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Research article

A tripled coincidence point technique for solving integral equations via an upper class of type II

Hasanen A. Hammad[1,2], **Hassen Aydi**[3,4,5,*] **and Aiman Mukheimer**[6,*]

[1] Department of Mathematics, Unaizah College of Sciences and Arts, Qassim University, Buraydah 52571, Saudi Arabia

[2] Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt

[3] Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, H. Sousse 4000, Tunisia

[4] China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

[5] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa

[6] Department of Mathematics and Sciences, College of Humanities and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia

* **Correspondence:** Email: hassen.aydi@isima.rnu.tn, mukheimer@psu.edu.sa.

Abstract: The goal of this paper is to obtain some tripled coincidence point results for generalized contraction mappings in the setting of JS-metric spaces endowed with a partial order. Furthermore, illustrative examples to support the theoretical results and the application are obtained. Finally, some theoretical results are applied to discuss the existence of a solution for a system of non-homogeneous and homogeneous integral equations as applications.

Keywords: tripled coincidence point; generalized contraction mapping; JS-metric spaces; Geraghty type contraction; upper class
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1. Background and preliminaries

The fixed point theory is one of the most powerful and productive tools from the nonlinear analysis and it can be considered to be the kernel of the nonlinear analysis since 1960. It has been classified into two major areas: metric fixed point theory and topological fixed point theory. The fixed point theory finds its roots with the method of successive approximations to prove the existence of solutions of differential equations introduced independently by Liouville [1] in 1837 and Picard [2] in 1890. While, officially, it was launched at the start of the 20th century as an important part of functional analysis. The best known result from the fixed point theory is Banach Contraction Principle [3] (1922), which can be considered to be the beginning of this theory. In a metric space, setting it can be briefly stated as follows: Every contraction on a complete metric space admits a unique fixed point. Furthermore, it provides a fixed point approximation algorithm as the limit of an iterated sequence. This result has been extended and generalized last years in various directions. Among these extensions, we cite the main result of Jleli and Samet [4], where they initiated a new generalized metric, called as JS-metric. It is given as follows:

Definition 1.1. [4] Let $\xi : \mho \times \mho \to [0, +\infty)$ be a mapping justifying, for each $\vartheta, \varpi \in \mho$,

(i) if $\xi(\vartheta, \varpi) = 0$, then $\vartheta = \varpi$;
(ii) $\xi(\vartheta, \varpi) = \xi(\varpi, \vartheta)$;
(iii) there is $Z > 0$ so that

$$\lim_{i\to\infty} \xi(\vartheta_i, \vartheta) \Rightarrow \xi(\vartheta, \varpi) \leq Z \limsup_{i\to\infty} \xi(\vartheta_i, \varpi).$$

Then ξ is a JS-metric on $\mho$ and the pair $(\mho, \xi)$ is called a JS-metric space.

Definition 1.2. Let $(\mho, \xi)$ be a JS-metric space and $\{x_n\}$ be a sequence in $(\mho, \xi)$.

(i) $\{x_n\}$ is said to be ξ-convergent to x in $\mho$ if $\lim_{n\to\infty} \xi(x_n, x) = 0$.

(ii) $\{x_n\}$ is said to be ξ-Cauchy sequence if $\lim_{n,m\to\infty} \xi(x_n, x_m) = 0$.

(iii) $(\mho, \xi)$ is said to be ξ-complete if every ξ-Cauchy sequence in $\mho$ is ξ-convergent to some point $x \in \mho$.

After the work of Jleli and Samet [4], there is an immense literature in fixed point theory and its applications in this setting. For more details, see [5–7].

In paper [8], the notions of mixed-monotone functions and coupled fixed points were initiated and studied. Under partially ordered metric spaces (POMSs) and abstract spaces, some main results in this direction have been driven, for broadening, see [9–14].

In 2011, Berinde and Borcut [15], introduced the definition of mixed monotone property and the definition of tripled fixed point for a mapping $T : \mho^3 =: \mho \times \mho \times \mho \to \mho$ and established tripled fixed point theorems for contractive type mappings having that property in partially ordered metric spaces. Later, Borcut [16] and Berinde and Borcut [17] have introduced the notion of a tripled coincidence point for a pair of nonlinear contractive mappings $T : \mho^3 \to \mho$ and $f : \mho \to \mho$. Subsequently, Aydi et al. [18] have proved some new tripled fixed point theorems in abstract metric spaces. Further works dealing in this direction have been appeared, see [19–21].

Combining the notion of triangular $\alpha-$admissible property and the concept of upper class of type II, the aim of this paper is to establish some tripled coincidence point results for a pair of generalized contraction type mappings $\Xi : \mho^3 \to \mho$ and $r : \mho \to \mho$ in the context of JS-metric spaces equipped with a partial order. The obtained results are supported by some concrete examples. At the end, we ensure the existence of a solution for a system of non-homogeneous and homogeneous integral equations.

2. Main results

We start this section with the following concepts.

Consider a nonempty set $\mho$. Assume that $\Xi : \mho^3 \to \mho$ and $r : \mho \to \mho$ are two mappings. We say that r commutes with Ξ if

$$r\Xi(\vartheta, \varpi, \varkappa) = \Xi(r\vartheta, r\varpi, r\varkappa), \ \forall \vartheta, \varpi, \varkappa \in \mho.$$

According to [4], for a partial order $\preceq$, define $\nabla_{\preceq} = \{(\vartheta, \varpi) \in \mho^2 : \vartheta \preceq \varpi\}$. Then Ξ has the $\preceq -r$ monotone property, if for each $\vartheta, \varpi, \varkappa \in \mho$,

$$\begin{aligned} \vartheta_1, \vartheta_2 &\in \mho, \ (r\vartheta_1, r\vartheta_2) \in \nabla_{\preceq} \text{ implies } (\Xi(\vartheta_1, \varpi, \varkappa), \Xi(\vartheta_2, \varpi, \varkappa)) \in \nabla_{\preceq}, \\ \varpi_1, \varpi_2 &\in \mho, \ (r\varpi_1, r\varpi_2) \in \nabla_{\preceq} \text{ implies } (\Xi(\vartheta, \varpi_1, \varkappa), \Xi(\vartheta, \varpi_2, \varkappa)) \in \nabla_{\preceq} \end{aligned}$$

and

$$\varkappa_1, \varkappa_2 \in \mho, \ (r\varkappa_1, r\varkappa_2) \in \nabla_{\preceq} \text{ implies } (\Xi(\vartheta, \varpi, \varkappa_1), \Xi(\vartheta, \varpi, \varkappa_2)) \in \nabla_{\preceq}.$$

After that, we generalize the notion of triangular $\alpha-$admissible property as follows:

Definition 2.1. Assume that $\Xi : \mho^3 \to \mho$, $r : \mho \to \mho$ and $\alpha : \mho^3 \times \mho^3 \to [0, +\infty]$ so that the postulates below hold:

(*i*) If $\alpha\left((r\vartheta, r\varpi, r\varkappa), \left(r\widetilde{\vartheta}, r\widetilde{\varpi}, r\widetilde{\varkappa}\right)\right) \geq 1$, then

$$\alpha\left((\Xi(\vartheta, \varpi, \varkappa), \Xi(\varpi, \varkappa, \vartheta), \Xi(\varkappa, \vartheta, \varpi)), \left(\Xi\left(\widetilde{\vartheta}, \widetilde{\varpi}, \widetilde{\varkappa}\right), \Xi\left(\widetilde{\varpi}, \widetilde{\varkappa}, \widetilde{\vartheta}\right), \Xi\left(\widetilde{\varkappa}, \widetilde{\vartheta}, \widetilde{\varpi}\right)\right)\right) \geq 1.$$

(*ii*) If $\alpha\left((r\vartheta, r\varpi, r\varkappa), \left(r\widetilde{\vartheta}, r\widetilde{\varpi}, r\widetilde{\varkappa}\right)\right) \geq 1$ and

$$\alpha\left(\left(r\widetilde{\vartheta}, r\widetilde{\varpi}, r\widetilde{\varkappa}\right), \left(\Xi\left(\widetilde{\vartheta}, \widetilde{\varpi}, \widetilde{\varkappa}\right), \Xi\left(\widetilde{\varpi}, \widetilde{\varkappa}, \widetilde{\vartheta}\right), \Xi\left(\widetilde{\varkappa}, \widetilde{\vartheta}, \widetilde{\varpi}\right)\right)\right) \geq 1,$$

then

$$\alpha\left((r\vartheta, r\varpi, r\varkappa), \left(\Xi\left(\widetilde{\vartheta}, \widetilde{\varpi}, \widetilde{\varkappa}\right), \Xi\left(\widetilde{\varpi}, \widetilde{\varkappa}, \widetilde{\vartheta}\right), \Xi\left(\widetilde{\varkappa}, \widetilde{\vartheta}, \widetilde{\varpi}\right)\right)\right) \geq 1.$$

Here, we say that Ξ and r is a generalized triangular $\alpha-$admissible (α_A^{GT}, for short).

Next, we extend the concept of upper class of type *II* [22] as follows.

Definition 2.2. Let $\wp : [0, +\infty]^3 \to \mathbb{R} \cup \{+\infty\}$ and $\ell : [0, +\infty]^2 \to \mathbb{R} \cup \{+\infty\}$ be two given mappings. We say that the pair $(\wp, \ell)$ is an upper class of type II (Ω_{II}, for short) if for all $\vartheta, \varpi, \varkappa, \widetilde{\vartheta}, \widetilde{\varpi}, \widetilde{\varkappa} \in [0, +\infty]$, the postulates below hold:

(*i*) $\wp(1, 1, \varpi) \leq \wp(\vartheta, \varpi, \varkappa)$, whenever $1 \leq \varpi, \varkappa$;
(*ii*) $\ell(\widetilde{\vartheta}, \widetilde{\varpi}) \leq \ell(1, \widetilde{\varpi})$, whenever $\widetilde{\vartheta} \leq 1$;
(*iii*) $\varkappa \leq \widetilde{\vartheta}\widetilde{\varpi}$, whenever $\wp(1, 1, \varkappa) \leq \ell(\widetilde{\vartheta}, \widetilde{\varpi})$.

Example 2.3. Let $\wp, \ell : [0, +\infty]^2 \to \mathbb{R} \cup \{+\infty\}$ be a function described as follows:

(1) $\wp(\vartheta, \varpi, \varkappa) = \begin{cases} (\varkappa + z)^{\vartheta\varpi}, & \text{if } \vartheta, \varpi, \varkappa \in [0, +\infty), \\ +\infty, & \text{otherwise,} \end{cases}$ $\quad \ell\left(\widetilde{\vartheta}, \widetilde{\varpi}\right) = \begin{cases} \widetilde{\vartheta}\widetilde{\varpi} + z, & \text{if } \widetilde{\vartheta}, \widetilde{\varpi} \in [0, +\infty), \\ +\infty, & \text{otherwise.} \end{cases}$

(2) $\wp(\vartheta,\varpi,\varkappa)=\begin{cases}(\vartheta+z)^{\varkappa\varpi}, & \text{if } \vartheta,\varpi,\varkappa\in[0,+\infty),\\ +\infty, & \text{otherwise,}\end{cases}$ $\ell\left(\widetilde{\vartheta},\widetilde{\varpi}\right)=\begin{cases}(1+z)^{\widetilde{\vartheta}\widetilde{\varpi}}, & \text{if } \widetilde{\vartheta},\widetilde{\varpi}\in[0,+\infty),\\ +\infty, & \text{otherwise.}\end{cases}$

(3) $\wp(\vartheta,\varpi,\varkappa)=\begin{cases}\varkappa, & \text{if } \vartheta,\varpi,\varkappa\in[0,+\infty),\\ +\infty, & \text{otherwise,}\end{cases}$ $\ell\left(\widetilde{\vartheta},\widetilde{\varpi}\right)=\begin{cases}\widetilde{\vartheta}\widetilde{\varpi}, & \text{if } \widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\in[0,+\infty),\\ +\infty, & \text{otherwise.}\end{cases}$

(4) $\wp(\vartheta,\varpi,\varkappa)=\begin{cases}\vartheta^i\varpi^j\varkappa^k & \text{if } \vartheta,\varpi,\varkappa\in[0,+\infty),\\ +\infty, & \text{otherwise,}\end{cases}$ $\ell\left(\widetilde{\vartheta},\widetilde{\varpi}\right)=\begin{cases}\left(\widetilde{\vartheta}\widetilde{\varpi}\right)^k, & \text{if } \widetilde{\vartheta},\widetilde{\varpi}\in[0,+\infty),\\ +\infty, & \text{otherwise,}\end{cases}$

for $z>1$ and $i,j,k\in\mathbb{N}$. Then the pair $(\wp,\ell)$ is Ω_{II}.

Here, the triplet $(\mho,\xi,\preceq)$ is a complete POJSM-space and $\widetilde{\Phi}$ consists of all $\phi:[0,+\infty]^3\to[0,1)$ satisfying:

(ϕ_1) For any $\vartheta,\varpi\in[0,+\infty]$, $\phi(\vartheta,\varpi)=\phi(\varpi,\vartheta)$.
(ϕ_2) For $\{\vartheta_i\},\{\varpi_i\}\subseteq[0,+\infty]$, $\lim_{i\to\infty}\phi(\vartheta_i,\varpi_i,\varkappa_i)=1\Longrightarrow\lim_{i\to\infty}\vartheta_i=\lim_{i\to\infty}\varpi_i=\lim_{i\to\infty}\varkappa_i=0$.

Also, we define the mappings $\Xi:\mho^3\to\mho$ and $r:\mho\to\mho$ so that the following properties hold:

(a) $\Xi\left(\mho^3\right)\subseteq r(\mho)$;
(b) Ξ is $\preceq -r$ monotone;
(c) r commutes with Ξ and is $\xi-$continuous.

Now, our first theorem becomes valid for presentation, which generalizes the results of [7].

Theorem 2.4. *Assume that the postulates below are fulfilled:*

($\triangle_i$) *For any Ω_{II} of the pair $(\wp,\ell)$, there exist $\alpha:\mho^3\times\mho^3\to[0,+\infty]$ and $\phi\in\Phi$ fulfilling, for any $\left(r\vartheta,r\widetilde{\vartheta}\right)\in\nabla_{\preceq}$, $(r\varpi,r\widetilde{\varpi})\in\nabla_{\preceq}$ and $(r\varkappa,r\widetilde{\varkappa})\in\nabla_{\preceq}$, the below inequality is verified:*

$$\begin{aligned}&\wp\left(1,\alpha\left((r\vartheta,r\varpi,r\varkappa),\left(r\widetilde{\vartheta},r\widetilde{\varpi},r\widetilde{\varkappa}\right)\right),\xi\left(\Xi(\vartheta,\varpi,\varkappa),\Xi\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)\right)\right)\\ \leq\ &\ell\left(\phi\left(\xi\left(r\vartheta,r\widetilde{\vartheta}\right),\xi(r\varpi,r\widetilde{\varpi}),\xi(r\varkappa,r\widetilde{\varkappa})\right),\aleph\left(\left(r\vartheta,r\widetilde{\vartheta}\right),(r\varpi,r\widetilde{\varpi}),(r\varkappa,r\widetilde{\varkappa})\right)\right),\end{aligned}$$

where

$$\begin{aligned}&\aleph\left(\left(r\vartheta,r\widetilde{\vartheta}\right),(r\varpi,r\widetilde{\varpi}),(r\varkappa,r\widetilde{\varkappa})\right)\\ =\ &\max\left\{\begin{array}{c}\xi\left(r\vartheta,r\widetilde{\vartheta}\right),\xi(r\varpi,r\widetilde{\varpi}),\xi(r\varkappa,r\widetilde{\varkappa}),\\ \xi(r\vartheta,\Xi(\vartheta,\varpi,\varkappa)),\xi(r\varpi,\Xi(\varpi,\varkappa,\vartheta)),\xi(r\varkappa,\Xi(\varkappa,\vartheta,\varpi)),\\ \xi\left(r\widetilde{\vartheta},\Xi\left(r\widetilde{\vartheta},r\widetilde{\varpi},r\widetilde{\varkappa}\right)\right),\xi\left(r\widetilde{\varpi},\Xi\left(r\widetilde{\varpi},r\widetilde{\varkappa},r\widetilde{\vartheta}\right)\right),\xi\left(r\varkappa,\Xi\left(r\widetilde{\varkappa},r\widetilde{\vartheta},r\widetilde{\varpi}\right)\right)\end{array}\right\};\end{aligned}$$

($\triangle_{ii}$) *Ξ and r is α_A^{GT}, and there exist $\vartheta_0,\varpi_0,\varkappa_0\in\mho$,*

$$(r\vartheta_0,\Xi(\vartheta_0,\varpi_0,\varkappa_0)),\ (r\varpi_0,\Xi(\varpi_0,\varkappa_0,\vartheta_0)),\ (r\varkappa_0,\Xi(\varkappa_0,\vartheta_0,\varpi_0))\in\nabla_{\preceq}$$

so that

$$\begin{aligned}\alpha((r\vartheta_0,r\varpi_0,r\varkappa_0),(\Xi(\vartheta_0,\varpi_0,\varkappa_0),\Xi(\varpi_0,\varkappa_0,\vartheta_0),\Xi(\vartheta_0,\varpi_0,\varkappa_0)))&\geq 1,\\ \alpha((r\varpi_0,r\varkappa_0,r\vartheta_0),(\Xi(\varpi_0,\varkappa_0,\vartheta_0),\Xi(\varkappa_0,\vartheta_0,\varpi_0),\Xi(\varpi_0,\varkappa_0,\vartheta_0)))&\geq 1\end{aligned}$$

and

$$\alpha((r\varkappa_0,r\vartheta_0,r\varpi_0),(\Xi(\varkappa_0,\vartheta_0,\varpi_0),\Xi(\vartheta_0,\varpi_0,\varkappa_0),\Xi(\varkappa_0,\vartheta_0,\varpi_0)))\geq 1;$$

($\triangle_{iii}$) *If* $\lim_{i\to\infty}\xi(r\vartheta_i, r\vartheta_{i+1}) = 0$, $\lim_{i\to\infty}\xi(r\varpi_i, r\varpi_{i+1}) = 0$ *and* $\lim_{i\to\infty}\xi(r\varkappa_i, r\varkappa_{i+1}) = 0$, *then*

$$\sup\{\xi(r\vartheta_0, r\vartheta_i), \xi(r\varpi_0, r\varpi_i), \xi(r\varkappa_0, r\varkappa_i)\} < \infty,$$

where $\{\vartheta_i\}$, $\{\varpi_i\}$ *and* $\{\varkappa_i\}$ *are sequences in* $\mho$;

($\triangle_{iv}$) Ξ *is* $\xi-$*continuous.*

Then Ξ *and* r *have a tripled coincidence point (TCP) in* $\mho$.

Proof. Assume that $\vartheta_0, \varpi_0, \varkappa_0 \in \mho$ justifying assertion ($\triangle_{ii}$). Because $\Xi\left(\mho^3\right) \subseteq r(\mho)$, we can select $\vartheta_1, \varpi_1, \varkappa_1 \in \mho$ so that $r\vartheta_1 = \Xi(\vartheta_0, \varpi_0, \varkappa_0)$, $r\varpi_1 = \Xi(\varpi_0, \varkappa_0, \vartheta_0)$ and $r\varkappa_1 = \Xi(\varkappa_0, \vartheta_0, \varpi_0)$. Analogously, $r\vartheta_2 = \Xi(\vartheta_1, \varpi_1, \varkappa_1)$, $r\varpi_2 = \Xi(\varpi_1, \varkappa_1, \vartheta_1)$ and $r\varkappa_2 = \Xi(\varkappa_1, \vartheta_1, \varpi_1)$. In the same scenario, $\{\vartheta_i\}$, $\{\varpi_i\}$ and $\{\varkappa_i\}$ are obtained with

$$r\vartheta_{i+1} = \Xi(\vartheta_i, \varpi_i, \varkappa_i),\ r\varpi_{i+1} = \Xi(\varpi_i, \varkappa_i, \vartheta_i) \text{ and } r\varkappa_{i+1} = \Xi(\varkappa_i, \vartheta_i, \varpi_i).$$

For some natural number i_0, if $r\vartheta_{i_0+1} = r\vartheta_{i_0}$, $r\varpi_{i_0+1} = r\varpi_{i_0}$ and $r\varkappa_{i_0+1} = r\varkappa_{i_0}$, then Ξ and r have a TCP. So, for some positive integer i, assume that

$$r\vartheta_{i+1} \neq r\vartheta_i,\ r\varpi_{i+1} \neq r\varpi_i \text{ and } r\varkappa_{i+1} \neq r\varkappa_i.$$

Based on assumption ($\triangle_{ii}$), we have

$$(r\vartheta_0, r\vartheta_1) \in \nabla_{\preceq}, (r\varpi_0, r\varpi_1) \in \nabla_{\preceq} \text{ and } (r\varkappa_0, r\varkappa_1) \in \nabla_{\preceq}.$$

Because Ξ is $\preceq -r$ monotone,

$$\begin{aligned}
(\Xi(\vartheta_0, \varpi_0, \varkappa_0), \Xi(\vartheta_1, \varpi_1, \varkappa_1)) &\in \nabla_{\preceq},\\
(\Xi(\varpi_0, \varkappa_0, \vartheta_0), \Xi(\varpi_1, \varkappa_1, \vartheta_1)) &\in \nabla_{\preceq},\\
(\Xi(\varkappa_0, \vartheta_0, \varpi_0), \Xi(\varkappa_1, \vartheta_1, \varpi_1)) &\in \nabla_{\preceq},
\end{aligned}$$

that is,

$$(r\vartheta_1, r\vartheta_2) \in \nabla_{\preceq}, (r\varpi_1, r\varpi_2) \in \nabla_{\preceq} \text{ and } (r\varkappa_1, r\varkappa_2) \in \nabla_{\preceq}.$$

Repeating the same approach, we get

$$(r\vartheta_i, r\vartheta_{i+1}) \in \nabla_{\preceq}, (r\varpi_i, r\varpi_{i+1}) \in \nabla_{\preceq} \text{ and } (r\varkappa_i, r\varkappa_{i+1}) \in \nabla_{\preceq},\ \forall i \in \mathbb{N}.$$

From the transitivity of $\preceq$, one can obtain

$$(r\vartheta_i, r\vartheta_{i+m}) \in \nabla_{\preceq}, (r\varpi_i, r\varpi_{i+m}) \in \nabla_{\preceq} \text{ and } (r\varkappa_i, r\varkappa_{i+m}) \in \nabla_{\preceq},\ \forall i, m \in \mathbb{N}.$$

Again, applying the postulate ($\triangle_{ii}$), we have

$$\begin{aligned}
&\alpha((r\vartheta_0, r\varpi_0, r\varkappa_0), (r\vartheta_1, r\varpi_1, r\varkappa_1))\\
=\ &\alpha((r\vartheta_0, r\varpi_0, r\varkappa_0), (\Xi(\vartheta_0, \varpi_0, \varkappa_0), \Xi(\varpi_0, \varkappa_0, \vartheta_0), \Xi(\varkappa_0, \vartheta_0, \varpi_0)))\\
\geq\ &1.
\end{aligned}$$

Since Ξ and r are α_A^{GT},

$$\begin{aligned} & \alpha\left(\left(r\vartheta_1, r\varpi_1, r\varkappa_1\right), \left(r\vartheta_2, r\varpi_2, r\varkappa_2\right)\right) \\ = & \alpha\begin{pmatrix} \left(\Xi\left(\vartheta_0, \varpi_0, \varkappa_0\right), \Xi\left(\varpi_0, \varkappa_0, \vartheta_0\right), \Xi\left(\varkappa_0, \vartheta_0, \varpi_0\right)\right), \\ \left(\Xi\left(\vartheta_1, \varpi_1, \varkappa_1\right), \Xi\left(\varpi_1, \varkappa_1, \vartheta_1\right), \Xi\left(\varkappa_1, \vartheta_1, \varpi_1\right)\right) \end{pmatrix}. \end{aligned}$$

By induction, one can deduce that

$$\alpha\left(\left(r\vartheta_i, r\varpi_i, r\varkappa_i\right), \left(r\vartheta_{i+1}, r\varpi_{i+1}, r\varkappa_{i+1}\right)\right) \geq 1, \ \forall i \in \mathbb{N}.$$

Analogously, we can obtain

$$\alpha\left(\left(r\varpi_i, r\varkappa_i, r\vartheta_i\right), \left(r\varpi_{i+1}, r\varkappa_{i+1}, r\vartheta_{i+1}\right)\right) \geq 1, \ \forall i \in \mathbb{N},$$

and

$$\alpha\left(\left(r\varkappa_i, r\vartheta_i, r\varpi_i\right), \left(r\varkappa_{i+1}, r\vartheta_{i+1}, r\varpi_{i+1}\right)\right) \geq 1, \ \forall i \in \mathbb{N}.$$

Since Ξ and r are α_A^{GT},

$$\begin{aligned} \alpha\left(\left(r\vartheta_i, r\varpi_i, r\varkappa_i\right), \left(r\vartheta_{i+m}, r\varpi_{i+m}, r\varkappa_{i+m}\right)\right) & \geq 1, \\ \alpha\left(\left(r\varpi_i, r\varkappa_i, r\vartheta_i\right), \left(r\varpi_{i+m}, r\varkappa_{i+m}, r\vartheta_{i+m}\right)\right) & \geq 1, \\ \alpha\left(\left(r\varkappa_i, r\vartheta_i, r\varpi_i\right), \left(r\varkappa_{i+m}, r\vartheta_{i+m}, r\varpi_{i+m}\right)\right) & \geq 1, \ \forall i, m \in \mathbb{N}. \end{aligned}$$

Next, we want to prove $\lim_{i\to\infty} \xi\left(r\vartheta_i, r\vartheta_{i+1}\right) = 0$, $\lim_{i\to\infty} \xi\left(r\varpi_i, r\varpi_{i+1}\right) = 0$ and $\lim_{i\to\infty} \xi\left(r\varkappa_i, r\varkappa_{i+1}\right) = 0$. For this, we use the opposite technique, suppose that either $\lim_{i\to\infty} \xi\left(r\vartheta_i, r\vartheta_{i+1}\right) \neq 0$, or $\lim_{i\to\infty} \xi\left(r\varpi_i, r\varpi_{i+1}\right) \neq 0$ or $\lim_{i\to\infty} \xi\left(r\varkappa_i, r\varkappa_{i+1}\right) \neq 0$. Then there is $\epsilon > 0$ for which we have a subsequence $\{i_s\}$ so that $s \leq i_s$ and

$$\max\left\{\xi\left(r\vartheta_{i_s}, r\vartheta_{i_s+1}\right), \xi\left(r\varpi_{i_s}, r\varpi_{i_s+1}\right), \xi\left(r\varkappa_{i_s}, r\varkappa_{i_s+1}\right), \xi\left(r\varkappa_{i_s}, r\varkappa_{i_s+1}\right)\right\} \geq \epsilon.$$

Consider

$$\begin{aligned} & \wp\left(1, 1, \xi\left(r\vartheta_{i_s}, r\vartheta_{i_s+1}\right)\right) \\ = & \wp\left(1, 1, \xi\left(\Xi\left(\vartheta_{i_s-1}, \varpi_{i_s-1}, \varkappa_{i_s-1}\right), \Xi\left(\vartheta_{i_s}, \varpi_{i_s}, \varkappa_{i_s}\right)\right)\right) \\ \leq & \wp\left(1, \alpha\left(\left(\vartheta_{i_s-1}, \varpi_{i_s-1}, \varkappa_{i_s-1}\right), \left(\vartheta_{i_s}, \varpi_{i_s}, \varkappa_{i_s}\right)\right), \xi\left(\Xi\left(\vartheta_{i_s-1}, \varpi_{i_s-1}, \varkappa_{i_s-1}\right), \Xi\left(\vartheta_{i_s}, \varpi_{i_s}, \varkappa_{i_s}\right)\right)\right) \\ \leq & \ell\begin{pmatrix} \phi\left(\xi\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right), \xi\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right), \xi\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right)\right), \\ \aleph\left(\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right), \left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right), \left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right)\right) \end{pmatrix}. \end{aligned} \tag{2.1}$$

Again,

$$\begin{aligned} & \wp\left(1, 1, \xi\left(r\varpi_{i_s}, r\varpi_{i_s+1}\right)\right) \\ = & \wp\left(1, 1, \xi\left(\Xi\left(\varpi_{i_s-1}, \varkappa_{i_s-1}, \vartheta_{i_s-1}\right), \Xi\left(\varpi_{i_s}, \varkappa_{i_s}, \vartheta_{i_s}\right)\right)\right) \\ \leq & \wp\left(1, \alpha\left(\left(\varpi_{i_s-1}, \varkappa_{i_s-1}, \vartheta_{i_s-1}\right), \left(\varpi_{i_s}, \varkappa_{i_s}, \vartheta_{i_s}\right)\right), \xi\left(\Xi\left(\varpi_{i_s-1}, \varkappa_{i_s-1}, \vartheta_{i_s-1}\right), \Xi\left(\varpi_{i_s}, \varkappa_{i_s}, \vartheta_{i_s}\right)\right)\right) \\ \leq & \ell\begin{pmatrix} \phi\left(\xi\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right), \xi\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right), \xi\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right)\right), \\ \aleph\left(\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right), \left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right), \left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right)\right) \end{pmatrix}, \end{aligned} \tag{2.2}$$

and

$$
\begin{aligned}
& \wp\left(1,1,\xi\left(r\varkappa_{i_s}, r\varkappa_{i_s+1}\right)\right) \\
= & \wp\left(1,1,\xi\left(\Xi\left(\varkappa_{i_s-1}, \vartheta_{i_s-1}, \varpi_{i_s-1}\right), \Xi\left(\varkappa_{i_s}, \vartheta_{i_s}, \varpi_{i_s}\right)\right)\right) \\
\leq & \wp\left(1, \alpha\left(\left(\varkappa_{i_s-1}, \vartheta_{i_s-1}, \varpi_{i_s-1}\right),\left(\varkappa_{i_s}, \vartheta_{i_s}, \varpi_{i_s}\right)\right), \xi\left(\Xi\left(\varkappa_{i_s-1}, \vartheta_{i_s-1}, \varpi_{i_s-1}\right), \Xi\left(\varkappa_{i_s}, \vartheta_{i_s}, \varpi_{i_s}\right)\right)\right) \\
\leq & \ell\left(\begin{array}{c}
\phi\left(\xi\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right), \xi\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right), \xi\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right)\right), \\
\aleph\left(\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right),\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right),\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right)\right)
\end{array}\right).
\end{aligned} \tag{2.3}
$$

The inequalities (2.1)–(2.3) lead to

$$
\begin{aligned}
\xi\left(r\vartheta_{i_s}, r\vartheta_{i_s+1}\right) \leq & \phi\left(\xi\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right), \xi\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right), \xi\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right)\right) \\
& \times\aleph\left(\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right),\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right),\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right)\right),
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
\xi\left(r\varpi_{i_s}, r\varpi_{i_s+1}\right) \leq & \phi\left(\xi\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right), \xi\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right), \xi\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right)\right) \\
& \times\aleph\left(\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right),\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right),\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right)\right),
\end{aligned} \tag{2.5}
$$

and

$$
\begin{aligned}
\xi\left(r\varkappa_{i_s}, r\varkappa_{i_s+1}\right) \leq & \phi\left(\xi\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right), \xi\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right), \xi\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right)\right) \\
& \times\aleph\left(\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right),\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right),\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right)\right).
\end{aligned} \tag{2.6}
$$

Because $\phi(\vartheta, \varpi, \varkappa) \in [0,1)$ for any $\vartheta, \varpi, \varkappa \in [0,+\infty]$, we get

$$
\begin{aligned}
& \aleph\left(\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right),\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right),\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right)\right) \\
= & \aleph\left(\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right),\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right),\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right)\right) \\
= & \aleph\left(\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right),\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right),\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right)\right) \\
= & \max\left\{\xi\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right), \xi\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right), \xi\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right)\right\}.
\end{aligned} \tag{2.7}
$$

It follows from (2.4)–(2.7) that $\max\left\{\xi\left(r\vartheta_{i_s}, r\vartheta_{i_s+1}\right), \xi\left(r\varpi_{i_s}, r\varpi_{i_s+1}\right), \xi\left(r\varkappa_{i_s}, r\varkappa_{i_s+1}\right)\right\}$

$$
\begin{aligned}
& \max\left\{\xi\left(r\vartheta_{i_s}, r\vartheta_{i_s+1}\right), \xi\left(r\varpi_{i_s}, r\varpi_{i_s+1}\right), \xi\left(r\varkappa_{i_s}, r\varkappa_{i_s+1}\right)\right\} \\
\leq & \phi\left(\xi\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right), \xi\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right), \xi\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right)\right) \\
& \times\max\left\{\xi\left(r\varkappa_{i_s-1}, r\varkappa_{i_s}\right), \xi\left(r\vartheta_{i_s-1}, r\vartheta_{i_s}\right), \xi\left(r\varpi_{i_s-1}, r\varpi_{i_s-1}\right)\right\}.
\end{aligned}
$$

Using this concept, we can write

$$
\begin{aligned}
& \max\left\{\xi\left(r\vartheta_{i_s}, r\vartheta_{i_s+1}\right), \xi\left(r\varpi_{i_s}, r\varpi_{i_s+1}\right), \xi\left(r\varkappa_{i_s}, r\varkappa_{i_s+1}\right)\right\} \\
\leq & \prod_{v=1}^{i_s} \phi\left(\xi\left(r\varkappa_{i_s-v}, r\varkappa_{i_s+1-v}\right), \xi\left(r\vartheta_{i_s-v}, r\vartheta_{i_s+1-v}\right), \xi\left(r\varpi_{i_s-v}, r\varpi_{i_s+1-v}\right)\right) \\
& \times\max\left\{\xi\left(r\varkappa_0, r\varkappa_1\right), \xi\left(r\vartheta_0, r\vartheta_1\right), \xi\left(r\varpi_0, r\varpi_1\right)\right\}.
\end{aligned}
$$

Select v_s so that

$$
\begin{aligned}
&\phi\left(\xi\left(r\varkappa_{i_s-v_s}, r\varkappa_{i_s+1-v_s}\right), \xi\left(r\vartheta_{i_s-v_s}, r\vartheta_{i_s+1-v_s}\right), \xi\left(r\varpi_{i_s-v_s}, r\varpi_{i_s+1-v_s}\right)\right) \\
= &\max_{1\le v\le i_s}\left\{\phi\left(\xi\left(r\varkappa_{i_s-v}, r\varkappa_{i_s+1-v}\right), \xi\left(r\vartheta_{i_s-v}, r\vartheta_{i_s+1-v}\right), \xi\left(r\varpi_{i_s-v}, r\varpi_{i_s+1-v}\right)\right)\right\}.
\end{aligned}
$$

Consider

$$
\nabla = \limsup_{s\to\infty}\left(\phi\left(\xi\left(r\varkappa_{i_s-v_s}, r\varkappa_{i_s+1-v_s}\right), \xi\left(r\vartheta_{i_s-v_s}, r\vartheta_{i_s+1-v_s}\right), \xi\left(r\varpi_{i_s-v_s}, r\varpi_{i_s+1-v_s}\right)\right)\right).
$$

If $\nabla < 1$, then

$$
\lim_{s\to\infty}\max\left\{\xi\left(r\vartheta_{i_s}, r\vartheta_{i_s+1}\right), \xi\left(r\varpi_{i_s}, r\varpi_{i_s+1}\right), \xi\left(r\varkappa_{i_s}, r\varkappa_{i_s+1}\right)\right\} = 0.
$$

This contradicts the hypothesis. If $\nabla = 1$, for suitability, we assume that

$$
\lim_{s\to\infty}\phi\left(\xi\left(r\varkappa_{i_s-v_s}, r\varkappa_{i_s+1-v_s}\right), \xi\left(r\vartheta_{i_s-v_s}, r\vartheta_{i_s+1-v_s}\right), \xi\left(r\varpi_{i_s-v_s}, r\varpi_{i_s+1-v_s}\right)\right) = 1.
$$

Because $\phi \in \widetilde{\Phi}$,

$$
\begin{aligned}
\lim_{s\to\infty}\xi\left(r\varkappa_{i_s-v_s}, r\varkappa_{i_s+1-v_s}\right) &= 0,\ \lim_{s\to\infty}\xi\left(r\vartheta_{i_s-v_s}, r\vartheta_{i_s+1-v_s}\right) = 0 \\
\text{and }\lim_{s\to\infty}\xi\left(r\varpi_{i_s-v_s}, r\varpi_{i_s+1-v_s}\right) &= 0.
\end{aligned}
$$

That is, there is an $s_0 \in \mathbb{N}$ so that

$$
\begin{aligned}
\xi\left(r\varkappa_{i_{s_0}-v_{s_0}}, r\varkappa_{i_{s_0}+1-v_{s_0}}\right) &< \frac{\epsilon}{3},\ \xi\left(r\vartheta_{i_{s_0}-v_{s_0}}, r\vartheta_{i_{s_0}+1-v_{s_0}}\right) < \frac{\epsilon}{3} \\
\text{and }\xi\left(r\varpi_{i_{s_0}-v_{s_0}}, r\varpi_{i_{s_0}+1-v_{s_0}}\right) &< \frac{\epsilon}{3}.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\epsilon &\le \max\left\{\xi\left(r\vartheta_{i_{s_0}}, r\vartheta_{i_{s_0}+1}\right), \xi\left(r\varpi_{i_{s_0}}, r\varpi_{i_{s_0}+1}\right), \xi\left(r\varkappa_{i_{s_0}}, r\varkappa_{i_{s_0}+1}\right)\right\} \\
&\le \prod_{j=1}^{v_{s_0}}\phi\left(\xi\left(r\varkappa_{i_{s_0}-j}, r\varkappa_{i_{s_0}+1-j}\right), \xi\left(r\vartheta_{i_{s_0}-j}, r\vartheta_{i_{s_0}+1-j}\right), \xi\left(r\varpi_{i_{s_0}-j}, r\varpi_{i_{s_0}+1-j}\right)\right) \\
&\quad\times\max\left\{\begin{array}{c}\xi\left(r\varkappa_{i_{s_0}-v_{s_0}}, r\varkappa_{i_{s_0}+1-v_{s_0}}\right), \xi\left(r\vartheta_{i_{s_0}-v_{s_0}}, r\vartheta_{i_{s_0}+1-v_{s_0}}\right), \\ \xi\left(r\varpi_{i_{s_0}-v_{s_0}}, r\varpi_{i_{s_0}+1-v_{s_0}}\right)\end{array}\right\} \\
&< \frac{\epsilon}{3},
\end{aligned}
$$

a contradiction, hence, we have

$$
\lim_{i\to\infty}\xi\left(r\vartheta_i, r\vartheta_{i+1}\right) = 0,\ \lim_{i\to\infty}\xi\left(r\varpi_i, r\varpi_{i+1}\right) = 0 \text{ and } \lim_{i\to\infty}\xi\left(r\varkappa_i, r\varkappa_{i+1}\right) = 0. \tag{2.8}
$$

Now, we claim that $\{r\vartheta_i\}$, $\{r\varpi_i\}$ and $\{r\varkappa_i\}$ are $\xi-$Cauchy sequences. Suppose that $\{r\vartheta_i\}$, $\{r\varpi_i\}$ and $\{r\varkappa_i\}$ are not $\xi-$Cauchy sequences, so for each $s \in \mathbb{N}$ with subsequences $i_s, j_s \ge s$ such that

$$
\max\left\{\xi\left(r\vartheta_{i_s}, r\vartheta_{i_s+j_s}\right), \xi\left(r\varpi_{i_s}, r\varpi_{i_s+j_s}\right), \xi\left(r\varkappa_{i_s}, r\varkappa_{i_s+j_s}\right)\right\} \ge \epsilon' \text{ for some } \epsilon' > 0.
$$

Consider

$$\begin{aligned}
&\wp\left(1,1,\xi\left(r\vartheta_{i_s},r\vartheta_{i_s+j_s}\right)\right)\\
=\;&\wp\left(1,1,\xi\left(\Xi\left(\vartheta_{i_s-1},\varpi_{i_s-1},\varkappa_{i_s-1}\right),\Xi\left(\vartheta_{i_s+j_s-1},\varpi_{i_s+j_s-1},\varkappa_{i_s+j_s}\right)\right)\right)\\
\leq\;&\wp\left(\begin{array}{c}1,\alpha\left(\left(\vartheta_{i_s-1},\varpi_{i_s-1},\varkappa_{i_s-1}\right),\left(\vartheta_{i_s+j_s-1},\varpi_{i_s+j_s-1},\varkappa_{i_s+j_s-1}\right)\right),\\ \xi\left(\Xi\left(\vartheta_{i_s-1},\varpi_{i_s-1},\varkappa_{i_s-1}\right),\Xi\left(\vartheta_{i_s+j_s-1},\varpi_{i_s+j_s-1},\varkappa_{i_s+j_s-1}\right)\right)\end{array}\right)\\
\leq\;&\ell\left(\begin{array}{c}\phi\left(\xi\left(r\vartheta_{i_s-1},r\vartheta_{i_s+j_s-1}\right),\xi\left(r\varpi_{i_s-1},r\varpi_{i_s+j_s-1}\right),\xi\left(r\varkappa_{i_s-1},r\varkappa_{i_s+j_s-1}\right)\right),\\ \aleph\left(\left(r\vartheta_{i_s-1},r\vartheta_{i_s+j_s-1}\right),\left(r\varpi_{i_s-1},r\varpi_{i_s+j_s-1}\right),\left(r\varkappa_{i_s-1},r\varkappa_{i_s+j_s-1}\right)\right)\end{array}\right),
\end{aligned}\tag{2.9}$$

$$\begin{aligned}
&\wp\left(1,1,\xi\left(r\varpi_{i_s},r\varpi_{i_s+j_s}\right)\right)\\
=\;&\wp\left(1,1,\xi\left(\Xi\left(\varpi_{i_s-1},\varkappa_{i_s-1},\vartheta_{i_s-1}\right),\Xi\left(\varpi_{i_s+j_s-1},\varkappa_{i_s+j_s},\vartheta_{i_s+j_s-1}\right)\right)\right)\\
\leq\;&\wp\left(\begin{array}{c}1,\alpha\left(\left(\varpi_{i_s-1},\varkappa_{i_s-1},\vartheta_{i_s-1}\right),\left(\varpi_{i_s+j_s-1},\varkappa_{i_s+j_s-1},\vartheta_{i_s+j_s-1}\right)\right),\\ \xi\left(\Xi\left(\varpi_{i_s-1},\varkappa_{i_s-1},\vartheta_{i_s-1}\right),\Xi\left(\varpi_{i_s+j_s-1},\varkappa_{i_s+j_s-1},\vartheta_{i_s+j_s-1}\right)\right)\end{array}\right)\\
\leq\;&\ell\left(\begin{array}{c}\phi\left(\xi\left(r\varpi_{i_s+j_s-1},r\varpi_{i_s-1}\right),\xi\left(r\varkappa_{i_s+j_s-1},r\varkappa_{i_s-1}\right),\xi\left(r\vartheta_{i_s-1},r\vartheta_{i_s+j_s-1}\right)\right),\\ \aleph\left(\left(r\varpi_{i_s-1},r\varpi_{i_s+j_s-1}\right),\left(r\varkappa_{i_s-1},r\varkappa_{i_s+j_s-1}\right),\left(r\vartheta_{i_s-1},r\vartheta_{i_s+j_s-1}\right)\right)\end{array}\right),
\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}
&\wp\left(1,1,\xi\left(r\varkappa_{i_s},r\varkappa_{i_s+j_s}\right)\right)\\
=\;&\wp\left(1,1,\xi\left(\Xi\left(\varkappa_{i_s-1},\vartheta_{i_s-1},\varpi_{i_s-1}\right),\Xi\left(\varkappa_{i_s+j_s},\vartheta_{i_s+j_s-1}\right),\varpi_{i_s+j_s-1}\right)\right)\\
\leq\;&\wp\left(\begin{array}{c}1,\alpha\left(\left(\varkappa_{i_s-1},\vartheta_{i_s-1},\varpi_{i_s-1}\right),\left(\varkappa_{i_s+j_s-1},\vartheta_{i_s+j_s-1},\varpi_{i_s+j_s-1}\right)\right),\\ \xi\left(\Xi\left(\varkappa_{i_s-1},\vartheta_{i_s-1},\varpi_{i_s-1}\right),\Xi\left(\varkappa_{i_s+j_s-1},\vartheta_{i_s+j_s-1},\varpi_{i_s+j_s-1}\right)\right)\end{array}\right)\\
\leq\;&\ell\left(\begin{array}{c}\phi\left(\xi\left(r\varkappa_{i_s+j_s-1},r\varkappa_{i_s-1}\right),\xi\left(r\vartheta_{i_s-1},r\vartheta_{i_s+j_s-1}\right),\xi\left(r\varpi_{i_s+j_s-1},r\varpi_{i_s-1}\right)\right),\\ \aleph\left(\left(r\varkappa_{i_s-1},r\varkappa_{i_s+j_s-1}\right),\left(r\vartheta_{i_s-1},r\vartheta_{i_s+j_s-1}\right),\left(r\varpi_{i_s-1},r\varpi_{i_s+j_s-1}\right)\right)\end{array}\right).
\end{aligned}\tag{2.11}$$

Based on (2.9)–(2.11) and the above properties, one can obtain

$$\begin{aligned}
\xi\left(r\vartheta_{i_s},r\vartheta_{i_s+j_s}\right)\;\leq\;&\phi\left(\xi\left(r\varkappa_{i_s+j_s-1},r\varkappa_{i_s-1}\right),\xi\left(r\vartheta_{i_s-1},r\vartheta_{i_s+j_s-1}\right),\xi\left(r\varpi_{i_s+j_s-1},r\varpi_{i_s-1}\right)\right)\\
&\times\aleph\left(\left(r\varkappa_{i_s-1},r\varkappa_{i_s+j_s-1}\right),\left(r\vartheta_{i_s-1},r\vartheta_{i_s+j_s-1}\right),\left(r\varpi_{i_s-1},r\varpi_{i_s+j_s-1}\right)\right),
\end{aligned}\tag{2.12}$$

$$\begin{aligned}
\xi\left(r\varpi_{i_s},r\varpi_{i_s+j_s}\right)\;\leq\;&\phi\left(\xi\left(r\varpi_{i_s+j_s-1},r\varpi_{i_s-1}\right),\xi\left(r\varkappa_{i_s+j_s-1},r\varkappa_{i_s-1}\right),\xi\left(r\vartheta_{i_s-1},r\vartheta_{i_s+j_s-1}\right)\right)\\
&\times\aleph\left(\left(r\varpi_{i_s-1},r\varpi_{i_s+j_s-1}\right),\left(r\varkappa_{i_s-1},r\varkappa_{i_s+j_s-1}\right),\left(r\vartheta_{i_s-1},r\vartheta_{i_s+j_s-1}\right)\right),
\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}
\xi\left(r\varkappa_{i_s},r\varkappa_{i_s+j_s}\right)\;\leq\;&\phi\left(\xi\left(r\varkappa_{i_s+j_s-1},r\varkappa_{i_s-1}\right),\xi\left(r\vartheta_{i_s-1},r\vartheta_{i_s+j_s-1}\right),\xi\left(r\varpi_{i_s+j_s-1},r\varpi_{i_s-1}\right)\right)\\
&\times\aleph\left(\left(r\varkappa_{i_s-1},r\varkappa_{i_s+j_s-1}\right),\left(r\vartheta_{i_s-1},r\vartheta_{i_s+j_s-1}\right),\left(r\varpi_{i_s-1},r\varpi_{i_s+j_s-1}\right)\right),
\end{aligned}\tag{2.14}$$

respectively. Using (2.8), we get

$$\begin{aligned}
&\aleph\left(\left(r\varkappa_{i_s-1}, r\varkappa_{i_s+j_s-1}\right), \left(r\vartheta_{i_s-1}, r\vartheta_{i_s+j_s-1}\right), \left(r\varpi_{i_s-1}, r\varpi_{i_s+j_s-1}\right)\right) \\
= \;& \aleph\left(\left(r\varpi_{i_s-1}, r\varpi_{i_s+j_s-1}\right), \left(r\varkappa_{i_s-1}, r\varkappa_{i_s+j_s-1}\right), \left(r\vartheta_{i_s-1}, r\vartheta_{i_s+j_s-1}\right)\right) \\
= \;& \aleph\left(\left(r\varkappa_{i_s-1}, r\varkappa_{i_s+j_s-1}\right), \left(r\vartheta_{i_s-1}, r\vartheta_{i_s+j_s-1}\right), \left(r\varpi_{i_s-1}, r\varpi_{i_s+j_s-1}\right)\right) \\
= \;& \max\{\xi\left(r\varkappa_{i_s-1}, r\varkappa_{i_s+j_s-1}\right), \xi\left(r\vartheta_{i_s-1}, r\vartheta_{i_s+j_s-1}\right), \xi\left(r\varpi_{i_s-1}, r\varpi_{i_s+j_s-1}\right). \qquad (2.15)
\end{aligned}$$

From (2.12)–(2.15), we have

$$\begin{aligned}
&\max\left\{\xi\left(r\vartheta_{i_s}, r\vartheta_{i_s+j_s}\right), \xi\left(r\varpi_{i_s}, r\varpi_{i_s+j_s}\right), \xi\left(r\varkappa_{i_s}, r\varkappa_{i_s+j_s}\right)\right\} \\
\leq \;& \phi\left(\xi\left(r\varkappa_{i_s+j_s-1}, r\varkappa_{i_s-1}\right), \xi\left(r\vartheta_{i_s-1}, r\vartheta_{i_s+j_s-1}\right), \xi\left(r\varpi_{i_s+j_s-1}, r\varpi_{i_s-1}\right)\right) \\
&\times\max\{\xi\left(r\varkappa_{i_s-1}, r\varkappa_{i_s+j_s-1}\right), \xi\left(r\vartheta_{i_s-1}, r\vartheta_{i_s+j_s-1}\right), \xi\left(r\varpi_{i_s-1}, r\varpi_{i_s+j_s-1}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\max\left\{\xi\left(r\vartheta_{i_s}, r\vartheta_{i_s+j_s}\right), \xi\left(r\varpi_{i_s}, r\varpi_{i_s+j_s}\right), \xi\left(r\varkappa_{i_s}, r\varkappa_{i_s+j_s}\right)\right\} \\
\leq \;& \prod_{v=1}^{i_s}\phi\left(\xi\left(r\varkappa_{i_s-v}, r\varkappa_{i_s+j_s-v}\right), \xi\left(r\vartheta_{i_s-v}, r\vartheta_{i_s+j_s-v}\right), \xi\left(r\varpi_{i_s-v}, r\varpi_{i_s+j_s-v}\right)\right) \\
&\times\max\left\{\xi\left(r\varkappa_0, r\varkappa_{j_s}\right), \xi\left(r\vartheta_0, r\vartheta_{j_s}\right), \xi\left(r\varpi_0, r\varpi_{j_s}\right)\right\}.
\end{aligned}$$

Select v_s so that

$$\begin{aligned}
&\phi\left(\xi\left(r\varkappa_{i_s-v_s}, r\varkappa_{i_s+j_s-v_s}\right), \xi\left(r\vartheta_{i_s-v_s}, r\vartheta_{i_s+j_s-v_s}\right), \xi\left(r\varpi_{i_s-v_s}, r\varpi_{i_s+j_s-v_s}\right)\right) \\
= \;& \max_{1\leq v\leq i_s}\left\{\phi\left(\xi\left(r\varkappa_{i_s-v}, r\varkappa_{i_s+j_s-v}\right), \xi\left(r\vartheta_{i_s-v}, r\vartheta_{i_s+j_s-v}\right), \xi\left(r\varpi_{i_s-v}, r\varpi_{i_s+j_s-v}\right)\right)\right\}.
\end{aligned}$$

Define

$$\nabla^* = \limsup_{s\to\infty}\left(\phi\left(\xi\left(r\varkappa_{i_s-v_s}, r\varkappa_{i_s+j_s-v_s}\right), \xi\left(r\vartheta_{i_s-v_s}, r\vartheta_{i_s+j_s-v_s}\right), \xi\left(r\varpi_{i_s-v_s}, r\varpi_{i_s+j_s-v_s}\right)\right)\right).$$

If $\nabla^* < 1$, then

$$\lim_{s\to\infty}\max\left\{\xi\left(r\vartheta_{i_s}, r\vartheta_{i_s+j_s}\right), \xi\left(r\varpi_{i_s}, r\varpi_{i_s+j_s}\right), \xi\left(r\varkappa_{i_s}, r\varkappa_{i_s+j_s}\right)\right\} = 0.$$

This is impossible to happen because of our hypothesis.

If $\nabla^* = 1$, for convenience, consider

$$\lim_{s\to\infty}\phi\left(\xi\left(r\varkappa_{i_s-v_s}, r\varkappa_{i_s+j_s-v_s}\right), \xi\left(r\vartheta_{i_s-v_s}, r\vartheta_{i_s+j_s-v_s}\right), \xi\left(r\varpi_{i_s-v_s}, r\varpi_{i_s+j_s-v_s}\right)\right) = 1.$$

Because $\phi \in \widetilde{\Phi}$,

$$\begin{aligned}
\lim_{s\to\infty}\xi\left(r\varkappa_{i_s-v_s}, r\varkappa_{i_s+j_s-v_s}\right) &= 0, \;\; \lim_{s\to\infty}\xi\left(r\vartheta_{i_s-v_s}, r\vartheta_{i_s+j_s-v_s}\right) = 0 \\
\text{and } \lim_{s\to\infty}\xi\left(r\varpi_{i_s-v_s}, r\varpi_{i_s+j_s-v_s}\right) &= 0.
\end{aligned}$$

That is, there is an $s_0 \in \mathbb{N}$ so that

$$\begin{aligned} \xi\left(r\varkappa_{i_{s_0}-v_{s_0}}, r\varkappa_{i_{s_0}+j_{s_0}-v_{s_0}}\right) &< \frac{\epsilon}{3},\ \xi\left(r\vartheta_{i_{s_0}-v_{s_0}}, r\vartheta_{i_{s_0}+j_{s_0}-v_{s_0}}\right) < \frac{\epsilon}{3} \\ \text{and } \xi\left(r\varpi_{i_{s_0}-v_{s_0}}, r\varpi_{i_{s_0}+j_{s_0}-v_{s_0}}\right) &< \frac{\epsilon}{3}. \end{aligned}$$

Thus, we get

$$\begin{aligned} \epsilon' &\leq \max\left\{\xi\left(r\vartheta_{i_{s_0}}, r\vartheta_{i_{s_0}+j_{s_0}}\right), \xi\left(r\varpi_{i_{s_0}}, r\varpi_{i_{s_0}+j_{s_0}}\right), \xi\left(r\varkappa_{i_{s_0}}, r\varkappa_{i_{s_0}+j_{s_0}}\right)\right\} \\ &\leq \prod_{j=1}^{v_{s_0}} \phi\left(\xi\left(r\varkappa_{i_{s_0}-j}, r\varkappa_{i_{s_0}+j_{s_0}-j}\right), \xi\left(r\vartheta_{i_{s_0}-j}, r\vartheta_{i_{s_0}+j_{s_0}-j}\right), \xi\left(r\varpi_{i_{s_0}-j}, r\varpi_{i_{s_0}+j_{s_0}-j}\right)\right) \\ &\quad \times \max\left\{\begin{array}{c} \xi\left(r\varkappa_{i_{s_0}-v_{s_0}}, r\varkappa_{i_{s_0}+j_{s_0}-v_{s_0}}\right), \xi\left(r\vartheta_{i_{s_0}-v_{s_0}}, r\vartheta_{i_{s_0}+j_{s_0}-v_{s_0}}\right), \\ \xi\left(r\varpi_{i_{s_0}-v_{s_0}}, r\varpi_{i_{s_0}+j_{s_0}-v_{s_0}}\right) \end{array}\right\} \\ &< \frac{\epsilon'}{3}. \end{aligned}$$

This is a contradiction, therefore, $\{r\vartheta_i\}$, $\{r\varpi_i\}$ and $\{r\varkappa_i\}$ are $\xi-$Cauchy sequences. The completeness of $(\mho, \xi)$ implies that for some $u, u', u'' \in \mho$,

$$\begin{aligned} \lim_{i\to\infty} \xi\left(\Xi\left(\vartheta_i, \varpi_i, \varkappa_i\right), u\right) &= \lim_{i\to\infty} \xi\left(r\vartheta_i, u\right) = 0, \\ \lim_{i\to\infty} \xi\left(\Xi\left(\varpi_i, \varkappa_i, \vartheta_i\right), u'\right) &= \lim_{i\to\infty} \xi\left(r\varpi_i, u'\right) = 0 \\ \text{and } \lim_{i\to\infty} \xi\left(\Xi\left(\varkappa_i, \vartheta_i, \varpi_i\right), u''\right) &= \lim_{i\to\infty} \xi\left(r\varkappa_i, u''\right) = 0. \end{aligned}$$

Since r is continuous, we have

$$\begin{aligned} \lim_{i\to\infty} \xi\left(r\left(\Xi\left(\vartheta_i, \varpi_i, \varkappa_i\right)\right), ru\right) &= 0, \\ \lim_{i\to\infty} \xi\left(r\left(\Xi\left(\varpi_i, \varkappa_i, \vartheta_i\right)\right), ru'\right) &= 0 \\ \text{and } \lim_{i\to\infty} \xi\left(r\left(\Xi\left(\varkappa_i, \vartheta_i, \varpi_i\right)\right), ru''\right) &= 0. \end{aligned}$$

Also, the continuity of Ξ leads to

$$\begin{aligned} \lim_{i\to\infty} \xi\left(\Xi\left(r\vartheta_i, r\varpi_i, r\varkappa_i\right), \Xi\left(u, u', u''\right)\right) &= 0, \\ \lim_{i\to\infty} \xi\left(\Xi\left(r\varpi_i, r\varkappa_i, r\vartheta_i\right), \Xi\left(u', u'', u\right)\right) &= 0 \\ \text{and } \lim_{i\to\infty} \xi\left(\Xi\left(r\varkappa_i, r\vartheta_i, r\varpi_i\right), \Xi\left(u'', u, u'\right)\right) &= 0. \end{aligned}$$

By alteration between Ξ and r, we have that $\Xi\left(u, u', u''\right) = ru$, $\Xi\left(u', u'', u\right) = ru'$ and $\Xi\left(u'', u, u'\right) = ru''$, therefore Ξ and r have a TCP. □

Corollary 2.5. *The results of Theorem 2.4 are still true if we replace the stipulation* $(\triangle_i)$ *with one of the hypotheses below: (note* $\aleph\left(\left(r\vartheta, r\widetilde{\vartheta}\right), \left(r\varpi, r\widetilde{\varpi}\right), \left(r\varkappa, r\widetilde{\varkappa}\right)\right)$ *is defined in the above theorem):*

($\dagger_1$) *There exist $\phi \in \Phi$ and $z > 1$, fulfilling, for any $\left(r\vartheta, r\widetilde{\vartheta}\right) \in \nabla_{\leq}$, $(r\varpi, r\widetilde{\varpi}) \in \nabla_{\leq}$ and $(r\varkappa, r\widetilde{\varkappa}) \in \nabla_{\leq}$, the following inequality is satisfied:*

$$\begin{aligned} &\left(\xi\left(\Xi(\vartheta,\varpi,\varkappa), \Xi\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)\right)+z\right)^{\alpha\left((r\vartheta,r\varpi,r\varkappa),\left(r\widetilde{\vartheta},r\widetilde{\varpi},r\widetilde{\varkappa}\right)\right)} \\ \leq\ & \phi\left(\xi\left(r\vartheta,r\widetilde{\vartheta}\right),\xi\left(r\varpi,r\widetilde{\varpi}\right),\xi\left(r\varkappa,r\widetilde{\varkappa}\right)\right)\aleph\left(\left(r\vartheta,r\widetilde{\vartheta}\right),\left(r\varpi,r\widetilde{\varpi}\right),\left(r\varkappa,r\widetilde{\varkappa}\right)\right)+z. \end{aligned}$$

($\dagger_2$) *There exist $\phi \in \Phi$ and $z > 1$, fulfilling, for any $\left(r\vartheta, r\widetilde{\vartheta}\right) \in \nabla_{\leq}$, $(r\varpi, r\widetilde{\varpi}) \in \nabla_{\leq}$ and $(r\varkappa, r\widetilde{\varkappa}) \in \nabla_{\leq}$, the following inequality is satisfied:*

$$\begin{aligned} &\left(\alpha\left((r\vartheta,r\varpi,r\varkappa),\left(r\widetilde{\vartheta},r\widetilde{\varpi},r\widetilde{\varkappa}\right)\right)+z\right)^{\xi\left(\Xi(\vartheta,\varpi,\varkappa),\Xi\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)\right)} \\ \leq\ & (1+z)^{\phi\left(\xi\left(r\vartheta,r\widetilde{\vartheta}\right),\xi\left(r\varpi,r\widetilde{\varpi}\right),\xi\left(r\varkappa,r\widetilde{\varkappa}\right)\right)\aleph\left(\left(r\vartheta,r\widetilde{\vartheta}\right),\left(r\varpi,r\widetilde{\varpi}\right),\left(r\varkappa,r\widetilde{\varkappa}\right)\right)}. \end{aligned}$$

($\dagger_3$) *There exists $\phi \in \Phi$, justifying, for any $\left(r\vartheta, r\widetilde{\vartheta}\right) \in \nabla_{\leq}$, $(r\varpi, r\widetilde{\varpi}) \in \nabla_{\leq}$ and $(r\varkappa, r\widetilde{\varkappa}) \in \nabla_{\leq}$, the following inequality holds:*

$$\begin{aligned} &\xi\left(\Xi(\vartheta,\varpi,\varkappa), \Xi\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)\right) \\ \leq\ & \phi\left(\xi\left(r\vartheta,r\widetilde{\vartheta}\right),\xi\left(r\varpi,r\widetilde{\varpi}\right),\xi\left(r\varkappa,r\widetilde{\varkappa}\right)\right)\aleph\left(\left(r\vartheta,r\widetilde{\vartheta}\right),\left(r\varpi,r\widetilde{\varpi}\right),\left(r\varkappa,r\widetilde{\varkappa}\right)\right). \end{aligned}$$

($\dagger_4$) *There exists $\phi \in \Phi$, satisfying, for any $\left(r\vartheta, r\widetilde{\vartheta}\right) \in \nabla_{\leq}$, $(r\varpi, r\widetilde{\varpi}) \in \nabla_{\leq}$ and $(r\varkappa, r\widetilde{\varkappa}) \in \nabla_{\leq}$, the inequality below is obtained:*

$$\begin{aligned} &\left\{\alpha\left((r\vartheta,r\varpi,r\varkappa),\left(r\widetilde{\vartheta},r\widetilde{\varpi},r\widetilde{\varkappa}\right)\right)\right\}^{j}\left\{\xi\left(\Xi(\vartheta,\varpi,\varkappa), \Xi\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)\right)\right\}^{k} \\ \leq\ & \left\{\phi\left(\xi\left(r\vartheta,r\widetilde{\vartheta}\right),\xi\left(r\varpi,r\widetilde{\varpi}\right),\xi\left(r\varkappa,r\widetilde{\varkappa}\right)\right)\right\}^{k}\left\{\aleph\left(\left(r\vartheta,r\widetilde{\vartheta}\right),\left(r\varpi,r\widetilde{\varpi}\right),\left(r\varkappa,r\widetilde{\varkappa}\right)\right)\right\}^{k}, \end{aligned}$$

for all positive integers j and k.

Proof. The proof follows immediately from Example 2.3 by applying the following in Theorem 2.4.
($\dagger_1$) Use the value of $\wp$ and ℓ from Example 2.3(1).
($\dagger_2$) Use the value of $\wp$ and ℓ from Example 2.3(2).
($\dagger_3$) Use the value of $\wp$ and ℓ from Example 2.3(3).
($\dagger_4$) Use the value of $\wp$ and ℓ from Example 2.3(4). □

Remark 2.6.
- In a $b-$metric and a $b-$metric-like space, Corollary 2.5 holds.
- The stipulation ($\dagger_3$) of Corollary 2.5 generalizes Theorem 3.2 of [23].
- The stipulation ($\dagger_4$) of Corollary 2.5 extends and unifies Theorem 3.1 of [7] when $j = k = 1$.

Example 2.7. Consider $\mho = [0, +\infty]$. Define $\xi(\vartheta, \varpi) = \max\{\vartheta, \varpi\}$, $\vartheta, \varpi \in \mho$. Describe the mappings $\Xi : \mho^3 \to \mho$, $r : \mho \to \mho$ and $\alpha : \mho^3 \times \mho^3 \to [0, +\infty]$ by

$$\Xi(\vartheta,\varpi,\varkappa) = \begin{cases} \frac{\vartheta+\varpi+\varkappa}{3}, & \text{if } \vartheta,\varpi,\varkappa \in [0,+\infty), \\ +\infty, & \text{otherwise,} \end{cases} \quad r\vartheta = \begin{cases} 2\vartheta, & \text{if } \vartheta \in [0,+\infty), \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\alpha\left((\vartheta,\varpi,\varkappa),\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)\right) = \begin{cases} 6, & \text{if } \vartheta \leq \varpi \leq \varkappa \text{ and } \widetilde{\vartheta} \leq \widetilde{\varpi} \leq \widetilde{\varkappa}, \\ 0, & \text{otherwise,} \end{cases}$$

respectively. Assume that $\vartheta \leq \widetilde{\vartheta}$, $\varpi \leq \widetilde{\varpi}$ and $\varkappa \leq \widetilde{\varkappa}$. Then

$$\begin{aligned}\xi\left(\Xi(\vartheta,\varpi,\varkappa),\Xi\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)\right) &= \max\left\{\frac{\vartheta+\varpi+\varkappa}{3},\frac{\widetilde{\vartheta}+\widetilde{\varpi}+\widetilde{\varkappa}}{3}\right\}\\ &\leq \frac{1}{3}\max\left\{2\widetilde{\vartheta},2\widetilde{\varpi},2\widetilde{\varkappa}\right\}\\ &= \phi\left(\xi\left(r\vartheta,r\widetilde{\vartheta}\right),\xi\left(r\varpi,r\widetilde{\varpi}\right),\xi\left(r\varkappa,r\widetilde{\varkappa}\right)\right)\aleph\left(\left(r\vartheta,r\widetilde{\vartheta}\right),\left(r\varpi,r\widetilde{\varpi}\right),\left(r\varkappa,r\widetilde{\varkappa}\right)\right).\end{aligned}$$

Hence, Stipulation ($\dagger_3$) of Corollary 2.5 holds for $\phi(m,z)=\frac{1}{3}$, where $m,z\in[0,+\infty]$. In an easy way, it can be shown that all the hypotheses of Theorem 2.4 are fulfilled, and then there is a TCP for the mappings Ξ and r.

Notice that, here ξ is not a metric on $\mho$ because, if we take $\vartheta \leq \varpi \leq \varkappa \leq \widetilde{\varkappa}$, and $\vartheta \leq \widetilde{\varpi} \leq \widetilde{\vartheta} \leq \widetilde{\varkappa}$, we have

$$\begin{aligned}&\alpha\left((r\vartheta,r\varpi,r\varkappa),\left(r\widetilde{\vartheta},r\widetilde{\varpi},r\widetilde{\varkappa}\right)\right)\xi\left(\Xi(\vartheta,\varpi,\varkappa),\Xi\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)\right)\\ =\ & 6\max\left\{\frac{\vartheta+\varpi+\varkappa}{3},\frac{\widetilde{\vartheta}+\widetilde{\varpi}+\widetilde{\varkappa}}{3}\right\}\\ \leq\ & 6\left(\frac{\widetilde{\vartheta}+\widetilde{\varpi}+\widetilde{\varkappa}}{3}\right)\\ =\ & 2\widetilde{\vartheta}+2\widetilde{\varpi}+2\widetilde{\varkappa}\\ >\ & \phi\left(\xi\left(r\vartheta,r\widetilde{\vartheta}\right),\xi\left(r\varpi,r\widetilde{\varpi}\right),\xi\left(r\varkappa,r\widetilde{\varkappa}\right)\right).2\widetilde{\varkappa}\\ =\ & \phi\left(\xi\left(r\vartheta,r\widetilde{\vartheta}\right),\xi\left(r\varpi,r\widetilde{\varpi}\right),\xi\left(r\varkappa,r\widetilde{\varkappa}\right)\right)\aleph\left(\left(r\vartheta,r\widetilde{\vartheta}\right),\left(r\varpi,r\widetilde{\varpi}\right),\left(r\varkappa,r\widetilde{\varkappa}\right)\right).\end{aligned}$$

Therefore, in the sense of TCPs, Theorem 3.1 in [23] and one of the hypotheses of Theorems 3.1 and 3.2 in [7] fails in studying the existence of a TCP for r and Ξ.

Let Φ^* represent the class of mappings $\phi^*:[0,+\infty]\to[0,1)$ so that

$$\phi^*(\vartheta_i)\to 1 \text{ implies } \vartheta_i\to 0 \text{ for all } \vartheta_i\in[0,+\infty].$$

The second part of this section is neglected the continuity condition for the mapping Ξ as follows:

Theorem 2.8. *Let the following postulates be satisfied:*

(h_i) *For any Ω_{II} of the pair $(\wp,\ell)$, there exist $\alpha:\mho^3\times\mho^3\to[0,+\infty]$ and $\phi^*\in\Phi^*$ fulfilling, for any $\left(r\vartheta,r\widetilde{\vartheta}\right)\in\nabla_{\leq}$, $(r\varpi,r\widetilde{\varpi})\in\nabla_{\leq}$ and $(r\varkappa,r\widetilde{\varkappa})\in\nabla_{\leq}$, the below inequality is justified:*

$$\begin{aligned}&\wp\left(1,\alpha\left((r\vartheta,r\varpi,r\varkappa),\left(r\widetilde{\vartheta},r\widetilde{\varpi},r\widetilde{\varkappa}\right)\right),\xi\left(\Xi(\vartheta,\varpi,\varkappa),\Xi\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)\right)\right)\\ \leq\ & \ell\left(\phi^*\left(Q\left(\left(r\vartheta,r\widetilde{\vartheta}\right),\left(r\varpi,r\widetilde{\varpi}\right),\left(r\varkappa,r\widetilde{\varkappa}\right)\right)\right),Q\left(\left(r\vartheta,r\widetilde{\vartheta}\right),\left(r\varpi,r\widetilde{\varpi}\right),\left(r\varkappa,r\widetilde{\varkappa}\right)\right)\right),\end{aligned}$$

where

$$\begin{aligned}&Q\left(\left(r\vartheta,r\widetilde{\vartheta}\right),\left(r\varpi,r\widetilde{\varpi}\right),\left(r\varkappa,r\widetilde{\varkappa}\right)\right)\\ =\ & \max\left\{\begin{array}{c}\xi\left(r\vartheta,r\widetilde{\vartheta}\right),\xi\left(r\varpi,r\widetilde{\varpi}\right),\xi\left(r\varkappa,r\widetilde{\varkappa}\right),\\ \xi\left(r\vartheta,\Xi(\vartheta,\varpi,\varkappa)\right),\xi\left(r\varpi,\Xi(\varpi,\varkappa,\vartheta)\right),\xi\left(r\varkappa,\Xi(\varkappa,\vartheta,\varpi)\right),\\ \xi\left(r\widetilde{\vartheta},\Xi\left(r\widetilde{\vartheta},r\widetilde{\varpi},r\widetilde{\varkappa}\right)\right),\xi\left(r\widetilde{\varpi},\Xi\left(r\widetilde{\varpi},r\widetilde{\varkappa},r\widetilde{\vartheta}\right)\right),\xi\left(r\varkappa,\Xi\left(r\widetilde{\varkappa},r\widetilde{\vartheta},r\widetilde{\varpi}\right)\right)\end{array}\right\};\end{aligned}$$

(h_{ii}) *Ξ and r is α_A^{GT} and there exist $\vartheta_0, \varpi_0, \varkappa_0 \in \mho$,*

$$(r\vartheta_0, \Xi(\vartheta_0, \varpi_0, \varkappa_0)), \ (r\varpi_0, \Xi(\varpi_0, \varkappa_0, \vartheta_0)), \ (r\varkappa_0, \Xi(\varkappa_0, \vartheta_0, \varpi_0)) \in \nabla_{\leq}$$

so that

$$\begin{aligned} \alpha((r\vartheta_0, r\varpi_0, r\varkappa_0), (\Xi(\vartheta_0, \varpi_0, \varkappa_0), \Xi(\varpi_0, \varkappa_0, \vartheta_0), \Xi(\vartheta_0, \varpi_0, \varkappa_0))) &\geq 1, \\ \alpha((r\varpi_0, r\varkappa_0, r\vartheta_0), (\Xi(\varpi_0, \varkappa_0, \vartheta_0), \Xi(\varkappa_0, \vartheta_0, \varpi_0), \Xi(\varpi_0, \varkappa_0, \vartheta_0))) &\geq 1 \end{aligned}$$

and

$$\alpha((r\varkappa_0, r\vartheta_0, r\varpi_0), (\Xi(\varkappa_0, \vartheta_0, \varpi_0), \Xi(\vartheta_0, \varpi_0, \varkappa_0), \Xi(\varkappa_0, \vartheta_0, \varpi_0))) \geq 1;$$

(h_{iii}) *If $\lim_{i\to\infty} \xi(r\vartheta_i, r\vartheta_{i+1}) = 0$, $\lim_{i\to\infty} \xi(r\varpi_i, r\varpi_{i+1}) = 0$ and $\lim_{i\to\infty} \xi(r\varkappa_i, r\varkappa_{i+1}) = 0$, then*

$$\sup\{\xi(r\vartheta_0, r\vartheta_i), \xi(r\varpi_0, r\varpi_i), \xi(r\varkappa_0, r\varkappa_i)\} < \infty,$$

where $\{\vartheta_i\}$, $\{\varpi_i\}$ and $\{\varkappa_i\}$ are sequences in $\mho$;

(h_{iv}) *If $\{\vartheta_i\}$, $\{\varpi_i\}$ and $\{\varkappa_i\}$ are sequences in $\mho$ with $(r\vartheta_i, r\vartheta_{i+1})$, $(r\varpi_i, r\varpi_{i+1})$, $(r\varkappa_i, r\varkappa_{i+1}) \in \nabla_{\leq}$,*

$$\begin{aligned} \alpha((r\vartheta_i, r\varpi_i, r\varkappa_i), (r\vartheta_{i+1}, r\varpi_{i+1}, r\varkappa_{i+1})) &\geq 1, \\ \alpha((r\varpi_i, r\varkappa_i, r\vartheta_i), (r\varpi_{i+1}, r\varkappa_{i+1}, r\vartheta_{i+1})) &\geq 1, \\ \alpha((r\varkappa_i, r\vartheta_i, r\varpi_i), (r\varkappa_{i+1}, r\vartheta_{i+1}, r\varpi_{i+1})) &\geq 1, \ \forall i \in \mathbb{N}, \end{aligned}$$

and $\lim_{i\to\infty} \xi(r\vartheta_i, u) = 0$, $\lim_{i\to\infty} \xi(r\varpi_i, u') = 0$, $\lim_{i\to\infty} \xi(r\varkappa_i, u'') = 0$, for each $u, u', u'' \in \mho$, then $(r\vartheta_i, ru)$, $(r\varpi_i, ru')$, $(r\varkappa_i, u'') \in \nabla_{\leq}$,

$$\begin{aligned} \alpha((r\vartheta_i, r\varpi_i, r\varkappa_i), (ru, ru', ru'')) &\geq 1, \\ \alpha((r\varpi_i, r\varkappa_i, r\vartheta_i), (ru', ru'', ru)) &\geq 1, \\ \alpha((r\varkappa_i, r\vartheta_i, r\varpi_i), (ru'', ru, ru')) &\geq 1, \ \forall i \in \mathbb{N}; \end{aligned}$$

(h_v) *There is $\Re \in (0, 1]$ so that*

$$\begin{aligned} \xi(ru, \Xi(u, u', u'')) &\leq \Re \limsup_{i\to\infty} \xi(\Xi(r\vartheta_i, r\varpi_i, r\varkappa_i), \Xi(u, u', u'')), \\ \xi(ru', \Xi(u', u'', u)) &\leq \Re \limsup_{i\to\infty} \xi(\Xi(r\varpi_i, r\varkappa_i, r\vartheta_i), \Xi(u', u'', u)) \ \textit{and} \\ \xi\left(ru'', \Xi(u'', u, u')\right) &\leq \Re \limsup_{i\to\infty} \xi(\Xi(r\varkappa_i, r\vartheta_i, r\varpi_i), \Xi(u'', u, u')). \end{aligned}$$

Then Ξ and r have a TCP.

Proof. Based on Theorem 2.4, the sequences $\{r\vartheta_i\}$, $\{r\varpi_i\}$ and $\{r\varkappa_i\}$ are obtained. Furthermore, replacing $\phi\left(\xi\left(r\vartheta, r\widetilde{\vartheta}\right), \xi(r\varpi, r\widetilde{\varpi}), \xi(r\varkappa, r\widetilde{\varkappa})\right)$ with $\phi^*\left(\xi\left(r\vartheta, r\widetilde{\vartheta}\right), \xi(r\varpi, r\widetilde{\varpi}), \xi(r\varkappa, r\widetilde{\varkappa})\right)$ in Theorem 2.4, where $\vartheta, \widetilde{\vartheta}, \varpi, \widetilde{\varpi}, \varkappa, \widetilde{\varkappa} \in \mho$, we conclude that the three sequences are Cauchy in $(\mho, \xi)$. The completness of this space leads to for some $u, u', u'' \in \mho$,

$$\lim_{i\to\infty} \xi(\Xi(r\vartheta_i, r\varpi_i, r\varkappa_i), u) = \lim_{i\to\infty} \xi(r\vartheta_i, u) = 0,$$

$$\begin{aligned}\lim_{i\to\infty}\xi\left(\Xi\left(r\varpi_i, r\varkappa_i, r\vartheta_i\right), u\right) &= \lim_{i\to\infty}\xi\left(r\varpi_i, u\right) = 0\\ \text{and } \lim_{i\to\infty}\xi\left(\Xi\left(r\varkappa_i, r\vartheta_i, r\varpi_i\right), u\right) &= \lim_{i\to\infty}\xi\left(r\varkappa_i, u\right) = 0.\end{aligned}$$

Since r is continuous,

$$\begin{aligned}\lim_{i\to\infty}\xi\left(r\Xi\left(r\vartheta_i, r\varpi_i, r\varkappa_i\right), ru\right) &= \lim_{i\to\infty}\xi\left(rr\vartheta_i, ru\right) = 0,\\ \lim_{i\to\infty}\xi\left(r\Xi\left(r\varpi_i, r\varkappa_i, r\vartheta_i\right), ru\right) &= \lim_{i\to\infty}\xi\left(r\varpi_i, ru\right) = 0\\ \text{and } \lim_{i\to\infty}\xi\left(r\Xi\left(r\varkappa_i, r\vartheta_i, r\varpi_i\right), ru\right) &= \lim_{i\to\infty}\xi\left(r\varkappa_i, ru\right) = 0.\end{aligned}$$

By postulates $(\triangle_i)$ and $(\triangle_{ii})$, we get

$$\begin{aligned}&\wp\left(1, 1, \xi\left(\Xi\left(r\vartheta_i, r\varpi_i, r\varkappa_i\right), \Xi\left(u, u', u''\right)\right)\right)\\ \leq\ &\wp\left(1, \alpha\left(\left(rr\vartheta_i, rr\varpi_i, rr\varkappa_i\right), \left(ru, ru', ru''\right)\right), \xi\left(\Xi\left(r\vartheta_i, r\varpi_i, r\varkappa_i\right), \Xi\left(u, u', u''\right)\right)\right)\\ \leq\ &\ell\begin{pmatrix}\phi^*\left(Q\left(\left(rr\vartheta_i, ru\right), \left(rr\varpi_i, ru'\right), \left(rr\varkappa_i, ru''\right)\right)\right),\\ Q\left(\left(rr\vartheta_i, ru\right), \left(rr\varpi_i, ru'\right), \left(rr\varkappa_i, ru''\right)\right)\end{pmatrix},\end{aligned}$$

$$\begin{aligned}&\wp\left(1, 1, \xi\left(\Xi\left(r\varpi_i, r\varkappa_i, r\vartheta_i\right), \Xi\left(u', u'', u\right)\right)\right)\\ \leq\ &\wp\left(1, \alpha\left(\left(rr\varpi_i, rr\varkappa_i, rr\vartheta_i\right), \left(ru', ru'', ru\right)\right), \xi\left(\Xi\left(r\varpi_i, r\varkappa_i, r\vartheta_i\right), \Xi\left(u', u'', u\right)\right)\right)\\ \leq\ &\ell\begin{pmatrix}\phi^*\left(Q\left(\left(rr\varpi_i, ru'\right), \left(rr\varkappa_i, ru''\right), \left(rr\vartheta_i, ru\right)\right)\right),\\ Q\left(\left(rr\varpi_i, ru'\right), \left(rr\varkappa_i, ru''\right), \left(rr\vartheta_i, ru\right)\right)\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}&\wp\left(1, 1, \xi\left(\Xi\left(r\varkappa_i, r\vartheta_i, r\varpi_i\right), \Xi\left(u'', u, u'\right)\right)\right)\\ \leq\ &\wp\left(1, \alpha\left(\left(rr\varkappa_i, rr\vartheta_i, rr\varpi_i\right), \left(ru'', ru, ru'\right)\right), \xi\left(\Xi\left(r\varkappa_i, r\vartheta_i, r\varpi_i\right), \Xi\left(u'', u, u'\right)\right)\right)\\ \leq\ &\ell\begin{pmatrix}\phi^*\left(Q\left(\left(rr\varkappa_i, ru''\right), \left(rr\vartheta_i, ru\right), \left(rr\varpi_i, ru'\right)\right)\right)\\ Q\left(\left(rr\varkappa_i, ru''\right), \left(rr\vartheta_i, ru\right), \left(rr\varpi_i, ru'\right)\right)\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}&Q\left(\left(rr\vartheta_i, ru\right), \left(rr\varpi_i, ru'\right), \left(rr\varkappa_i, ru''\right)\right)\\ =\ &Q\left(\left(rr\varpi_i, ru'\right), \left(rr\varkappa_i, ru''\right), \left(rr\vartheta_i, ru\right)\right)\\ =\ &Q\left(\left(rr\varkappa_i, ru''\right), \left(rr\vartheta_i, ru\right), \left(rr\varpi_i, ru'\right)\right)\\ =\ &\max\left\{\begin{array}{c}\xi\left(rr\vartheta_i, ru\right), \xi\left(rr\varpi_i, ru'\right), \xi\left(rr\varkappa_i, ru''\right),\\ \xi\left(rr\vartheta_i, \Xi\left(r\vartheta_i, r\varpi_i, r\varkappa_i\right)\right), \xi\left(rr\varpi_i, \Xi\left(r\varpi_i, r\varkappa_i, r\vartheta_i\right)\right),\\ \xi\left(rr\varkappa_i, \Xi\left(r\varkappa_i, r\vartheta_i, r\varpi_i\right)\right), \xi\left(ru, \Xi\left(u, u', u''\right)\right),\\ \xi\left(ru', \Xi\left(u', u'', u\right)\right), \xi\left(ru'', \Xi\left(u'', u', u\right)\right)\end{array}\right\}.\end{aligned}\tag{2.16}$$

Accordingly,

$$\begin{aligned}&\xi\left(\Xi\left(r\varpi_i, r\varkappa_i, r\vartheta_i\right), \Xi\left(u', u'', u\right)\right)\\ \leq\ &\phi^*\left(Q\left(\left(rr\varpi_i, ru'\right), \left(rr\varkappa_i, ru''\right), \left(rr\vartheta_i, ru\right)\right)\right)Q\left(\left(rr\varpi_i, ru'\right), \left(rr\varkappa_i, ru''\right), \left(rr\vartheta_i, ru\right)\right),\end{aligned}\tag{2.17}$$

$$\begin{aligned}&\xi\left(\Xi\left(r\varkappa_i, r\vartheta_i, r\varpi_i\right), \Xi\left(u'', u, u'\right)\right)\\ \leq\ & \phi^*\left(Q\left(\left(rr\varkappa_i, ru''\right), \left(rr\vartheta_i, ru\right), \left(rr\varpi_i, ru'\right)\right)\right) Q\left(\left(rr\varkappa_i, ru''\right), \left(rr\vartheta_i, ru\right), \left(rr\varpi_i, ru'\right)\right)\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}&\xi\left(\Xi\left(r\vartheta_i, r\varpi_i, r\varkappa_i\right), \Xi\left(u, u', u''\right)\right)\\ \leq\ & \phi^*\left(Q\left(\left(rr\vartheta_i, ru\right), \left(rr\varpi_i, ru'\right), \left(rr\varkappa_i, ru''\right)\right)\right) Q\left(\left(rr\vartheta_i, ru\right), \left(rr\varpi_i, ru'\right), \left(rr\varkappa_i, ru''\right)\right).\end{aligned} \tag{2.19}$$

Assume that $ru \neq \Xi\left(u, u', u''\right)$ or $ru' \neq \Xi\left(u', u'', u\right)$ or $ru'' \neq \Xi\left(u'', u, u'\right)$, that is,

$$W = \max\left\{\xi\left(ru, \Xi\left(u, u', u''\right)\right), \xi\left(ru', \Xi\left(u', u'', u\right)\right), \xi\left(ru'', \Xi\left(u'', u, u'\right)\right)\right\} > 0.$$

Applying hypothesis (h_v), there is $\Re \in (0, 1]$ so that

$$\begin{aligned}\xi\left(ru, \Xi\left(u, u', u''\right)\right) &\leq \Re \limsup_{i\to\infty} \xi\left(\Xi\left(r\vartheta_i, r\varpi_i, r\varkappa_i\right), \Xi\left(u, u', u''\right)\right) \leq \Re W,\\ \xi\left(ru', \Xi\left(u', u'', u\right)\right) &\leq \Re \limsup_{i\to\infty} \xi\left(\Xi\left(r\varpi_i, r\varkappa_i, r\vartheta_i\right), \Xi\left(u', u'', u\right)\right) \leq \Re W\end{aligned}$$

and

$$\xi\left(ru'', \Xi\left(u'', u, u'\right)\right) \leq \Re \limsup_{i\to\infty} \xi\left(\Xi\left(r\varkappa_i, r\vartheta_i, r\varpi_i\right), \Xi\left(u'', u, u'\right)\right) \leq \Re W.$$

Hence

$$\begin{aligned}W &= \max\left\{\xi\left(ru, \Xi\left(u, u', u''\right)\right), \xi\left(ru', \Xi\left(u', u'', u\right)\right), \xi\left(ru'', \Xi\left(u'', u, u'\right)\right)\right\}\\ &\leq \Re \limsup_{i\to\infty} \max\left\{\begin{array}{c}\xi\left(\Xi\left(r\vartheta_i, r\varpi_i, r\varkappa_i\right), \Xi\left(u, u', u''\right)\right),\\ \xi\left(\Xi\left(r\varpi_i, r\varkappa_i, r\vartheta_i\right), \Xi\left(u', u'', u\right)\right),\\ \xi\left(\Xi\left(r\varkappa_i, r\vartheta_i, r\varpi_i\right), \Xi\left(u'', u, u'\right)\right)\end{array}\right\}\\ &\leq \Re W.\end{aligned}$$

Because $1 \leq \frac{1}{\Re}$, we have

$$\begin{aligned}W &\leq \frac{1}{\Re} W\\ &\leq \limsup_{i\to\infty} \max\left\{\begin{array}{c}\xi\left(\Xi\left(r\vartheta_i, r\varpi_i, r\varkappa_i\right), \Xi\left(u, u', u''\right)\right),\\ \xi\left(\Xi\left(r\varpi_i, r\varkappa_i, r\vartheta_i\right), \Xi\left(u', u'', u\right)\right),\\ \xi\left(\Xi\left(r\varkappa_i, r\vartheta_i, r\varpi_i\right), \Xi\left(u'', u, u'\right)\right)\end{array}\right\}\\ &\leq W,\end{aligned}$$

this implies that

$$W = \limsup_{i\to\infty} \max\left\{\begin{array}{c}\xi\left(\Xi\left(r\vartheta_i, r\varpi_i, r\varkappa_i\right), \Xi\left(u, u', u''\right)\right),\\ \xi\left(\Xi\left(r\varpi_i, r\varkappa_i, r\vartheta_i\right), \Xi\left(u', u'', u\right)\right),\\ \xi\left(\Xi\left(r\varkappa_i, r\vartheta_i, r\varpi_i\right), \Xi\left(u'', u, u'\right)\right)\end{array}\right\}.$$

Then there exists a subsequence $\max\left\{\begin{array}{c}\xi\left(\Xi\left(r\vartheta_{i_s}, r\varpi_{i_s}, r\varkappa_{i_s}\right), \Xi\left(u, u', u''\right)\right),\\ \xi\left(\Xi\left(r\varpi_{i_s}, r\varkappa_{i_s}, r\vartheta_{i_s}\right), \Xi\left(u', u'', u\right)\right),\\ \xi\left(\Xi\left(r\varkappa_{i_s}, r\vartheta_{i_s}, r\varpi_{i_s}\right), \Xi\left(u'', u, u'\right)\right)\end{array}\right\}$ so that

$$\lim_{s\to\infty} \max\left\{\begin{array}{c}\xi\left(\Xi\left(r\vartheta_{i_s}, r\varpi_{i_s}, r\varkappa_{i_s}\right), \Xi\left(u, u', u''\right)\right),\\ \xi\left(\Xi\left(r\varpi_{i_s}, r\varkappa_{i_s}, r\vartheta_{i_s}\right), \Xi\left(u', u'', u\right)\right),\\ \xi\left(\Xi\left(r\varkappa_{i_s}, r\vartheta_{i_s}, r\varpi_{i_s}\right), \Xi\left(u'', u, u'\right)\right)\end{array}\right\} = W.$$

Passing $i \to \infty$ in (2.16), we obtain that

$$\lim_{s\to\infty} Q\left(\left(rr\vartheta_i, ru\right), \left(rr\varpi_i, ru'\right), \left(rr\varkappa_i, ru''\right)\right) = W. \tag{2.20}$$

Using (2.17)–(2.19), one can write

$$\begin{aligned}&\frac{\max\left\{\begin{array}{c}\xi\left(\Xi\left(r\vartheta_{i_s}, r\varpi_{i_s}, r\varkappa_{i_s}\right), \Xi\left(u, u', u''\right)\right), \xi\left(\Xi\left(r\varpi_{i_s}, r\varkappa_{i_s}, r\vartheta_{i_s}\right), \Xi\left(u', u'', u\right)\right),\\ \xi\left(\Xi\left(r\varkappa_{i_s}, r\vartheta_{i_s}, r\varpi_{i_s}\right), \Xi\left(u'', u, u'\right)\right)\end{array}\right\}}{Q\left(\left(rr\vartheta_{i_s}, ru\right), \left(rr\varpi_{i_s}, ru'\right), \left(rr\varkappa_{i_s}, ru''\right)\right)}\\ \leq\ & \phi^*\left(Q\left(\left(rr\vartheta_{i_s}, ru\right), \left(rr\varpi_{i_s}, ru'\right), \left(rr\varkappa_{i_s}, ru''\right)\right)\right).\end{aligned}$$

Putting $s \to \infty$ on both sides, we get

$$\lim_{s\to\infty} \phi^*\left(Q\left(\left(rr\vartheta_{i_s}, ru\right), \left(rr\varpi_{i_s}, ru'\right), \left(rr\varkappa_{i_s}, ru''\right)\right)\right) = 1.$$

Thus, $\lim_{s\to\infty} Q\left(\left(rr\vartheta_{i_s}, ru\right), \left(rr\varpi_{i_s}, ru'\right), \left(rr\varkappa_{i_s}, ru''\right)\right) = 0$. This contradicts (2.20). Hence $ru = \Xi\left(u, u', u''\right)$ and $ru' = \Xi\left(u', u'', u\right)$ and $ru'' = \Xi\left(u'', u, u'\right)$. Therefore, the element$(u, u', u'')$ is a TCP of r and Ξ. □

Corollary 2.9. *The statements of Theorem 2.8 are still valid if we replace the stipulation* (h_i) *with one of the assumptions below: (note* $Q\left(\left(r\vartheta, r\widetilde{\vartheta}\right), \left(r\varpi, r\widetilde{\varpi}\right), \left(r\varkappa, r\widetilde{\varkappa}\right)\right)$ *is defined in Theorem 2.8):*

(1) *There exist* $\phi^* \in \Phi^*$ *and* $z > 1$, *fulfilling, for any* $\left(r\vartheta, r\widetilde{\vartheta}\right) \in \nabla_{\leq}$, $\left(r\varpi, r\widetilde{\varpi}\right) \in \nabla_{\leq}$ *and* $\left(r\varkappa, r\widetilde{\varkappa}\right) \in \nabla_{\leq}$, *the following inequality is obtained:*

$$\begin{aligned}&\left(\xi\left(\Xi\left(\vartheta, \varpi, \varkappa\right), \Xi\left(\widetilde{\vartheta}, \widetilde{\varpi}, \widetilde{\varkappa}\right)\right) + z\right)^{\alpha\left(\left(r\vartheta, r\varpi, r\varkappa\right), \left(r\widetilde{\vartheta}, r\widetilde{\varpi}, r\widetilde{\varkappa}\right)\right)}\\ \leq\ & \phi^*\left(Q\left(\xi\left(r\vartheta, r\widetilde{\vartheta}\right), \xi\left(r\varpi, r\widetilde{\varpi}\right), \xi\left(r\varkappa, r\widetilde{\varkappa}\right)\right)\right) Q\left(\left(r\vartheta, r\widetilde{\vartheta}\right), \left(r\varpi, r\widetilde{\varpi}\right), \left(r\varkappa, r\widetilde{\varkappa}\right)\right) + z.\end{aligned}$$

(2) *There exist* $\phi^* \in \Phi^*$ *and* $z > 1$, *fulfilling, for any* $\left(r\vartheta, r\widetilde{\vartheta}\right) \in \nabla_{\leq}$, $\left(r\varpi, r\widetilde{\varpi}\right) \in \nabla_{\leq}$ *and* $\left(r\varkappa, r\widetilde{\varkappa}\right) \in \nabla_{\leq}$, *the following inequality is verified:*

$$\begin{aligned}&\left(\alpha\left(\left(r\vartheta, r\varpi, r\varkappa\right), \left(r\widetilde{\vartheta}, r\widetilde{\varpi}, r\widetilde{\varkappa}\right)\right) + z\right)^{\xi\left(\Xi\left(\vartheta, \varpi, \varkappa\right), \Xi\left(\widetilde{\vartheta}, \widetilde{\varpi}, \widetilde{\varkappa}\right)\right)}\\ \leq\ & (1+z)^{\phi^*\left(Q\left(\xi\left(r\vartheta, r\widetilde{\vartheta}\right), \xi\left(r\varpi, r\widetilde{\varpi}\right), \xi\left(r\varkappa, r\widetilde{\varkappa}\right)\right)\right) Q\left(\left(r\vartheta, r\widetilde{\vartheta}\right), \left(r\varpi, r\widetilde{\varpi}\right), \left(r\varkappa, r\widetilde{\varkappa}\right)\right)}.\end{aligned}$$

(3) *There is* $\phi^* \in \Phi^*$, *fulfilling, for any* $\left(r\vartheta, r\widetilde{\vartheta}\right) \in \nabla_{\leq}$, $\left(r\varpi, r\widetilde{\varpi}\right) \in \nabla_{\leq}$ *and* $\left(r\varkappa, r\widetilde{\varkappa}\right) \in \nabla_{\leq}$ *so that*

$$\begin{aligned}&\xi\left(\Xi\left(\vartheta, \varpi, \varkappa\right), \Xi\left(\widetilde{\vartheta}, \widetilde{\varpi}, \widetilde{\varkappa}\right)\right)\\ \leq\ & \phi^*\left(Q\left(\xi\left(r\vartheta, r\widetilde{\vartheta}\right), \xi\left(r\varpi, r\widetilde{\varpi}\right), \xi\left(r\varkappa, r\widetilde{\varkappa}\right)\right)\right) Q\left(\left(r\vartheta, r\widetilde{\vartheta}\right), \left(r\varpi, r\widetilde{\varpi}\right), \left(r\varkappa, r\widetilde{\varkappa}\right)\right).\end{aligned}$$

(4) *There is* $\phi^* \in \Phi^*$, *satisfying, for any* $\left(r\vartheta, r\widetilde{\vartheta}\right) \in \nabla_{\leq}$, $\left(r\varpi, r\widetilde{\varpi}\right) \in \nabla_{\leq}$ *and* $\left(r\varkappa, r\widetilde{\varkappa}\right) \in \nabla_{\leq}$, *the inequality below is obtained:*

$$\begin{aligned}&\left\{\alpha\left(\left(r\vartheta, r\varpi, r\varkappa\right), \left(r\widetilde{\vartheta}, r\widetilde{\varpi}, r\widetilde{\varkappa}\right)\right)\right\}^j \left\{\xi\left(\Xi\left(\vartheta, \varpi, \varkappa\right), \Xi\left(\widetilde{\vartheta}, \widetilde{\varpi}, \widetilde{\varkappa}\right)\right)\right\}^k\\ \leq\ & \left\{Q\left(\phi\left(\xi\left(r\vartheta, r\widetilde{\vartheta}\right), \xi\left(r\varpi, r\widetilde{\varpi}\right), \xi\left(r\varkappa, r\widetilde{\varkappa}\right)\right)\right)\right\}^k \left\{Q\left(\left(r\vartheta, r\widetilde{\vartheta}\right), \left(r\varpi, r\widetilde{\varpi}\right), \left(r\varkappa, r\widetilde{\varkappa}\right)\right)\right\}^k,\end{aligned}$$

for all positive integers j *and* k.

Proof. The proof follows immediately from Example 2.3 by applying the following in Theorem 2.8:
(1) Use the value of $\wp$ and ℓ from Example 2.3(1).
(2) Use the value of $\wp$ and ℓ from Example 2.3(2).
(3) Use the value of $\wp$ and ℓ from Example 2.3(3).
(4) Use the value of $\wp$ and ℓ from Example 2.3(4). □

Example 2.10. Consider $\mho = [0, +\infty]$. Define $\xi(\vartheta, \varpi) = |\vartheta| + |\varpi|$, for all $\vartheta, \varpi \in \mho$. Define $\Xi : \mho^3 \to \mho$, $r : \mho \to \mho$ and $\alpha : \mho^3 \times \mho^3 \to [0, +\infty]$ by

$$\Xi(\vartheta, \varpi, \varkappa) = \begin{cases} \frac{|\vartheta-\varpi|+|\varpi-\varkappa|+|\varkappa-\vartheta|}{8}, & \text{if } \vartheta, \varpi, \varkappa \in [0, +\infty), \\ +\infty, & \text{otherwise,} \end{cases} \quad r(\vartheta) = \begin{cases} 4\vartheta, & \text{if } \vartheta \in [0, +\infty), \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\alpha\left((\vartheta, \varpi, \varkappa), \left(\widetilde{\vartheta}, \widetilde{\varpi}, \widetilde{\varkappa}\right)\right) = \begin{cases} 1, & \text{if } \vartheta \le \varpi \le \varkappa \text{ and } \widetilde{\vartheta} \le \widetilde{\varpi} \le \widetilde{\varkappa}, \\ 0, & \text{otherwise,} \end{cases}$$

respectively. Suppose that $\vartheta \le \widetilde{\vartheta}$, $\varpi \le \widetilde{\varpi}$ and $\varkappa \le \widetilde{\varkappa}$. When $z > 1$, we have

$$\begin{aligned} & \left(\alpha\left((r\vartheta, r\varpi, r\varkappa), \left(r\widetilde{\vartheta}, r\widetilde{\varpi}, r\widetilde{\varkappa}\right)\right) + z\right)^{\xi\left(\Xi(\vartheta,\varpi,\varkappa), \Xi\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)\right)\times 1} \\ \le\ & (1+z)^{\left(\left|\frac{|\vartheta-\varpi|+|\varpi-\varkappa|+|\varkappa-\vartheta|}{6}\right| + \left|\frac{\left|\widetilde{\vartheta}-\widetilde{\varpi}\right|+\left|\widetilde{\varpi}-\widetilde{\varkappa}\right|+\left|\widetilde{\varkappa}-\widetilde{\vartheta}\right|}{6}\right|\right)} \\ \le\ & (1+z)^{\frac{1}{4}\left(|\vartheta|+|\varpi|+|\varkappa|+\left|\widetilde{\vartheta}\right|+\left|\widetilde{\varpi}\right|+\left|\widetilde{\varkappa}\right|\right)} \\ \le\ & (1+z)^{\frac{3}{4}\max\left\{|4\vartheta|+\left|4\widetilde{\vartheta}\right|, |4\varpi|+|4\widetilde{\varpi}|, |4\varkappa|+\left|4\widetilde{\varkappa}\right|\right\}} \\ =\ & (1+z)^{\phi^*\left(Q\left(\xi\left(r\vartheta, r\widetilde{\vartheta}\right), \xi(r\varpi, r\widetilde{\varpi}), \xi(r\varkappa, r\widetilde{\varkappa})\right)\right) Q\left(\left(r\vartheta, r\widetilde{\vartheta}\right), (r\varpi, r\widetilde{\varpi}), (r\varkappa, r\widetilde{\varkappa})\right)} \\ =\ & (1+z)^{\phi^*\left(Q\left(\xi\left(r\vartheta, r\widetilde{\vartheta}\right), \xi(r\varpi, r\widetilde{\varpi}), \xi\left(r\varkappa, r\widetilde{\varkappa}\right)\right)\right) Q\left(\left(r\vartheta, r\widetilde{\vartheta}\right), (r\varpi, r\widetilde{\varpi}), \left(r\varkappa, r\widetilde{\varkappa}\right)\right)}. \end{aligned}$$

So, the condition (2) of Corollary 2.9 is fulfilled for $\phi^*(m) = \frac{3}{4}$, where $m \in [0, +\infty]$. In an easy way, it can be shown that all assumptions of Theorem 2.8 are satisfied, therefore Ξ and r have a TCP.

3. Solve a system of integral equations

This section serves as the basis and core of our paper in which the existence of a solution to a system of integral equations is discussed. This system takes the following form:

$$\begin{cases} \vartheta(b) = \Lambda(b) + \int_0^1 \Im_1(b, c)\, \eth_1(c, \vartheta(c))dc \int_0^1 \Im_2(b, c)\, \eth_2(c, \varpi(c))dc \int_0^1 \Im_3(b, c)\, \eth_3(c, \varkappa(c))dc, \\ \varpi(b) = \Lambda(b) + \int_0^1 \Im_1(b, c)\, \eth_1(c, \varpi(c))dc \int_0^1 \Im_2(b, c)\, \eth_2(c, \varkappa(c))dc \int_0^1 \Im_3(b, c)\, \eth_3(c, \vartheta(c))dc, \\ \varkappa(b) = \Lambda(b) + \int_0^1 \Im_1(b, c)\, \eth_1(c, \varkappa(c))dc \int_0^1 \Im_2(b, c)\, \eth_2(c, \vartheta(c))dc \int_0^1 \Im_3(b, c)\, \eth_3(c, \varpi(c))dc, \end{cases} \tag{3.1}$$

where $b \in [0, 1]$. Consider $\mho = C[0, 1]$ equipped with $\xi(\vartheta, \varpi) = \sup_{b\in[0,1]} |\vartheta(b) - \varpi(b)|$, for each $\vartheta, \varpi \in \mho$. Clearly, $(\mho, \xi, \preceq)$ is a complete POJSM-space. Based on the theoretical results presented in the upper section of the paper, we are able to present theorems related to the existence of a solution to system (3.1) as follows:

Theorem 3.1. *Consider the integral equations (3.1) via the following postulates:*

($\heartsuit_i$) *The functions* $\Lambda : [0,1] \to \mathbb{R}$, $\Im_s : [0,1]\times[0,1] \to \mathbb{R}^+$ *and* $\eth_s : [0,1]\times\mathbb{R} \to \mathbb{R}^+$ *are continuous, where* $s = 1,2,3$;

($\heartsuit_{ii}$) *For* $\vartheta_1, \vartheta_2 \in \mho$, *if* $\vartheta_1 \leq \vartheta_2$, *then* $\eth_1(b,\vartheta_1(b)) \leq \eth_1(b,\vartheta_2(b))$, $\eth_2(b,\vartheta_1(b)) \leq \eth_2(b,\vartheta_2(b))$ *and* $\eth_3(b,\vartheta_1(b)) \leq \eth_3(b,\vartheta_2(b))$;

($\heartsuit_{iii}$) *There exist* $\widehat{A}$, $\widehat{B}$ *and* $\widehat{C}$ *in* $\mathbb{R}^+$ *so that* $3\widehat{ABC} < 1$, $\int_0^1 \Im_s(b,c)\,dc \leq \widehat{A}$, $\int_0^1 \Im_s(b,c)\,\eth_s(c,\vartheta(c))dc \leq \widehat{C}$ *and*

$$\left|\eth_s(b,\vartheta(b)) - \eth_s(b,\widetilde{\vartheta}(b)\right| \leq \widehat{B}\left|\vartheta(b) - \widetilde{\vartheta}(b)\right|,$$

where $\vartheta, \widetilde{\vartheta} \in \mho$, $b \in [0,1]$ *and* $s = 1,2,3$;

($\heartsuit_{iv}$) *There exist* $\vartheta_0, \varpi_0, \varkappa_0 \in \mho$ *such that* $\vartheta_0 \leq \Xi(\vartheta_0,\varpi_0,\varkappa_0)$, $\varpi_0 \leq \Xi(\varpi_0,\varkappa_0,\vartheta_0)$ *and* $\varkappa_0 \leq \Xi(\varkappa_0,\vartheta_0,\varpi_0)$;

($\heartsuit_v$) *If* $\{\vartheta_i\}, \{\varpi_i\}, \{\varkappa_i\} \in \mho$ *so that* $\lim_{i\to\infty} |\vartheta_i - \vartheta_{i+1}| = 0$, $\lim_{i\to\infty} |\varpi_i - \varpi_{i+1}| = 0$ *and* $\lim_{i\to\infty} |\varkappa_i - \varkappa_{i+1}| = 0$, *then*

$$\sup\{|\vartheta_0 - \vartheta_i|, |\varpi_0 - \varpi_i|, |\varkappa_0 - \varkappa_i| : i \geq 1\} < \infty,$$

where $\vartheta_0, \varpi_0, \varkappa_0 \in \mho$.

Then Problem (3.1) possesses a solution.

Proof. Define the mappings $\Xi : \mho^3 \to \mho$ and $r : \mho \to \mho$ by

$$\begin{aligned} &\Xi(\vartheta,\varpi,\varkappa)(b) \\ = \;& \Lambda(b) + \int_0^1 \Im_1(b,c)\,\eth_1(c,\vartheta(c))dc \int_0^1 \Im_2(b,c)\,\eth_2(c,\varpi(c))dc \int_0^1 \Im_3(b,c)\,\eth_3(c,\varkappa(c))dc \end{aligned}$$

and $r\vartheta = \vartheta$ for $\vartheta, \varpi, \varkappa \in \mho$. Clearly Ξ is $\xi-$continuous $\Xi\left(\mho^3\right) \subseteq r(\mho)$ and r is $\xi-$continuous and commutes with Ξ. After that, we claim that Ξ is $\preceq -r$ monotone. Suppose that $\vartheta_1, \vartheta_2, \varpi, \varkappa \in \mho$, $\vartheta_1 \leq \vartheta_2$. Based on postulate ($\heartsuit_{ii}$), we get

$$\begin{aligned} &\Xi(\vartheta_1,\varpi,\varkappa)(b) \\ = \;& \Lambda(b) + \int_0^1 \Im_1(b,c)\,\eth_1(c,\vartheta_1(c))dc \int_0^1 \Im_2(b,c)\,\eth_2(c,\varpi(c))dc \int_0^1 \Im_3(b,c)\,\eth_3(c,\varkappa(c))dc \\ \leq \;& \Lambda(b) + \int_0^1 \Im_1(b,c)\,\eth_1(c,\vartheta_2(c))dc \int_0^1 \Im_2(b,c)\,\eth_2(c,\varpi(c))dc \int_0^1 \Im_3(b,c)\,\eth_3(c,\varkappa(c))dc \\ = \;& \Xi(\vartheta_2,\varpi,\varkappa)(b), \end{aligned}$$

$$\begin{aligned} &\Xi(\varpi,\varkappa,\vartheta_1)(b) \\ = \;& \Lambda(b) + \int_0^1 \Im_1(b,c)\,\eth_1(c,\varpi(c))dc \int_0^1 \Im_2(b,c)\,\eth_2(c,\varkappa(c))dc \int_0^1 \Im_3(b,c)\,\eth_3(c,\vartheta_1(c))dc, \end{aligned}$$

$$\leq \Lambda(b) + \int_0^1 \Im_1(b,c)\,\eth_1(c,\varpi(c))dc \int_0^1 \Im_2(b,c)\,\eth_2(c,\varkappa(c))dc \int_0^1 \Im_3(b,c)\,\eth_3(c,\vartheta_2(c))dc,$$
$$= \Xi(\varpi,\varkappa,\vartheta_2)(b).$$

Similarly, one can show that $\Xi(\varkappa,\vartheta_1,\varpi)(b) \leq \Xi(\varkappa,\vartheta_2,\varpi)(b)$. Hence, Ξ has the $\preceq -r$ monotone property.

Now, for $(\vartheta,\varpi,\varkappa), \left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right) \in \mho^3$, describe $\alpha : \mho^3 \times \mho^3 \to [0,+\infty]$ by $\alpha\left((\vartheta,\varpi,\varkappa), \left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)\right) = 1$. Obviously, Ξ and r is α_A^{GT}. Also, assumptions $(\triangle_{ii})$ and $(\triangle_{iii})$ of Theorem 2.4 follow immediately from postulates $(\heartsuit_{iv})$ and $(\heartsuit_v)$, respectively. In order to be able to validate Theorem 2.4, it remains to fulfill condition $(\triangle_i)$, which can be replaced by condition $(\dagger_3)$ of Corollary 2.5. Therefore, let $\vartheta,\varpi,\varkappa,\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa} \in \mho$. If $\vartheta \preceq \widetilde{\vartheta}$, $\varpi \preceq \widetilde{\varpi}$ and $\varkappa \preceq \widetilde{\varkappa}$, then using postulate $(\heartsuit_{iii})$, we have

$$\begin{aligned}
&\left|\Xi(\vartheta,\varpi,\varkappa)(b) - \Xi\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)(b)\right| \\
=& \left|\int_0^1 \Im_1(b,c)\,\eth_1(c,\vartheta(c))dc \int_0^1 \Im_2(b,c)\,\eth_2(c,\varpi(c))dc \int_0^1 \Im_3(b,c)\,\eth_3(c,\varkappa(c))dc\right. \\
&\left. - \int_0^1 \Im_1(b,c)\,\eth_1(c,\widetilde{\vartheta}(c))dc \int_0^1 \Im_2(b,c)\,\eth_2(c,\widetilde{\varpi}(c))dc \int_0^1 \Im_3(b,c)\,\eth_3(c,\widetilde{\varkappa}(c))dc\right| \\
\leq& \left|\int_0^1 \Im_1(b,c)\,\eth_1(c,\vartheta(c))dc \int_0^1 \Im_2(b,c)\,\eth_2(c,\varpi(c))dc \int_0^1 \Im_3(b,c)\left[\eth_3(c,\varkappa(c)) - \eth_3(c,\widetilde{\varkappa}(c))\right]dc\right| \\
&+ \left|\int_0^1 \Im_1(b,c)\,\eth_1(c,\vartheta(c))dc \int_0^1 \Im_2(b,c)\left[\eth_2(c,\varpi(c)) - \eth_2(c,\widetilde{\varpi}(c))\right]dc \int_0^1 \Im_3(b,c)\,\eth_3(c,\varkappa(c))dc\right| \\
&+ \left|\int_0^1 \Im_1(b,c)\left[\eth_1(c,\vartheta(c)) - \eth_1(c,\widetilde{\vartheta}(c))\right]dc \int_0^1 \Im_2(b,c)\,\eth_2(c,\varpi(c))dc \int_0^1 \Im_3(b,c)\,\eth_3(c,\varkappa(c))dc\right| \\
\leq& \widetilde{ABC} \sup_{b\in[0,1]} \left|\varkappa(b) - \widetilde{\varkappa}(b)\right| + \widetilde{ABC} \sup_{b\in[0,1]} |\varpi(b) - \widetilde{\varpi}(b)| + \widetilde{ABC} \sup_{b\in[0,1]} \left|\vartheta(b) - \widetilde{\vartheta}(b)\right| \\
\leq& 3\widetilde{ABC} \max\left\{\sup_{b\in[0,1]} \left|\varkappa(b) - \widetilde{\varkappa}(b)\right|, \sup_{b\in[0,1]} |\varpi(b) - \widetilde{\varpi}(b)|, \sup_{b\in[0,1]} \left|\vartheta(b) - \widetilde{\vartheta}(b)\right|\right\}.
\end{aligned}$$

Consider $\phi = 3\widetilde{ABC} < 1$, then $\phi \in \widetilde{\Phi}$. Hence the condition $(\dagger_3)$ of Corollary 2.5 is fulfilled. Therefore, a tripled fixed point of Ξ exists. Thus, there is a solution to the problem (3.1). □

It should be noted that in the above theorem, JS-metric ξ is a usual metric, therefore, the results seem easy to obtain, for this reason, we define JS-metric $\xi : \mho^2 \to [0,\infty]$ by

$$\xi(\vartheta,\varpi) = \sup_{b\in[0,1]} \left(|\vartheta(b)| + |\varpi(b)|\right).$$

Clearly, under this distance, JS-metric ξ is not a metric. As a result, $(\mho,\xi,\preceq)$ is complete partially ordered. Moreover, we can guarantee the existence of the solution to system (3.1) only if it is homogeneous.

Theorem 3.2. *From the integral equations (3.1), let* $\Xi : \mho^3 \to \mho$ *be described as*

$$\begin{aligned}&\Xi(\vartheta,\varpi,\varkappa)(b)\\ =\ & \int_0^1 \Im_1(b,c)\eth_1(c,\vartheta(c))dc \int_0^1 \Im_2(b,c)\eth_2(c,\varpi(c))dc \int_0^1 \Im_3(b,c)\eth_3(c,\varkappa(c))dc,\end{aligned}$$

where $b \in [0,1]$. *Assume also,*

($\clubsuit_1$) *the functions* $\Im_s : [0,1]\times[0,1] \to \mathbb{R}^+$ *and* $\eth_s : [0,1]\times\mathbb{R} \to \mathbb{R}^+$ *are continuous, where* $s = 1,2,3$;
($\clubsuit_2$) *for* $\vartheta_1,\vartheta_2 \in \mho$, *if* $\vartheta_1 \le \vartheta_2$, *then* $\eth_1(b,\vartheta_1(b)) \le \eth_1(b,\vartheta_2(b))$, $\eth_2(b,\vartheta_1(b)) \le \eth_2(b,\vartheta_2(b))$ *and* $\eth_3(b,\vartheta_1(b)) \le \eth_3(b,\vartheta_2(b))$;
($\clubsuit_3$) *there exist* $\widehat{A}$, $\widehat{B}$ *and* $\widehat{C}$ *in* $\mathbb{R}^+$ *so that* $3\widehat{A}\widehat{B}\widehat{C} < 1$, $\int_0^1 \Im_s(b,c)\,dc \le \widehat{A}$, $\int_0^1 \Im_s(b,c)\eth_s(c,\vartheta(c))dc \le \widehat{C}$ *and*

$$|\eth_s(b,\vartheta(b)| + \left|\eth_s(b,\widetilde{\vartheta}(b)\right| \le \widehat{B}\left(|\vartheta(b)| + \left|\widetilde{\vartheta}(b)\right|\right),$$

where $\vartheta,\widetilde{\vartheta} \in \mho$, $b \in [0,1]$ *and* $s = 1,2,3$;
($\clubsuit_4$) *there exist* $\vartheta_0,\varpi_0,\varkappa_0 \in \mho$ *so that* $\vartheta_0 \le \Xi(\vartheta_0,\varpi_0,\varkappa_0)$, $\varpi_0 \le \Xi(\varpi_0,\varkappa_0,\vartheta_0)$ *and* $\varkappa_0 \le \Xi(\varkappa_0,\vartheta_0,\varpi_0)$.

Then Problem (3.1) has a solution provided that the integral equations are homogeneous.

Proof. Assume that $\alpha\left((\vartheta,\varpi,\varkappa),\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)\right) = 1$ for any $(\vartheta,\varpi,\varkappa),\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right) \in \mho^3$. Repeating the same arguments used in Theorem 3.1, under this JS-metric ξ and considering r is the identity mapping, we have the ξ−continuities of r and Ξ and all assumption except condition $(\triangle_i)$ of Theorem 2.4 are valid.

So, our task is to prove the condition $(\triangle_i)$ of Theorem 2.4. Again, this condition can be replaced by assumption $(\dagger_3)$ of Corollary 2.5. Let $\vartheta,\varpi,\varkappa,\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa} \in \mho$. If $\vartheta \le \widetilde{\vartheta}$, $\varpi \le \widetilde{\varpi}$ and $\varkappa \le \widetilde{\varkappa}$, then by ($\clubsuit_3$), we get

$$\begin{aligned}&|\Xi(\vartheta,\varpi,\varkappa)(b)| + \left|\Xi\left(\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa}\right)(b)\right|\\
=\ & \left|\int_0^1 \Im_1(b,c)\eth_1(c,\vartheta(c))dc \int_0^1 \Im_2(b,c)\eth_2(c,\varpi(c))dc \int_0^1 \Im_3(b,c)\eth_3(c,\varkappa(c))dc\right|\\
&+\left|\int_0^1 \Im_1(b,c)\eth_1(c,\widetilde{\vartheta}(c))dc \int_0^1 \Im_2(b,c)\eth_2(c,\widetilde{\varpi}(c))dc \int_0^1 \Im_3(b,c)\eth_3(c,\widetilde{\varkappa}(c))dc\right|\\
\le\ & \left|\int_0^1 \Im_1(b,c)\eth_1(c,\vartheta(c))dc \int_0^1 \Im_2(b,c)\eth_2(c,\varpi(c))dc \int_0^1 \Im_3(b,c)\left[\eth_3(c,\varkappa(c)) + \eth_3(c,\widetilde{\varkappa}(c))\right]dc\right|\\
&+\left|\int_0^1 \Im_1(b,c)\eth_1(c,\vartheta(c))dc \int_0^1 \Im_2(b,c)\left[\eth_2(c,\varpi(c)) + \eth_2(c,\widetilde{\varpi}(c))\right]dc \int_0^1 \Im_3(b,c)\eth_3(c,\varkappa(c))dc\right|\\
&+\left|\int_0^1 \Im_1(b,c)\left[\eth_1(c,\vartheta(c)) + \eth_1(c,\widetilde{\vartheta}(c))\right]dc \int_0^1 \Im_2(b,c)\eth_2(c,\varpi(c))dc \int_0^1 \Im_3(b,c)\eth_3(c,\varkappa(c))dc\right|\end{aligned}$$

$$\leq \widehat{ABC} \sup_{b\in[0,1]} \left(|\varkappa(b)| + \left|\widetilde{\varkappa}(b)\right|\right) + \widehat{ABC} \sup_{b\in[0,1]} \left(|\varpi(b)| + |\widetilde{\varpi}(b)|\right) + \widehat{ABC} \sup_{b\in[0,1]} \left(|\vartheta(b)| + \left|\widetilde{\vartheta}(b)\right|\right)$$

$$\leq 3\widehat{ABC} \max\left\{ \sup_{b\in[0,1]} \left(|\varkappa(b)| + \left|\widetilde{\varkappa}(b)\right|\right), \sup_{b\in[0,1]} \left(|\varpi(b)| + |\widetilde{\varpi}(b)|\right), \sup_{b\in[0,1]} \left(|\vartheta(b)| + \left|\widetilde{\vartheta}(b)\right|\right)\right\}.$$

Analogously, if we take $\phi = 3\widehat{ABC} < 1$, then $\phi \in \widetilde{\Phi}$. Hence, the assumption ($\dagger_3$) of Corollary 2.5 is satisfied. Hence, Ξ possesses a tripled fixed point, which is a unique solution for the problem (3.1) under the condition of homogeneity for these equations. □

The following example supports Theorem 3.2:

Example 3.3. Consider the problem below:

$$\begin{cases} \vartheta(b) = \int_0^1 \frac{c^2}{1+b^4} \cdot \frac{1}{1+c^3} \cdot \frac{|\vartheta(c)|}{1+|\vartheta(c)|} dc \int_0^1 ce^{-b^2} \cdot \frac{c^2}{1+c^4} \cdot \frac{|\varpi(c)|}{2+|\varpi(c)|} dc \int_0^1 c^3 e^{-b^2} \cdot \frac{c^3}{1+c^7} \cdot \frac{|\varkappa(c)|}{3+|\varkappa(c)|} dc \\ \varpi(b) = \int_0^1 \frac{c^2}{1+b^4} \cdot \frac{1}{1+c^3} \cdot \frac{|\varpi(c)|}{1+|\varpi(c)|} dc \int_0^1 ce^{-b^2} \cdot \frac{c^2}{1+c^4} \cdot \frac{|\varkappa(c)|}{2+|\varkappa(c)|} dc \int_0^1 c^3 e^{-b^2} \cdot \frac{c^3}{1+c^7} \cdot \frac{|\vartheta(c)|}{3+|\vartheta(c)|} dc \\ \varkappa(b) = \int_0^1 \frac{c^2}{1+b^4} \cdot \frac{1}{1+c^3} \cdot \frac{|\varkappa(c)|}{1+|\varkappa(c)|} dc \int_0^1 ce^{-b^2} \cdot \frac{c^2}{1+c^4} \cdot \frac{|\vartheta(c)|}{2+|\vartheta(c)|} dc \int_0^1 c^3 e^{-b^2} \cdot \frac{c^3}{1+c^7} \cdot \frac{|\varpi(c)|}{3+|\varpi(c)|} dc \end{cases}, \tag{3.2}$$

where $b \in [0, 1]$. By comparing this system with system (3.1) in the homogeneous case, we can write

$$\begin{array}{ll} \Im_1(b,c) = \frac{c^2}{1+b^4}, & \eth_1(c,\vartheta(c)) = \frac{1}{1+c^3} \cdot \frac{|\vartheta(c)|}{1+|\vartheta(c)|}, \\ \Im_2(b,c) = c^2 e^{-b^2}, & \eth_2(c,\vartheta(c)) = \frac{c}{1+c^4} \cdot \frac{|\vartheta(c)|}{2+|\vartheta(c)|}, \\ \Im_3(b,c) = c^2 e^{-b^3}, & \eth_3(c,\vartheta(c)) = \frac{c^4}{1+c^7} \cdot \frac{|\vartheta(c)|}{3+|\vartheta(c)|}, \end{array}$$

for $b, c \in [0, 1]$. It is easy to see that $\Im_s$ and $\eth_s$ are continuous, and $\eth_s(c, \vartheta) \geq 0$ for $s = 1, 2, 3$. Furthermore, $\eth_s(b, \vartheta(b)) \leq \eth_s(b, \varpi(b))$ whenever $\vartheta \leq \varpi$ for all $s = 1, 2, 3$. Moreover, for non-positive-valued $\vartheta_0, \varpi_0, \varkappa_0$, the assumption ($\clubsuit_4$) of Theorem 3.2 holds.

Now, consider

$$|\eth_1(b, \vartheta(b))| + \left|\eth_1(b, \widetilde{\vartheta}(b))\right| = \frac{1}{1+b^3} \left(\frac{|\vartheta(b)|}{1+|\vartheta(b)|} + \frac{\left|\widetilde{\vartheta}(b)\right|}{1+\left|\widetilde{\vartheta}(b)\right|} \right) \leq \frac{1}{2} \left(|\vartheta(b)| + \left|\widetilde{\vartheta}(b)\right|\right),$$

$$|\eth_2(b, \varpi(b))| + |\eth_2(b, \widetilde{\varpi}(b))| = \frac{b}{1+b^4} \left(\frac{|\varpi(b)|}{2+|\varpi(b)|} + \frac{|\widetilde{\varpi}(b)|}{2+|\widetilde{\varpi}(b)|} \right) \leq \frac{1}{2} \left(|\varpi(b)| + |\widetilde{\varpi}(b)|\right),$$

and

$$|\eth_3(b, \varkappa(b))| + \left|\eth_3(b, \widetilde{\varkappa}(b))\right| = \frac{b^4}{1+b^7} \left(\frac{|\varkappa(b)|}{3+|\varkappa(b)|} + \frac{\left|\widetilde{\varkappa}(b)\right|}{3+\left|\widetilde{\varkappa}(b)\right|} \right) \leq \frac{1}{2} \left(|\varkappa(b)| + \left|\widetilde{\varkappa}(b)\right|\right),$$

so, assume that $\widehat{B} = \frac{1}{2}$. Also, one can write

$$\int_0^1 \Im_1(b,c)\, \eth_1(c, \vartheta(c)) dc \leq \int_0^1 \frac{c^2}{1+b^4} \cdot \frac{1}{1+c^3} dc = \frac{1}{1+b^4} \cdot \frac{1}{3} \ln(2) \leq \frac{1}{3} \ln(2),$$

$$\int_0^1 \Im_2(b,c)\,\eth_2(c,\vartheta(c))dc \le \int_0^1 c^2 e^{-b^2}.\frac{c}{1+c^4}dc = e^{-b^2}.\frac{1}{4}\ln(2) \le \frac{1}{4}\ln(2),$$
$$\int_0^1 \Im_2(b,c)\,\eth_2(c,\vartheta(c))dc \le \int_0^1 c^2 e^{-b^3}.\frac{c^4}{1+c^7}dc = e^{-b^3}.\frac{1}{7}\ln(2) \le \frac{1}{7}\ln(2).$$

Putting $\widehat{C} = \frac{1}{3}\ln(2)$, we have $\int_0^1 \Im_s(b,c)\,\eth_s(c,\vartheta(c))dc \le \widehat{C}$ for $s = 1,2,3$. Moreover,

$$\int_0^1 \Im_1(b,c)\,dc = \int_0^1 \frac{c^2}{1+b^4}dc = \frac{1}{3(1+b^4)} \le \frac{1}{3},$$
$$\int_0^1 \Im_2(b,c)\,dc = \int_0^1 c^2 e^{-b^2}dc = \frac{1}{3}e^{-b^2} \le \frac{1}{3},$$
$$\int_0^1 \Im_3(b,c)\,dc = \int_0^1 c^2 e^{-b^3}dc = \frac{1}{3}e^{-b^3} \le \frac{1}{3},$$

select $\widehat{A} = \frac{1}{3}$, then we have $\int_0^1 \Im_s(b,c)\,dc \le \widehat{A}$. Consequently, $3\widehat{A}\widehat{B}\widehat{C} = \frac{1}{6}\ln(2) < 1$. Therefore, all hypotheses of Theorem 3.2 are fulfilled. This implies that Problem (3.2) owns a solution in $C[0,1]$.

4. Conclusions and future works

The existence of a tripled coincidence point of a generalized contraction type mapping, which is considered in JS-metric spaces endowed with a partial order, was investigated in this article. Furthermore, the theoretical results have been supported by illustrative examples. Ultimately, as an application, the existence of a solution for a system of non-homogeneous and homogeneous integral equations is provided. Moreover, a numerical example of the system is derived. Along with the works presented in [4, 24, 25] as future works, we launch the following two inquiries: What would the results look like if the JS-metric space was replaced by a Modular space? What if we used the variational principle?

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Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

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