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Research article

Fixed point equations for superlinear operators with strong upper or strong lower solutions and applications

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Abstract: It is well known that sublinear operators and superlinear operators are two classes of important nonlinear operators in nonlinear analysis and dynamical systems. Since sublinear operators have only weak nonlinearity, this advantage makes it easy to deal with them. However, superlinear operators have strong nonlinearity, and there are only a few results about them. In this paper, the convergence of Picard iteration for the superlinear operator A is obtained based on the conditions that the fixed point equation $Ax = x$ has a strong upper solution and a lower solution (or alternatively, an upper solution and a strong lower solution). Besides, the uniqueness of the fixed point of strongly increasing operators as well as the global attractivity of strongly monotone dynamical systems are also discussed. In addition, the main results are applied to monotone dynamics of superlinear operators and nonlinear integral equations. The method used in our work develops the traditional method of upper and lower solutions. Since a strong upper (upper) solution and a lower (strong lower) solution are easily checked, the obtained results are effective and practicable in the study of nonlinear equations and dynamical systems. The main novelty is that this paper provides new fixed point results for increasing superlinear operators and the obtained results are applied to strongly monotone systems to investigate their global attractivity.

Keywords: cone and partial order; superlinear operator; fixed point equations; strong upper or strong lower solutions; monotone dynamics; nonlinear integral equations
Mathematics Subject Classification: 54H25, 47H10

1. Introduction

Fixed point theory plays an important role in our life. In the real world, we are faced with a lot of nonlinear phenomenon. Naturally, all kinds of nonlinear problems arise around us. In a wide range of mathematical, computational, economic, modelling, and engineering problems, the existence of a solution to a theoretical or real-world problem is equivalent to the existence of a fixed point for a suitable map or operator. Fixed points are thus crucial in many areas of mathematics, science and engineering. In terms of the theory itself, topology, geometry, and pure and applied analysis are all beautifully incorporated. Fixed point results have been revealed as a very powerful and significant tool in the research of nonlinear phenomena over the last sixty years or so. Fixed point techniques, in particular, have been widely used in fields of biology, chemistry, physics, engineering, game theory and economics [1–6]. Recently, fixed point method is well used in solving nonlinear equations, including Volterra integral equations [1], nonlinear Telegraph equation [2], fractional integral equations [4,5] and Urysohn integral equations [6].

As is mentioned above, when dealing with such nonlinear problems, we need to find the solutions to nonlinear operator equations. In order to solve the fixed point equations involving the nonlinear operator in practical applications, we have to explore a number of nonlinear operators, which include two classes of significant ones, namely, superlinear operators and sublinear operators [7]. Many results involving sublinear operators, especially in the aspect of sublinear dynamics, can he found in rich literature such as Dafermos and Slemrod [8], Krawse and Nussbaum [9], Smith [10], Takáč [11, 12], Zhao [13] and Hirsch and Smith [14].

However, as for superlinear operators, few results can be found in the existing literature (see [7], p.63). The reason for this is that sublinear operators have strong weak nonlinearity. Recently, Xu and Han [15] studied a class of superlinear operators and obtained the existence and uniqueness of fixed point for such operators. In this paper, we further investigate superlinear operators. By virtue of the strong upper or strong lower solution of the fixed point equation, we get some new fixed point results about the superlinear operators. Besides, we also discuss the strongly monotone operator, and obtain the fixed point's existence and uniqueness, the iteration convergence and the error estimation of the Picard approximation. In addition, we also discuss strongly monotone dynamical systems and obtain some new global attractivity results of superlinear dynamics, making an addition in the field of monotone dynamical systems.

In what follows, Section 2 presents a review of basic definitions and results as preliminaries. Section 3 deals with superlinear operators with the strong upper or strong lower solution. Section 4 copes with strongly monotone dynamical systems involving superlinear operators. In the last section, an example involving superlinear operators is presented to show that the results obtained are powerful to solve the nonlinear integral equations.

2. Preliminaries

Suppose E is a real Banach space and P is a cone of E with int$P \neq \emptyset$. The notation θ expresses the null element of E and $\leq$ represents the partial order in terms of P. Cone and partial order are the basic concepts in ordered Banach spaces, which own the standard definitions. For more details, the readers may refer to [7].

Let D be a subset of E and the operator $A : D \to E$. If there is an element $x \in D$ satisfying $Ax = x$, then x is said to be a fixed point of A in D. Let $x_0, y_0 \in D$, x_0 is said to be a lower solution of the fixed point equation $Ax = x$ if $x_0 \leq Ax_0$, while y_0 is called an upper solution if $Ay_0 \leq y_0$. Similarly, x_0 is called a strong lower solution of the fixed point equation $Ax = x$ if $x_0 \ll Ax_0$ when int$P \neq \emptyset$, while y_0

is called a strong upper solution if $Ay_0 \ll y_0$.

For any $u_0, v_0 \in E$ with $u_0 \leq v_0$, then,

$$[u_0, v_0] = \{x \in E | u_0 \leq x \leq v_0\}$$

is named an ordering interval. The operator $A : D \to E$ is named increasing, if for any $x, y \in D$, $x \leq y$ implies $Ax \leq Ay$; A is said to be strongly increasing (or alternatively, strongly monotone) if for any $x, y \in D$, $x \leq y$ implies $Ax \ll Ay$ (see [13,14,16].

Suppose (X, d) is a metric space and $A : X \to X$ is a continuous operator. The omega limit set of $x \in X$ is defined by

$$\omega(x) = \{y \in X : A^{n_k} x \to y \ (n_k \to \infty)\}.$$

Let z be a fixed point of A (i.e. $Az = z$), then z is called globally attractive for A in X if $\omega(x) = \{z\}$ for all $x \in X$ (see [13], p.42).

Let z be a fixed point of A, then the basin of attraction of z is defined as (see [14], p.95)

$$K = \{x \in E : A^n x \to z \ (n \to \infty)\}.$$

Definition 2.1. [12] For any set $D \subset E$. D is named a star-type subset of E, if for any $x \in D$ and $0 < t < 1$, we have $tx \in D$.

It is clear that a convex subset $D \subset E$ with the null element $\theta \in D$ is a star-type subset of E. Especially, each cone P in E is a star-type subset of E.

Definition 2.2. [7] Assume D is a star-type subset of E and $A : D \to D$ is an operator, then,
(1) A is said to be sublinear, if for any $x \in D$ and $0 < t < 1$, $A(tx) \geq tAx$;
(2) A is said to be superlinear, if for any $x \in D$ and $0 < t < 1$, $tAx \geq A(tx)$.

Definition 2.3. [7] Assume $e > \theta$. $A : P \to P$ is named an e-convex operator, if
(i) A is e-positive, that is, $A(P - \{\theta\}) \subset P_e$, where

$$P_e = \{x \in E | \text{ there exist } \lambda, \mu > 0, \text{such that } \lambda e \leq x \leq \mu e\};$$

(ii) for any $x \in P_e$ and $0 < t < 1$, there is a function $\eta = \eta(t, x) > 0$ such that

$$A(tx) \leq (1 - \eta)tAx,$$

where $\eta = \eta(t, x)$ is named the characteristic function of A.

Definition 2.4. [17] Assume $e > \theta$. $A : P \to P$ is called a generalized e-convex operator, if
(i) $Ae \in P_e$, where

$$P_e = \{x \in E | \text{there exist } \lambda, \mu > 0, \text{such that } \lambda e \leq x \leq \mu e\};$$

(ii) for all $x \in P_e$ and $0 < t < 1$, there is a function $\eta = \eta(t, x) > 0$ such that

$$A(tx) \leq ((1 + \eta)t)^{-1} Ax,$$

where $\eta = \eta(t, x)$ is named the characteristic function of A.

Definition 2.5. [8] Assume the operator $A : P \to P$ and $\alpha > 0$. A is named an α-convex operator, if for each $x \in P$ and $0 < t < 1$, $A(tx) \leq t^{\alpha} Ax$.

The relationships among these operators have been given in our previous paper, see [15]. Now, we need the following results.

Proposition 2.1. Assume P is a cone in the real Banach space E and $a, b \in E$. Then,
(i) If $a \leq b$, then $b < a$ is not true;
(ii) If $b < a$, then $a \leq b$ is not true.

Proof. (i) Suppose $a \leq b$, i.e., $b - a \in P$, we can assert that $b < a$ does not hold. Otherwise, assume that $b < a$, then $b \leq a$, i.e., $a - b \in P$ or $b - a \in -P$. So we have $b - a \in P \cap (-P) = \{\theta\}$, i.e., $b = a$, in contradiction to $b < a$. Hence, $b < a$ is not true.

(ii) Similar to (i).

Proposition 2.2. [18,19] Assume P, E are the same as above and $a, b, c, d \in E$. Then,
(i) If $a \leq b$ and $b < c$, then $a < c$;
(ii) If $a < b$ and $b \leq c$, then $a < c$;
(iii) If $a \leq b$ and $\lambda \geq 0$, then $\lambda a \leq \lambda b$;
(iv) If $a < b$ and $\lambda > 0$, then $\lambda a < \lambda b$;
(v) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$;
(vi) If $a \leq b$ and $c < d$, then $a + c < b + d$.

Proposition 2.3. [20] Assume P is a cone with $\text{int}P \neq \emptyset$ in E. Then,
(i) $\theta \notin \text{int}P$;
(ii) $\text{int}P \subset P$;
(iii) $P + \text{int}P \subset P$;
(iv) $\lambda \text{int}P \subset P(\lambda > 0)$.

Proposition 2.4. [18,19] Assume P, E are the same as above and $a, b, c, d \in E$. Then,
(i) If $a \ll b$ and $b \ll c$, then $a \ll c$;
(ii) If $a \ll b$ and $b \leq c$, then $a \ll c$;
(iii) If $a \ll b$ and $\lambda > 0$, then $\lambda a \ll \lambda b$;
(iv) If $a \ll b$ and $b \ll c$, then $a \ll c$;
(v) If $a \ll b$, then $a + c \ll b + c$.

3. Superlinear operators

In this section, we would like to discuss the fixed point results as well as the error estimations for the Picard iteration of superlinear operators under the condition that there exist a strong upper solution and a lower solution (or alternatively, a strong lower solution and an upper solution).

Let us begin with a useful lemma.

Lemma 3.1. Let P be a cone with $\text{int}P \neq \emptyset$ in E and $a, b \in E$. If $a \ll b$ $(b \neq \theta)$, then there exists $0 < \varepsilon < 1$ such that $a \leq \varepsilon b$.

Proof. Since $a \ll b$, $b - a \in \text{int}P$. By the definition of interior point, there exists $0 < r < \|b\|$ such that

$$N_r(b - a) = \{x \in E : \|x - (b - a)\| < r\} \subset P. \tag{3.1}$$

Taking $\varepsilon \in (1 - \frac{r}{\|b\|}, 1)$, we see $\varepsilon b - a \in N_r(b - a)$. In fact,

$$\|\varepsilon b - a - (b - a)\| = \|(\varepsilon - 1)b\| = (1 - \varepsilon)\|b\| < r,$$

i.e., $\varepsilon b - a \in N_r(b - a)$, which implies by (3.1) that $\varepsilon b - a \in P$. Hence, $a \le \varepsilon b$.

Using Lemma 3.1 we immediately obtain the following result.

Corollary 3.1. [16,19] If $a \ll b$, then $a < b$.

Based on Theorems 10 and 11 in [15], we further investigate the error estimations of the iterative approximations for superlinear operators. In the following Lemmas 3.2–3.5 and Theorems 3.1–3.3, we always suppose P is a normal cone in E with $\text{int}P \neq \emptyset$ and $A : P \to P$ is an increasing superlinear operator, while M denotes the normal constant of the cone P.

Lemma 3.2. [15] If there exist $a \in (0,\ 1)$ and $u_0,\ v_0 \in P$ with $u_0 < v_0$ such that $u_0 \le Au_0, Av_0 \le av_0$, then the operator A has a unique fixed point $\hat{x} \in [u_0, v_0]$. For any $x_0 \in [u_0, v_0]$ and iterated sequence $x_n = Ax_{n-1} (n = 1, 2, \cdots)$, we have $\|x_n - \hat{x}\| \to 0\ (n \to \infty)$.

Lemma 3.3. Assume all conditions of Lemma 3.2 hold. Then, the error estimation is such that

$$\|x_n - \hat{x}\| \le 2M^2 \|u_0 - v_0\| a^n. \tag{3.2}$$

Proof. The proof of the existence and uniqueness of the fixed point $\hat{x}$ can be seen in [9]. Now we prove the error estimation. Set $u_n = Au_{n-1},\ v_n = Av_{n-1}$. Let $x_n = Ax_{n-1}$ $(n = 1, 2, \cdots)$ for any $x_0 \in [u_0, v_0]$. Then by the arguments from (18), (19) and (21) in [9] we find the unique fixed point $\hat{x} \in [u_0, v_0]$ and have

$$\theta \le u_n - v_n \le a^n (v_0 - u_0) \quad (n = 1, 2, \cdots), \tag{3.3}$$

$$\theta \le u_{n+p} - u_n \le v_n - u_n \quad (n, p = 1, 2, \cdots), \tag{3.4}$$

$$u_n \le x_n \le v_n \quad (n = 1, 2, \cdots), \tag{3.5}$$

and

$$u_n \to \hat{x}, v_n \to \hat{x}, x_n \to \hat{x} \quad (n \to \infty). \tag{3.6}$$

By (3.3) and the normality of P, we see

$$\|u_n - v_n\| \le M\|v_0 - u_0\| a^n, \tag{3.7}$$

where M is the normal constant of P. By (3.5) we have

$$\theta \le x_n - u_n \le (v_n - u_n) \quad (n = 1, 2, \cdots),$$

which implies

$$\|x_n - u_n\| \le M\|v_n - u_n\| \quad (n = 1, 2, \cdots). \tag{3.8}$$

Letting $p \to \infty$ in (3.4), by (3.6) we have

$$\theta \le \hat{x} - u_n \le v_n - u_n,$$

which implies that

$$\|\hat{x} - u_n\| \le M\|v_n - u_n\|. \tag{3.9}$$

Hence, by (3.7)–(3.9), we get

$$\|x_n - \hat{x}\| \le \|x_n - u_n\| + \|u_n - \hat{x}\| \le M\|v_n - u_n\| + M\|v_n - u_n\| \le 2M^2 \|v_0 - u_0\| a^n,$$

as desired. So, the error estimation (3.2) holds.

Similar to Lemma 3.2 and Theorem 11 in [15], we have the following lemma.

Lemma 3.4. If there exist $a > 1$ and $u_0, v_0 \in P$ with $u_0 < v_0$ such that $au_0 \leq Au_0, Av_0 \leq v_0$, then the equation $Ax = ax$ has a unique fixed point $\hat{x} \in [u_0, v_0]$. For any $x_0 \in [u_0, v_0]$ and the iterated sequence $x_n = \frac{1}{a}Ax_{n-1}(n = 1, 2, \cdots)$, we have $\|x_n - \hat{x}\| \to 0$ $(n \to \infty)$. Moreover, the error estimation is such that

$$\|x_n - \hat{x}\| \leq 2M^2\|v_0 - u_0\|\left(\frac{1}{a}\right)^n. \tag{3.10}$$

Proof. Set $B = a^{-1}A$, then

$$Bu_0 = a^{-1}Au_0 \geq a^{-1}au_0 = u_0,$$

and

$$Bv_0 = a^{-1}Av_0 \leq a^{-1}v_0.$$

For any $x \in P$ and $t \in (0, 1)$, we get

$$B(tx) = a^{-1}A(tx) \leq a^{-1}tAx = tBx.$$

By Lemma 3.2, B has a unique fixed point $\hat{x} \in [u_0, v_0]$. So the equation $Ax = ax$ has a unique fixed point $\hat{x} \in [u_0, v_0]$. For any $x_0 \in [u_0, v_0]$, set $x_n = a^{-1}Ax_{n-1} = Bx_{n-1}$, then by (3.2) we gain

$$\|x_n - \hat{x}\| \leq 2M^2\|v_0 - u_0\|\left(\frac{1}{a}\right)^n,$$

as desired. Hence (3.10) is true.

Remark 3.1. Compared to Theorem 11 in [15], Lemma 3.4 not only presents the error estimation of the iterative approximation, but also corrects the typos "$a \in (0, 1)$" and "$x_n = Ax_{n-1}$" in [15] by "$a > 1$" and "$x_n = \frac{1}{a}Ax_{n-1}$" respectively.

Remark 3.2. Similarly, the typo "$x_n = Ax_{n-1}$" appearing in Corollaries 16, 18, 20, 22 and 24 in [15] should be replaced by "$x_n = \frac{1}{a}Ax_{n-1}$".

Now by virtue of the condition that the fixed point equation has a strong lower or strong upper solution and the lemmas above, we give the convergence of the iterated sequence as well as the error estimation of the successive approximation for superlinear operators.

Theorem 3.1. If there exist $u_0, v_0 \in P$, $u_0 < v_0$ such that $u_0 \leq Au_0$, $Av_0 \ll v_0$, then the operator A has a unique fixed point $\hat{x} \in [u_0, v_0]$. For any $x_0 \in [u_0, v_0]$ and the iterated sequence $x_n = Ax_{n-1}(n = 1, 2, \cdots)$, we have $\|x_n - \hat{x}\| \to 0$ $(n \to \infty)$. Moreover, the error estimation is such that

$$\|x_n - \hat{x}\| \leq 2M^2\|v_0 - u_0\|\varepsilon^n,$$

where $\varepsilon \in (0, 1)$ is a constant only dependent on A and v_0.

Proof. Since $Av_0 \ll v_0$, by Lemma 3.1, there exists $\varepsilon \in (0, 1)$ such that $Av_0 \leq \varepsilon v_0$. All the conditions of Lemma 3.2 are satisfied, so the result follows from Lemmas 3.2 and 3.3.

Theorem 3.2. If there exist $u_0, v_0 \in P$, $u_0 < v_0$ such that $u_0 \ll Au_0$, $Av_0 \leq v_0$, then there exists $\lambda > 1$ such that the operator equation $Ax = \lambda x$ has a unique fixed point $\hat{x}$ in $[u_0, v_0]$. For any $x_0 \in [u_0, v_0]$ and the iterated sequence $x_n = \frac{1}{\lambda}Ax_{n-1}(n = 1, 2, \cdots)$, we have $\|x_n - \hat{x}\| \to 0$ $(n \to \infty)$. Moreover, the error estimation is such that

$$\|x_n - \hat{x}\| \leq 2M^2\|v_0 - u_0\|\left(\frac{1}{\lambda}\right)^n.$$

Proof. Since $u_0 \ll Au_0$, by Lemma 3.1, there exists $\varepsilon \in (0, 1)$ such that $u_0 \leq \varepsilon Au_0$. Set $\lambda = \frac{1}{\varepsilon}$, then $\lambda > 1$, and $\lambda u_0 \leq Au_0$. All the conditions of Lemma 3.4 are satisfied, so the result follows from Lemma 3.4.

Similar to Lemmas 3.2, 3.3 and Theorem 3.1, we immediately get the following two results. We omit the proofs.

Lemma 3.5. If there exist $\varepsilon \in (0, 1)$ such that $A\theta > \theta$, $A^3\theta \leq \varepsilon A^2\theta$, then the operator A has a unique fixed point $\hat{x} \in [A\theta, A^2\theta]$. For any $x_0 \in [A\theta, A^2\theta]$ and the iterated sequence $x_n = Ax_{n-1}$ $(n = 1, 2, \cdots)$, we have $\|x_n - \hat{x}\| \to 0$ $(n \to \infty)$. Moreover, the error estimation is such that

$$\|x_n - \hat{x}\| \leq 2M^2\|A^2\theta - A\theta\|\varepsilon^n.$$

Theorem 3.3. Suppose that $A\theta > \theta$, $A^3\theta \ll A^2\theta$, then the operator A has a unique fixed point $\hat{x} \in [A\theta, A^2\theta]$. For any $x_0 \in [A\theta, A^2\theta]$ and the iterated sequence $x_n = Ax_{n-1}$ $(n = 1, 2, \cdots)$, we have $\|x_n - \hat{x}\| \to 0$ $(n \to \infty)$. Moreover, the error estimation is such that

$$\|x_n - \hat{x}\| \leq 2M^2\|A^2\theta - A\theta\|\varepsilon^n,$$

where $\varepsilon \in (0, 1)$ is a constant only dependent on A.

Now we discuss the strongly monotone and superlinear operators. Suppose P is a normal cone in E with $\text{int}P \neq \emptyset$ and $A : P \to P$ is strongly monotone and superlinear.

Theorem 3.4. If there exist $u_0, v_0 \in P$, $u_0 < v_0$ such that $u_0 \leq Au_0$, $Av_0 < v_0$, then the operator A has a unique fixed point $\hat{x}$ in $[u_0, v_0]$. For any $x_0 \in [u_0, v_0]$ and the iterated sequence $x_n = Ax_{n-1}$ $(n = 1, 2, \cdots)$, we have $\|x_n - \hat{x}\| \to 0$ $(n \to \infty)$. Moreover, the error estimation is such that

$$\|x_n - \hat{x}\| \leq 2M^2\|v_0 - u_0\|\varepsilon^n,$$

where $\varepsilon \in (0, 1)$ is a constant only dependent on A and v_0.

Proof. We use Theorem 3.1 to prove the existence of the fixed point of A. Let $v_1 = Av_0$. Since $Av_0 < v_0$, $v_1 < v_0$. Then by the fact that A is strongly monotone, we get $Av_1 \ll Av_0 = v_1$. By Theorem 3.1, A has a unique fixed point $\hat{x} \in [u_0, v_1]$.

Next, we prove A has a unique fixed point in $[u_0, v_0]$. Suppose $\bar{x}$ is any fixed point in $[u_0, v_0]$, then $u_0 \leq \bar{x} < v_0$, so $u_0 \leq Au_0 \leq \bar{x} \ll Av_0 = v_1$. Hence by Corollary 3.1, we get $u_0 \leq \bar{x} < v_1$, which implies that $\bar{x} = \hat{x}$. Therefore, the operator A has a unique fixed point $\hat{x} \in [u_0, v_0]$. For $x_1 \in [u_0, v_1]$ and the iterated sequence $x_{n+1} = Ax_n$ $(n = 1, 2, \cdots)$, we gain $\|x_n - \hat{x}\| \to 0$ $(n \to \infty)$.

At last, we prove the convergence of the Picard iteration. In fact, for any $x_0 \in [u_0, v_0]$, by the Picard iteration $x_n = Ax_{n-1}$ $(n = 1, 2, \cdots)$, we obtain $u_0 \leq Au_0 \leq Ax_0 \leq Av_0 = v_1$, so $x_1 = Ax_0 \in [u_0, v_1]$. Hence, by the arguments above, we also get $\|x_n - \hat{x}\| \to 0$ $(n \to \infty)$ for any $x_0 \in [u_0, v_0]$. Therefore, all the conclusions of Theorem 3.4 are true.

Similar to Theorem 3.4, we have the next result, omitting the proof.

Theorem 3.5. If there exist $u_0, v_0 \in P$, $u_0 < v_0$ such that $u_0 < Au_0$, $Av_0 \leq v_0$, then there exists $\lambda > 1$ such that the operator equation $Ax = \lambda x$ has a unique fixed point $\hat{x} \in [u_0, v_0]$. For any $x_0 \in [u_0, v_0]$ and the iterated sequence $x_n = \frac{1}{\lambda}Ax_{n-1}$ $(n = 1, 2, \cdots)$, we have $\|x_n - \hat{x}\| \to 0$ $(n \to \infty)$. Moreover, the error estimation is such that

$$\|x_n - \hat{x}\| \le 2M^2 \|v_0 - u_0\| \left(\frac{1}{\lambda}\right)^n .$$

4. Monotone dynamics of superlinear operators

In this section, we will discuss the monotone dynamics of superlinear operators by using the main results obtained in above sections, while P is a normal cone in E with $\mathrm{int}P \neq \emptyset$.

Lemma 4.1. Let $A : P \to P$ be superlinear. If A is continuous at $x = \theta$, then $A\theta = \theta$.

Proof. Because the operator A is superlinear, we have

$$\theta \le A(tx) \le tAx \ (x \in P,\ 0 < t < 1),$$

so it follows that

$$\theta \le A\left(\frac{1}{n}x\right) \le \frac{1}{n}Ax \ (x \in P,\ n = 1, 2, \cdots). \tag{4.1}$$

Letting $n \to \infty$ in (4.1), we gain $\theta \le A\theta \le \theta$, so $A\theta = \theta$ since $P \cap (-P) = \theta$.

Theorem 4.1. Let $A : P \to P$ be superlinear. Suppose that A is monotone and continuous and there exist $0 < \varepsilon < 1$ and $v_0 \in P - \{\theta\}$ such that $Av_0 \le \varepsilon v_0$. Then, A has a unique fixed point $\hat{x} \in [\theta, v_0]$ satisfying $\omega(x) = \{\hat{x}\}$ for any $x \in [\theta, v_0]$. So $\hat{x}$ is globally attractive for A in $[\theta, v_0]$. Moreover, for any $x \in [\theta, v_0]$, we have

$$\|A^n x - \hat{x}\| \le 2M^2\|v_0\|\varepsilon^n,$$

where M is the normal constant of the cone P, and the basin of the attraction of $\hat{x}$

$$K_{\hat{x}} = \{x \in E : A^n x \to \hat{x}(n \to \infty)\}$$

satisfies $K_{\hat{x}} \supset [\theta, v_0]$.

Proof. Let $u_0 = \theta$. Then by Lemma 4.1, we get $Au_0 = u_0$. It is easy to check that all the conditions of Theorem 3.1 are satisfied. Thus, the conclusions of Theorem 4.1 are true.

Similar to Theorem 4.1, the following result about strongly monotone dynamics of superlinear operators can be easily proven, so we omit its proof.

Theorem 4.2. Let $A : P \to P$ be strongly monotone, continuous and superlinear. Suppose there exists $v_0 \in P - \{\theta\}$ such that $Av_0 < v_0$. Then, A has a unique fixed point $\hat{x} \in [\theta, v_0]$ satisfying $\omega(x) = \{\hat{x}\}$ for any $x \in [\theta, v_0]$. So $\hat{x}$ is globally attractive for A in $[\theta, v_0]$. Moreover, the basin of the attraction of $\hat{x}$:

$$K_{\hat{x}} = \{x \in E : A^n x \to \hat{x}(n \to \infty)\}$$

satisfies $K_{\hat{x}} \supset [\theta, v_0]$.

Remark 4.1. Theorems 4.1 and 4.2 present new results about global attractivity for monotone or strongly monotone dynamical systems of superlinear operators, which is a valuable addition to the existing literature in this field.

5. Applications

In this section, we give an example to show that the main results obtained may be powerful to solve the nonlinear integral equations.

Example 5.1. Consider Hammerstein nonlinear integral equation on $\mathbb{R}^n$

$$x(w) = (Ax)(w) = \int_{\mathbb{R}^n} H(w,z)f(z,x(z))dz, \tag{5.1}$$

where $H(w,z)$ is nonnegative measurable on $\mathbb{R}^n \times \mathbb{R}^n$ and

$$\lim_{w\to w_0} \int_{\mathbb{R}^n} |H(w,z) - H(w_0,z)|dz = 0,$$

and there exist constants L and l with $L > l > 0$ such that

$$l \leq \int_{\mathbb{R}^n} H(w,z)dz \leq L, \quad \forall w \in \mathbb{R}^n.$$

For each $x \geq 0$, $f(\cdot,x)$ is measurable on $\mathbb{R}^n$; for each $w \in \mathbb{R}^n$, $f(w,\cdot)$ is continuous on $(0,+\infty)$. Moreover, suppose that there exist constants r and R with $0 < r < R$ such that for any $w \in \mathbb{R}^n$, $f(w,\cdot) : [r,R] \to \mathbb{R}^1$ is increasing and superlinear, namely, $f(w,\lambda x) \leq \lambda f(w,x)$ $(0 < \lambda < 1)$ satisfying

$$f(w,r) \geq \frac{1}{l}r$$

and

$$f(w,R) \leq \left(\frac{1}{L} - \varepsilon\right)R,$$

where $\varepsilon > 0$ is a constant.

Then integral equation (5.1) has a unique continuous solution $\hat{x}(w)$ satisfying $r \leq \hat{x}(w) \leq R$ $(\forall w \in \mathbb{R}^n)$. Moreover, for any initial continuous function $x_0(w) \in [r,R]$, the iterated sequence

$$x_n(w) = \int_{\mathbb{R}^n} H(w,z)f(z,x_{n-1}(z))dz \; (w \in \mathbb{R}^n, \; n = 1,2,\cdots)$$

uniformly converges to $\hat{x}(w)$, and

$$\sup_{w\in\mathbb{R}^n} |x_n(w) - \hat{x}(w)| \leq M_0\tau^n \to 0 \; (n \to \infty),$$

where $M_0 > 0,\ 0 < \tau < 1$ are constants which are independent on $x_0(w)$.

Proof. Suppose

$$E = C_B(\mathbb{R}^n) = \{x \in C(\mathbb{R}^n) : \sup_{w\in\mathbb{R}^n} |x(w)| < \infty\}$$

is a bounded continuous function space in $\mathbb{R}^n$. Put $\| x \| = \sup_{w\in\mathbb{R}^n} |x(w)|$, then E is a Banach space. Set $P = \{x \in C_B(\mathbb{R}^n) : x(w) \geq 0, w \in \mathbb{R}^n\}$ denote all nonnegative continuous functions in E, then P is a normal solid cone in E and $\text{int}P = \{x \in C_B(\mathbb{R}^n) : inf_{w\in\mathbb{R}^n} x(w) \geq 0\}$. Consider the operator A defined as

$$(Ax)(w) = \int_{w\in\mathbb{R}^n} H(w,z)f(z,x_{n-1}(z))dz.$$

Let $u_0(w) \equiv r(w \in \mathbb{R}^n), v_0(w) \equiv R(w \in \mathbb{R}^n)$. It is easy to check that $A : [u_0, v_0] \to E$ is a superlinear increasing operator and satisfies $u_0 \leq Au_0, Av_0 \ll v_0$. In fact, for $x = x(w),\ y = y(w) \in C_B(\mathbb{R}^n)$, if $x \leq y$, i.e., $x(w) \leq y(w)$ $(w \in \mathbb{R}^n)$, since $f(z, \cdot)$ is increasing, for any $w \in \mathbb{R}^n$, we have

$$Ay - Ax = (Ay)(w) - (Ax)(w) = \int_{\mathbb{R}^n} H(w,z)(f(z, y(z)) - f(z, x(z)))dz \geq 0.$$

So, A is increasing. For any $0 < \lambda < 1, x(w) \in C_B(\mathbb{R}^n)$, since $f(z, \lambda w) \leq \lambda f(z, w)$, we get

$$\begin{aligned} A(\lambda x) &= \int_{\mathbb{R}^n} H(w,z) f(z, \lambda x(z))dz \\ &\leq \lambda \int_{\mathbb{R}^n} H(w,z) f(z, x(z))dz \\ &= \lambda Ax(w) = \lambda Ax. \end{aligned}$$

Thus, A is superlinear.

For $u_0 = u_0(w) \equiv r, w \in \mathbb{R}^n$,

$$Au_0 = \int_{\mathbb{R}^n} H(w,z) f(z, r)dz \geq \frac{r}{m} \int_{\mathbb{R}^n} H(w,z)dz \geq m \cdot \frac{r}{m} = r \equiv u_0,$$

i.e., $Au_0 \geq u_0$.

For $v_0 = v_0(w) \equiv R, w \in \mathbb{R}^n$, we see

$$Av_0 = \int_{\mathbb{R}^n} H(w,z) f(z, R)dz \leq \int_{\mathbb{R}^n} \left(\frac{1}{M} - \varepsilon\right) Rdz \leq \left(\frac{1}{M} - \varepsilon\right) RM,$$

so

$$v_0 - Av_0 \geq R - \left(\frac{1}{M} - \varepsilon\right) RM = R\varepsilon M > 0.$$

Thus, $\inf_{w\in\mathbb{R}^n}(v_0 - Av_0) \geq R\varepsilon M > 0$, i.e., $v_0 - Av_0 \in \text{int}P$, so $Av_0 \ll v_0$. That is, all conditions of Theorem 3.1 are satisfied. So, the result follows from Theorem 3.1.

6. Conclusions

In this article, by virtue of the theory of cone and partial order, we investigate the fixed point equation $Ax = x$ involving the superlinear operator A in the setting of the ordered real Banach space. Under the crucial condition that the fixed point equation $Ax = x$ has a strong upper solution and a lower solution (or alternatively, an upper solution and a strong lower solution), we obtain the convergence and the error estimation of the Picard iteration for the superlinear operator A via the monotone iterative technique. This method develops the classical method of upper and lower solutions. Since in the context of a real Banach space with a normal and solid cone, a strong upper (upper) solution and a lower (strong lower) solution are easily checked, we provide some practicable examples involving nonlinear integral equations as well as dynamical systems that elaborated on the usability of our main results. Nevertheless, once the positive cone in the real Banach space is not either normal or solid, this method is invalid and of useless. The main contribution is that we obtain the new results on the fixed point equations involving superlinear increasing operators with a strong upper (upper) solution and a lower (strong lower) solution as well as the new ones about global attractivity of strongly monotone dynamical systems.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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