Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 9761–9781.
DOI: 10.3934/math.2023492
Received: 11 December 2022
Revised: 07 February 2023
Accepted: 15 February 2023
Published: 21 February 2023

Research article

Soft order topology and graph comparison based on soft order

Kemal Taşköprü*

Department of Mathematics, Faculty of Science, Bilecik Şeyh Edebali University, Bilecik 11100, Turkey

* **Correspondence:** Email: kemal.taskopru@bilecik.edu.tr; Tel: +902282141764.

Abstract: Soft sets provide a suitable framework for representing and dealing with vagueness. A scenario for vagueness can be that alternatives are composed of specific factors and these factors have specific attributes. Towards this scenario, this paper introduces soft order and its associated order topology on the soft sets with a novel approach. We first present the definitions and properties of the soft order relations on the soft sets via soft elements. Next, we define soft order topology on any soft set and provide some properties of this topology. In order to implement what we introduced about the soft orders, we describe soft preference and soft utility mapping on the soft sets and we finally demonstrate a decision-making application over the soft orders intended for comparing graphs.

Keywords: soft order; soft order topology; soft preference; soft utility; graph comparison
Mathematics Subject Classification: 03E72, 06F30, 05C90

1. Introduction

Soft sets were introduced as a mathematical tool to be used to identify alternatives with desired attributes. After its foundations were laid in 1999 [1], soft set theory has been integrated into various branches of mathematics, hybridized with other set theories, and has developed in interaction with other fields. Some of the recent works that have contributed to this development are [2–14] among others. On the other hand, there can be situations where the alternatives consist of specific factors and these factors are requested to have specific attributes. The concept of soft element within a soft set provides a model that represents the factors determining the alternative with their desired attributes [15]. There are various works that use the soft elements especially on the topology and (generalized) metric structures [16–23].

Order is a broad topic of research in mathematics, economics, informatics, and many other applications. The relationships between order and topology are required to provide a theoretical underpinning for the related studies. There have been numerous works on the connections between order and topology, most of which are relevant to the order theory and the utility theory (see [24–30] and others).

On the (hybrid) soft sets, there have also been several works on the relations and their properties, as well as their applications to decision-making, where the decision is made as a single element among alternatives specifying their attributes and weights [31–37]. By analysing the relationships between order relations and topology, it is seen that soft topology can be considered as a special case of ordered soft topology. By this means, the studies of ordered soft topology lead to the construction of different concepts and classes of soft topologies [38–41]. In addition to the lack of studies about the order and its associated topology on the soft sets, there has been no discussion of this topic in the context of soft elements. In this paper, we propose a novel approach to define an order relation and its associated order topology on the soft sets. While a relation on the soft sets is actually described as corresponding to a soft set in the mentioned works, a soft relation is recently defined in [42] over a collection of soft elements. Hence, we describe the orders on the soft sets through these relations and we construct the order topology on the soft sets from these orders. Furthermore, a utility function has been interpreted conventionally as a way of quantifying preferences numerically, i.e., in real-valued. However, it is emphasized that a utility function need not necessarily be real-valued and can be extended to more general totally ordered sets [26, 30]. Inspired by this, we explain soft preference and soft utility mapping on the soft sets. Moreover, comparing graphs and interpreting that comparison is a valuable aspect in the applications of graph theory (see [43]). We illustrate the feasibility of the soft orders that we offer by presenting a decision-making application in this aspect.

The paper is organized as follows: Section 2 contains the basic background about soft set, soft element, soft relation, and ε-soft topology. In Section 3, we first provide the definitions and some properties of soft order relations with the concept of soft relation defined by using the soft elements on any soft set. Unlike the soft order relation defined and used for the metric structures [15, 16], we conceive a soft order relation with the totality property for the soft real numbers. Hence, the studies about metric structures on the soft sets can be extended. Next, we show that an ε-soft topology can be built from the total preorder relations on any set. After that, we define the ε-soft order topology with regard to a soft total preorder relation on any soft set, provide some properties of this topology, and give some illustrative examples. Moreover, we define soft preference and soft utility mapping on a soft totally preordered set as soft real-valued by using the soft total order defined in the previous section to discuss the notions of preference and utility in terms of the soft sets. In Section 4, to put into practice what we introduced, by using the soft set representation of a graph described in [2], we demonstrate a decision-making application for graph comparison, which has an essential role in the applications of graph theory. Section 5 outlines the implications of this paper.

2. Preliminaries

The definitions and properties in this section are recalled from [20, 21, 42] to be utilized in later. For more detailed discussion, see these papers.

Definition 2.1. [20] Let U be a universal set, P be a parameters set and $\mathrm{P}(U)$ be the power set of U. A pair (G, P) is called a soft set on U, where G: $P \to \mathrm{P}(U)$ is a set-valued mapping.

A function ε: $P \to U$ is called a soft element of U and ε is said to be member of (G, P) if $\varepsilon(\alpha) \in G(\alpha)$ for each $\alpha \in P$. The class of soft elements of (G, P) are denoted by $SE(G, P)$ and the soft elements are denoted by $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, etc. Also, the belonging of a soft element $\tilde{x}$ to a soft set (G, P) is denoted by $\tilde{x}\tilde{\in}(G, P)$ or $\tilde{x} \in SE(G, P)$.

Definition 2.2. [20] Let (G, P) and (H, P) be two soft sets on U. The soft set (G, P) is said to be a null soft set and denoted by Φ (an absolute soft set and denoted by $\tilde{U}$) if $G(\alpha) = \emptyset$ ($G(\alpha) = U$) for each $\alpha \in P$.

The soft set (G, P) is said to be a soft subset of (H, P) if $G(\alpha) \subseteq H(\alpha)$ for every $\alpha \in P$ and denoted by $(G, P)\tilde{\subseteq}(H, P)$. Then, $(G, P) = (H, P)$ if and only if

$$(G, P)\tilde{\subseteq}(H, P),$$

and

$$(H, P)\tilde{\subseteq}(G, P).$$

Throughout the work, the soft sets (G, P) on U such that $G(\alpha) \neq \emptyset$ for every $\alpha \in P$ and the null soft set Φ will be considered. The class of these soft sets is denoted by $S(\tilde{U})$.

A soft set (G, P) produced by a collection of soft elements $\mathfrak{B}$ is defined by

$$(G, P) = SS(\mathfrak{B}) = \{(\alpha, G(\alpha)) : \forall \alpha \in P,\ G(\alpha) = \bigcup_{\tilde{x} \in \mathfrak{B}} \{\tilde{x}(\alpha)\}\}.$$

Notice that $\mathfrak{B}$ and $SE(SS(\mathfrak{B}))$ are not the same in general, but $\mathfrak{B} \subseteq SE(SS(\mathfrak{B}))$.

The ε-union and ε-intersection of

$$(G, P),\ (H, P) \in S(\tilde{U})$$

are defined by

$$(G, P) \Cup (H, P) = SS(SE(G, P) \cup SE(H, P)),$$

and

$$(G, P) \Cap (H, P) = SS(SE(G, P) \cap SE(H, P)),$$

respectively. The ε-complement of (G, P) is defined

$$(G, P)^{\mathbb{C}} = SS(SE(G, P)^{c}),$$

where $(G, P)^c = (G^c, P)$ is soft complement of (G, P) and G^c: $P \to \mathrm{P}(U)$ is a mapping given by

$$G^c(\alpha) = U \backslash G(\alpha)$$

for each $\alpha \in P$.

From now on, we will use G instead of (G, P) for simplicity and the parameters set P will be referred to as a finite set.

Definition 2.3. [42] Let U and U' be two universal sets and P be a parameters set. A soft relation $\mathfrak{R}$ from $\tilde{U}$ to $\tilde{U}'$ is a subclass of $SE(\tilde{U}) \times SE(\tilde{U}')$ and then a soft relation $\mathfrak{R}$ on $\tilde{U}$ is a subclass of $SE(\tilde{U}) \times SE(\tilde{U})$.

Definition 2.4. [42] Let $\tilde{U}$ be an absolute soft set with parameter set P and $\mathfrak{R}$ be a soft relation on $\tilde{U}$. The soft relation $\mathfrak{R}$ is called

- *reflexive* if $\tilde{x}\Re\tilde{x}$ for each $\tilde{x} \in SE(\tilde{U})$;
- *symmetric* if $\tilde{x}\Re\tilde{y} \Rightarrow \tilde{y}\Re\tilde{x}$ for each $\tilde{x}, \tilde{y} \in SE(\tilde{U})$;
- *antisymmetric* if $\tilde{x}\Re\tilde{y}$ and $\tilde{y}\Re\tilde{x} \Rightarrow \tilde{x} = \tilde{y}$ for each $\tilde{x}, \tilde{y} \in SE(\tilde{U})$;
- *transitive* if $\tilde{x}\Re\tilde{y}$ and $\tilde{y}\Re\tilde{z} \Rightarrow \tilde{x}\Re\tilde{z}$ for each $\tilde{x}, \tilde{y}, \tilde{z} \in SE(\tilde{U})$;
- *total (complete, connected, comparable or connex)* if $\tilde{x}\Re\tilde{y}$ or $\tilde{y}\Re\tilde{x}$ for each $\tilde{x}, \tilde{y} \in SE(\tilde{U})$;

where $\tilde{x}R\tilde{y}$ to mean that $(\tilde{x}, \tilde{y}) \in \Re$. Also, a soft relation $\Re$ is called

- *pre-order* if it is reflexive and transitive;
- *total pre-order (weak order, preference)* if it is reflexive, total, and transitive;
- *strict preorder (strict partial order)* if it is irreflexive, asymmetric, and transitive;
- *partial order* if it is reflexive, antisymmetric, and transitive;
- *total (complete, linear) order* if it is reflexive, antisymmetric, total, and transitive;
- *equivalence relation* if it is reflexive, symmetric, and transitive.

Example 2.5. Suppose that a company wants to set up franchises and supply products also between them.

Let

$$U = \{x, y, z\}$$

be a set of warehouses, where the products will be supplied. Let

$$P = \{\alpha_1 := \text{Market proximity}, \alpha_2 := \text{Prompt delivery}\}$$

be a parameters set which corresponds to the attributes of the warehouses that the company wants to supply. Then, the warehouses from which the company can supply products according to the parameters are represented by a soft set such as

$$G = \{(\alpha_1, \{x, y\}), (\alpha_2, \{x, z\})\}.$$

The soft elements of $\tilde{U}$ are as follows:

$$\begin{aligned} &\tilde{x}_1 = \{(\alpha_1, x), (\alpha_2, x)\}, && \tilde{x}_4 = \{(\alpha_1, y), (\alpha_2, x)\}, && \tilde{x}_7 = \{(\alpha_1, z), (\alpha_2, x)\},\\ &\tilde{x}_2 = \{(\alpha_1, x), (\alpha_2, y)\}, && \tilde{x}_5 = \{(\alpha_1, y), (\alpha_2, y)\}, && \tilde{x}_8 = \{(\alpha_1, z), (\alpha_2, y)\},\\ &\tilde{x}_3 = \{(\alpha_1, x), (\alpha_2, z)\}, && \tilde{x}_6 = \{(\alpha_1, y), (\alpha_2, z)\}, && \tilde{x}_9 = \{(\alpha_1, z), (\alpha_2, z)\}, \end{aligned}$$

and the soft elements of G,

$$SE(G) = \{\tilde{x}_1, \tilde{x}_3, \tilde{x}_4, \tilde{x}_6\},$$

present all of the franchising scenarios of the company according to the set of parameters. Naturally, the company can consider other conditions, e.g. safety or quality control of the products, to determine its preferences. Next, the connections between the franchises can exist that are represented by a soft relation such as

$$\Re = \{(\tilde{x}_1, \tilde{x}_3), (\tilde{x}_1, \tilde{x}_4), (\tilde{x}_4, \tilde{x}_6), (\tilde{x}_6, \tilde{x}_3)\},$$

where the pairs of soft elements represent the product supply between the franchises.

Proposition 2.6. *[42] Let $\tilde{U}$ be an absolute soft set with parameter set P, $\Re$ be any soft relation on $\tilde{U}$ and $\mathscr{R} = \{R_\alpha \colon \alpha \in P\}$ be any parametrized family of classical relations on U.*

(1) $\mathscr{R}$ can be considered as $\Re$ with

$$\Re(\alpha) = \{(\tilde{x}, \tilde{y})(\alpha) = (\tilde{x}(\alpha), \tilde{y}(\alpha)) \in R_\alpha : \tilde{x}, \tilde{y} \in SE(\tilde{U})\}$$

and vice versa.

(2) If $\mathscr{R}$ is family of classical reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and transitive relations, then it can be considered as a reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and transitive soft relation, respectively.

(3) If $\Re$ is reflexive, symmetric, and total soft relation, then it can be considered as a parametrized family of reflexive, symmetric, and total classical relations, respectively.

Remark 2.7. *If $\mathscr{R}$ is a family of total classical relations, then it cannot be considered as a total soft relation. In addition, if the parametrized family of classical relations with various properties cannot be considered as a soft relation with the same properties.*

On the other hand, if $\Re$ is irreflexive, transitive, asymmetric, and antisymmetric soft relation, then it cannot be considered as a parametrized family of irreflexive, transitive, antisymmetric, and asymmetric classical relations, respectively (for details, see [42]).

Definition 2.8. [20] Let $\mathcal{T} \subseteq S(\tilde{U})$ be a family of soft sets over U with parameter set P. Then, $\mathcal{T}$ is called an ε-soft topology and $(\tilde{U}, \mathcal{T}, P)$ is called an ε-soft topological space according to the ε-operations if it satisfies the followings:

(ε1) $\tilde{U}, \Phi \in \mathcal{T}$.
(ε2) If $G, H \in \mathcal{T}$, then $G \Cap H \in \mathcal{T}$.
(ε3) If $\forall i \in I$, $G_i \in \mathcal{T}$, then $\underset{i\in I}{\Cup} G_i \in \mathcal{T}$.

Also, the members of $\mathcal{T}$ is said to be soft open sets and for $K \in S(\tilde{U})$, K is said to be soft closed set if $K^c \in S(\tilde{U})$ and $K^{\mathbb{C}} \in \mathcal{T}$.

Definition 2.9. [20, 21] Let $\mathcal{B} \subseteq S(\tilde{U})$. Then, $\mathcal{B}$ is called a soft basis for an ε-soft topology on $\tilde{U}$ if the followings are met:

(B1) For all $\tilde{x} \in SE(\tilde{U})$, there exists at least one soft set $B \in \mathcal{B}$ such that $\tilde{x} \tilde{\in} B$.
(B2) If $\tilde{x} \tilde{\in} \tilde{U}$ and $\tilde{x} \tilde{\in} B_1 \Cap B_2$ for $B_1, B_2 \in \mathcal{B}$, there is $B_3 \in \mathcal{B}$ such that $\tilde{x} \tilde{\in} B_3 \tilde{\subseteq} B_1 \Cap B_2$.

Also, the ε-soft topology

$$\mathcal{T}_{\mathcal{B}} = \{G \in S(\tilde{U}) : \forall \tilde{x} \tilde{\in} G,\ \exists B \in \mathcal{B},\ \tilde{x} \tilde{\in} B \tilde{\subseteq} G\}$$

is said to be the ε-soft topology produced by $\mathcal{B}$ and $(\tilde{U}, \mathcal{T}, P)$ is called second-countable ε-soft space if there is a countable soft basis for $\mathcal{T}$.

Proposition 2.10. *[20] Let $(\tilde{U}, \mathcal{T}, P)$ be an ε-soft space and $\mathcal{B}$ be a soft basis for $\mathcal{T}$ (i.e., $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$). Then, every soft open set could be stated as the ε-union of some members of $\mathcal{B}$.*

Remark 2.11. *The converse of Proposition 2.10 is not true. For example, suppose that $U = \{x, y, z\}$, $P = \{\alpha, \beta\}$ and $\mathcal{T} = \{\Phi, \tilde{U}, G_1, G_2, G_3, G_4, G_5\}$ is a ε-soft topology on $\tilde{U}$, where*

$$G_1 = \{(\alpha, \{x\}), (\beta, \{x, y\})\}, \quad G_2 = \{(\alpha, \{z\}), (\beta, \{x, y\})\},$$

$$G_3 = \{(\alpha, \{x, z\}), (\beta, \{x, y\})\}, \quad G_4 = \{(\alpha, \{y, z\}), (\beta, \{z\})\},$$
$$G_5 = \{(\alpha, \{y, z\}), (\beta, U)\}.$$

Then, the collection $\mathcal{B} = \{G_1, G_2, G_4\}$ satisfies the condition of Proposition 2.10. However, for the soft element

$$\tilde{x} = \{(\alpha, y), (\beta, x)\} \tilde{\in} G_5,$$

there is not any $B \in \mathcal{B}$ such that $\tilde{x} \tilde{\in} B \tilde{\subseteq} G_5$. Thus, the collection $\mathcal{B}$ is not a soft basis for $\mathcal{T}$.

Theorem 2.12. *[20] Let $(\tilde{U}, \mathcal{T}, P)$ be an ε-soft space and $\mathcal{B} \subseteq \mathcal{T}$. If the following condition (B_*) is satisfied, then $\mathcal{B}$ is a soft basis for the $\mathcal{T}$.*

(B_*) For all $G \in \mathcal{T}$ and all $\tilde{x} \tilde{\in} G$, there exists $B \in \mathcal{B}$ such that $\tilde{x} \tilde{\in} B \tilde{\subseteq} G$.

Definition 2.13. [20] Let $(\tilde{U}, \mathcal{T}, P)$ be an ε-soft space and $\mathcal{S} \subseteq \mathcal{T}$. $\mathcal{S}$ is called soft subbasis for the $\mathcal{T}$, if the collection of all finite ε-intersection of members $\mathcal{S}$ is a soft basis for $\mathcal{T}$.

Definition 2.14. [21] Let $(\tilde{U}, \mathcal{T}, P)$ be an ε-soft space.

- Let $\tilde{x} \in SE(\tilde{U})$. A soft set $N \in S(\tilde{U})$ is called a soft neighbourhood of $\tilde{x}$ if there is a $\tilde{G} \in \mathcal{T}$ such that $\tilde{x} \tilde{\in} G \tilde{\subset} N$. The family of all soft neighbourhoods of $\tilde{x}$ is indicated by $\mathcal{N}(\tilde{x})$.
- For any $G \in S(\tilde{U})$, a soft element $\tilde{x} \in SE(\tilde{U})$ is called a soft closure element of G, if $G \Cap N \neq \Phi$, for every $N \in \mathcal{N}(\tilde{x})$. The collection of all soft closure elements of G is indicated by clG and the soft closure of G is indicated by $\bar{G} = SS(clG)$.
- A soft set $G \in S(\tilde{U})$ is called countable if $SE(G)$ is countable. Accordingly, if a collection of soft elements $\mathfrak{B} \subseteq SE(\tilde{U})$ is countable, then the soft set $G \in S(\tilde{U})$ generated by $\mathfrak{B}$ is countable.
- A soft set $G \in S(\tilde{U})$ is called dense soft set if $\bar{G} = \tilde{U}$.
- $(\tilde{U}, \mathcal{T}, P)$ is called separable if there exists a countable dense soft set in $G \in S(\tilde{X})$.

3. Soft order and soft topology

Before discussing the relationships between the soft order and the ε-soft topology, we first present some definitions about the soft order that we will utilize later. Next, we introduce the ε-soft order topology and provide some properties of this topology.

Definition 3.1. Let $\Re$ be a soft preorder on $\tilde{U}$, $G \in S(\tilde{U})$ and $\tilde{x} \tilde{\in} G$.

If $\tilde{x} \Re \tilde{y}$ implies $\tilde{x} = \tilde{y}$ for every $\tilde{y} \tilde{\in} G$, $\tilde{x}$ is called a maximal element in G. If $\tilde{y} \Re \tilde{x}$ implies $\tilde{x} = \tilde{y}$ for every $\tilde{y} \tilde{\in} \tilde{G}$, $\tilde{x}$ is called a minimal element in G.

If $\tilde{y} \Re \tilde{x}$ for every $\tilde{y} \tilde{\in} G$, $\tilde{x}$ is called a greatest element in G. If $\tilde{x} \Re \tilde{y}$ for every $\tilde{y} \tilde{\in} G$, $\tilde{x}$ is called a least element in G. For a soft total preorder, a soft element $\tilde{x}$ to be a maximal (minimal) in G is the same thing as for $\tilde{x}$ to be a greatest (least) in G.

Now, suppose that the soft preorder $\Re$ is antisymmetric (i.e., it is a soft partial order). Then, G can have at most one greatest (or least) element. If $\tilde{x}$ is a greatest (or least) element in G, then it is the unique maximal (or minimal) element of G. In addition, a soft total order is called a soft well-order provided every non-empty soft subset of $\tilde{U}$ has a least element.

After that, the usual notation $\tilde{\lesssim}$ instead of $\Re$ is used for any kind of soft order relation and the soft order notation can be inverted and written $\tilde{y} \tilde{\gtrsim} \tilde{x}$ $(\tilde{y} \tilde{>} \tilde{x})$ to mean the same thing as $\tilde{x} \tilde{\lesssim} \tilde{y}$ $(\tilde{x} \tilde{<} \tilde{y})$. The same setup will apply for the notations of all other kind of order relations.

Example 3.2. The soft order on the soft real numbers in [15] defined as follows: For any two soft element $\tilde{x}, \tilde{y}$ in $\tilde{\mathbb{R}}(P)$,

$$\tilde{x} \tilde{\leq} \tilde{y} \Leftrightarrow \tilde{x}(\alpha) \leq \tilde{y}(\alpha) \text{ for all } \alpha \in P,$$

where $\tilde{\mathbb{R}}(P)$ is denoted the set of all soft real numbers in [15], i.e., $SE(\tilde{\mathbb{R}})$ with a parameters set P, and $\leq$ is usual total order on $\mathbb{R}$. This order is actually the product order on the Cartesian product of the image sets under P of the soft elements, which are the subsets of $\mathbb{R} \times \mathbb{R}$. So, it is obvious that this relation is soft partial order.

In the soft set theory, the parameters set can be any set and weights are determined for the parameters in many papers, especially involving decision-making applications. As a result of this, it is possible to order the parameters according to their weights or more different approaches.

Here, we propose to determine the parameters set more specifically as a totally ordered set and accordingly, we define a new ordering among the soft elements in $SE(\tilde{\mathbb{R}})$.

Definition 3.3. Let

$$P = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

be a totally ordered parameters set by a relation $\leq$, where $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$, and $\tilde{\leqslant}$ be a relation on $SE(\tilde{\mathbb{R}})$ with the parameters set P defined as

$$\tilde{x} \tilde{\leqslant} \tilde{y} \Leftrightarrow \exists i, \forall j < i, \tilde{x}(\alpha_j) = \tilde{y}(\alpha_j) \text{ and } \tilde{x}(\alpha_i) \leq \tilde{y}(\alpha_i),$$

where $\leq$ is usual total order on $\mathbb{R}$. The relation $\tilde{\leqslant}$ is actually the lexicographic order on the Cartesian products of the image sets under P of the soft elements, which are the subsets of $\mathbb{R} \times \mathbb{R}$, and hence it is a total order relation. The relation $\tilde{\leqslant}$ is called soft total order on $\tilde{\mathbb{R}}$.

Definition 3.4. Let $\tilde{\lesssim}$ be a soft preorder on $\tilde{U}$ with a parameters set P. The triplet $(\tilde{U}, \tilde{\lesssim}, P)$ is said to be a soft preordered set. Also, the soft equivalence relation $\tilde{\sim}$ associated with $\tilde{\lesssim}$ is denoted by

$$\tilde{x} \tilde{\sim} \tilde{y} \Leftrightarrow \tilde{x} \tilde{\lesssim} \tilde{y} \text{ and } \tilde{y} \tilde{\lesssim} \tilde{x},$$

and the soft asymmetric relation $\tilde{\prec}$ associated with $\tilde{\lesssim}$ is denoted by

$$\tilde{x} \tilde{\prec} \tilde{y} \Leftrightarrow \tilde{x} \tilde{\lesssim} \tilde{y} \text{ and } \tilde{y} \tilde{\not\lesssim} \tilde{x}.$$

Moreover, a soft set $G \in S(\tilde{U})$ is called soft order-dense according to $\tilde{\lesssim}$ if, for each $\tilde{x}, \tilde{y} \tilde{\in} \tilde{U}$ with $\tilde{x} \tilde{\prec} \tilde{y}$, there exists $\tilde{z} \in SE(G)$ such that $\tilde{x} \tilde{\lesssim} \tilde{z} \tilde{\lesssim} \tilde{y}$. Then, $\tilde{U}$ is called soft order separable according to $\tilde{\lesssim}$ if there is a countable soft order-dense set in $S(\tilde{U})$.

Definition 3.5. Let $(\tilde{U}, \tilde{\lesssim}, P)$ be a soft preordered set. The collections of soft elements of $\tilde{U}$

$$\begin{aligned}
\mathfrak{L}(\tilde{a}) &= \{\tilde{x} \in SE(\tilde{U}) : \tilde{x} \tilde{\lesssim} \tilde{a}\}, \\
\mathfrak{SL}(\tilde{a}) &= \{\tilde{x} \in SE(\tilde{U}) : \tilde{x} \tilde{\prec} \tilde{a}\}, \\
\mathfrak{U}(\tilde{a}) &= \{\tilde{x} \in SE(\tilde{U}) : \tilde{x} \tilde{\gtrsim} \tilde{a}\}, \\
\mathfrak{SU}(\tilde{a}) &= \{\tilde{x} \in SE(\tilde{U}) : \tilde{x} \tilde{\succ} \tilde{a}\},
\end{aligned}$$

are called lower, strict lower, upper, and strict upper contour set of a soft element $\tilde{a} \in SE(\tilde{U})$, respectively.

Then, the soft subsets $SS(\mathfrak{L}(\tilde{x}))$, $SS(\mathfrak{S}\mathfrak{L}(\tilde{x}))$, $SS(\mathfrak{U}(\tilde{x}))$ and $SS(\mathfrak{S}\mathfrak{U}(\tilde{x}))$ are called soft lower, soft strict lower, soft upper, and soft strict upper contour set of $\tilde{x}$, respectively.

Also, the soft sets in $S(\tilde{U})$

$$\begin{aligned}
SS(\tilde{a}, \tilde{b}) &= SS\left(\mathfrak{S}\mathfrak{U}(\tilde{a}) \cap \mathfrak{S}\mathfrak{L}(\tilde{b})\right), \\
SS[\tilde{a}, \tilde{b}] &= SS\left(\mathfrak{U}(\tilde{a}) \cap \mathfrak{L}(\tilde{b})\right), \\
SS(\tilde{a}, \tilde{b}] &= SS\left(\mathfrak{S}\mathfrak{U}(\tilde{a}) \cap \mathfrak{L}(\tilde{b})\right), \\
SS[\tilde{a}, \tilde{b}) &= SS\left(\mathfrak{U}(\tilde{a}) \cap \mathfrak{S}\mathfrak{L}(\tilde{b})\right),
\end{aligned}$$

are called soft open interval, soft closed interval, and soft half open intervals with the extreme soft elements $\tilde{a}$ and $\tilde{b}$, respectively.

In addition, for any $\tilde{a}, \tilde{b} \in SE(\tilde{U})$, if the soft open interval $SS(\tilde{a}, \tilde{b})$ does not have any soft elements, it is called a soft jump, denoted $J(\tilde{a}, \tilde{b})$.

Now, we begin to determine the relationships between the soft order and the ε-soft topology.

Definition 3.6. Let $(\tilde{U}, \mathcal{T}, P)$ be an ε-soft topological space and $\tilde{\lesssim}$ be a soft preorder on $\tilde{U}$. $\tilde{\lesssim}$ is called soft continuous on $\tilde{U}$ according to $\mathcal{T}$ if $SS(\mathfrak{U}(\tilde{x}))$ and $SS(\mathfrak{L}(\tilde{x}))$ are soft open set in $\tilde{U}$ for each $\tilde{x} \in SE(\tilde{U})$. Then, an ε-soft topology in which a soft preoerder $\tilde{\lesssim}$ is soft continuous is called compatible with $\tilde{\lesssim}$.

The next proposition states that an ε-soft topology can be constructed from any parametrized family of total preorders on any set.

Proposition 3.7. *Let U be a universal set, P be a parameters set and*

$$\mathscr{P} = \{\lesssim_\alpha : \alpha \in P\}$$

be any parametrized family of total preorders on U. Then, the family of soft sets

$$\mathcal{T}_{\mathscr{P}} = \{G \in S(\tilde{U}) : \forall \alpha \in P,\, G(\alpha) \in \tau_{\lesssim_\alpha}\}$$

is an ε-soft topology on $\tilde{U}$, where $\tau_{\lesssim_\alpha}$ is the order topology on U with respect to $\lesssim_\alpha$ for each $\alpha \in P$.

Proof. (ε1) Since $\emptyset, U \in \tau_{\lesssim_\alpha}$ for each $\alpha \in P$, we get $\Phi, \tilde{U} \in \mathcal{T}_{\mathscr{P}}$.
(ε2) Suppose that $G, H \in \mathcal{T}_{\mathscr{P}}$. If $G \Cap H = \Phi$, then it is clear that $G \Cap H \in \mathcal{T}_{\mathscr{P}}$. If $G \Cap H \neq \Phi$, then

$$G(\tilde{\alpha}) \cap H(\alpha) \neq \emptyset \text{ and } G(\tilde{\alpha}) \cap H(\alpha) \in \tau_{\lesssim_\alpha}$$

for each $\alpha \in P$. Hence, it is obtained that $G \Cap H \in \mathcal{T}_{\mathscr{P}}$.
(ε3) Suppose that $\forall i \in I$, $G_i \in \mathcal{T}_{\mathscr{P}}$. Since $\bigcup_{i \in I} G_i(\tilde{\alpha}) \in \tau_{\lesssim_\alpha}$ for each $\alpha \in P$, we have $\Cup_{i \in I} G_i \in \mathcal{T}_{\mathscr{P}}$.

Remark 3.8. *From Proposition 2.6 and Remark 2.7, one can see that any parametrized family of total preorders*

$$\mathscr{P} = \{\lesssim_\alpha : \alpha \in P\}$$

on U can be considered as at least a soft preorder on $\tilde{U}$. Additionally, it can be observed that the family of all soft sets of the form

$$B = \{(\alpha, (a, b)_{\precsim_\alpha}) : \alpha \in P\},$$

where $(a, b)_{\precsim_\alpha}$ is an open interval in U with respect to $\precsim_\alpha$ for each $\alpha \in P$, is a soft basis for $\mathcal{T}_{\mathscr{P}}$ and these are also the soft open sets in $\mathcal{T}_{\mathscr{P}}$.

Furthermore, the family of all soft sets of the form

$$\{(\alpha, \mathfrak{S}\mathfrak{U}(x)_{\precsim_\alpha}) : \alpha \in P\} \text{ and } \{(\alpha, \mathfrak{S}\mathfrak{L}(x)_{\precsim_\alpha}) : \alpha \in P\},$$

where $\mathfrak{S}\mathfrak{U}(x)_{\precsim_\alpha}$ and $\mathfrak{S}\mathfrak{L}(x)_{\precsim_\alpha}$ are the strict upper and lower contour sets in U with respect to $\precsim_\alpha$ for each $\alpha \in P$ and $x \in U$, respectively, is a soft subbasis for $\mathcal{T}_{\mathscr{P}}$.

Note that these soft sets are actually the soft strict upper and lower contour sets of any $\tilde{x} \in SE(\tilde{U})$ with respect to the soft preorder to which $\mathscr{P}$ corresponds. Thus, this soft preorder is soft continuous on $\tilde{U}$ according to $\mathcal{T}_{\mathscr{P}}$.

Example 3.9. Suppose that P is a parameters set and the soft partial order $\tilde{\leq}$ on $\tilde{\mathbb{R}}$ is considered from Example 3.2.

Then, this order can be considered as generated by the usual total order on $\mathbb{R}$, i.e., it can be considered as a family of parametrized relations $\mathscr{P}$ in which each parameter corresponds to the usual total order on $\mathbb{R}$. Hence, the family of all soft sets of the form

$$B = \{(\alpha, (a, b)) : \alpha \in P \text{ and } a, b \in \mathbb{R}\}$$

is a soft basis for $\mathcal{T}_{\mathscr{P}}$ on $\tilde{\mathbb{R}}$ and each soft open set $G \in S(\tilde{\mathbb{R}})$ in $\mathcal{T}_{\mathscr{P}}$ is obtained as

$$G = \{(\alpha_1, V_1), (\alpha_2, V_2), \ldots, (\alpha_{n-1}, V_{n-1}), (\alpha_n, V_n)\}\},$$

where each V_i is an open set in the usual topology on $\mathbb{R}$ for $i \in \{1, 2, \ldots, n\}$.

Moreover, it is clear that $\tilde{\leq}$ is soft continuous since for any $\tilde{a} \in SE(\tilde{\mathbb{R}})$, the soft strict upper and lower contour sets, i.e., the soft open rays, are soft open sets obtained as

$$\{(\alpha, (\tilde{a}(\alpha), \infty)) : \alpha \in P\} \text{ and } \{(\alpha, (-\infty, \tilde{a}(\alpha))) : \alpha \in P\}.$$

Thus, it can be seen that all of the above soft open rays form the soft subbasis for $\mathcal{T}_{\mathscr{P}}$.

Theorem 3.10. *Let $(\tilde{U}, \tilde{\precsim}, P)$ be a soft totally preordered set. A collection $\mathcal{B} \subseteq S(\tilde{U})$ of all of the following types of soft sets forms a soft basis for an ε-soft topology on $\tilde{U}$.*

- *All soft open intervals $SS(\tilde{a}, \tilde{b})$ in $\tilde{U}$.*
- *All soft intervals of the form $SS[\tilde{a}_0, \tilde{b})$, where $\tilde{a}_0$ is the least element (if any) of $\tilde{U}$.*
- *All soft intervals of the form $SS(\tilde{a}, \tilde{b}_0]$, where $\tilde{b}_0$ is the greatest element (if any) of $\tilde{U}$.*

Proof. (B1) If $\tilde{x} = \tilde{a}_0$ or $\tilde{x} = \tilde{b}_0$, then it can be seen that there exists a soft set $B \in \mathcal{B}$ containing $\tilde{x}$. On the other hand, if $\tilde{x} \neq \tilde{a}_0$ and $\tilde{x} \neq \tilde{b}_0$, then $\tilde{x} \in \mathfrak{S}\mathfrak{U}(\tilde{a}) \cap \mathfrak{S}\mathfrak{L}(\tilde{b})$ and so $\tilde{x} \tilde{\in} SS(\tilde{a}, \tilde{b})$ for some $\tilde{a}, \tilde{b} \tilde{\in} \tilde{U}$.

(B2) Without loss of generality for any kind of contour sets, the intersections of them have the form $(\tilde{a}, \tilde{b})$ and hence $SS(\tilde{a}, \tilde{b}) \in \mathcal{B}$.

Thus, if $\tilde{x} \tilde{\in} B_1 \Cap B_2$ for any $B_1, B_2 \in \mathcal{B}$, there exists a basis element containing $\tilde{x}$.

Definition 3.11. Let $(\tilde{U}, \tilde{\lesssim}, P)$ be a soft totally preordered set. An ε-soft topology on $\tilde{U}$, the soft basis of which is the collection of all soft open intervals, is called ε-soft order topology on $\tilde{U}$ and denoted by $\mathcal{T}_{\tilde{\lesssim}}$.

Then, each soft open interval is a soft open set in the ε-soft order topology. In addition, each soft closed interval whose soft complement belongs to $S(\tilde{U})$ is a soft closed set in the ε-soft order topology.

Example 3.12. Suppose that the soft total order $\tilde{\leqslant}$ on $\tilde{\mathbb{R}}$ is given with a totally ordered parameters set $P = \{\alpha_i : i \in \{1, 2, \dots, n\}\}$, where $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$. Then, the soft open intervals $SS(\tilde{a}, \tilde{b})$ according to this order are obtained as following types:

- If $\tilde{a}(\alpha_1) < \tilde{b}(\alpha_1)$, then
$$\{(\alpha_1, [\tilde{a}(\alpha_1), \tilde{b}(\alpha_1)]), (\alpha_i, \mathbb{R}) : i \in \{2, \dots, n\}\}.$$
- If $\tilde{a}(\alpha_1) = \tilde{b}(\alpha_1)$, $\tilde{a}(\alpha_2) = \tilde{b}(\alpha_2)$, $\dots$, $\tilde{a}(\alpha_i) = \tilde{b}(\alpha_i)$ and $\tilde{a}(\alpha_{i+1}) < \tilde{b}(\alpha_{i+1})$, then
$$\{(\alpha_j, \{\tilde{a}(\alpha_j)\}), (\alpha_{i+1}, [\tilde{a}(\alpha_{i+1}), \tilde{b}(\alpha_{i+1})]), (\alpha_k, \mathbb{R}) : j \in \{1, \dots, i\} \text{ and } k \in \{i+2, \dots, n\}\}.$$
- If $\tilde{a}(\alpha_1) = \tilde{b}(\alpha_1)$, $\tilde{a}(\alpha_2) = \tilde{b}(\alpha_2)$, $\dots$, $\tilde{a}(\alpha_{n-1}) = \tilde{b}(\alpha_{n-1})$ and $\tilde{a}(\alpha_n) < \tilde{b}(\alpha_n)$, then
$$\{(\alpha_i, \{\tilde{a}(\alpha_i)\}), (\alpha_n, (\tilde{a}(\alpha_n), \tilde{b}(\alpha_n))) : i \in \{1, \dots, n-1\}\}.$$

Hence, each soft open set $G \in S(\tilde{\mathbb{R}})$ in the ε-soft order topology on $\tilde{\mathbb{R}}$, which takes these soft open intervals as a soft basis, is obtained as
$$G = \{(\alpha_i, V_i), (\alpha_n, V_n) : i \in \{1, 2, \dots, n-1\}\},$$
where each V_i is an open set in the discrete topology on $\mathbb{R}$ and V_n is an open set in the usual topology on $\mathbb{R}$.

Also, it is obvious that $\tilde{\leqslant}$ is soft continuous since the soft strict upper and lower contour sets, i.e., the soft open rays, are soft open sets obtained as follows:
$$SS(\mathfrak{U}(\tilde{a})) = SS(\tilde{a}, \infty) = \{(\alpha_1, [\tilde{a}(\alpha_1), \infty)), (\alpha_i, \mathbb{R}) : i \in \{2, \dots, n\}\},$$
$$SS(\mathfrak{L}(\tilde{a})) = SS(\infty, \tilde{b}) = \{(\alpha_1, (-\infty, \tilde{b}(\alpha_1)]), (\alpha_i, \mathbb{R}) : i \in \{2, \dots, n\}\}.$$

On the other hand, the finite ε-intersections of the soft sets just above are obtained as follows:
$$B = \{(\alpha_1, V), (\alpha_i, \mathbb{R}) : i \in \{2, \dots, n\}\},$$
where V is an open set in the discrete topology on $\mathbb{R}$.

However, the collection of these soft sets $\mathcal{B}$ are not a soft basis for the ε-soft order topology since this collection is not satisfied the condition (B_*) in Theorem 2.12. For example, assume that G is a soft open set in $\mathcal{T}_{\tilde{\leqslant}}$ such that $G(\alpha_n)$ is an open interval in $\mathbb{R}$, and $\tilde{x}\tilde{\in}G$.

Then, there exists a soft set $B \in \mathcal{B}$ such that $\tilde{x} \in B$, but G does not contain B, i.e., $B\tilde{\nsubseteq}G$. Therefore, the collection of soft open rays is not a soft subbasis for $\mathcal{T}_{\tilde{\leqslant}}$.

Remark 3.13. *Similarly to the last part of Example 3.12, one can obtain the soft strict upper and lower contour sets for a given soft total preorder $\tilde{\lesssim}$ over an absolute soft set $\tilde{U}$ with a parameters set P.*

Then, it can be seen that if the collection C of finite ε-intersections of them satisfies the conditions to be a soft basis for an ε-soft topology $\mathcal{T}_C$ on $\tilde{U}$, then $\mathcal{T}_C$ has fewer soft open sets than $\mathcal{T}_{\tilde{\precsim}}$, i.e., $\mathcal{T}_C \subseteq \mathcal{T}_{\tilde{\precsim}}$, since

$$SS(\tilde{a}, \tilde{b}) \tilde{\subseteq} SS(\Subset\mathfrak{U}(\tilde{a})) \Cap SS(\Subset\mathfrak{L}(\tilde{b})),$$

for any $\tilde{a}, \tilde{b} \in SE(\tilde{U})$ provided that $\tilde{a} \tilde{\prec} \tilde{b}$.

Also, it is clear that the soft strict upper and lower contour sets are again soft open sets in $\mathcal{T}_C$ and hence $\tilde{\precsim}$ is soft continuous on $\tilde{U}$ according to $\mathcal{T}_C$.

Definition 3.14. An ε-soft space $(\tilde{U}, \mathcal{T}, P)$ is called (pre)orderable soft space if there exists a soft total preorder on $\tilde{U}$ such that the ε-soft order topology induced by that soft order and the given ε-soft topology on $\tilde{U}$ coincide.

Example 3.15. In [16] and in metric-related studies based on this work, the soft partial order relation $\tilde{\leq}$ on $\tilde{\mathbb{R}}$ was used in the soft metric axioms when defining a soft metric space. Accordingly, if it is assumed that $(\tilde{\mathbb{R}}, d, P)$ is a soft metric space such that

$$d(\tilde{x}, \tilde{y})(\alpha) = |\tilde{x}(\alpha) - \tilde{y}(\alpha)|,$$

for each $\alpha \in P$, then by considering Example 3.9, the soft metric topology corresponds to $\mathcal{T}_{\mathscr{P}}$ on $\tilde{\mathbb{R}}$.

On the other hand, it is possible to define a soft metric space by using the soft total order $\tilde{\leqslant}$ on $\tilde{\mathbb{R}}$ since the linear extension of a partial order is a total order that is compatible with the partial order. Then, the soft metric topology corresponds to $\mathcal{T}_{\tilde{\leqslant}}$ on $\tilde{\mathbb{R}}$ and thus $(\tilde{\mathbb{R}}, d, P)$ is an orderable soft space. In addition, it is clear that $\mathcal{T}_{\mathscr{P}} \subset \mathcal{T}_{\tilde{\leqslant}}$ and $\mathcal{T}_{\mathscr{P}}$ is not comparable with $\mathcal{T}_C$ on $\tilde{\mathbb{R}}$.

Furthermore, if it is assumed that $(\tilde{U}, d_1, P)$ and $(\tilde{U}, d_2, P)$ are any two soft metric spaces, one can show that $(\tilde{U}, d, P)$ is a soft metric space with

$$d(\tilde{x}, \tilde{y}) = \max\{d_1(\tilde{x}, \tilde{y}), d_2(\tilde{x}, \tilde{y})\} \text{ for all } \tilde{x}, \tilde{y} \in SE(\tilde{U}),$$

by using the soft total order $\tilde{\leqslant}$ on $\tilde{\mathbb{R}}$. But, it is clear that if the soft partial order $\tilde{\leq}$ on $\tilde{\mathbb{R}}$ is used, then d will not be a soft metric.

Theorem 3.16. *Let $(\tilde{U}, \tilde{\precsim}, P)$ be a soft totally preordered set. A soft set $H \in S(\tilde{U})$ generated by the collection of the extreme soft elements of every soft jump is soft order-dense if and only if H is a dense soft set in $(\tilde{U}, \mathcal{T}_{\tilde{\precsim}}, P)$.*

Proof. Let H be soft order-dense according to $\tilde{\precsim}$. Assume that H is not a dense soft set in $(\tilde{U}, \mathcal{T}_{\tilde{\precsim}}, P)$. Hence, there exists a soft open set G such that

$$H \Cap G = \Phi.$$

Then, for $\tilde{x} \tilde{\in} G$, there is a soft open interval I such that $\tilde{x} \tilde{\in} I \tilde{\subset} G$. So, since each soft open interval I is a soft open set and H is generated by the collection of the extreme soft elements of every soft jump, I covers a soft open interval with the extreme soft elements not in H.

Hence, H is not soft order dense which is a contradiction. Thus, H is a dense soft set in $(\tilde{U}, \mathcal{T}_{\tilde{\precsim}}, P)$.

Conversely, let H be a dense soft set in $(\tilde{U}, \mathcal{T}_{\tilde{\precsim}}, P)$ and $\tilde{a}, \tilde{b} \in SE(\tilde{U})$ with $\tilde{a} \tilde{\prec} \tilde{b}$. Then, there exists a soft open interval I with the extreme soft elements $\tilde{a}$ and $\tilde{b}$ such that

$$H \Cap I \neq \Phi.$$

Hence, if I is a soft jump, then $\tilde{a}, \tilde{b} \tilde{\in} H$, and if I is not a soft jump, then there exists $\tilde{x} \tilde{\in} H$ such that $\tilde{a} \tilde{\prec} \tilde{x} \tilde{\prec} \tilde{b}$. Thus, this implies that H is soft order-dense.

Theorem 3.17. *Let $(\tilde{U}, \tilde{\lesssim}, P)$ be a soft totally preordered set. Then, the followings are equivalent:*

(1) $\tilde{U}$ is soft order separable.
(2) There are only countably many soft jumps in $\tilde{U}$ and $(\tilde{U}, \mathcal{T}_{\tilde{\lesssim}}, P)$ is separable.
(3) $(\tilde{U}, \mathcal{T}_{\tilde{\lesssim}}, P)$ is second countable.

Proof.

(1)⇔(2) Suppose that $\tilde{U}$ is soft order separable, $G \in S(\tilde{U})$ is a countable soft order-dense set and $H \in S(\tilde{U})$ is a soft set generated by the collection of the extreme soft elements of every soft jump. For a soft jump $J(\tilde{a}, \tilde{b})$, there exists a soft element $\tilde{x} \in SE(G)$ such that $\tilde{a} \tilde{\sim} \tilde{x}$ or $\tilde{b} \tilde{\sim} \tilde{x}$ since $\tilde{U}$ is soft order separable according to $\tilde{\lesssim}$.

This allows each soft element $\tilde{x} \in SE(G)$ to be associated to a maximum of two soft jumps as $\tilde{\lesssim}$ is total. Hence, H is countable soft set, because G is countable. Thus, there are only countably many soft jumps in $\tilde{U}$.

Then,

$$F = G \Cup H$$

is also countable and soft order dense. So, from Theorem 3.16, $(\tilde{U}, \mathcal{T}_{\tilde{\lesssim}}, P)$ is separable.

Conversely, suppose that there are only countably many soft jumps in $\tilde{U}$ and $(\tilde{U}, \mathcal{T}_{\tilde{\lesssim}}, P)$ is separable. Let $G \in S(\tilde{U})$ be a dense soft set, and $H \in S(\tilde{U})$ be a countable soft set generated by the collection of the extreme soft elements of every soft jump.

Then, $F = G \Cup H$ is countable and also soft order dense from Theorem 3.16. Thus, $\tilde{U}$ is soft order separable.

(2)⇔(3) Suppose that there are only countably many soft jumps in $\tilde{U}$ and $(\tilde{U}, \mathcal{T}_{\tilde{\lesssim}}, P)$ is separable. From first part of the proof, $\tilde{U}$ is soft order separable.

Let $H \in S(\tilde{U})$ be a countable soft order dense set and for all $\tilde{a}_k, \tilde{a}_l \tilde{\in} H$, $\mathcal{D}$ be a collection of soft order intervals in the following:

$$SS(\tilde{a}_k, \tilde{a}_l), \text{ where } \tilde{a}_k \tilde{\prec} \tilde{a}_l,$$
$$SS(\tilde{a}_k, \tilde{b}_0], \text{ where } \tilde{b}_0 \text{ is greatest soft element (if any) of } \tilde{U} \text{ and } \tilde{a}_k \tilde{\prec} \tilde{b}_0,$$
$$SS[\tilde{a}_0, \tilde{a}_l), \text{ where } \tilde{a}_0 \text{ is least soft element (if any) of } \tilde{U} \text{ and } \tilde{a}_0 \tilde{\prec} \tilde{a}_l.$$

So, $\mathcal{D}$ is countable and each soft set in $\mathcal{D}$ is a soft open in $\mathcal{T}_{\tilde{\lesssim}}$. Now, let G be a soft open set in $\mathcal{T}_{\tilde{\lesssim}}$ and $\tilde{x} \tilde{\in} G$. Then, there exists $B \in \mathcal{B}$ such that $\tilde{x} \tilde{\in} B \tilde{\subset} G$. Hence, without losing generality, if

$$B = SS(\tilde{a}, \tilde{b}),$$

then there exist $\tilde{a}_k, \tilde{a}_l \tilde{\in} H$ such that $\tilde{a} \tilde{\lesssim} \tilde{a}_k \tilde{\lesssim} \tilde{x} \tilde{\lesssim} \tilde{a}_l \tilde{\lesssim} \tilde{b}$ since H is soft order dense. Thus, $\mathcal{D}$ is a countable soft basis for $\mathcal{T}_{\tilde{\lesssim}}$ since

$$\tilde{x} \tilde{\in} SS(\tilde{a}_k, \tilde{a}_l) \tilde{\subset} G,$$

and so $(\tilde{U}, \mathcal{T}_{\tilde{\lesssim}}, P)$ is second countable.

On the contrary, suppose that $(\tilde{U}, \mathcal{T}_{\tilde{\lesssim}}, P)$ is second countable and $\mathcal{B}$ is a countable soft basis for $\mathcal{T}_{\tilde{\lesssim}}$. Then, it is clear that $(\tilde{U}, \mathcal{T}_{\tilde{\lesssim}}, P)$ is separable.

Afterwards, let $\mathcal{J}$ be the collection of all soft jumps and $\Psi : \mathcal{J} \to \mathbb{N}$ be a mapping such that

$$\tilde{a} \tilde{\in} B_{\Psi(J(\tilde{a},\tilde{b}))} \tilde{\subseteq} SS(\mathfrak{L}(\tilde{b})).$$

Since $SS(\mathfrak{L}(\tilde{b}))$ is soft open, we can always make such a choice of $\Psi(J(\tilde{a}, \tilde{b}))$. For any

$$J(\tilde{a}, \tilde{b}) \neq J(\tilde{c}, \tilde{d}),$$

either $\tilde{b} \tilde{<} \tilde{d}$ or $\tilde{d} \tilde{<} \tilde{b}$, it is obtained that

$$\Psi(J(\tilde{a}, \tilde{b})) \neq \Psi(J(\tilde{c}, \tilde{d})).$$

Thus, Ψ is an injection from $\mathcal{J}$ into $\mathcal{B}$ and so $\mathcal{J}$ is countable.

- **Soft preference and soft utility**

It is a typical approach to model a preference relation as a total preorder relation. Through the same approach, we describe a soft preference relation as a soft total preorder. Accordingly, from Definition 3.4, for a soft preference relation $\tilde{\lesssim}$ on $\tilde{U}$ with a parameters set P, the associated soft equivalence relation $\tilde{\sim}$ is called soft indifference relation and the associated soft asymmetric relation $\tilde{<}$ is called soft strict preference relation.

Moreover, we define a soft utility mapping on a soft totally preordered set as soft real-valued by using the soft total order $\tilde{\leqslant}$ on $\tilde{\mathbb{R}}$ in the following.

Definition 3.18. Let $\tilde{\lesssim}$ be a soft preference relation on $\tilde{U}$ with totally ordered parameters set P. The mapping $\mathfrak{u} : SE(\tilde{U}) \to SE(\tilde{\mathbb{R}})$ is said to represent $\tilde{\lesssim}$ and it is called a soft utility mapping if it satisfies

$$\mathfrak{u}(\tilde{x}) \tilde{\leqslant} \mathfrak{u}(\tilde{y}) \Leftrightarrow \tilde{x} \tilde{\lesssim} \tilde{y} \text{ for all } \tilde{x}, \tilde{y} \in SE(\tilde{U}).$$

Example 3.19. Suppose that a garden owner wants to grow some crops in the garden. Let

$$U = \{x, y, z\}$$

be a set of crops and

$$P = \{\alpha_1 := \text{Marketable}, \alpha_2 := \text{Resistant}\}$$

be a parameters set which corresponds to the attributes of the crops that the garden owner wants to grow.

Also, assume that the garden owner wants the crops to be marketable more than to be resistant to insects, i.e., $\alpha_1 \geq \alpha_2$.

Then, the crops that the garden owner can grow according to the parameters are represented by a soft set such as

$$F = \{(\alpha_1, \{x, y\}), (\alpha_2, \{y, z\})\}.$$

The soft elements of $\tilde{U}$ are as follows:

$$\tilde{x}_1 = \{(\alpha_1, x), (\alpha_2, x)\}, \qquad \tilde{x}_4 = \{(\alpha_1, y), (\alpha_2, x)\}, \qquad \tilde{x}_7 = \{(\alpha_1, z), (\alpha_2, x)\},$$

$$\tilde{x}_2 = \{(\alpha_1, x), (\alpha_2, y)\}, \qquad \tilde{x}_5 = \{(\alpha_1, y), (\alpha_2, y)\}, \qquad \tilde{x}_8 = \{(\alpha_1, z), (\alpha_2, y)\},$$
$$\tilde{x}_3 = \{(\alpha_1, x), (\alpha_2, z)\}, \qquad \tilde{x}_6 = \{(\alpha_1, y), (\alpha_2, z)\}, \qquad \tilde{x}_9 = \{(\alpha_1, z), (\alpha_2, z)\},$$

and the soft elements of F,

$$SE(F) = \{\tilde{x}_2, \tilde{x}_3, \tilde{x}_5, \tilde{x}_6\},$$

present all of the crop-growing scenarios of the garden according to the set parameters. Here, if the parameters correspond to different crops, they will be grown together; if they correspond to the same crop, that crop will be grown only.

The garden owner can consider other circumstances, for example, perhaps weather conditions or interactions of the crops with each other, to determine her or his preferences. Hence, the particular preferences for the soft elements of F can be given as $\tilde{x}_2 \tilde{\lesssim} \tilde{x}_3 \tilde{\lesssim} \tilde{x}_5 \tilde{\lesssim} \tilde{x}_6$.

Following that, for example; a soft utility mapping $\mathfrak{u}$: $SE(F) \to SE(\tilde{\mathbb{R}})$ that represents these preferences can be defined by

$$\begin{aligned} \mathfrak{u}(\tilde{x}_2) = \{(\alpha_1, 1), (\alpha_2, 3)\} &\tilde{\leqslant} \mathfrak{u}(\tilde{x}_3) = \{(\alpha_1, 2), (\alpha_2, 1)\} \\ &\tilde{\leqslant} \mathfrak{u}(\tilde{x}_5) = \{(\alpha_1, 2), (\alpha_2, 4)\} \\ &\tilde{\leqslant} \mathfrak{u}(\tilde{x}_6) = \{(\alpha_1, 3), (\alpha_2, 5)\}. \end{aligned}$$

Remark 3.20. *Suppose that U is a universal set, P is a totally ordered parameters set and*

$$\mathscr{U} = \{u_\alpha : U \to \mathbb{R} : \alpha \in P\}$$

is a parametrized family of the utility functions that represent the preference relations $\lesssim_\alpha$ on U for each $\alpha \in P$. If the soft mapping $\mathfrak{u} : SE(\tilde{U}) \to SE(\tilde{\mathbb{R}})$ is defined as

$$\mathfrak{u}(\tilde{x})(\alpha) = u_\alpha(\tilde{x}(\alpha)) \text{ for each } \alpha \in P, \tag{3.1}$$

then from Remark 2.7, $\mathfrak{u}$ cannot be a soft utility mapping that represents a soft preference relation $\tilde{\lesssim}$ on $\tilde{U}$, i.e., any parametrized family of the utility functions cannot be considered as a soft utility mapping.

Conversely, suppose that $\tilde{\lesssim}$ is a soft preference relation on $\tilde{U}$ and $\mathfrak{u}$ is a soft utility mapping that represents $\tilde{\lesssim}$. Again from Remark 2.7, $\mathfrak{u}$ cannot be considered as a parametrized family of the utility functions $u_\alpha \in \mathscr{U}$ for each $\alpha \in P$.

Example 3.21. In Example 3.19, if we set the preferences on U according to the parameters as $x \lesssim_{\alpha_1} y \lesssim_{\alpha_1} z$ and $y \lesssim_{\alpha_2} x \lesssim_{\alpha_2} z$, then the utility functions u_{α_1} and u_{α_2} on U that represent $\lesssim_{\alpha_1}$ and $\lesssim_{\alpha_2}$ can be defined in a way that is appropriate, respectively. As noted in Remark 2.7, the soft relation on the soft set F generated by these preferences

$$\mathfrak{R} = \{(\tilde{x}_2, \tilde{x}_2), (\tilde{x}_3, \tilde{x}_3), (\tilde{x}_5, \tilde{x}_5), (\tilde{x}_6, \tilde{x}_6), (\tilde{x}_2, \tilde{x}_3), (\tilde{x}_2, \tilde{x}_5), (\tilde{x}_2, \tilde{x}_6), (\tilde{x}_3, \tilde{x}_6), (\tilde{x}_5, \tilde{x}_6)\}$$

is not a soft preference relation since the soft elements $\tilde{x}_3$ and $\tilde{x}_5$ of F are not comparable according to this soft relation. As a similar conclusion, the utility functions u_{α_1} and u_{α_2} cannot generate a soft utility mapping.

On the other hand, if we consider the soft utility mapping in Example 3.19, then we can decompose $\mathfrak{u}$ considering (3.1). But, u_{α_1} and u_{α_2} are not even well defined mappings. Indeed,

$$\mathfrak{u}(\tilde{x}_2)(\alpha_1) = u_{\alpha_1}(\tilde{x}_2(\alpha_1)) = u_{\alpha_1}(x) = 1,$$

$$\mathfrak{u}(\tilde{x}_3)(\alpha_1) = u_{\alpha_1}(\tilde{x}_3(\alpha_1)) = u_{\alpha_1}(x) = 2,$$

and

$$\mathfrak{u}(\tilde{x}_2)(\alpha_2) = u_{\alpha_2}(\tilde{x}_2(\alpha_2)) = u_{\alpha_2}(y) = 1,$$
$$\mathfrak{u}(\tilde{x}_5)(\alpha_2) = u_{\alpha_2}(\tilde{x}_5(\alpha_2)) = u_{\alpha_2}(y) = 3.$$

Theorem 3.22. *Let $\tilde{U}$ be a finite absolute soft set with a totally ordered parameters set P and $\tilde{\prec}$ be a soft strict preference relation on $\tilde{U}$. Then, there exists a soft utility mapping that represents $\tilde{\prec}$.*

Proof. Suppose that $\mathfrak{u}\colon SE(\tilde{U}) \to SE(\tilde{\mathbb{R}})$ is a soft mapping defined by for each $\tilde{x} \in SE(\tilde{X})$

$$\mathfrak{u}(\tilde{x}) = \{(\alpha, |SS(\mathfrak{L}(\tilde{x}))(\alpha)|) : \alpha \in P\},$$

where $|SS(\mathfrak{L}(\tilde{x}))(\alpha)|$ denotes the number of elements of

$$SS(\mathfrak{L}(\tilde{x}))(\alpha) \subset U \text{ for each } \alpha \in P.$$

This mapping is well defined since $\tilde{U}$ being finite depends on U and P being finite.

Let

$$\tilde{x}\tilde{\prec}\tilde{y} \text{ for } \tilde{x}, \tilde{y} \in SE(\tilde{U}).$$

So, $\tilde{y} \in \mathfrak{L}(\tilde{y})$ by totality and $\tilde{y} \notin \mathfrak{L}(\tilde{x})$ since $\tilde{x}\tilde{\precsim}\tilde{y}$ and $\tilde{y}\tilde{\not\precsim}\tilde{x}$. If $\tilde{z} \in \mathfrak{L}(\tilde{x})$, then $\tilde{z} \in \mathfrak{L}(\tilde{y})$ by transitivity.

Hence,

$$\mathfrak{L}(\tilde{x}) \subseteq \mathfrak{L}(\tilde{y})$$

and so

$$SS(\mathfrak{L}(\tilde{x}))\tilde{\subseteq}SS(\mathfrak{L}(\tilde{y})).$$

This implies that $\mathfrak{L}(\tilde{x})$ and $\{\tilde{y}\}$ are disjoint and both subsets of $\mathfrak{L}(\tilde{y})$. Then,

$$\mathfrak{L}(\tilde{x}) \cup \{\tilde{y}\} \subseteq \mathfrak{L}(\tilde{y})$$

and hence

$$\mathfrak{u}(\tilde{x}) + \bar{1}\tilde{\leqslant}\mathfrak{u}(\tilde{y}).$$

Therefore, $\mathfrak{u}(\tilde{x})\tilde{<}\mathfrak{u}(\tilde{y})$.

The above theorem can be proved similarly in the case of soft preference relation $\tilde{\precsim}$. Hence, we can state the following corollary.

Corollary 3.23. *Let $\tilde{U}$ be a finite absolute soft set with a totally ordered parameters set P and $\tilde{\precsim}$ be a soft preference relation on $\tilde{U}$. Then, there exists a soft utility mapping that represents $\tilde{\precsim}$.*

4. A decision-making application: graph comparison based on the soft order

Graph theory is of substantial interest to researchers due to its diverse applications spanning many fields. In particular, the comparison of graphs finds applications in fields such as bio-cheminformatics and network analysis [43].

From this perspective, the weights of the vertices can be expected to be decisive in evaluating the compatibility or correspondence of the graphs having these vertices. Apart from this, Ali et al. [2] showed that the adjacency of vertices and soft set theory were used to represent a graph and gave some features and examples related to this representation.

Here, by using the soft set representation of a graph, we present a decision-making application, when a graph is given as a particular model or sample, which can be used to obtain the most relevant graph or to compare it with other available ones.

First of all, the following concepts are recalled from [2] in order to have a framework about the soft set representation of a graph.

Definition 4.1. Let

$$U = \{u_1, u_2, \ldots, u_n\}$$

be a finite universal set. A multi-set A of U is characterized by a function

$$c_A : U \to \{0, 1, 2, \ldots\},$$

such that $c_A(u_i)$ is the number of occurrences of the element u_i in the multi-set for each $i = 1, 2 \ldots, n$ and then it is expressed by

$$A = \{u_1^{c_A(u_1)}, \ldots, u_n^{c_A(u_n)}\}.$$

Here, both U and A will be considered as finite sets. Then, a pair (G, P) or simply G is called a soft multi-set on U, when P is a parameters set and $G : P \to \text{PM}(U)$ is a multi-set valued mapping, where $\text{PM}(U)$ denotes the set of all multi-sets of U.

Definition 4.2. Let V and E be sets such that $E \subseteq V \times V$. Then (V, E) is called a *graph*, the elements of V are called *vertices* of this graph, and the elements of E are called *edges* of this graph. Also, a graph is said to be edge (vertex)-weighted if each edge (vertex) is associated with a weight (usually a real number) in the graph.

So, one can associate a weight to each vertex in a graph based on their relative utility, i.e., it can be studied on the vertex-weighted graphs, and hence the vertices can be totally ordered.

Definition 4.3. A soft set representation of a graph is a soft multi-set G on the set of vertices V, where $G : V \to \text{PM}(V)$ with

$$G(v) = \begin{cases} \{v' \in V : \quad v' \text{ is adjacent to } v\}, \\ \emptyset, \qquad\qquad \text{otherwise,} \end{cases}$$

for each $v \in V$.

Notice that, according to the above definitions, the set of all soft elements of $\widetilde{\text{PM}}(V)$, i.e., $SE(\widetilde{\text{PM}}(V)))$, is considered to be all of the soft set representations of graphs with the set of vertices V.

Now, the following Algorithm is created for a decision-making application that uses the soft element, the soft order, and the soft set representation of a graph.

Algorithm Determining the correspondence of graphs.

Step 1. Input a set of vertices as $V = \{v_1, v_2, \ldots, v_n\}$.
Step 2. Input the preferences on the vertices as totally ordered, i.e., $V = \{v_i : i \in \{1, \ldots, n\}\}$, where $v_1 \succeq v_2 \succeq \ldots \succeq v_n$ and $\preceq$ is a total order on the vertices.
Step 3. Input a graph $G^* \in SE(\tilde{PM}(V))$ as a sample and input a set of graphs $\mathcal{G} = \{G_1, \ldots, G_k\}$ within $SE(\tilde{PM}(V))$ to be checked.
Step 4. Construct a soft mapping $\mathfrak{u} : SE(\tilde{PM}(V)) \times SE(\tilde{PM}(V)) \rightarrow SE(\tilde{\mathbb{R}})$ to measure the correspondence of two graphs.
Step 5. For G^* and each $G_k \in \mathcal{G}$, compute the soft mapping $\mathfrak{u}$ and order each soft element $\mathfrak{u}(G_k, G^*)$ with respect to the soft total order $\tilde{\leqslant}$ on $\tilde{\mathbb{R}}$.
Step 6. The decision is a graph $G_{k^*} \in \mathcal{G}$ if $\mathfrak{u}(G_{k^*}, G^*)$ is the greatest soft element according to $\tilde{\leqslant}$.
Step 7. If k^* has more than one value then any one of G_{k^*} can be chosen.

Suppose that

$$V = \{v_i : i \in \{1, 2, 3, 4\}\}$$

is a set of vertices, where $v_1 \succeq v_2 \succeq v_3 \succeq v_4$ and $\preceq$ is a total order on the vertices based on their relative utility, that is, the vertices are weighted.

By these vertices, let G^* be a vertex-weighted graph drawn in Figure 1 as a sample, where the size of the vertices is a representation of the order of the vertex-weights. So, the soft set representation of G^* is

$$G^* = \{(v_1, \{v_2^2, v_3, v_4\}), (v_2, \{v_1^2, v_3\}), (v_3, \{v_1, v_2\}), (v_4, \{v_1\})\}.$$

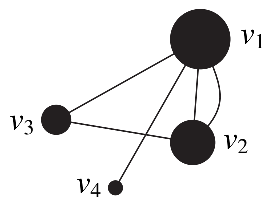

Figure 1. Visualisation of G^*.

Let

$$\mathcal{G} = \{G_1, \ldots, G_7\}$$

be a set of graphs relevant to G^* or desired to be associated with G^* drawn in Figure 2. The soft set representations of these graphs are as follows:

$$G_1 = \{(v_1, \{v_2, v_3^2, v_4\}), (v_2, \{v_1\}), (v_3, \{v_1^2\}), (v_4, \{v_1\})\},$$
$$G_2 = \{(v_1, \{v_2^2, v_4\}), (v_2, \{v_1^2, v_4\}), (v_3, \{v_4\}), (v_4, \{v_1, v_2, v_3\})\},$$
$$G_3 = \{(v_1, \{v_2^3, v_3\}), (v_2, \{v_1^3, v_4\}), (v_3, \{v_1, v_4\}), (v_4, \{v_2, v_3\})\},$$
$$G_4 = \{(v_1, \{v_3^2, v_4\}), (v_2, \{v_3\}), (v_3, \{v_1^2, v_2, v_4\}), (v_4, \{v_1, v_3\})\},$$
$$G_5 = \{(v_1, \{v_3^2, v_4^2\}), (v_2, \{v_4\}), (v_3, \{v_1^2\}), (v_4, \{v_1^2, v_2\})\},$$
$$G_6 = \{(v_1, \{v_2^2, v_4\}), (v_2, \{v_1^2\}), (v_3, \emptyset), (v_4, \{v_1\})\},$$

$$G_7 = \{(v_1, \{v_2, v_3^3, v_4\}), (v_2, \{v_1, v_4\}), (v_3, \{v_1^3\}), (v_4, \{v_1, v_2\})\}.$$

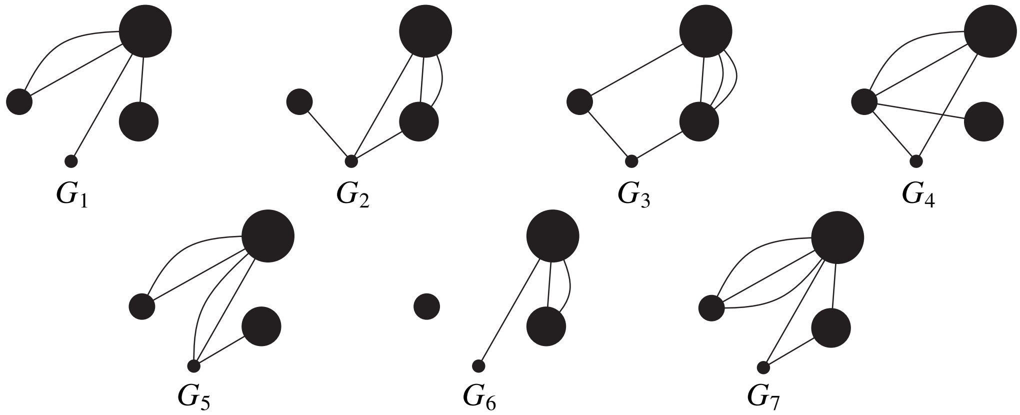

Figure 2. Graphs relevant to G^*.

Assume that the soft mapping $\mathfrak{u}$ mentioned in Step 4 is defined by for each $G, G' \in SE(P\tilde{M}(V))$,

$$\mathfrak{u}(G, G') = \{(v_i, \sum_{v_j \in V} \min\{c_{G(v_j)}, c_{G'(v_j)}\}) : i, j \in \{1, \ldots, n\}\}.$$

Then, if we compute the soft mapping $\mathfrak{u}$ for G^* and each of the graph in $\mathcal{G}$, we have

$$\begin{aligned}
\mathfrak{u}(G_1, G^*) &= \{(v_1, 3), (v_2, 1), (v_3, 1), (v_4, 1)\},\\
\mathfrak{u}(G_2, G^*) &= \{(v_1, 3), (v_2, 2), (v_3, 0), (v_4, 1)\},\\
\mathfrak{u}(G_3, G^*) &= \{(v_1, 3), (v_2, 2), (v_3, 1), (v_4, 0)\},\\
\mathfrak{u}(G_4, G^*) &= \{(v_1, 2), (v_2, 1), (v_3, 2), (v_4, 1)\},\\
\mathfrak{u}(G_5, G^*) &= \{(v_1, 2), (v_2, 0), (v_3, 1), (v_4, 1)\},\\
\mathfrak{u}(G_6, G^*) &= \{(v_1, 3), (v_2, 2), (v_3, 0), (v_4, 1)\},\\
\mathfrak{u}(G_7, G^*) &= \{(v_1, 3), (v_2, 1), (v_3, 1), (v_4, 1)\}.
\end{aligned}$$

Hence, the soft order of correspondence of these graphs with G^* is obtained as

$$G_5 \tilde{\lesssim} G_4 \tilde{\lesssim} G_1 \tilde{\simeq} G_7 \tilde{\lesssim} G_2 \tilde{\simeq} G_6 \tilde{\lesssim} G_3.$$

Therefore, one can choose G_3 as the most relevant graph to G^* and it is observed that there is indifference between G_1 and G_7 and between G_2 and G_6.

5. Conclusions

In this work, we present a novel approach by gathering the concepts of order and topology on the soft sets via the soft elements. It is shown that a soft topology can be obtained with a parametrized family of total orders and that a soft (order) topology can always be constructed in a soft ordered set. Since a soft order cannot be considered as a parametrized family of classical orders, a soft topology on the soft ordered set is found to have different characteristics.

The study of soft topology in the ordered set-up leads to the construction of various concepts and classes of soft topologies. The notions of preference and utility are described over the soft sets via the soft elements and, contrary to usual practice, a non-real valued utility function is depicted. By using this approach, further investigations can be carried out to combine the order theory and the utility theory with the soft set theory and for other topological properties. Also, it will allow to the extension of soft metric structures and their applications. In addition, the ideas mentioned in the paper can be integrated with the hybrid soft sets and more affirmative solutions can be provided in the decision-making applications.

Acknowledgments

We thank the referees for taking the time to review and for their contributions to improve the clarity and impact of this paper.

Conflict of interest

The author declares no conflict of interest.

References

1. D. Molodtsov, Soft set theory-first results, *Comput. Math. Appl.*, **37** (1999), 19–31. http://doi.org/10.1016/S0898-1221(99)00056-5
2. M. I. Ali, M. Shabir, F. Feng, Representation of graphs based on neighborhoods and soft sets, *Int. J. Mach. Learn. Cyber.*, **8** (2017), 1525–1535. http://doi.org/10.1007/s13042-016-0525-z
3. M. B. Kandemir, The concept of σ-algebraic soft set, *Soft Comput.*, **22** (2018), 4353–43607. http://doi.org/10.1007/s00500-017-2901-3
4. E. Aygün, H. Kamacı, Some generalized operations in soft set theory and their role in similarity and decision making, *J. Intell. Fuzzy Syst.*, **36** (2019), 6537–6547. http://doi.org/10.3233/JIFS-182924
5. V. Çetkin, E. Güner, H. Aygün, On 2S-metric spaces, *Soft Comput.*, **24** (2020), 12731–12742. http://doi.org/10.1007/s00500-020-05134-w
6. S. A. Ghour, W. Hamed, On two classes of soft sets in soft topological spaces, *Symmetry*, **12** (2020), 265. http://doi.org/10.3390/sym12020265
7. J. C. R. Alcantud, Soft open bases and a novel construction of soft topologies from bases for topologies, *Mathematics*, **8** (2020), 672. http://doi.org/10.3390/math8050672
8. J. C. R. Alcantud, An operational characterization of soft topologies by crisp topologies, *Mathematics*, **9** (2021), 1656. http://doi.org/10.3390/math9141656
9. G. Muhiuddin, D. Al-Kadi, K. P. Shum, A. M. Alanazi, Generalized ideals of BCK/BCI-algebras based on fuzzy soft set theory, *Adv. Fuzzy Syst.*, **2021** (2021), 8869931. http://doi.org/10.1155/2021/8869931
10. İ. Zorlutuna, Soft set-valued mappings and their application in decision making problems, *Filomat*, **35** (2021), 1725–1733. http://doi.org/10.2298/FIL2105725Z

11. T. M. Al-shami, E. A. Abo-Tabl, Soft α-separation axioms and α-fixed soft points, *AIMS Math.*, **6** (2021), 5675–5694. http://doi.org/10.3934/math.2021335
12. S. A. Ghour, On soft generalized ω-closed sets and soft $T_{1/2}$ spaces in soft topological spaces, *Axioms*, **11** (2022), 194. http://doi.org/10.3390/axioms11050194
13. G. Ali, M. N. Ansari, Multiattribute decision-making under Fermatean fuzzy bipolar soft framework, *Granular Comput.*, **7** (2022), 337–352. http://doi.org/10.1007/s41066-021-00270-6
14. T. M. Al-shami, J. C. R. Alcantud, A. Mhemdi, New generalization of fuzzy soft sets: (a, b)-fuzzy soft sets, *AIMS Math.*, **8** (2023), 2995–3025. http://doi.org/10.3934/math.2023155
15. S. Das, S. K. Samanta, Soft real sets, soft real numbers and their properties, *J. Fuzzy Math.*, **20** (2012), 551–576.
16. S. Das, S. K. Samanta, On soft metric spaces, *J. Fuzzy Math.*, **21** (2013), 707–734.
17. A. Ç. Güler, E. D. Yıldırım, O. B. Özbakır, A fixed point theorem on soft G-metric spaces, *J. Nonlinear Sci. Appl.*, **9** (2016), 885–894. http://doi.org/10.22436/jnsa.009.03.18
18. M. Chiney, S. K. Samanta, Soft topology redefined, *J. Fuzzy Math.*, **27** (2019), 459–486.
19. İ. Altıntaş, K. Taşköprü, Compactness of soft cone metric space and fixed point theorems related to diametrically contractive mapping, *Turk. J. Math.*, **44** (2020), 2199–2216. http://doi.org/10.3906/mat-2004-63
20. K. Taşköprü, İ. Altıntaş, A new approach for soft topology and soft function via soft element, *Math. Meth. Appl. Sci.*, **44** (2021), 7556–7570. http://doi.org/10.1002/mma.6354
21. İ. Altıntaş, K. Taşköprü, B. Selvi, Countable and separable elementary soft topological space, *Math. Meth. Appl. Sci.*, **44** (2021), 7811–7819. http://doi.org/10.1002/mma.6976
22. İ. Demir, Some soft topological properties and fixed soft element results in soft complex valued metric spaces, *Turk. J. Math.*, **45** (2021), 971–987. http://doi.org/10.3906/mat-2101-15
23. İ. Altıntaş, K. Taşköprü, P. Esengul kyzy, Soft partial metric spaces, *Soft Comput.*, **26** (2022), 8997–9010. http://doi.org/10.1007/s00500-022-07313-3
24. D. S. Bridges, G. B. Mehta, *Representations of preferences orderings*, Springer, 1995. http://dx.doi.org/10.1007/978-3-642-51495-1
25. S. Barberà, P. J. Hammond, C. Seidl, *Handbook of utility theory*, Springer, 1999.
26. G. Herden, G. B. Mehta, The Debreu Gap Lemma and some generalizations, *J. Math. Econ.*, **40** (2004), 747–769. http://doi.org/10.1016/j.jmateco.2003.06.002
27. M. J. Campión, J. C. Candeal, E. Induráin, Preorderable topologies and order-representability of topological spaces, *Topol. Appl.*, **156** (2009), 2971–2978. http://doi.org/10.1016/j.topol.2009.01.018
28. Ö. Evren, E. A. Ok, On the multi-utility representation of preference relations, *J. Math. Econ.*, **47** (2011), 554–563. http://doi.org/10.1016/j.jmateco.2011.07.003
29. J. C. R. Alcantud, G. Bosi, M. Zuanon, Richter-Peleg multi-utility representations of preorders, *Theory Decis.*, **80** (2016), 443–450. http://doi.org/10.1007/s11238-015-9506-z
30. A. F. Beardon, *Topology and preference relations*, Springer, 2020. http://doi.org/10.1007/978-3-030-34226-5-1

31. M. I. Ali, T. Mahmood, M. M. U. Rehman, M. F. Aslam, On lattice ordered soft sets, *Appl. Soft Comput.*, **36** (2015), 499–505. http://doi.org/10.1016/j.asoc.2015.05.052

32. A. Ali, M. I. Ali, N. Rehman, A more efficient conflict analysis based on soft preference relation, *J. Intell. Fuzzy Syst.*, **34** (2018), 283–293. http://doi.org/10.3233/JIFS-171172

33. M. A. Qamar, N. Hassan, Q-neutrosophic soft relation and its application in decision making, *Entropy*, **20** (2018), 1–14. http://doi.org/10.3390/e20030172

34. R. S. Kanwal, M. Shabir, Rough approximation of a fuzzy set in semigroups based on soft relations, *Comput. Appl. Math.*, **38** (2019), 89. http://doi.org/10.1007/s40314-019-0851-3

35. M. E. El-Shafei, T. M. Al-shami, Applications of partial belong and total non-belong relations on soft separation axioms and decision-making problem, *Comput. Appl. Math.*, **39** (2020), 138. http://doi.org/10.1007/s40314-020-01161-3

36. O. Dalkılıç, Relations on neutrosophic soft set and their application in decision making, *J. Appl. Math. Comput.*, **67** (2021), 257–273. http://doi.org/10.1007/s12190-020-01495-5

37. O. Dalkılıç, N. Demirtaş, A novel perspective for Q-neutrosophic soft relations and their application in decision making, *Artif. Intell. Rev.*, **56** (2022), 1493–1513. http://doi.org/10.1007/s10462-022-10207-3

38. G. Yaylalı, N. Ç. Polat, B. Tanay, Soft intervals and soft ordered topology, *CBU Fen Derg.*, **13** (2017), 81–89. http://doi.org/10.18466/cbayarfbe.302645

39. T. M. Al-Shami, M. E. El-Shafei, M. Abo-Elhamayel, On soft topological ordered spaces, *J. King Saud Univ. Sci.*, **31** (2019), 556–566. http://doi.org/10.1016/j.jksus.2018.06.005

40. T. M. Al-Shami, M. E. El-Shafei, Two new forms of ordered soft separation axioms, *Demonstr. Math.*, **53** (2020), 8–26. http://doi.org/10.1515/dema-2020-0002

41. S. Jafari, A. E. F. El-Atik, R. M. Latif, M. K. El-Bably, Soft topological spaces induced via soft relations, *WSEAS Trans. Math.*, **20** (2021), 1–8. http://doi.org/10.37394/23206.2021.20.1

42. K. Taşköprü, E. Karaköse, A soft set approach to relations and its application to decision making, *Math. Sci. Appl. E-Notes*, **11** (2023), 1–13. http://doi.org/10.36753/mathenot.1172408

43. N. M. Kriege, F. D. Johansson, C. Morris, A survey on graph kernels, *Appl. Network Sci.*, **5** (2020), 1–42. http://doi.org/10.1007/s41109-019-0195-3

AIMS Press