Mathematics

AIMS Mathematics, 8(4): 9832–9839.
DOI: 10.3934/math.2023496
Received: 07 November 2022
Revised: 30 January 2023
Accepted: 06 February 2023
Published: 22 February 2023

http://www.aimspress.com/journal/Math

Research article

Some conditions for sequences to be minimal completely monotonic

Xifeng Wang* and Senlin Guo

College of Science, Zhongyuan University of Technology, Zhengzhou, Henan 450007, China

* **Correspondence:** Email: wang-xifeng@hotmail.com.

Abstract: In this article, we establish some necessary conditions for sequences to be minimal completely monotonic. We also present some properties for completely monotonic sequences.

Keywords: completely monotonic sequence; completely monotonic function; Hausdorff Theorem; minimal completely monotonic sequence
Mathematics Subject Classification: 44A60, 44A10

1. Introduction and main results

We will first introduce some definitions and some basic results on completely monotonic sequences. Recall that [8] a sequence $\{\mu_n\}_{n=0}^{\infty}$ is called to be completely monotonic if

$$(-1)^k \Delta^k \mu_n \geq 0, \quad n, k \in \mathbb{N}_0, \tag{1}$$

where

$$\Delta^0 \mu_n = \mu_n \tag{2}$$

and

$$\Delta^{k+1} \mu_n = \Delta^k \mu_{n+1} - \Delta^k \mu_n. \tag{3}$$

Here, in (1) and throughout the paper, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and $\mathbb{N}$ is the set of all positive integers. Such a sequence is called totally monotonic in [21].

In [9] the authors showed that for a completely monotonic sequence $\{\mu_n\}_{n=0}^{\infty}$, we always have

$$(-1)^k \Delta^k \mu_n > 0, \quad n, k \in \mathbb{N}_0,$$

unless $\mu_n = c$, a constant for all $n \in \mathbb{N}$.

Hausdorff [8] proved the following fundamental result, in terms of Stieltjes integrals, for completely monotonic sequences (Hausdorff Theorem): a sequence $\{\mu_n\}_{n=0}^{\infty}$ is completely monotonic if and only if there exists a non-decreasing and bounded function $\alpha(t)$ on the interval $[0, 1]$ such that

$$\mu_n = \int_0^1 t^n d\alpha(t), \quad n \in \mathbb{N}_0. \tag{4}$$

Also recall that [19] a sequence $\{\mu_n\}_{n=0}^{\infty}$ is called minimal completely monotonic if it is completely monotonic and if it will not be completely monotonic when μ_0 is replaced by a number which is less than μ_0.

In [5] the author proved that for each completely monotonic sequence $\{\mu_n\}_{n=0}^{\infty}$ there exists one and only one number μ_0^* such that the sequence

$$\{\mu_0^*, \mu_1, \mu_2, \mu_3, \cdots\}$$

is minimal completely monotonic.

One of the results of [6] is that: suppose that the sequence $\{\mu_n\}_{n=1}^{\infty}$ is completely monotonic and that the series

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1$$

converges. Let μ_0 be such that

$$\mu_0 \geq \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1.$$

Then the sequence $\{\mu_n\}_{n=0}^{\infty}$ is completely monotonic.

In [15] the following result, among others, was established. Suppose that the sequence $\{\mu_n\}_{n=1}^{\infty}$ is completely monotonic and that the series

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1 \tag{5}$$

converges. Let

$$\mu_0^* := \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1. \tag{6}$$

Then the sequence

$$\{\mu_0^*, \mu_1, \mu_2, \mu_3, \cdots\}$$

is minimal completely monotonic.

We also note that in [15] the authors gave a counterexample showing that complete monotonicity of the sequence $\{\mu_n\}_{n=1}^{\infty}$ cannot guarantee the convergence of the series

$$\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1. \tag{7}$$

A function *f* is said [1] to be completely monotonic on an interval *I*, if *f* is continuous on *I*, has derivatives of all orders on I^o (the interior of *I*) and for all $n\in\mathbb{N}_0$

$$(-1)^n f^{(n)}(x)\geq 0,\quad x\in I^o. \tag{8}$$

In [10] the authors showed that if the function f is completely monotonic on the interval $[a,\infty)$, the sequence $\{\Delta x_k\}_{k=0}^{\infty}$ is completely monotonic and $x_0\geq a$, then the sequence $\{f(x_k)\}_{k=0}^{\infty}$ is completely monotonic.

The following result was obtained in [5]. Suppose that the sequence $\{\mu_n\}_{n=0}^{\infty}$ is completely monotonic. Then for any $\varepsilon\in(0,1)$, there exists a continuous interpolating function $f(x)$ on the interval $[0,\infty)$ such that $f|_{[0,\varepsilon]}$ and $f|_{[\varepsilon,\infty)}$ are both completely monotonic and $f(n)=\mu_n,\quad n\in\mathbb{N}_0$.

For the operation of pointwise convergence, the author [20] showed that suppose that for $n\in\mathbb{N}$, the function f_n is completely monotonic on the interval *I*, where $I=(a,b)$ or $(a,b]$. If the limit function

$$f(x)=\lim_{n\to\infty}f_n(x)$$

exists on the interval *I*, then f is completely monotonic on *I*.

Here we would like to point out that in the result above the interval *I* cannot be $[a,b)$ or $[a,b]$. For example, let

$$f_n(x)=\frac{1}{x^n},\quad n\in\mathbb{N}$$

and $I_1=[1,b)$ or $I_1=[1,b]$.

It is easy to verify that

$$f_n^{(k)}(x)=\frac{(n+k-1)!}{(n-1)!}\frac{(-1)^k}{x^{n+k}},\quad k\in\mathbb{N}_0.$$

Hence $f_n(x)$ is completely monotonic on the interval I_1. The limit function

$$f(x)=\lim_{n\to\infty}f_n(x)=\begin{cases}1, & x=1;\\ 0, & x\in I_1-\{1\}.\end{cases}$$

Clearly the function f is not completely monotonic on the interval I_1 because f is not continuous on I_1.

There is rich literature on completely monotonic functions and completely monotonic sequences, and their applications. For more recent works, see, for example, [2–7,11–18].

In this article, we shall further investigate completely monotonic and minimal completely monotonic sequences. By using the Hausdorff Theorem, some necessary conditions for sequences to be completely monotonic are presented and proved. Also some properties of minimal completely monotonic sequences are established. The main results of the article are as follows.

Theorem 1. *Suppose that the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is completely monotonic. Then, for any* $k \in \mathbb{N}_0$ *and any* $m \in \mathbb{N}$*, the series*

$$\sum_{j=k}^{\infty}(-1)^j \Delta^j \mu_{m+1}$$

converges and

$$(-1)^k \Delta^k \mu_m = \sum_{j=k}^{\infty}(-1)^j \Delta^j \mu_{m+1}. \tag{9}$$

Theorem 2. *Suppose that the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is completely monotonic. Then, for any* $m \in \mathbb{N}$*, the series*

$$\sum_{j=0}^{\infty}(-1)^j \Delta^j \mu_{m+1}$$

converges and

$$\mu_m = \sum_{j=0}^{\infty}(-1)^j \Delta^j \mu_{m+1}. \tag{10}$$

Theorem 3. *Suppose that the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is minimal completely monotonic. Then, for any* $m \in \mathbb{N}_0$ *, the series*

$$\sum_{j=0}^{\infty}(-1)^j \Delta^j \mu_{m+1}$$

converges and

$$\mu_m = \sum_{j=0}^{\infty}(-1)^j \Delta^j \mu_{m+1}. \tag{11}$$

Theorem 4. *Suppose that the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is minimal completely monotonic. Then, for any* $k, m \in \mathbb{N}_0$ *, the series*

$$\sum_{j=k}^{\infty}(-1)^j \Delta^j \mu_{m+1}$$

converges and

$$(-1)^k \Delta^k \mu_m = \sum_{j=k}^{\infty}(-1)^j \Delta^j \mu_{m+1}. \tag{12}$$

2. Lemmas

Lemma 1. *If the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is completely monotonic, then for all* $i, n \in \mathbb{N}_0$

$$(-1)^i \Delta^i \mu_n = \int_0^1 (1-t)^i t^n d\alpha(t), \tag{13}$$

where $\alpha(t)$ *is a non-decreasing and bounded function on the interval* $[0,1]$.

Proof of Lemma 1. Let $n \in \mathbb{N}_0$. When $i = 0$, by (2) and the Haudorff Theorem, we see that (13) is true. Now suppose that (13) is true for $i \in \mathbb{N}_0$. By (3) and (4) we have

$$\begin{aligned}(-1)^{i+1}\Delta^{i+1}\mu_n &= (-1)^{i+1}(\Delta^i \mu_{n+1} - \Delta^i \mu_n)\\ &= \int_0^1 (1-t)^i t^n d\alpha(t) - \int_0^1 (1-t)^i t^{n+1} d\alpha(t)\\ &= \int_0^1 (1-t)^{i+1} t^n d\alpha(t),\end{aligned}$$

which means that (13) is also true for $i+1$. By induction we see that (13) is true for all $i, n \in \mathbb{N}_0$. The proof of Lemma 1 is completed.

Lemma 2 ([15]). *Suppose that the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is minimal completely monotonic. Then the series*

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1$$

is convergent and

$$\mu_0 = \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1.$$

3. Proof of the results

We now prove the main results of the paper.

Proof of Theorem 1. For $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$, by Lemma 1, we have

$$(-1)^k \Delta^k \mu_m - \sum_{j=k}^{n} (-1)^j \Delta^j \mu_{m+1}$$

$$\begin{aligned}&= \int_0^1 (1-t)^k t^m d\alpha(t) - \sum_{j=k}^{n} \int_0^1 (1-t)^j \, t^{m+1} d\alpha(t)\\ &= \int_0^1 (1-t)^k t^m d\alpha(t) - \int_0^1 \sum_{j=k}^{n} (1-t)^j \cdot t^{m+1} d\alpha(t)\\ &= \int_0^1 (1-t)^k t^m d\alpha(t) - \int_0^1 \frac{(1-t)^k - (1-t)^{n+1}}{t} \cdot t^{m+1} d\alpha(t)\\ &= \int_0^1 t(1-t)^{n+1} t^{m-1} d\alpha(t), \quad n > k.\end{aligned} \tag{14}$$

Since the function $t(1-t)^{n+1}$ attains its maximum on the interval $[0,1]$ at $1/(n+2)$, we have

$$0 \leq \int_0^1 t(1-t)^{n+1} t^{m-1} d\alpha(t) \leq \frac{1}{n+2}\left(1-\frac{1}{n+2}\right)^{n+1} \int_0^1 t^{m-1} d\alpha(t).$$

By the Hausdorff Theorem we obtain

$$0 \leq \int_0^1 t(1-t)^{n+1} t^{m-1} d\alpha(t) \leq \frac{1}{n+2}\left(1-\frac{1}{n+2}\right)^{n+1} \mu_{m-1}.$$

Since

$$\left(1-\frac{1}{n+2}\right)^{n+1} \to 1/e \quad \text{as } n \to \infty,$$

we have

$$\lim_{n\to\infty} \frac{1}{n+2}\left(1-\frac{1}{n+2}\right)^{n+1} \mu_{m-1} = 0.$$

Consequently,

$$\lim_{n\to\infty} \int_0^1 t(1-t)^{n+1} t^{m-1} d\alpha(t) = 0.$$

Then from (14) we obtain

$$\lim_{n\to\infty} \left((-1)^k \Delta^k \mu_m - \sum_{j=k}^{n} (-1)^j \Delta^j \mu_{m+1} \right) = 0. \tag{15}$$

That is,

$$\lim_{n\to\infty} \sum_{j=k}^{n} (-1)^j \Delta^j \mu_{m+1} = (-1)^k \Delta^k \mu_m. \tag{16}$$

Therefore the series

$$\sum_{j=k}^{\infty} (-1)^j \Delta^j \mu_{m+1}$$

converges and

$$(-1)^k \Delta^k \mu_m = \sum_{j=k}^{\infty} (-1)^j \Delta^j \mu_{m+1}. \tag{17}$$

The proof of Theorem 1 is completed.

Proof of Theorem 2. Let k be zero. Then from Theorem 1 we can acquire the conclusion.

The proof of Theorem 2 is hence completed.

Proof of Theorem 3. If $m = 0$, then from Lemma 2 we can obtain the conclusion. If $m \in \mathbb{N}$, in view that a minimal completely monotonic sequence is also completely monotonic, then from Theorem 2 we can obtain the conclusion.

The proof of Theorem 3 is thus completed.

Proof of Theorem 4. Let m be a fixed non-negative integer. Then from Theorem 3 we see that

$$\mu_m = \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_{m+1}, \tag{18}$$

which means that (12) is valid for $k = 0$.

Suppose that (12) is valid for $k = r$. Then

$$\begin{aligned} (-1)^{r+1}\Delta^{r+1}\mu_m &= (-1)^{r+1}(\Delta^r \mu_{m+1} - \Delta^r \mu_m) \\ &= \sum_{j=r}^{\infty} (-1)^j \Delta^j \mu_{m+1} - (-1)^r \Delta^r \mu_{m+1} \\ &= \sum_{j=r+1}^{\infty} (-1)^j \Delta^j \mu_{m+1}, \end{aligned} \tag{19}$$

which means that (12) is also valid for $k = r + 1$. Therefore by induction, (12) is valid for all $k \in \mathbb{N}_0$.

The proof of Theorem 4 is thus completed.

4. Conclusions

In this article, we presented some necessary conditions for a sequence to be a completely monotonic sequence. We also established some necessary conditions for a sequence to be minimal completely monotonic. The Hausdorff Theorem plays the key role when we prove our results.

Acknowledgments

The authors thank the editor and the referees for their valuable suggestions which significantly improve the quality of this paper.

Conflict of interest

The authors declare that they have no competing interests.

References

1. S. Bernstein, Sur la d′efinition et les propri′et′es des fonctions analytiques d’une variable r′eelle, *Math. Ann.*, **75** (1914), 449–468.
2. S. Guo, Some properties of functions related to completely monotonic functions, *Filomat*, **31** (2017), 247–254. https://doi.org/10.2298/FIL1702247G
3. S. Guo, On completely monotonic and related functions, *Filomat*, **30** (2016), 2083–2090. https://doi.org/10.2298/FIL1607083G

4. S. Guo, Some conditions for a class of functions to be completely monotonic, *J. Inequal. Appl.*, **2015** (2015), 11. https://doi.org/10.1186/s13660-014-0534-y
5. S. Guo, Some properties of completely monotonic sequences and related interpolation, *Appl. Math. Comput.*, **219** (2013), 4958–4962. https://dx.doi.org/10.1016/j.amc.2012.11.073
6. S. Guo, H. M. Srivastava, N. Batir, A certain class of completely monotonic sequences, *Adv. Differ. Equ.*, **2013** (2013), 294. https://doi.org/10.1186/1687-1847-2013-294
7. S. Guo, H. M. Srivastava, W. S. Cheung, Some properties of functions related to certain classes of completely monotonic functions and logarithmically completely monotonic functions, *Filomat*, **28** (2014), 821–828. https://doi.org/10.2298/FIL1404821G
8. F. Hausdorff, Summationsmethoden und momentfolgen I, *Math. Z.*, **9** (1921), 74–109.
9. L. Lorch, L. Moser, A remark on completely monotonic sequences, with an application to summability, *Canad. Math. Bull.*, **6** (1963), 171–173.
10. L. Lorch, D. J. Newman, On the composition of completely monotonic functions, completely monotonic sequences and related questions, *J. London Math. Soc.*, **28** (1983), 31–45.
11. F. Qi, Necessary and sufficient conditions for a difference constituted by four derivatives of a function involving trigamma function to be completely monotonic, *Math. Inequal. Appl.*, **24** (2021), 845–855. https://doi.org/10.7153/mia-2021-24-58
12. F. Qi, Completely monotonic degree of a function involving trigamma and tetragamma functions, *AIMS Math.*, **5** (2020), 3391–3407. https://doi.org/10.3934/math.2020219
13. F. Qi, R. P. Agarwal, On complete monotonicity for several classes of functions related to ratios of gamma functions, *J. Inequal. Appl.*, **2019** (2019), 36. https://doi.org/10.1186/s13660-019-1976-z
14. F. Qi, D. W. Niu, Some logarithmically completely monotonic functions and inequalities for multinomial coefficients and multivariate beta functions, *Appl. Anal. Discrete Math.*, **14** (2020), 512–527. https://doi.org/10.2298/AADM191111033Q
15. X. F.Wang, M. Ismail, N. Batir, S. Guo, A necessary and sufficient condition for sequences to be minimal completely monotonic, *Adv. Difference Equ.*, **2020** (2020), 665. https://doi.org/10.1186/s13662-020-03051-8
16. A. M. Xu, Z. D.Cen, Qi's conjectures on completely monotonic degrees of remainders of asymptotic formulas of di- and tri-gamma functions, *J. Inequal. Appl.*, **2020** (2020), 83. https://doi.org/10.1186/s13660-020-02345-5
17. Z. H. Yang, J. F. Tian, Monotonicity rules for the ratio of two Laplace transforms with applications, *J. Math. Anal. Appl.*, **470** (2019), 821–845. https://doi.org/10.1016/j.jmaa.2018.10.034
18. Z. H. Yang, Y. M. Chu, Monotonicity and inequalities involving the modified Bessel functions of the second kind, *J. Math. Anal. Appl.*, **508** (2022), 125889. https://doi.org/10.1016/j.jmaa.2021.125889
19. D. V. Widder, Necessary and sufficient conditions for the representation of a function as a Laplace integral, *Trans. Amer. Math. Soc.*, **33** (1931), 851–892.
20. D. V. Widder, *The Laplace transform*, Princeton: Princeton University Press, 1941.
21. J. Wimp, *Sequence transformations and their applications*, New York: Academic Press, 1981.