ has at most one more neighbor deleted than the center vertex of each S_m in $D_{2,n-1}$. Since by the structure of $D_{2,n-1}$ and $D_{2,n}$, for any vertex v in $D_{2,n-1}$, $d(v) = n$, and for any vertex u in $D_{2,n}$, $d(u) = n+1$. Thus, $D_{2,n} - F$ is connected. □

Lemma 3.3. *Let* $F = \{T | T \cong K_1$ *or* $T \cong S_m, 1 \leq m \leq n-3\}$*. Then* $D_{2,n} - F$ *is connected for* $n \geq 4$ *and* $|F| \leq \lceil \frac{n-1}{m+1} \rceil + 1$*.*

Proof. To prove this lemma by induction n. When $n = 4$, we have $m = 1$, $F = \{T | T \cong K_1$ or $T \cong S_1\}$, where $S_1 \cong K_2$ and $\lceil \frac{n-1}{m+1} \rceil + 1 = \lceil \frac{3}{2} \rceil + 1 = 3$. It is easy to check that $D_{2,4} - F$ is connected when $|F| \leq 3$. Suppose that $D_{2,n-1} - F$ is connected when $|F| \leq \lceil \frac{n-2}{m+1} \rceil + 1$ for $1 \leq m \leq n-4$. It suffices to show that $D_{2,n} - F$ is connected when $|F| \leq \lceil \frac{n-1}{m+1} \rceil + 1$ for $1 \leq m \leq n-3$.

If $\lceil \frac{n-1}{m+1} \rceil + 1 = \lceil \frac{n-2}{m+1} \rceil + 1$, then the conclusion obviously holds.

Suppose that $\lceil \frac{n-1}{m+1} \rceil + 1 - (\lceil \frac{n-2}{m+1} \rceil + 1) = 1$. When $F = \{T | T \cong K_1\}$, we have $|F| \leq \lceil \frac{n-1}{m+1} \rceil + 1 = n-1+1 = n$. Since $\kappa(D_{2,n}) = n+1$, by Lemma 2.5, $D_{2,n} - F$ is connected. When $F = \{T | T \cong S_m, 1 \leq m \leq n-3\}$, by inductive hypothesis, $D_{2,n-1} - F$ is connected for $|F| \leq \lceil \frac{n-2}{m+1} \rceil + 1$ and $1 \leq m \leq n-4$. Since $\lceil \frac{n-1}{m+1} \rceil + 1 - (\lceil \frac{n-2}{m+1} \rceil + 1) = 1$, it means that only more one S_m is deleted in $D_{2,n}$ than in $D_{2,n-1}$. Let the center vertex of this S_m be u.

Assume that u is in $D^i_{1,n}$ for $i \in I_{2,n}$. Let $F^i = F \cap D^i_{1,n}$. By the structure of $D_{k,n}$, we know that $D_{1,n}$ is made up of $n+1$ copies of $D_{0,n}$, where $D_{0,n} \cong K_n$ and $D_{1,n-1}$ is made up of n copies of $D_{0,n-1}$, where $D_{0,n-1} \cong K_{n-1}$. When $D_{1,n-1}$ goes to $D_{1,n}$, each copy of $D_{0,n-1}$ adds a vertex to $D_{0,n}$, and another copy of $D_{0,n}$ is added. In this case, u is a new vertex from $D_{1,n-1}$ to $D_{1,n}$. By the structure of $D_{2,n}$, u has only one out neighbor $u' \in V(D^k_{1,n})$, it is clearly that $D^k_{1,n} - F^k$ is connected, so $G[\cup_{i \neq l \in I_{1,n}} V(D^l_{1,n} - F^l)]$ is connected for $i \in I_{2,n}$. Since $D^i_{1,n} \cong D_{1,n}$, u is in a $D_{0,n}$, denoted by $D'_{0,n}$. For any a vertex v in $D^i_{1,n} - F^i$, if $v \notin V(D'_{0,n})$, then it is clearly that v connects $G[\cup_{i \neq l \in I_{1,n}} V(D^l_{1,n} - F^l)]$. If $v \in V(D'_{0,n})$, since $D'_{0,n} \cong K_n$, then we have that v' which is a neighbor of v connects $G[\cup_{i \neq l \in I_{1,n}} V(D^l_{1,n} - F^l)]$. So $D_{2,n} - F$ is connected.

Assume that u is in $D^i_{1,n-1}$ for $i \in I_{2,n-1}$. Let $F_i = F \cap D^i_{1,n-1}$. By the structure of $D_{2,n}$, u has only one out neighbor $u' \in V(D^j_{1,n-1})$. If $D_{2,n-1} - F$ is disconnected, then $D^i_{1,n-1} - F_i$ or $D^j_{1,n-1} - F_j$ is disconnected and $G[\cup_{l \in I_{2,n-1}} V(D^l_{1,n} - F^l)]$ is connected for $i \neq j, i \neq l, j \neq l$. Without loss of generality, suppose that $D^i_{1,n-1} - F_i$ is disconnected. For any vertex w of each component of $D^i_{1,n-1} - F_i$ adds a new neighbor w', when $D^i_{1,n-1}$ becomes $D^i_{1,n}$. We have that w' has an out neighbor w'' which is in $G[\cup_{l \in I_{2,n}} V(D^l_{1,n} - F^l)]$ for $i \neq j, i \neq l, j \neq l$. (See Figure 4.) It is clearly that $G[\cup_{j \neq l \in I_{1,n}} V(D^l_{1,n} - F^l)]$ is connected for $j \in I_{2,n}$.

$$\begin{aligned} |V(F)| &\leq (\lceil \tfrac{n-1}{m+1} \rceil + 1) * (m+1) \\ &= \lceil \tfrac{n-1}{m+1} \rceil * (m+1) + m + 1 \\ &\leq \tfrac{n-1+m}{m+1} * (m+1) + m + 1 \\ &= n + 2m \\ &\leq n + 2(n-3) \\ &= 3n - 6. \end{aligned}$$

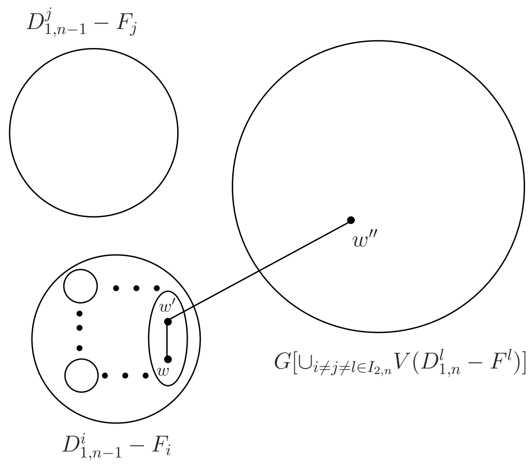

Figure 4. An illustration for "w'' is in $G[\cup_{i \neq j \neq l \in I_{2,n}} V(D^l_{1,n} - F^l)]$" in Lemma 3.3.

By Lemma 2.6, there exist $t_{1,n}$ disjoint paths (in which any two paths have no common vertices) joining $D^i_{1,n}$ and $D^j_{1,n}$ for $i, j \in I_{2,n}$, then we can get that $t_{1,n} \geq (n - \frac{1}{2})^2 + \frac{1}{2}$ for $n \geq 4$, furthermore $t_{1,n} \geq (n + \frac{1}{2})^2 + \frac{1}{2} > 3n - 6 \geq |V(F)|$. This implies that there is at least a path between $D^i_{1,n}$ and $D^j_{1,n}$ in $D_{2,n} - F$. So $D_{2,n} - F$ is connected when $|F| \leq \lceil \frac{n-1}{m+1} \rceil + 1$.

□

Lemma 3.4. $\kappa^s(D_{k,n}; S_m) \geq \lceil \frac{n-1}{m+1} \rceil + k$ *for* $n \geq 4$, $k \geq 2$ *and* $1 \leq m \leq n + k - 2$.

Proof. For an positive integer t, let $F = \{T_j | T_j \cong K_1$ or $T_j \cong S_m, 1 \leq m \leq n + k - 2, 1 \leq j \leq t\}$ and $|F| = t$. Let $F^i = \{T_j | T_j \cong K_1$ or $T_j \cong S_m, T_j \cap D^i_{k-1,n}, 1 \leq m \leq n + k - 2, 1 \leq j \leq t\}$ and C^i be the set of the center vertex of F in $D^i_{k-1,n}$ for $i \in I_{k,n}$. Divide it into the following two cases:

Case 1. $n - 2 \leq m \leq n + k - 2$.

Note that $n - 2 \leq m \leq n + k - 2$, it is clearly that $\lceil \frac{n-1}{m+1} \rceil = 1$. Thus, $\kappa^s(D_{k,n}, S_m) \geq \lceil \frac{n-1}{m+1} \rceil + k = 1 + k$ for $n \geq 4$ and $k \geq 2$. We need to show that $D_{k,n} - F$ is connected when $|F| \leq k$. To prove it by induction on k. When $k = 2$, $D_{2,n} - F$ is connected by Lemma 3.2. For each S_m $(n-2 \leq m \leq n+k-2)$ in $D_{k,n}$, there might be one more vertex than the S_m $(n-2 \leq m \leq n+k-3)$ in $D_{k-1,n}$, but each vertex in $D_{k,n}$ has one more neighbor than the S_m in $D_{k-1,n}$, so we don't have to think about the size of S_m that we delete here, we think about the number of S_m that we delete. Suppose that $D_{k-1,n} - F$ is connected when $|F| \leq k-1$. In the following, we prove that $D_{k,n} - F$ is connected when $|F| \leq k$ for $k \geq 3$.

Case 1.1 $|C^i| = k$.

By the structure of $D_{k,n}$, each center vertex of S_m in $D^i_{k-1,n}$ has at most an out neighbor in $D^j_{k-1,n}$, thus $|F^j| \leq 1$ for $i \neq j \in I_{k,n}$, so the subgraph induced by $\bigcup_{i \neq j \in I_{k,n}} V(D^j_{k-1,n} - F^j)$ is connected. For any vertex $u \in V(D^i_{k-1,n} - F^i)$, we have that u has an out neighbor u' in $G[\cup_{i \neq j \in I_{k,n}}(D^j_{k-1,n} - F^j)]$. By Lemma 2.4(2), we know that $u' \notin V(F)$. It means that u connects $G[\cup_{i \neq j \in I_{k,n}}(D^j_{k-1,n} - F^j)]$. Thus, $D_{k,n} - F$ is connected when $|F| \leq k$.

Case 1.2 $|C^i| = k-1$.

Let w be the center vertex of S_m in $D^l_{k-1,n}$ for $i \neq l \in I_{k,n}$.

Suppose that w has no out neighbor in $D^i_{k-1,n}$. If w has an out neighbor in $D^j_{k-1,n}$ and a center vertex of S_m in $D^i_{k-1,n}$ also has an out neighbor in $D^j_{k-1,n}$, then $|F^j| = 2$ for $i \neq l \neq j \in I_{k,n}$. By the induction hypothesis, $D^j_{k-1,n}$ is connected for $j \in I_{k,n}$. By Lemma 2.4(2) and Lemma 2.5(4), we can get that each copy has $t_{k-1,n}$ out edges and $t_{k-1,n} \geq (n+\frac{1}{2})^{2^{k-1}} - \frac{1}{2} > 2$ for $n \geq 4$, $k \geq 3$. Thus, the subgraph induced by $\bigcup_{i \neq j \in I_{k,n}} V(D^j_{k-1,n} - F^j)$ is connected. For any vertex $u \in V(D^i_{k-1,n} - F^i)$, we have that u has an out neighbor u' in $G[\cup_{i \neq j \in I_{k,n}}(D^j_{k-1,n} - F^j)]$. By Lemma 2.4(2), we know that $u' \notin V(F)$. It implies that u connects $G[\cup_{i \neq j \in I_{k,n}}(D^j_{k-1,n} - F^j)]$. Thus, $D_{k,n} - F$ is connected.

Suppose that w has an out neighbor in $D^i_{k-1,n}$. So w has no out neighbor in $D^j_{k-1,n}$, it follows that $|F^j| \leq 1$ for $i \neq l \neq j \in I_{k,n}$. By induction hypothesis, $D^i_{k-1,n}$ may be disconnected but $D^j_{k-1,n}$ is connected for $i \neq j \in I_{k,n}$. So the subgraph induced by $\bigcup_{i \neq j \in I_{k,n}} V(D^j_{k-1,n} - F^j)$ is connected. For any vertex $u \in V(D^i_{k-1,n} - F^i)$, we have that u has an out neighbor u' in $G[\cup_{i \neq j \in I_{k,n}}(D^j_{k-1,n} - F^j)]$. By Lemma 2.4(2), we know that $u' \notin V(F)$. It means that u connects $G[\cup_{i \neq j \in I_{k,n}}(D^j_{k-1,n} - F^j)]$. Thus, $D_{k,n} - F$ is connected.

Case 1.3 $|C^i| \leq k-2$.

Suppose that all center vertices of S_m's which are outside of $D^i_{k-1,n}$ have an out neighbor in $D^i_{k-1,n}$. Hence, $|F^i| = k$, then $D^i_{k-1,n} - F^i$ may be disconnected. Since each vertex has only an out neighbor, we know that $D^j_{k-1,n} - F^j$ is connected for $i \neq j \in I_{k,n}$. So the subgraph induced by $\bigcup_{i \neq j \in I_{k,n}} V(D^j_{k-1,n} - F^j)$ is connected. For any vertex $u \in V(D^i_{k-1,n} - F^i)$, we have that u has an out neighbor u' in $G[\cup_{i \neq j \in I_{k,n}}(D^j_{k-1,n} - F^j)]$. By Lemma 2.4(2), we know that $u' \notin V(F)$. It means that u connects $G[\cup_{i \neq j \in I_{k,n}}(D^j_{k-1,n} - F^j)]$. Thus, $D_{k,n} - F$ is connected.

Suppose that at least one center vertex of S_m which is outside of $D^i_{k-1,n}$ has no out neighbor in $D^i_{k-1,n}$. By induction hypothesis, $D^i_{k-1,n}$ is connected for $i \in I_{k,n}$. When $|F| \leq k$, we have $|V(F)| \leq k*(n+k-2)$. By the structure of $D_{k,n}$, it has $t_{k-1}+1$ copies of $D_{k-1,n}$. By Lemma 2.5(4), we get that $t_{k-1,n}+1 \geq (n+\frac{1}{2})^{2^{k-1}} + \frac{1}{2}$ and $t_{k-1,n}+1 \geq (n+\frac{1}{2})^{2^{k-1}} + \frac{1}{2} > k*(n+k-2)$ when $n \geq 4$, $k \geq 3$. It means that there is at least a copy $D^h_{k-1,n}$ which is not deleted the vertices, so $|F^h| = 0$. By Lemma 2.6, there exist $t_{k-1,n}$ disjoint paths joining $D^h_{k-1,n}$ and $D^i_{k-1,n}$ for $i, h \in I_{k,n}$. Thus, $D_{k,n} - F$ is connected.

Case 2. $1 \leq m \leq n-3$.

We prove it by induction on k. When $k = 2$, $D_{2,n} - F$ is connected by Lemma 3.3. Suppose that

$D_{k-1,n} - F$ is connected for $|F| \leq \lceil \frac{n-1}{m+1} \rceil + k - 2$. Divide it into the three subcases to prove that $D_{k,n} - F$ is connected when $|F| \leq \lceil \frac{n-1}{m+1} \rceil + k - 1$ for $k \geq 3$.

Case 2.1 $|C^i| = \lceil \frac{n-1}{m+1} \rceil + k - 1$ for $i \in I_{k,n}$.

By the structure of $D_{k,n}$, each center vertex of S_m in $D^i_{k-1,n}$ has at most an out neighbor in $D^j_{k-1,n}$, so $|F^j| \leq 1$ for $i \neq j \in I_{k,n}$, furthermore, the subgraph induced by $G[\cup_{i\neq j\in I_{k,n}}(D^j_{k-1,n} - F^j)]$ is connected. For any vertex $u \in V(D^i_{k-1,n} - F^i)$, we have that u has an out neighbor u' in $G[\cup_{i\neq j\in I_{k,n}}(D^j_{k-1,n} - F^j)]$. By Lemma 2.4(2), we know that $u' \notin V(F)$. It means that u connects $G[\cup_{i\neq j\in I_{k,n}}(D^j_{k-1,n} - F^j)]$. Thus, $D_{k,n} - F$ is connected when $|F| \leq \lceil \frac{n-1}{m+1} \rceil + k - 1$ for $k \geq 3$.

Case 2.2 $|C^i| = \lceil \frac{n-1}{m+1} \rceil + k - 2$ for $i \in I_{k,n}$.

Let w be the center vertex of S_m in $D^h_{k-1,n}$ for $i \neq h \in I_{k,n}$.

Suppose that w has no out neighbor in $D^i_{k-1,n}$. If w has an out neighbor in $D^j_{k-1,n}$ and a center vertex of S_m in $D^i_{k-1,n}$ also has an out neighbor in $D^j_{k-1,n}$, then $|F^j| = 2$ for $i \neq h \neq j \in I_{k,n}$. By induction hypothesis, $D^j_{k-1,n}$ is connected for $j \in I_{k,n}$. By Lemma 2.4(2) and Lemma 2.5(4), we can get that each copy has $t_{k-1,n}$ out edges and $t_{k-1,n} \geq (n+\frac{1}{2})^{2^{k-1}} - \frac{1}{2} > 2$ for $n \geq 4$, $k \geq 3$. It means that the graph induced by $\bigcup_{i\neq j\in I_{k,n}} V(D^j_{k-1,n} - F^j)$ is connected. For any vertex $u \in V(D^i_{k-1,n} - F^i)$, we have that u has an out neighbor u' in $G[\cup_{i\neq j\in I_{k,n}}(D^j_{k-1,n} - F^j)]$. By Lemma 2.4(2), we know that $u' \notin V(F)$. It implies that the vertex u connects $G[\cup_{i\neq j\in I_{k,n}}(D^j_{k-1,n} - F^j)]$. Thus, $D_{k,n} - F$ is connected.

Suppose that w has an out neighbor in $D^i_{k-1,n}$. So w has no out neighbor in $D^j_{k-1,n}$, it follows that $|F^j| \leq 1$ for $i \neq h \neq j \in I_{k,n}$. By induction hypothesis, $D^i_{k-1,n}$ may be disconnected, but $D^j_{k-1,n}$ is connected for $i \neq j \in I_{k,n}$. So the subgraph induced by $\bigcup_{i\neq j\in I_{k,n}} V(D^j_{k-1,n} - F^j)$ is connected. For any vertex $u \in V(D^i_{k-1,n} - F^i)$, we have that u has an out neighbor u' in $G[\cup_{i\neq j\in I_{k,n}}(D^j_{k-1,n} - F^j)]$. By Lemma 2.4(2), we know that $u' \notin V(F)$. It means that u connects $G[\cup_{i\neq j\in I_{k,n}}(D^j_{k-1,n} - F^j)]$. Thus, $D_{k,n} - F$ is connected.

Case 2.3 $|C^i| \leq \lceil \frac{n-1}{m+1} \rceil + k - 3$ for $i \in I_{k,n}$.

Suppose that the center vertices of S_m's which are outside of $D^i_{k-1,n}$ have an out neighbor in $D^i_{k-1,n}$. Hence, $|F^i| = \lceil \frac{n-1}{m+1} \rceil + k - 1$, furthermore, $D^i_{k-1,n} - F^i$ may be disconnected. Since each vertex has only an out neighbor, we have that $D^j_{k-1,n} - F^j$ is connected for $i \neq j \in I_{k,n}$. So the subgraph induced by $\bigcup_{i\neq j\in I_{k,n}} V(D^j_{k-1,n} - F^j)$ is connected. For any vertex $u \in V(D^i_{k-1,n} - F^i)$, we have that u has an out neighbor u' in $G[\cup_{i\neq j\in I_{k,n}}(D^j_{k-1,n} - F^j)]$. By Lemma 2.4(2), we know that $u' \notin V(F)$. It means that u connects $G[\cup_{i\neq j\in I_{k,n}}(D^j_{k-1,n} - F^j)]$. Thus, $D_{k,n} - F$ is connected.

Next, we consider that $|F^i| \leq \lceil \frac{n-1}{m+1} \rceil + k - 2$. By induction hypothesis, $D^i_{k-1,n}$ is connected for $i \in I_{k,n}$. Hence,

$$
\begin{aligned}
|V(F)| &= (\lceil \tfrac{n-1}{m+1} \rceil + k - 1) * (m+1) \\
&= \lceil \tfrac{n-1}{m+1} \rceil * (m+1) + (k-1) * (m+1) \\
&< 2(n-1) + (k-1) * (m+1) \\
&\leq 2(n-1) + (k-1) * (n-2) \\
&< 2(n-1) + (k-1) * (n-1) \\
&= (n-1) * (k+1).
\end{aligned}
$$

By the structure of $D_{k,n}$ and Lemma 2.5(4), we can get that $t_{k-1,n} + 1 \geq (n+\frac{1}{2})^{2^{k-1}} + \frac{1}{2}$. It is easy to check that $t_{k-1,n} + 1 \geq (n+\frac{1}{2})^{2^{k-1}} + \frac{1}{2} > (n-1) * (k+1) > |V(F)|$ for $n \geq 4$ and $k \geq 3$. It implies that at least one copy $D^s_{k-1,n}$ is not deleted a vertex for $s \in I_{k,n}$. By Lemma 2.6, there exist $t_{k-1,n}$ disjoint paths joining $D^s_{k-1,n}$ and $D^i_{k-1,n}$ for $i, s \in I_{k,n}$, so $D_{k,n} - F$ is connected.

□

By Lemma 3.1 and Lemma 3.4, we obtain the following result.

Theorem 3.5. *Let $n \geq 4$, $k \geq 2$ and $1 \leq m \leq n+k-2$. Then $\kappa(D_{k,n}; S_m) = \kappa^s(D_{k,n}; S_m) = \lceil \frac{n-1}{m+1} \rceil + k$.*

4. Results of S_{23}-structure and substructure connectivity of $D_{k,n}$

For any vertex u in $D_{k,n}$, it has $(n-1+k)$ neighbors: $(n-1)$ neighbors in a copy of $D_{0,n}$, denoted by $D'_{0,n}$ and k neighbors outside of $D'_{0,n}$, denoted by $u^1, u^2, \ldots, u^k$. In $D_{1,n}$, the vertex u^1 is called an out neighbor of u; in $D_{2,n}$, the vertex u^2 is called an out neighbor of u, moreover, u and u^1 are in the same copy $D^i_{1,n}$ for $i \in I_{2,n}$. So in $D_{k,n}$, the vertex u^k is called an out neighbor of u and $u, u^1, u^2, \ldots, u^{k-1}$ are in the same copy $D^i_{k-1,n}$ for $j \in I_{k,n}$. In the same dimensional copy, each vertex has only one out neighbor, so there is no edge (u^i, u^j). Thus, u^i and u^j have no other common neighbors except for vertex u for $u^i, u^j \in \{u^1, u^2, \ldots, u^k\}$.

In this part, we prove the results of S_{23} structure and substructure connectivity of $D_{k,n}$.

Lemma 4.1. *Let S_{23} be a 2-step star with 7 vertices. For any vertex v in $D_{k,n}$, it has k neighbors outside of a $D_{0,n}$, denoted by $\{v^1, v^2, \ldots, v^k\}$. Let $T = \{v^1, v^2, \ldots, v^k\}$. Then $|V(S_{23})| \cap |T| \leq 2$.*

Proof. Assume that $v \in V(D^i_{k-1,n})$ for $i \in I_{k,n}$. Let w be the center vertex of the S_{23} in $D^l_{k-1,n}$ for $i \neq l \in I_{k,n}$. (The case of w in $D^i_{k-1,n}$ is similar to the case of w in $D^l_{k-1,n}$.) Let w^k be the out neighbor of w, furthermore, w^1 and w^2 be neighbors of w in $D^l_{k-1,n}$. If w^k is in $D^i_{k-1,n}$, then the S_{23} has two vertices in $D^i_{k-1,n}$. Since each vertex has only one out neighbor, it is clearly that v^k is not an out neighbor of w^1 or w^2. Since there is no edge (v^i, v^j) for $v^i, v^j \in \{v^1, v^2, \ldots, v^{k-1}\}$, we have $|V(S_{23})| \cap |T| \leq 1$. If w^k is in $D^j_{k-1,n}$ for $i \neq l \neq j \in I_{k,n}$, then the S_{23} has two vertices in $D^j_{k-1,n}$. In this case, v^k can be a neighbor of w^k and the out neighbor of w^1 or w^2 can be v^i for $v^i \in \{v^1, v^2, \ldots, v^{k-1}\}$. (See Figure 5.) So we have $|V(S_{23})| \cap |T| \leq 2$. Next, we show that $|V(S_{23})| \cap |T| \geq 3$ does not hold. It is clearly that $v^1, v^2, \ldots, v^k \in V(D^i_{k-1,n}) \cup V(D^j_{k-1,n})$. The vertices of the S_{23} has at most 3 out neighbors and there is only one edge between any two copies, so at most two out neighbors of an S_{23} are in $D^i_{k-1,n}$ and $D^j_{k-1,n}$. Thus, $|V(S_{23})| \cap |T| \leq 2$.

□

Lemma 4.2. *Let $n \geq 8$ and $k \geq 3$. Then $\kappa(D_{k,n}, S_{23}) \leq \lceil \frac{n-1}{7} \rceil + \frac{k}{2}$ for even k and $\kappa(D_{k,n}; S_{23}) \leq \lceil \frac{n-2}{7} \rceil + \frac{k+1}{2}$ for odd k.*

Proof. For any vertex v in $D^i_{k-1,n}$, let v be in $D'_{0,n}$, where $D'_{0,n} \cong K_n$, then v has k neighbors outside of $D'_{0,n}$, denoted by $v^1, v^2, \ldots, v^k$ and $n-1$ neighbors in $D'_{0,n}$.

When k is even. By Lemma 4.1, an S_{23} contains at most two vertices of $v^1, v^2, \ldots, v^k$, so there are $\frac{k}{2}$ S_{23}'s connecting to v outside of $D'_{0,n}$. Let $p \geq 0$, $q \geq 0$ be two positive integers such that $n-1 = 7p+q$, where $q \leq 6$. If $q = 0$, then there are p S_{23}'s connecting to v in $D'_{0,n}$ and $\frac{k}{2}$ S_{23}'s connecting to v outside of $D'_{0,n}$. If $1 \leq q \leq 6$, then there are p S_{23}'s connecting to v and q neighbors of v are left in $D'_{0,n}$ and $\frac{k}{2}$ S_{23}'s connecting to v outside of $D'_{0,n}$. Here we only illustrate the case when $q = 1$, denoted by u, other cases are similar. In $D_{k,n}$, the vertex u has at least three neighbors outside of $D'_{0,n}$, denoted by x, y, w, because $k \geq 3$. Let x', y', w' be the neighbors of x, y, w, respectively. Then u, x, y, w, x', y', w' constitute an S_{23}. Hence, there are $(p+1)$ S_{23}'s connecting to v in $D'_{0,n}$ when $1 \leq q \leq 6$. The graph $D_{k,n}$ will be disconnected by deleting $\lceil \frac{n-1}{7} \rceil + \frac{k}{2}$ S_{23}'s.

When k is odd. By Lemma 4.1, there are $\frac{k-1}{2}$ S_{23}'s connecting to the vertex v outside of $D'_{0,n}$ and v^k are left in $D^j_{k-1,n}$. We construct an S_{23} which contains v^k and v', where v' is the neighbor of v in $D'_{0,n}$. (See Figure 6.) Then there are $\lceil \frac{n-2}{7} \rceil$ S_{23}'s connecting to v in $D'_{0,n}$ and $\frac{k-1}{2} + 1$ S_{23}'s connecting to the vertex v outside of $D'_{0,n}$. The graph $D_{k,n}$ will be disconnected by deleting $(\lceil \frac{n-2}{7} \rceil + \frac{k+1}{2})$ S_{23}'s. □

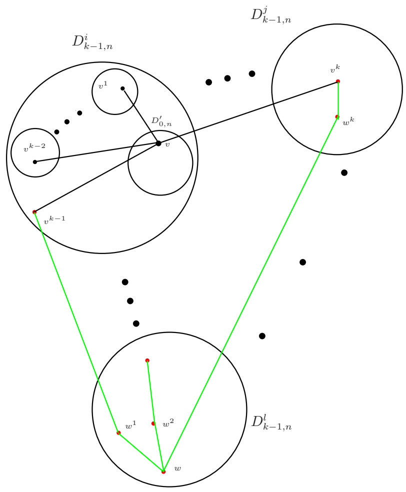

Figure 5. An illustration for "w^k is in $D^j_{k-1,n}$" in Lemma 4.1.

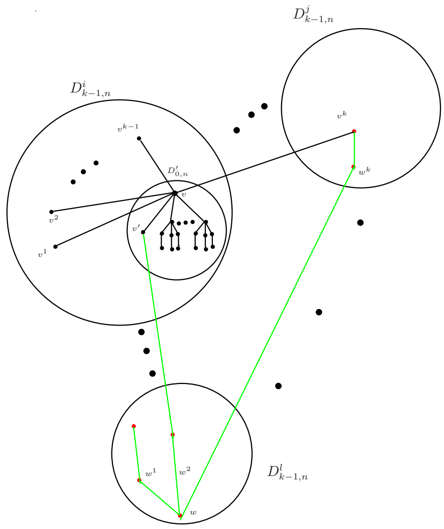

Figure 6. An illustration for the case which is "k is odd" in Lemma 4.2.

Lemma 4.3. *Let $n \geq 8$ and $k \geq 8$. Then $\kappa^s(D_{k,n}; S_{23}) \geq \lceil \frac{n-1}{7} \rceil + \frac{k}{2}$ for even k, and $\kappa^s(D_{k,n}; S_{23}) \geq \lceil \frac{n-2}{7} \rceil + \frac{k+1}{2}$ for odd k.*

Proof. We show that $\kappa^s(D_{k,n}, S_{23}) \geq \lceil\frac{n-1}{7}\rceil + \frac{k}{2}$ when k is even. Let $F = \{T|T \leq S_{23}\}$ and $F^i = \{T_i|T_i \leq S_{23}, T_i \cap D^i_{k-1,n}\}$ for $i \in I_{k,n}$. In the following, we prove that $D_{k,n} - F$ is connected when $|F| \leq \lceil\frac{n-1}{7}\rceil + \frac{k}{2} - 1$. To the contrary, suppose that $D_{k,n} - F$ is disconnected and G_0 is a smallest component of $D_{k,n} - F$.

$|V(F)| = (\lceil\frac{n-1}{7}\rceil + \frac{k}{2} - 1) * 7 \leq (\frac{n-1+6}{7} + \frac{k}{2} - 1) * 7 = \frac{7}{2}k + n - 2 < 4k + n - 4 = \kappa_3(D_{k,n})$.

By Lemma 2.7, we have $|V(G_0)| \leq 3$, thus discussion as follows:

Case 1. $|V(G_0)| = 1$.

Set $V(G_0) = \{v\}$. Thus $N(v) \subseteq V(F)$. To make the number of subgraphs of S_{23}'s minimum which contain the vertices in $N(v)$, we should construct as many S_{23}'s as possible and each S_{23} needs to contain as many vertices in $N(v)$ as possible. Since v has $n-1$ neighbors in a $D_{0,n}$ which is denoted by $D'_{0,n}$ and has k neighbors $v^1, v^2, \ldots, v^k$ outside of the $D'_{0,n}$, each S_{23} contains at most seven vertices in $D'_{0,n}$ or each S_{23} contains at most two vertices of $v^1, v^2, \ldots, v^k$ by Lemma 4.1. Then $|F| \geq \lceil\frac{n-1}{7}\rceil + \frac{k}{2} > \lceil\frac{n-1}{7}\rceil + \frac{k}{2} - 1 \geq |F|$, a contradiction.

Case 2. $|V(G_0)| = 2$.

Set $V(G_0) = \{u, w\}$. Thus $N(\{u, w\}) \subseteq V(F)$. Let u be in a $D_{0,n}$, denoted by $D''_{0,n}$. If w is in $D''_{0,n}$, then w and u have $(n-2)$ common neighbors in $D''_{0,n}$. The vertex w has k neighbors outside of $D''_{0,n}$ and v also has k neighbors outside of $D''_{0,n}$. Furthermore, each S_{23} contains at most seven vertices in $D''_{0,n}$ or each S_{23} contains at most two vertices of the neighbors outside of the $D''_{0,n}$, by Lemma 4.1. So $|F| \geq \lceil\frac{n-2}{7}\rceil + \frac{2k}{2} = \lceil\frac{n-2}{7}\rceil + k > \lceil\frac{n-1}{7}\rceil + \frac{k}{2} - 1 \geq |F|$ for $n \geq 8$ and $k \geq 8$, a contradiction. If w is neighbor of u outside of $D''_{0,n}$, then w and u have no common neighbors. The vertex u has $n-1$ neighbors in $D''_{0,n}$ and $k-1$ neighbors outside of $D''_{0,n}$ except for w. Furthermore, each S_{23} contains at most seven vertices in $D''_{0,n}$ or each S_{23} contains at most two vertices of the neighbors outside of the $D''_{0,n}$, by Lemma 4.1. (The same situation for w.) So $|F| \geq 2 * \lceil\frac{n-1}{7}\rceil + 2 * \frac{k-1}{2} = 2 * \lceil\frac{n-2}{7}\rceil + (k-1) > \lceil\frac{n-1}{7}\rceil + \frac{k}{2} - 1 \geq |F|$ for $n \geq 8$ and $k \geq 8$, a contradiction.

Case 3. $|V(G_0)| = 3$.

Set $V(G_0) = \{x, y, z\}$. Thus $N(\{x, y, z\}) \subseteq V(F)$. To make the number of subgraphs of S_{23}'s minimum which contain the vertices in $N(\{x, y, z\})$, we should construct as many S_{23}'s as possible and each S_{23} needs to contain as many vertices in $N(\{x, y, z\})$ as possible. When x, y and z are in a same $D_{0,n}$, denoted by $D'''_{0,n}$, they have $(n-3)$ common neighbors in $D'''_{0,n}$ and each of x, y, z has k neighbors outside of $D'''_{0,n}$. Each S_{23} contains at most seven vertices in $D'''_{0,n}$ or an S_{23} contains at most two vertices of their neighbors outside of $D'''_{0,n}$ by Lemma 4.1. Then $|F| \geq \lceil\frac{n-3}{7}\rceil + 3 * \frac{k}{2} > \lceil\frac{n-1}{7}\rceil + \frac{k}{2} - 1 \geq |F|$ for $n \geq 8$ and $k \geq 8$, a contradiction. When x, y and z are in two different $D_{0,n}$, without loss of generality, assume that x and y are in $D'''_{0,n}$ and z is in another $D_{0,n}$. Then x and y have $(n-2)$ common neighbors, each of x, y has k neighbors outside of $D'''_{0,n}$. And z has $(n-1)$ neighbors in a $D_{0,n}$ and $(k-1)$ neighbors outside of a $D_{0,n}$ except for x or y. Then $|F| \geq \lceil\frac{n-2}{7}\rceil + \lceil\frac{n-1}{7}\rceil + 2 * \frac{k}{2} + \frac{k-1}{2} > \lceil\frac{n-1}{7}\rceil + \frac{k}{2} - 1 \geq |F|$ for $n \geq 8$ and $k \geq 8$, a contradiction. When x, y and z are in three different $D_{0,n}$, each of x, y, z has $(n-1)$ neighbors in a $D_{0,n}$ and $(k-1)$ neighbors outside of a $D_{0,n}$. Then $|F| \geq 3 * \lceil\frac{n-1}{7}\rceil + 3 * \frac{k-1}{2} > \lceil\frac{n-1}{7}\rceil + \frac{k}{2} - 1 \geq |F|$ for $n \geq 8$ and $k \geq 8$, a contradiction.

The proof of $\kappa^s(D_{k,n}, S_{23}) \geq \lceil\frac{n-2}{7}\rceil + \frac{k+1}{2}$ when k is odd is similar to the case when k is even.

□

By Lemma 4.2 and Lemma 4.3, we have the following result.

Theorem 4.4. *Let $n \geq 8$, $k \geq 8$. Then $\kappa(D_{k,n}; S_{23}) = \kappa^s(D_{k,n}; S_{23}) = \lceil\frac{n-1}{7}\rceil + \frac{k}{2}$ for even k, and $\kappa(D_{k,n}; S_{23}) = \kappa^s(D_{k,n}; S_{23}) = \lceil\frac{n-2}{7}\rceil + \frac{k+1}{2}$ for odd k.*

5. Conclusions

Structure connectivity and substructure connectivity are important parameters for measuring network fault tolerance. In this paper, we obtain that $\kappa(D_{k,n}; S_m) = \kappa^s(D_{k,n}; S_m) = \lceil \frac{n-1}{m+1} \rceil + k$ for $n \geq 4$, $k \geq 2$ and $1 \leq m \leq n + k - 2$. And when $n \geq 8$, $k \geq 8$, we prove that $\kappa(D_{k,n}; S_{23}) = \kappa^s(D_{k,n}; S_{23}) = \lceil \frac{n-1}{7} \rceil + \frac{k}{2}$ for even k, and $\kappa(D_{k,n}; S_{23}) = \kappa^s(D_{k,n}; S_{23}) = \lceil \frac{n-2}{7} \rceil + \frac{k+1}{2}$ for odd k.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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