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Research article

A novel class of forward-backward explicit iterative algorithms using inertial techniques to solve variational inequality problems with quasi-monotone operators

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Abstract: The theory of variational inequalities is an important tool in physics, engineering, finance, and optimization theory. The projection algorithm and its variants are useful tools for determining the approximate solution to the variational inequality problem. This paper introduces three distinct extragradient algorithms for dealing with variational inequality problems involving quasi-monotone and semistrictly quasi-monotone operators in infinite-dimensional real Hilbert spaces. This problem is a general mathematical model that incorporates a set of applied mathematical models as an example, such as equilibrium models, optimization problems, fixed point problems, saddle point problems, and Nash equilibrium point problems. The proposed algorithms employ both fixed and variable stepsize rules that are iteratively transformed based on previous iterations. These algorithms are based on the fact that no prior knowledge of the Lipschitz constant or any line-search framework is required. To demonstrate the convergence of the proposed algorithms, some simple conditions are used. Numerous experiments have been conducted to highlight the numerical capabilities of algorithms.

Keywords: variational inequalities; Tseng's extragradient method; inertial-type iterative scheme; quasi-monotone operator; Lipschitz continuous operators
Mathematics Subject Classification: 47H05, 47H10, 47J20, 47J25, 65J15, 91B50

1. Introduction

The primary objective of this study is to look into the iterative methods used to solve the variational inequality problem [13] including quasi-monotone operators in any real Hilbert space. Consider that $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$ is a mapping. Assume $\mathcal{H}$ is an arbitrary Hilbert space and $\mathcal{K}$ is a nonempty, closed, and convex subset of $\mathcal{H}$. The variational inequality problem for mapping $\mathcal{A}$ on $\mathcal{K}$ is defined as follows:

$$\text{Find } u^* \in \mathcal{K} \text{ such that } \langle \mathcal{A}(u^*), r_2 - u^* \rangle \geq 0, \ \forall\, r_2 \in \mathcal{K}. \tag{1.1}$$

Conditions of convergence study. The following conditions must be met in order to investigate the strong convergence of proposed algorithms.
($\mathcal{A}1$) The solution set of a problem (1.1) is denoted by $VI(\mathcal{K}, \mathcal{A})$ and nonempty it is non-empty;
($\mathcal{A}2$) $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$ is said to be quasi-monotone if

$$\langle \mathcal{A}(r_1), r_2 - r_1 \rangle > 0 \Longrightarrow \langle \mathcal{A}(r_2), r_2 - r_1 \rangle \geq 0, \ \forall\, r_1, r_2 \in \mathcal{K};$$

($\mathcal{A}3$) $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$ is said to be Lipschitz continuous with $L > 0$ such that

$$\|\mathcal{A}(r_1) - \mathcal{A}(r_2)\| \leq L\|r_1 - r_2\|, \ \forall\, r_1, r_2 \in \mathcal{K};$$

($\mathcal{A}4$) $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$ is sequentially weakly continuous if $\{\mathcal{A}(r_i)\}$ weakly converges to $\mathcal{A}(r)$ for each $\{r_i\}$ weakly converges to r.

The formal formulation of the variational inequality problem is an important topic in nonlinear analysis. It is a key mathematical model that advances many important ideas in modern mathematics, such as optimization conditions for problems with the optimization process, nonlinear systems of equations, complementarity problems, finance, and network equilibrium problems (see for more details [12, 17–20, 27, 30]). As a result, this concept has a wide range of applications in mathematics, engineering, transportation studies, network economics, game theory, and computer science. The regularized approach and the projection algorithm are both useful and general solutions for dealing with variational inequalities. It should also be noted that the first algorithm is most commonly used to deal with variational inequalities that are largely determined by the monotone operator class. The regularized subproblem is strongly monotone in this approach, and its unique solution is determined to be more convenient than the primary problem. In this study, we will look into projection methods that are well-known for their simplicity of numerical calculation. Many researchers have examined not just the theory of solution existence and stability but also iterative procedures for dealing with variational inequality problems (see for more details [2, 7, 15, 44–46]) and others in [24, 29, 32–36, 38].

Furthermore, projection methods are appropriate for finding a numerical solution to variational inequalities. Many researchers have created novel projection techniques to deal with various types of variational inequalities [3, 5, 8–10, 16, 21, 26, 28, 37, 47–49] and others in [4, 6, 39–43]. All algorithms for resolving the (1.1) problem are based on computing a projection on the appropriate set $\mathcal{K}$. The corresponding extragradient method was developed by Korpelevich [21] and Antipin [1]. Their algorithm is designed as follows:

$$\begin{cases} u_1 \in \mathcal{K}, \\ v_i = P_{\mathcal{K}}[u_i - \varkappa \mathcal{A}(u_i)], \\ u_{i+1} = P_{\mathcal{K}}[u_i - \varkappa \mathcal{A}(v_i)], \end{cases} \tag{1.2}$$

where $0 < \varkappa < \frac{1}{L}$. In accordance with the previous algorithm, we used two projections on the underlying set $\mathcal{K}$ for each iteration. If the feasible set $\mathcal{K}$ has a complicated structure, the method's computing capabilities may be decreased. This section will go over various methods for getting around this limitation. The first is Censor et al. [8] subgradient extragradient algorithm. This algorithm is defined as follows:

$$\begin{cases} u_1 \in \mathcal{K}, \\ v_i = P_{\mathcal{K}}[u_i - \varkappa \mathcal{A}(u_i)], \\ u_{i+1} = P_{\mathcal{H}_i}[u_i - \varkappa \mathcal{A}(v_i)], \end{cases} \tag{1.3}$$

where $0 < \varkappa < \frac{1}{L}$ with

$$\mathcal{H}_i = \{z \in \mathcal{H} : \langle u_i - \varkappa \mathcal{A}(u_i) - v_i, z - v_i \rangle \leq 0\}.$$

In this study, we will focus on Tseng's extragradient algorithm [37], which uses only one projection each iteration. This algorithm is defined as follows:

$$\begin{cases} u_1 \in \mathcal{K}, \\ v_i = P_{\mathcal{K}}[u_i - \varkappa \mathcal{A}(u_i)], \\ u_{i+1} = v_i + \varkappa[\mathcal{A}(u_i) - \mathcal{A}(v_i)], \end{cases} \tag{1.4}$$

while $0 < \varkappa < \frac{1}{L}$. It is worth noting that the previous two algorithms have two major flaws: a fixed stepsize rule that is dependent on the Lipschitz constant of the cost operator and generates a weakly convergent iterative sequence. The Lipschitz constant of a mapping is often unknown or difficult to approximate. A fixed stepsize criterion that influences the method's efficacy and speed of convergence. Furthermore, in the case of an infinite-dimensional Hilbert space, the investigation of a strongly convergent iterative sequence is important.

The purpose of this research is to solve variational inequalities in an infinite-dimensional Hilbert space using a quasi-monotone operator. Furthermore, the proposed algorithms take into account the extension and modification of the previous results [11, 31]. We show that Tseng's iterative sequence for solving variational inequalities with quasi-monotone operators is strongly convergent to a solution. Tseng's inertial-type extragradient approach makes use of both monotone and non-monotone variable stepsize rules. For each algorithm, we created a series of computational studies.

The following is the format of the paper: Preliminary results are provided in Section 2. All new algorithms and their convergence analysis are described in Section 3. Finally, Section 4 presents some numerical results to show the practical usefulness of the proposed algorithms.

2. Preliminaries

This section contains various important identities as well as significant lemmas. For any $F_1, F_2 \in \mathcal{H}$, we have

$$\|e_1 + e_2\|^2 = \|e_1\|^2 + 2\langle e_1, e_2 \rangle + \|e_2\|^2.$$

Furthermore, a set is defined as follows:

$$VI(\mathcal{K}, \mathcal{A})_+ = \{u^* \in \mathcal{K} : \langle \mathcal{A}(u^*), r_2 - u^* \rangle > 0, \ \forall\, r_2 \in \mathcal{K}\}$$

A metric projection $P_{\mathcal{K}}(e_1)$ of $e_1 \in \mathcal{H}$ is defined by

$$P_{\mathcal{K}}(e_1) = \arg\min\{\|e_1 - e_2\| : e_2 \in \mathcal{K}\}.$$

Lemma 2.1. *[23] Let $\{a_i\} \subset [0, +\infty)$ be a sequence satisfying the following inequality*

$$a_{i+1} \leq (1 - \alpha_i)a_i + \alpha_i\beta_i, \;\; n \geq 0,$$

where $\{\alpha_i\} \subset (0, 1)$ and $\{\beta_i\} \subset \mathbb{R}$ are two sequences such that

$$\lim_{i\to+\infty} \alpha_i = 0, \;\; \sum_{i=1}^{+\infty} \alpha_i = +\infty \;\; and \;\; \limsup_{i\to+\infty} \beta_i \leq 0.$$

Then, $\lim_{i\to+\infty} a_i = 0$.

Lemma 2.2. *[25] Let $\{a_i\} \subset \mathbb{R}$ be a sequence with $\{i_j\}$ is a subsequence of $\{i\}$ such that*

$$a_{i_j} < a_{i_{j+1}} \quad \forall \quad j \in \mathbb{N}.$$

Then, there exists a nondecreasing sequence $m_k \subset \mathbb{N}$ with

$$m_k \to +\infty \; with \; k \to +\infty,$$

such that

$$a_{m_k} \leq a_{m_{k+1}} \quad and \quad a_k \leq a_{m_{k+1}}, \; \forall\, k \in \mathbb{N}.$$

Eventually, $m_k = \max\{j \leq k : a_j \leq a_{j+1}\}$.

3. Main results

In this section, we present three new inertial methods for solving quasi-monotone variational inequalities in a real Hilbert space. We verified that the sequences generated by the proposed algorithms are strongly convergent. The following three methods are modified variants of Tseng's extragradient method [37]. The nature of their use of the stepsize rule differentiates these three algorithms between each other. The primary objective of this study is to extend and modify Tseng's extragradient method [37] and compare the numerical efficiency of the algorithms in terms of their stepsize rule.

Lemma 3.1. *Let $\{\varkappa_i\}$ sequence is created due to* (3.7) *is diminishing and converges monotonically to* $\varkappa > 0$.

Proof. The proof is similar to the proof of Lemma 3.1 in [22]. □

Lemma 3.2. *Let $\{\varkappa_i\}$ is a sequence is created due to* (3.14) *is convergent to $\varkappa$ with*

$$\min\left\{\frac{\chi}{L}, \varkappa_1\right\} \leq \varkappa_i \leq \varkappa_1 + P \quad where \quad P = \sum_{i=1}^{+\infty} \varphi_i.$$

Proof. The proof is similar to the proof of Lemma 3.1 in [22]. □

Lemma 3.3. *Suppose $\mathcal{A}:\mathcal{H}\to\mathcal{H}$ satisfies the conditions $(\mathcal{A}1)$–$(\mathcal{A}4)$ and sequence $\{u_i\}$ generated due to Algorithm 1. For any $u^*\in VI(\mathcal{K},\mathcal{A})_+$, we have*

$$\|u_{i+1}-u^*\|^2 \le \left\|w_i-u^*\right\|^2-(1-\varkappa^2L^2)\left\|w_i-v_i\right\|^2.$$

Algorithm 1

Step 0. Take $u_0,u_1\in\mathcal{K}$, $\theta\in(0,1)$ and $0<\varkappa<\frac{1}{L}$. Moreover, $\{\vartheta_i\}\subset(0,1)$ satisfies the following conditions:

$$\lim_{i\to+\infty}\vartheta_i=0 \text{ and } \sum_{i=1}^{+\infty}\vartheta_i=+\infty.$$

Step 1. Compute

$$w_i=u_i+\theta_i(u_i-u_{i-1})-\vartheta_i[u_i+\theta_i(u_i-u_{i-1})]$$

with θ_i such that

$$0\le\theta_i\le\hat{\theta}_i \quad\text{and}\quad \hat{\theta}_i=\begin{cases}\min\left\{\frac{\theta}{2},\frac{\epsilon_i}{\|u_i-u_{i-1}\|}\right\} & \text{if} \quad u_i\neq u_{i-1},\\ \frac{\theta}{2} & \text{else,}\end{cases}$$

where $\epsilon_i=\circ(\vartheta_i)$ is satisfying the condition $\lim_{i\to+\infty}\frac{\epsilon_i}{\vartheta_i}=0$.
Step 2. Compute

$$v_i=P_{\mathcal{K}}(w_i-\varkappa\mathcal{A}(w_i)).$$

If $w_i=v_i$, STOP. Otherwise, go to Step 3.
Step 3. Compute

$$u_{i+1}=v_i+\varkappa[\mathcal{A}(w_i)-\mathcal{A}(v_i)].$$

Set $i:=i+1$ and go back to Step 1.

Proof. It is given that $u^*\in VI(\mathcal{K},\mathcal{A})$, such that

$$\begin{aligned}\left\|u_{i+1}-u^*\right\|^2 &= \left\|v_i+\varkappa[\mathcal{A}(w_i)-\mathcal{A}(v_i)]-u^*\right\|^2\\ &= \left\|v_i-u^*\right\|^2+\varkappa^2\left\|\mathcal{A}(w_i)-\mathcal{A}(v_i)\right\|^2+2\varkappa\langle v_i-u^*,\mathcal{A}(w_i)-\mathcal{A}(v_i)\rangle\\ &= \left\|v_i+w_i-w_i-u^*\right\|^2+\varkappa^2\left\|\mathcal{A}(w_i)-\mathcal{A}(v_i)\right\|^2+2\varkappa\langle v_i-u^*,\mathcal{A}(w_i)-\mathcal{A}(v_i)\rangle\\ &= \left\|v_i-w_i\right\|^2+\left\|w_i-u^*\right\|^2+2\langle v_i-w_i,w_i-u^*\rangle\\ &\quad+\varkappa^2\left\|\mathcal{A}(w_i)-\mathcal{A}(v_i)\right\|^2+2\varkappa\langle v_i-u^*,\mathcal{A}(w_i)-\mathcal{A}(v_i)\rangle\\ &= \left\|w_i-u^*\right\|^2+\left\|v_i-w_i\right\|^2+2\langle v_i-w_i,v_i-u^*\rangle+2\langle v_i-w_i,w_i-v_i\rangle\\ &\quad+\varkappa^2\left\|\mathcal{A}(w_i)-\mathcal{A}(v_i)\right\|^2+2\varkappa\langle v_i-u^*,\mathcal{A}(w_i)-\mathcal{A}(v_i)\rangle. \end{aligned}\tag{3.1}$$

It provides that $v_i=P_{\mathcal{K}}[w_i-\varkappa\mathcal{A}(w_i)]$ and it implies that

$$\langle w_i-\varkappa\mathcal{A}(w_i)-v_i,v-v_i\rangle\le0,\ \forall\, v\in\mathcal{K}.\tag{3.2}$$

Thus, we have

$$\langle w_i-v_i,u^*-v_i\rangle\le\varkappa\langle\mathcal{A}(w_i),u^*-v_i\rangle.\tag{3.3}$$

Combining expressions (3.1) and (3.3), we obtain

$$\begin{aligned}\left\|u_{i+1}-u^*\right\|^2 &\leq \left\|w_i-u^*\right\|^2+\left\|v_i-w_i\right\|^2+2\varkappa\langle\mathcal{A}(w_i),u^*-v_i\rangle-2\langle w_i-v_i,w_i-v_i\rangle \\ &\quad+\varkappa^2\left\|\mathcal{A}(w_i)-\mathcal{A}(v_i)\right\|^2-2\varkappa\langle\mathcal{A}(w_i)-\mathcal{A}(v_i),u^*-v_i\rangle \\ &=\left\|w_i-u^*\right\|^2-\left\|w_i-v_i\right\|^2+\varkappa^2\left\|\mathcal{A}(w_i)-\mathcal{A}(v_i)\right\|^2-2\varkappa\langle\mathcal{A}(v_i),v_i-u^*\rangle. \end{aligned} \tag{3.4}$$

It is given that u^* is the solution of the problem (1.1) implies that

$$\langle\mathcal{A}(u^*),v-u^*\rangle>0,\ \ \forall\, v\in\mathcal{K}.$$

It implies that

$$\langle\mathcal{A}(v),v-u^*\rangle\geq 0,\ \ \forall\, v\in\mathcal{K}.$$

By substituting $v=v_i\in\mathcal{K}$, we have

$$\langle\mathcal{A}(v_i),v_i-u^*\rangle\geq 0. \tag{3.5}$$

From expressions (3.4) and (3.5), we obtain

$$\begin{aligned}\left\|u_{i+1}-u^*\right\|^2 &\leq \left\|w_i-u^*\right\|^2-\left\|w_i-v_i\right\|^2+\varkappa^2L^2\left\|w_i-v_i\right\|^2 \\ &=\left\|w_i-u^*\right\|^2-(1-\varkappa^2L^2)\left\|w_i-v_i\right\|^2. \end{aligned} \tag{3.6}$$

□

Lemma 3.4. *Consider that $\mathcal{A}:\mathcal{H}\to\mathcal{H}$ satisfies the conditions ($\mathcal{A}$1)–($\mathcal{A}$4). Let $\{u_i\}$ generated due to Algorithm 2. For any $u^*\in VI(\mathcal{K},\mathcal{A})_+$, we have*

$$\|u_{i+1}-u^*\|^2\leq\left\|w_i-u^*\right\|^2-\left(1-\chi^2\frac{\varkappa_i^2}{\varkappa_{i+1}^2}\right)\left\|w_i-v_i\right\|^2.$$

Algorithm 2

Step 0. Take $u_0, u_1 \in \mathcal{K}$, $\chi \in (0,1)$, $\theta \in (0,1)$ and $\varkappa_1 > 0$. Moreover, $\{\vartheta_i\} \subset (0,1)$ satisfies the following conditions:

$$\lim_{i \to +\infty} \vartheta_i = 0 \text{ and } \sum_{i=1}^{+\infty} \vartheta_i = +\infty.$$

Step 1. Compute

$$w_i = u_i + \theta_i(u_i - u_{i-1}) - \vartheta_i[u_i + \theta_i(u_i - u_{i-1})]$$

with θ_i such that

$$0 \leq \theta_i \leq \hat{\theta}_i \quad \text{and} \quad \hat{\theta}_i = \begin{cases} \min\left\{\frac{\theta}{2}, \frac{\epsilon_i}{\|u_i - u_{i-1}\|}\right\} & \text{if} \quad u_i \neq u_{i-1}, \\ \frac{\theta}{2} & \text{else,} \end{cases}$$

where $\epsilon_i = \circ(\vartheta_i)$ is satisfying the condition $\lim_{i \to +\infty} \frac{\epsilon_i}{\vartheta_i} = 0$.
Step 2. Compute

$$v_i = P_{\mathcal{K}}(w_i - \varkappa_i \mathcal{A}(w_i)).$$

If $w_i = v_i$, STOP. Otherwise, go to Step 3.
Step 3. Compute

$$u_{i+1} = v_i + \varkappa_i[\mathcal{A}(w_i) - \mathcal{A}(v_i)].$$

Step 4. Compute

$$\varkappa_{i+1} = \begin{cases} \min\left\{\varkappa_i, \frac{\chi\|w_i - v_i\|}{\|\mathcal{A}(w_i) - \mathcal{A}(v_i)\|}\right\} & \text{if} \quad \mathcal{A}(w_i) - \mathcal{A}(v_i) \neq 0, \\ \varkappa_i & \text{else.} \end{cases} \tag{3.7}$$

Set $i := i + 1$ and go back to Step 1.

Proof. Let $u^* \in VI(\mathcal{K}, \mathcal{A})$ and by description of u_{i+1}, we obtain

$$\begin{aligned}
\left\|u_{i+1} - u^*\right\|^2 &= \left\|v_i + \varkappa_i[\mathcal{A}(w_i) - \mathcal{A}(v_i)] - u^*\right\|^2 \\
&= \left\|v_i - u^*\right\|^2 + \varkappa_i^2\left\|\mathcal{A}(w_i) - \mathcal{A}(v_i)\right\|^2 + 2\varkappa_i\langle v_i - u^*, \mathcal{A}(w_i) - \mathcal{A}(v_i)\rangle \\
&= \left\|v_i + w_i - w_i - u^*\right\|^2 + \varkappa_i^2\left\|\mathcal{A}(w_i) - \mathcal{A}(v_i)\right\|^2 + 2\varkappa_i\langle v_i - u^*, \mathcal{A}(w_i) - \mathcal{A}(v_i)\rangle \\
&= \left\|v_i - w_i\right\|^2 + \left\|w_i - u^*\right\|^2 + 2\langle v_i - w_i, w_i - u^*\rangle \\
&\quad + \varkappa_i^2\left\|\mathcal{A}(w_i) - \mathcal{A}(v_i)\right\|^2 + 2\varkappa_i\langle v_i - u^*, \mathcal{A}(w_i) - \mathcal{A}(v_i)\rangle \\
&= \left\|w_i - u^*\right\|^2 + \left\|v_i - w_i\right\|^2 + 2\langle v_i - w_i, v_i - u^*\rangle + 2\langle v_i - w_i, w_i - v_i\rangle \\
&\quad + \varkappa_i^2\left\|\mathcal{A}(w_i) - \mathcal{A}(v_i)\right\|^2 + 2\varkappa_i\langle v_i - u^*, \mathcal{A}(w_i) - \mathcal{A}(v_i)\rangle.
\end{aligned} \tag{3.8}$$

It is given that $v_i = P_{\mathcal{K}}[w_i - \varkappa_i \mathcal{A}(w_i)]$ and indicates that

$$\langle w_i - \varkappa_i \mathcal{A}(w_i) - v_i, v - v_i\rangle \leq 0, \ \forall \, v \in \mathcal{K} \tag{3.9}$$

or, in certain cases, equivalently $u^* \in VI(\mathcal{K}, \mathcal{A})$, we are able to write

$$\langle w_i - v_i, u^* - v_i\rangle \leq \varkappa_i\langle \mathcal{A}(w_i), u^* - v_i\rangle. \tag{3.10}$$

From (3.8) and (3.10), we have

$$\begin{aligned}
&\left\|u_{i+1} - u^*\right\|^2 \\
&\leq \left\|w_i - u^*\right\|^2 + \left\|v_i - w_i\right\|^2 + 2\varkappa_i\langle \mathcal{A}(w_i), u^* - v_i\rangle - 2\langle w_i - v_i, w_i - v_i\rangle \\
&\quad + \varkappa_i^2\left\|\mathcal{A}(w_i) - \mathcal{A}(v_i)\right\|^2 - 2\varkappa_i\langle \mathcal{A}(w_i) - \mathcal{A}(v_i), u^* - v_i\rangle \\
&= \left\|w_i - u^*\right\|^2 - \left\|w_i - v_i\right\|^2 + \varkappa_i^2\left\|\mathcal{A}(w_i) - \mathcal{A}(v_i)\right\|^2 - 2\varkappa_i\langle \mathcal{A}(v_i), v_i - u^*\rangle. \qquad (3.11)
\end{aligned}$$

Thus, we obtain

$$\langle \mathcal{A}(u^*), v - u^*\rangle > 0, \ \ \forall\, v \in \mathcal{K}.$$

Because the property of $\mathcal{A}$ on $\mathcal{K}$, such as

$$\langle \mathcal{A}(v), v - u^*\rangle \geq 0, \ \ \forall\, v \in \mathcal{K}.$$

Substituting $v = v_i \in \mathcal{K}$, we have

$$\langle \mathcal{A}(v_i), v_i - u^*\rangle \geq 0. \qquad (3.12)$$

Combining expressions (3.11) and (3.12), we obtain

$$\begin{aligned}
\left\|u_{i+1} - u^*\right\|^2 &\leq \left\|w_i - u^*\right\|^2 - \left\|w_i - v_i\right\|^2 + \chi^2\frac{\varkappa_i^2}{\varkappa_{i+1}^2}\left\|w_i - v_i\right\|^2 \\
&= \left\|w_i - u^*\right\|^2 - \left(1 - \chi^2\frac{\varkappa_i^2}{\varkappa_{i+1}^2}\right)\left\|w_i - v_i\right\|^2. \qquad (3.13)
\end{aligned}$$

□

Lemma 3.5. *Consider that $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ satisfies the conditions ($\mathcal{A}$1)–($\mathcal{A}$4). Let $\{u_i\}$ generated due to Algorithm 3. For each $u^* \in VI(\mathcal{K}, \mathcal{A})$, we have*

$$\|u_{i+1} - u^*\|^2 \leq \left\|w_i - u^*\right\|^2 - \left(1 - \chi^2\frac{\varkappa_i^2}{\varkappa_{i+1}^2}\right)\left\|w_i - v_i\right\|^2.$$

Algorithm 3

Step 0. Take $u_0, u_1 \in \mathcal{K}$, $\chi \in (0,1)$, $\theta \in (0,1)$, $\varkappa_1 > 0$ and sequence $\{\varphi_i\}$ satisfying $\sum_{i=1}^{+\infty} \varphi_i < +\infty$. Moreover, $\{\vartheta_i\} \subset (0,1)$ satisfies the following conditions:

$$\lim_{i\to+\infty} \vartheta_i = 0 \text{ and } \sum_{i=1}^{+\infty} \vartheta_i = +\infty.$$

Step 1. Compute

$$w_i = u_i + \theta_i(u_i - u_{i-1}) - \vartheta_i[u_i + \theta_i(u_i - u_{i-1})]$$

with θ_i such that

$$0 \le \theta_i \le \hat{\theta}_i \quad \text{and} \quad \hat{\theta}_i = \begin{cases} \min\left\{\frac{\theta}{2}, \frac{\epsilon_i}{\|u_i - u_{i-1}\|}\right\} & \text{if} \quad u_i \neq u_{i-1}, \\ \frac{\theta}{2} & \text{else,} \end{cases}$$

where $\epsilon_i = \circ(\vartheta_i)$ is satisfying the condition $\lim_{i\to+\infty} \frac{\epsilon_i}{\vartheta_i} = 0$.
Step 2. Compute

$$v_i = P_{\mathcal{K}}(w_i - \varkappa_i \mathcal{A}(w_i)).$$

If $w_i = v_i$, STOP. Otherwise, go to Step 3.
Step 3. Compute

$$u_{i+1} = v_i + \varkappa_i[\mathcal{A}(w_i) - \mathcal{A}(v_i)].$$

Step 4. Compute

$$\varkappa_{i+1} = \begin{cases} \min\left\{\varkappa_i + \varphi_i, \frac{\chi\|w_i - v_i\|}{\|\mathcal{A}(w_i) - \mathcal{A}(v_i)\|}\right\} & \text{if} \quad \mathcal{A}(w_i) - \mathcal{A}(v_i) \neq 0, \\ \varkappa_i + \varphi_i & \text{else.} \end{cases} \tag{3.14}$$

Set $i := i + 1$ and go back to Step 1.

Proof. The proof is similar to the proof of Lemma 3.4. □

Theorem 3.1. *Let a mapping $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ satisfies the conditions* ($\mathcal{A}$1)–($\mathcal{A}$4). *Then, the sequence $\{u_i\}$ generated due to Algorithm 3 converges strongly to $u^* = P_{VI(\mathcal{K},\mathcal{A})}(0)$.*

Proof. Due to the value of $\{w_i\}$ such that

$$\begin{aligned} \left\|w_i - u^*\right\| &= \left\|u_i + \theta_i(u_i - u_{i-1}) - \vartheta_i u_i - \theta_i\vartheta_i(u_i - u_{i-1}) - u^*\right\| \\ &= \left\|(1-\vartheta_i)(u_i - u^*) + (1-\vartheta_i)\theta_i(u_i - u_{i-1}) - \vartheta_i u^*\right\| \qquad (3.15) \\ &\le (1-\vartheta_i)\left\|u_i - u^*\right\| + (1-\vartheta_i)\theta_i\left\|u_i - u_{i-1}\right\| + \vartheta_i\left\|u^*\right\| \\ &\le (1-\vartheta_i)\|u_i - u^*\| + \vartheta_i K_1, \qquad (3.16) \end{aligned}$$

where

$$(1-\vartheta_i)\frac{\theta_i}{\vartheta_i}\left\|u_i - u_{i-1}\right\| + \left\|u^*\right\| \le K_1.$$

Since $\varkappa_i \to \varkappa$ to use a certain number $\Im \in (0, 1-\chi^2)$ such that

$$\lim_{i\to+\infty}\Big(1-\frac{\chi^2\varkappa_i^2}{\varkappa_{i+1}^2}\Big) = 1-\chi^2 > \Im > 0.$$

As a result, a finite number $N_1 \in \mathbb{N}$ appears here such that

$$\Big(1-\frac{\chi^2\varkappa_i^2}{\varkappa_{i+1}^2}\Big) > \Im > 0, \;\; \forall\, i \geq N_1. \tag{3.17}$$

By the use of Lemma 3.4, we obtain

$$\|u_{i+1}-u^*\|^2 \leq \|w_i-u^*\|^2, \;\; \forall\, i \geq N_1. \tag{3.18}$$

Due to Eqs (3.16) and (3.18), we have

$$\begin{aligned}
\|u_{i+1}-u^*\| &\leq (1-\vartheta_i)\|u_i-u^*\| + \vartheta_i K_1 \\
&\leq \max\{\|u_i-u^*\|, K_1\} \\
&\;\;\vdots \\
&\leq \max\{\|u_{N_1}-u^*\|, K_1\}.
\end{aligned} \tag{3.19}$$

As a result, we may conclude that $\{u_i\}$ is a bounded sequence.

Apparently, relation (3.16) gives that

$$\begin{aligned}
\left\|w_i-u^*\right\|^2 &\leq (1-\vartheta_i)^2\|u_i-u^*\|^2 + \vartheta_i^2K_1^2 + 2K_1\vartheta_i(1-\vartheta_i)\|u_i-u^*\| \\
&\leq \|u_i-u^*\|^2 + \vartheta_i[\vartheta_iK_1^2 + 2K_1(1-\vartheta_i)\|u_i-u^*\|] \\
&\leq \|u_i-u^*\|^2 + \vartheta_iK_2,
\end{aligned} \tag{3.20}$$

for some $K_2 > 0$. Combining expressions (3.13) and (3.20) we have

$$\|u_{i+1}-u^*\|^2 \leq \|u_i-u^*\|^2 + \vartheta_iK_2 - \Big(1-\frac{\chi^2\varkappa_i^2}{\varkappa_{i+1}^2}\Big)\|w_i-v_i\|^2. \tag{3.21}$$

From expression (3.15), we can write

$$\begin{aligned}
&\left\|w_i-u^*\right\|^2 \\
&= \left\|u_i+\theta_i(u_i-u_{i-1})-\vartheta_iu_i-\theta_i\vartheta_i(u_i-u_{i-1})-u^*\right\|^2 \\
&= \left\|(1-\vartheta_i)(u_i-u^*)+(1-\vartheta_i)\theta_i(u_i-u_{i-1})-\vartheta_iu^*\right\|^2 \\
&\leq \left\|(1-\vartheta_i)(u_i-u^*)+(1-\vartheta_i)\theta_i(u_i-u_{i-1})\right\|^2 + 2\vartheta_i\langle -u^*, w_i-u^*\rangle \\
&= (1-\vartheta_i)^2\left\|u_i-u^*\right\|^2 + (1-\vartheta_i)^2\theta_i^2\left\|u_i-u_{i-1}\right\|^2 \\
&\quad + 2\theta_i(1-\vartheta_i)^2\left\|u_i-u^*\right\|\left\|u_i-u_{i-1}\right\| + 2\vartheta_i\langle -u^*, w_i-u_{i+1}\rangle + 2\vartheta_i\langle -u^*, u_{i+1}-u^*\rangle \\
&\leq (1-\vartheta_i)\left\|u_i-u^*\right\|^2 + \theta_i^2\left\|u_i-u_{i-1}\right\|^2 + 2\theta_i(1-\vartheta_i)\left\|u_i-u^*\right\|\left\|u_i-u_{i-1}\right\| \\
&\quad + 2\vartheta_i\left\|u^*\right\|\left\|w_i-u_{i+1}\right\| + 2\vartheta_i\langle -u^*, u_{i+1}-u^*\rangle
\end{aligned}$$

$$= (1-\vartheta_i)\left\|u_i - u^*\right\|^2 + \vartheta_i\Big[\theta_i\left\|u_i - u_{i-1}\right\|\frac{\theta_i}{\vartheta_i}\left\|u_i - u_{i-1}\right\|$$

$$+ 2(1-\vartheta_i)\left\|u_i - u^*\right\|\frac{\theta_i}{\vartheta_i}\left\|u_i - u_{i-1}\right\| + 2\left\|u^*\right\|\left\|w_i - u_{i+1}\right\| + 2\langle u^*, u^* - u_{i+1}\rangle\Big]. \tag{3.22}$$

Due to Eqs (3.18) and (3.22), we obtain

$$\left\|u_{i+1} - u^*\right\|^2$$

$$\leq (1-\vartheta_i)\left\|u_i - u^*\right\|^2 + \vartheta_i\Big[\theta_i\left\|u_i - u_{i-1}\right\|\frac{\theta_i}{\vartheta_i}\left\|u_i - u_{i-1}\right\|$$

$$+ 2(1-\vartheta_i)\left\|u_i - u^*\right\|\frac{\theta_i}{\vartheta_i}\left\|u_i - u_{i-1}\right\| + 2\left\|u^*\right\|\left\|w_i - u_{i+1}\right\| + 2\langle u^*, u^* - u_{i+1}\rangle\Big]. \tag{3.23}$$

Case 1. Consider there is a definite integer $N_2 \in \mathbb{N}$ ($N_2 \geq N_1$) such that

$$\|u_{i+1} - u^*\| \leq \|u_i - u^*\|, \ \ \forall\, i \geq N_2. \tag{3.24}$$

The relation implies that $\lim_{i\to+\infty}\|u_i - u^*\|$ exists and let $\lim_{i\to+\infty}\|u_i - u^*\| = l$, for $l \geq 0$. Hence, we have

$$\Big(1 - \frac{\chi^2\varkappa_i^2}{\varkappa_{i+1}^2}\Big)\|w_i - v_i\|^2 \leq \|u_i - u^*\|^2 + \vartheta_i K_2 - \|u_{i+1} - u^*\|^2. \tag{3.25}$$

Because of the existence of the limits $\|u_i - u^*\|$ and $\vartheta_i \to 0$, we may derive that

$$\|w_i - v_i\| \to 0 \quad \text{as} \quad i \to +\infty. \tag{3.26}$$

It follows that

$$\|u_{i+1} - v_i\| = \|v_i + \varkappa_i[\mathcal{A}(w_i) - \mathcal{A}(v_i)] - v_i\| \leq \varkappa_1 L\|w_i - v_i\|.$$

The preceding statement suggests that

$$\lim_{i\to+\infty}\|u_{i+1} - v_i\| = 0. \tag{3.27}$$

It continues from expression (3.26) that

$$\lim_{i\to+\infty}\|w_i - u_{i+1}\| \leq \lim_{i\to+\infty}\|w_i - v_i\| + \lim_{i\to+\infty}\|v_i - u_{i+1}\| = 0. \tag{3.28}$$

Following that, we evaluate

$$\begin{aligned}\|w_i - u_i\| &= \|u_i + \theta_i(u_i - u_{i-1}) - \vartheta_i[u_i + \theta_i(u_i - u_{i-1})] - u_i\| \\ &\leq \theta_i\|u_i - u_{i-1}\| + \vartheta_i\|u_i\| + \theta_i\vartheta_i\|u_i - u_{i-1}\| \\ &= \vartheta_i\frac{\theta_i}{\vartheta_i}\|u_i - u_{i-1}\| + \vartheta_i\|u_i\| + \vartheta_i^2\frac{\theta_i}{\vartheta_i}\|u_i - u_{i-1}\| \longrightarrow 0.\end{aligned} \tag{3.29}$$

The accompanying statement indicates that

$$\lim_{i\to+\infty}\|u_i - u_{i+1}\| \leq \lim_{i\to+\infty}\|u_i - w_i\| + \lim_{i\to+\infty}\|w_i - u_{i+1}\| = 0. \tag{3.30}$$

Since $\{w_{i_k}\}$ weakly convergent to $\hat{u}$ and thus $\lim_{k\to+\infty} \|w_{i_k} - v_{i_k}\| = 0$, then $\{v_{i_k}\}$ also convergent weakly to $\hat{u}$. Then, we must demonstrate that is $\hat{u} \in VI(\mathcal{K}, \mathcal{A})$. From the value of v_i, we get

$$v_{i_k} = P_{\mathcal{K}}[w_{i_k} - \varkappa_{i_k}\mathcal{A}(w_{i_k})]$$

that is equivalent to

$$\langle w_{i_k} - \varkappa_{i_k}\mathcal{A}(w_{i_k}) - v_{i_k}, v - v_{i_k}\rangle \leq 0, \ \forall\, v \in \mathcal{K}. \tag{3.31}$$

The above inequality implies that

$$\langle w_{i_k} - v_{i_k}, v - v_{i_k}\rangle \leq \varkappa_{i_k}\langle \mathcal{A}(w_{i_k}), v - v_{i_k}\rangle, \ \forall\, v \in \mathcal{K}. \tag{3.32}$$

Thus, we obtain

$$\frac{1}{\varkappa_{i_k}}\langle w_{i_k} - v_{i_k}, v - v_{i_k}\rangle + \langle \mathcal{A}(w_{i_k}), v_{i_k} - w_{i_k}\rangle \leq \langle \mathcal{A}(w_{i_k}), v - w_{i_k}\rangle, \ \forall\, v \in \mathcal{K}. \tag{3.33}$$

By the use of $\lim_{k\to+\infty} \|w_{i_k} - v_{i_k}\| = 0$ and $k \to +\infty$ in (3.33), we have

$$\liminf_{k\to+\infty}\langle \mathcal{A}(w_{i_k}), v - w_{i_k}\rangle \geq 0, \ \forall\, v \in \mathcal{K}. \tag{3.34}$$

Furthermore, it implies that

$$\begin{aligned}&\langle \mathcal{A}(v_{i_k}), v - v_{i_k}\rangle \\ &= \langle \mathcal{A}(v_{i_k}) - \mathcal{A}(w_{i_k}), v - w_{i_k}\rangle + \langle \mathcal{A}(w_{i_k}), v - w_{i_k}\rangle + \langle \mathcal{A}(v_{i_k}), w_{i_k} - v_{i_k}\rangle.\end{aligned} \tag{3.35}$$

Since $\lim_{k\to+\infty} \|w_{i_k} - v_{i_k}\| = 0$. Thus, we have

$$\lim_{k\to+\infty} \|\mathcal{A}(w_{i_k}) - \mathcal{A}(v_{i_k})\| = 0, \tag{3.36}$$

which together with (3.35) and (3.36), we obtain

$$\liminf_{k\to+\infty}\langle \mathcal{A}(v_{i_k}), v - v_{i_k}\rangle \geq 0, \ \forall\, v \in \mathcal{K}. \tag{3.37}$$

Furthermore, consider a positive sequence $\{\epsilon_k\}$ that would be decreasing and converging to zero. There is indeed a least positive integer represented by m_k for each $\{\epsilon_k\}$ such that

$$\langle \mathcal{A}(w_{i_j}), v - w_{i_j}\rangle + \epsilon_k > 0, \ \forall\, j \geq m_k. \tag{3.38}$$

Because $\{\epsilon_k\}$ is a diminishing sequence, it is straightforward to observe that $\{m_k\}$ is increasing.

Thus, we have

$$\zeta_{i_{m_k}} = \frac{\mathcal{A}(w_{i_{m_k}})}{\|\mathcal{A}(w_{i_{m_k}})\|^2}, \ \forall\, i_{m_k} \geq N_0. \tag{3.39}$$

Due to the above definition, we have

$$\langle \mathcal{A}(w_{i_{m_k}}), \zeta_{i_{m_k}}\rangle = 1, \ \forall\, i_{m_k} \geq N_0. \tag{3.40}$$

Moreover, from expressions (3.38) and (3.40) for all $i_{m_k} \geq N_0$, we have

$$\langle \mathcal{A}(w_{i_{m_k}}),\ v + \epsilon_k \zeta_{i_{m_k}} - w_{i_{m_k}} \rangle > 0. \tag{3.41}$$

By the definition of quasi-monotone, we have

$$\langle \mathcal{A}(v + \epsilon_k \zeta_{i_{m_k}}),\ v + \epsilon_k \zeta_{i_{m_k}} - w_{i_{m_k}} \rangle > 0. \tag{3.42}$$

Due to all integer $i_{m_k} \geq N_0$, gives that

$$\langle \mathcal{A}(v), v - w_{i_{m_k}} \rangle \geq \langle \mathcal{A}(v) - \mathcal{A}(v + \epsilon_k \zeta_{i_{m_k}}),\ v + \epsilon_k \zeta_{i_{m_k}} - w_{i_{m_k}} \rangle - \epsilon_k \langle \mathcal{A}(v), \zeta_{i_{m_k}} \rangle. \tag{3.43}$$

Furthermore, we might draw the conclusion that

$$\|\mathcal{A}(\hat{u})\| \leq \liminf_{k \to +\infty} \|\mathcal{A}(w_{i_k})\|. \tag{3.44}$$

It is given that $\{w_{i_{m_k}}\} \subset \{w_{i_k}\}$ with $\lim_{k \to +\infty} \epsilon_k = 0$. It follows from this that

$$0 \leq \lim_{k \to +\infty} \|\epsilon_k \zeta_{i_{m_k}}\| = \lim_{k \to +\infty} \frac{\epsilon_k}{\|\mathcal{A}(w_{i_{m_k}})\|} \leq \frac{0}{\|\mathcal{A}(\hat{u})\|} = 0. \tag{3.45}$$

By letting $k \to +\infty$ in expression (3.43), we get

$$\langle \mathcal{A}(v), v - \hat{u} \rangle \geq 0,\ \ \forall\, v \in \mathcal{K}. \tag{3.46}$$

Consider the case when $u \in \mathcal{K}$ is an arbitrary element and $0 < \varkappa \leq 1$. Thus, we have

$$\hat{u}_{\varkappa} = \varkappa u + (1 - \varkappa)\hat{u}. \tag{3.47}$$

Then $\hat{u}_{\varkappa} \in \mathcal{K}$ and from expression (3.46), we have

$$\varkappa \langle \mathcal{A}(\hat{u}_{\varkappa}), u - \hat{u} \rangle \geq 0. \tag{3.48}$$

Hence, we have

$$\big\langle \mathcal{A}(\hat{u}_{\varkappa}), u - \hat{u} \big\rangle \geq 0. \tag{3.49}$$

Let $\varkappa \to 0$. Then $\hat{u}_{\varkappa} \to \hat{u}$ along a line segment. By the continuity of an operator, $\mathcal{A}(\hat{u}_{\varkappa})$ converges to $\mathcal{A}(\hat{u})$ as $\varkappa \to 0$. It follows from expression (3.49) such that

$$\big\langle \mathcal{A}(\hat{u}), u - \hat{u} \big\rangle \geq 0. \tag{3.50}$$

Hence $\hat{u} \in VI(\mathcal{K}, \mathcal{A})$. It is given that $u^* = P_{VI(\mathcal{K},\mathcal{A})}(0)$, we have

$$\langle 0 - u^*, y - u^* \rangle \leq 0,\ \ \forall\, y \in VI(\mathcal{K}, \mathcal{A}). \tag{3.51}$$

As a result, we obtain

$$\limsup_{i \to +\infty} \langle u^*, u^* - u_i \rangle$$

$$= \lim_{k\to+\infty} \langle u^*, u^* - u_{i_k}\rangle = \langle u^*, u^* - \hat{u}\rangle \le 0. \tag{3.52}$$

By using the fact that $\lim_{i\to+\infty} \left\|u_{i+1} - u_i\right\| = 0$. As a result of employing the formula (3.52), we receive

$$\begin{aligned}&\limsup_{i\to+\infty}\langle u^*, u^* - u_{i+1}\rangle \\ &\le \limsup_{i\to+\infty}\langle u^*, u^* - u_i\rangle + \limsup_{i\to+\infty}\langle u^*, u_i - u_{i+1}\rangle \le 0.\end{aligned} \tag{3.53}$$

We may derive from the Eqs (3.23) and (3.53), and Lemma 2.1 that $\left\|u_i - u^*\right\| \to 0$ as $i \to +\infty$.
Case 2. suppose the fact that somehow there arises a subsequence $\{i_j\}$ of $\{i\}$ in order that

$$\|u_{i_j} - u^*\| \le \|u_{i_{j+1}} - u^*\|, \ \forall\, j \in \mathbb{N}.$$

A sequence exists $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \to +\infty$ by applying Lemma 2.2, such that

$$\|u_{m_k} - u^*\| \le \|u_{m_{k+1}} - u^*\| \quad \text{and} \quad \|u_k - u^*\| \le \|u_{m_{k+1}} - u^*\|, \ \forall\, k \in \mathbb{N}. \tag{3.54}$$

As in Case 1, the Eq (3.25) suggests that

$$\Big(1 - \frac{\chi^2 \varkappa_{m_k}^2}{\varkappa_{m_k+1}^2}\Big)\|w_{m_k} - v_{m_k}\|^2 \le \|u_{m_k} - u^*\|^2 + \vartheta_{m_k} K_2 - \|u_{m_k+1} - u^*\|^2. \tag{3.55}$$

Due to $\vartheta_{m_k} \to 0$, and following step as in Case 1

$$\lim_{k\to+\infty} \|w_{m_k} - v_{m_k}\| = \lim_{k\to+\infty} \|u_{m_k+1} - v_{m_k}\| = 0. \tag{3.56}$$

Furthermore, it follows that

$$\lim_{k\to+\infty} \|u_{m_{k+1}} - w_{m_k}\| \le \lim_{k\to+\infty} \|u_{m_{k+1}} - v_{m_k}\| + \lim_{k\to+\infty} \|v_{m_k} - w_{m_k}\| = 0. \tag{3.57}$$

Following that, we must evaluate

$$\begin{aligned}\|w_{m_k} - u_{m_k}\| &= \|u_{m_k} + \alpha_{m_k}(u_{m_k} - u_{m_k-1}) - \vartheta_{m_k}[u_{m_k} + \alpha_{m_k}(u_{m_k} - u_{m_k-1})] - u_{m_k}\| \\ &\le \alpha_{m_k}\|u_{m_k} - u_{m_k-1}\| + \vartheta_{m_k}\|u_{m_k}\| + \alpha_{m_k}\vartheta_{m_k}\|u_{m_k} - u_{m_k-1}\| \\ &= \vartheta_{m_k}\frac{\alpha_{m_k}}{\vartheta_{m_k}}\|u_{m_k} - u_{m_k-1}\| + \vartheta_{m_k}\|u_{m_k}\| + \vartheta_{m_k}^2\frac{\alpha_{m_k}}{\vartheta_{m_k}}\|u_{m_k} - u_{m_k-1}\| \longrightarrow 0.\end{aligned} \tag{3.58}$$

The preceding statement suggests that

$$\lim_{k\to+\infty} \|u_{m_k} - u_{m_k+1}\| \le \lim_{k\to+\infty} \|u_{m_k} - w_{m_k}\| + \lim_{k\to+\infty} \|w_{m_k} - u_{m_k+1}\| = 0. \tag{3.59}$$

By adopting the same argument as in Case 1, in order that

$$\limsup_{k\to+\infty}\langle u^*, u^* - u_{m_k+1}\rangle \le 0. \tag{3.60}$$

Using merging the formulae (3.23) and (3.54). Thus

$$\left\|u_{m_k+1} - u^*\right\|^2$$

$$\leq (1-\vartheta_{m_k})\big\|u_{m_k}-u^*\big\|^2+\vartheta_{m_k}\Big[\alpha_{m_k}\big\|u_{m_k}-u_{m_k-1}\big\|\frac{\alpha_{m_k}}{\vartheta_{m_k}}\big\|u_{m_k}-u_{m_k-1}\big\|$$
$$+2(1-\vartheta_{m_k})\big\|u_{m_k}-u^*\big\|\frac{\alpha_{m_k}}{\vartheta_{m_k}}\big\|u_{m_k}-u_{m_k-1}\big\|+2\big\|u^*\big\|\big\|w_{m_k}-u_{m_k+1}\big\|+2\langle u^*,u^*-u_{m_k+1}\rangle\Big]$$
$$\leq (1-\vartheta_{m_k})\big\|u_{m_{k+1}}-u^*\big\|^2+\vartheta_{m_k}\Big[\alpha_{m_k}\big\|u_{m_k}-u_{m_k-1}\big\|\frac{\alpha_{m_k}}{\vartheta_{m_k}}\big\|u_{m_k}-u_{m_k-1}\big\|$$
$$+2(1-\vartheta_{m_k})\big\|u_{m_k}-u^*\big\|\frac{\alpha_{m_k}}{\vartheta_{m_k}}\big\|u_{m_k}-u_{m_k-1}\big\|+2\big\|u^*\big\|\big\|w_{m_k}-u_{m_k+1}\big\|+2\langle u^*,u^*-u_{m_k+1}\rangle\Big]. \tag{3.61}$$

Consequently, we acquire

$$\big\|u_{m_k+1}-u^*\big\|^2$$
$$\leq \Big[\alpha_{m_k}\big\|u_{m_k}-u_{m_k-1}\big\|\frac{\alpha_{m_k}}{\vartheta_{m_k}}\big\|u_{m_k}-u_{m_k-1}\big\|$$
$$+2(1-\vartheta_{m_k})\big\|u_{m_k}-u^*\big\|\frac{\alpha_{m_k}}{\vartheta_{m_k}}\big\|u_{m_k}-u_{m_k-1}\big\|+2\big\|u^*\big\|\big\|w_{m_k}-u_{m_k+1}\big\|+2\langle u^*,u^*-u_{m_k+1}\rangle\Big]. \tag{3.62}$$

Since $\vartheta_{m_k}\to 0$ and $\big\|u_{m_k}-u^*\big\|$ is a bounded sequence. Thus, Eqs (3.60) and (3.62), we get

$$\|u_{m_k+1}-u^*\|^2\to 0 \text{ as } k\to+\infty. \tag{3.63}$$

The accompanying phrase suggests

$$\lim_{k\to+\infty}\|u_k-u^*\|^2\leq\lim_{k\to+\infty}\|u_{m_k+1}-u^*\|^2\leq 0. \tag{3.64}$$

As a result, the sequence $u_i\to u^*$ is represented as $i\to+\infty$. □

Remark 3.1. *The convergence studies for the Algorithms 1 and 2 are analogous to the proof of Theorem 3.1.*

3.1. Methods to solve semistrictly quasi-monotone variational inequalities

In this part, we present two new inertial methods for solving quasi-monotone variational inequalities in a real Hilbert space. Let $\mathcal{H}$ be a real Hilbert space. Given $u,v\in\mathcal{H}$ and closed line segment is defined as follows:

$$[u,v]=\{tu+(1-t)v:0\leq t\leq 1\}.$$

The segments $(u,v]$, $[u,v)$, and (u,v) are defined similarly. A mapping $\mathcal{A}:\mathcal{H}\to\mathcal{H}$ is said to be semistrictly quasi-monotone [14] on $\mathcal{K}$ if $\mathcal{A}$ is quasi-monotone on $\mathcal{K}$ and for all distinct of points $u,v\in\mathcal{K}$, we have that

$$\langle\mathcal{A}(v),u-v\rangle>0 \quad\Longrightarrow\quad \langle\mathcal{A}(z),u-v\rangle\geq 0, \text{ for some } z\in\Big(\frac{u+v}{2},u\Big).$$

The first method (Algorithm 4) is a modified variant of the two-step extragradient method [21] with a non-monotone stepsize rule to solve variational inequalities involving semistrictly quasi-monotone operators.

Algorithm 4

Step 0. Take $u_0, u_1 \in \mathcal{K}$, $\chi \in (0,1)$, $\theta \in (0,1)$, $\varkappa_1 > 0$ and sequence $\{\varphi_i\}$ satisfying $\sum_{i=1}^{+\infty} \varphi_i < +\infty$. Moreover, $\{\vartheta_i\} \subset (0,1)$ with $\lim_{i\to+\infty} \vartheta_i = 0$ and $\sum_{i=1}^{+\infty} \vartheta_i = +\infty$.
Step 1. Compute $w_i = u_i + \theta_i(u_i - u_{i-1}) - \vartheta_i[u_i + \theta_i(u_i - u_{i-1})]$ with θ_i such as

$$0 \le \theta_i \le \hat{\theta}_i \quad \text{and} \quad \hat{\theta}_i = \begin{cases} \min\left\{\frac{\theta}{2}, \frac{\epsilon_i}{\|u_i - u_{i-1}\|}\right\} & \text{if} \quad u_i \neq u_{i-1}, \\ \frac{\theta}{2} & \text{else,} \end{cases}$$

where $\epsilon_i = \circ(\vartheta_i)$ is satisfying $\lim_{i\to+\infty} \frac{\epsilon_i}{\vartheta_i} = 0$.
Step 2. Compute $v_i = P_{\mathcal{K}}(w_i - \varkappa_i \mathcal{A}(w_i))$.
Step 3. Compute $u_{i+1} = P_{\mathcal{K}}(w_i - \varkappa \mathcal{A}(v_i))$.
Step 4. Compute

$$\varkappa_{i+1} = \begin{cases} \min\left\{\varkappa_i + \varphi_i, \frac{\chi\|w_i - v_i\|}{\|\mathcal{A}(w_i) - \mathcal{A}(v_i)\|}\right\} & \text{if} \quad \mathcal{A}(w_i) - \mathcal{A}(v_i) \neq 0, \\ \varkappa_i + \varphi_i & \text{else.} \end{cases}$$

The second method (Algorithm 5) is a modified variant of the Tseng's extragradient method [37] with a non-monotone stepsize rule to solve variational inequalities involving semistrictly quasi-monotone operators.

Algorithm 5

Step 0. Take $u_0, u_1 \in \mathcal{K}$, $\chi \in (0,1)$, $\theta \in (0,1)$, $\varkappa_1 > 0$ and sequence $\{\varphi_i\}$ satisfying $\sum_{i=1}^{+\infty} \varphi_i < +\infty$. Moreover, $\{\vartheta_i\} \subset (0,1)$ wiht $\lim_{i\to+\infty} \vartheta_i = 0$ and $\sum_{i=1}^{+\infty} \vartheta_i = +\infty$.
Step 1. Compute $w_i = u_i + \theta_i(u_i - u_{i-1}) - \vartheta_i[u_i + \theta_i(u_i - u_{i-1})]$ with θ_i such as

$$0 \le \theta_i \le \hat{\theta}_i \quad \text{and} \quad \hat{\theta}_i = \begin{cases} \min\left\{\frac{\theta}{2}, \frac{\epsilon_i}{\|u_i - u_{i-1}\|}\right\} & \text{if} \quad u_i \neq u_{i-1}, \\ \frac{\theta}{2} & \text{else,} \end{cases}$$

where $\epsilon_i = \circ(\vartheta_i)$ is satisfying $\lim_{i\to+\infty} \frac{\epsilon_i}{\vartheta_i} = 0$.
Step 2. Compute $v_i = P_{\mathcal{K}}(w_i - \varkappa_i \mathcal{A}(w_i))$.
Step 3. Compute $u_{i+1} = v_i + \varkappa_i[\mathcal{A}(w_i) - \mathcal{A}(v_i)]$.
Step 4. Compute

$$\varkappa_{i+1} = \begin{cases} \min\left\{\varkappa_i + \varphi_i, \frac{\chi\|w_i - v_i\|}{\|\mathcal{A}(w_i) - \mathcal{A}(v_i)\|}\right\} & \text{if} \quad \mathcal{A}(w_i) - \mathcal{A}(v_i) \neq 0, \\ \varkappa_i + \varphi_i & \text{else.} \end{cases}$$

4. Numerical illustration

This section evaluates the numerical performance of the proposed algorithms to some relevant work in the literature, as well as how changes in control parameters affect the numerical validity of the proposed algorithms.

All computations are done in MATLAB R2018b and run on HP i-5 Core(TM)i3-6200 6.00 GB (5.78 GB usable) RAM laptop.

Example 4.1. *Assume that a real Hilbert space $\mathcal{H} = l_2$ square summable sequences*

$$\|u_1\|^2 + \|u_2\|^2 + \cdots + \|u_i\|^2 + \cdots < +\infty. \tag{4.1}$$

Let $\mathcal{A} : \mathcal{K} \to \mathcal{K}$ be an operator is defined by

$$\mathcal{A}(u) = (5 - \|u\|)u, \ \forall \, u \in \mathcal{H},$$

where $\mathcal{K} = \{u \in \mathcal{H} : \|u\| \leq 3\}$. It is simple to demonstrate that $\mathcal{A}$ is sequentially weakly continuous on $\mathcal{H}$ with $VI(\mathcal{K}, \mathcal{A}) = \{0\}$. Take any two elements $u, v \in \mathcal{H}$, such that

$$\begin{aligned} \left\|\mathcal{A}(u) - \mathcal{A}(v)\right\| &= \left\|(5 - \|u\|)u - (5 - \|v\|)v\right\| \\ &= \left\|5(u - v) - \|u\|(u - v) - (\|u\| - \|v\|)v\right\| \\ &\leq 5\|u - v\| + \|u\|\|u - v\| + \left|\|u\| - \|v\|\right|\|v\| \\ &\leq 5\|u - v\| + 3\|u - v\| + 3\|u - v\| \\ &\leq 11\|u - v\|. \end{aligned} \tag{4.2}$$

The mappings L is Lipschitz continuous through a parameter value of $L = 11$. Take any two elements $u, v \in \mathcal{H}$, and consider $\langle \mathcal{A}(u), v - u \rangle > 0$, with

$$(5 - \|u\|)\langle u, v - u \rangle > 0.$$

Because $\|u\| \leq 3$, as a result of

$$\langle u, v - u \rangle > 0.$$

Take this into consideration:

$$\begin{aligned} \langle \mathcal{A}(v), v - u \rangle &= (5 - \|v\|)\langle v, v - u \rangle \\ &\geq (5 - \|v\|)\langle v, v - u \rangle - (5 - \|v\|)\langle u, v - u \rangle \\ &\geq 2\|u - v\|^2 \geq 0. \end{aligned} \tag{4.3}$$

It demonstrates that $\mathcal{A}$ is quasi-monotone on the set $\mathcal{K}$. The projection formula is defined by:

$$P_{\mathcal{K}}(u) = \begin{cases} u \quad if \quad \|u\| \leq 3, \\ \frac{3u}{\|u\|}, \quad otherwise. \end{cases}$$

The numerical findings are shown in Table 1 and Figures 1–3. While defining the control conditions, the following factors are taken into account: (1) *Algorithm 1:* $\varkappa = \frac{0.5}{L}, \theta = 0.50, \epsilon_i = \frac{1}{(i+1)^2}, u_1 = (1, 1, \cdots, 1_{50000}, 0, 0, \cdots), D_i = \|w_i - v_i\|$; (2) *Algorithm 2:* $\varkappa_1 = 0.55, \chi = 0.33, \theta = 0.50, u_1 = (1, 1, \cdots, 1_{50000}, 0, 0, \cdots), \epsilon_i = \frac{1}{(i+1)^2}, D_i = \|w_i - v_i\|$; (3) *Algorithm 3:* $\varkappa_1 = 0.55, \chi = 0.33, \theta = 0.50, \varphi_i = \frac{100}{(i+1)^2}, u_1 = (1, 1, \cdots, 1_{50000}, 0, 0, \cdots), \epsilon_i = \frac{1}{(i+1)^2}, D_i = \|w_i - v_i\|$.

Table 1. The obtained from the numerical calculations for Example 4.1.

	Number of iterations			Execution time in seconds		
ϑ_i	Algorithm 1	Algorithm 2	Algorithm 3	Algorithm 1	Algorithm 2	Algorithm 3
$\frac{1}{(i+2)}$	28	18	22	1.12	1.33	1.47
$\frac{1}{2(i+2)}$	34	25	19	1.25	1.64	1.27
$\frac{1}{5(i+2)}$	45	32	34	1.85	2.01	2.11

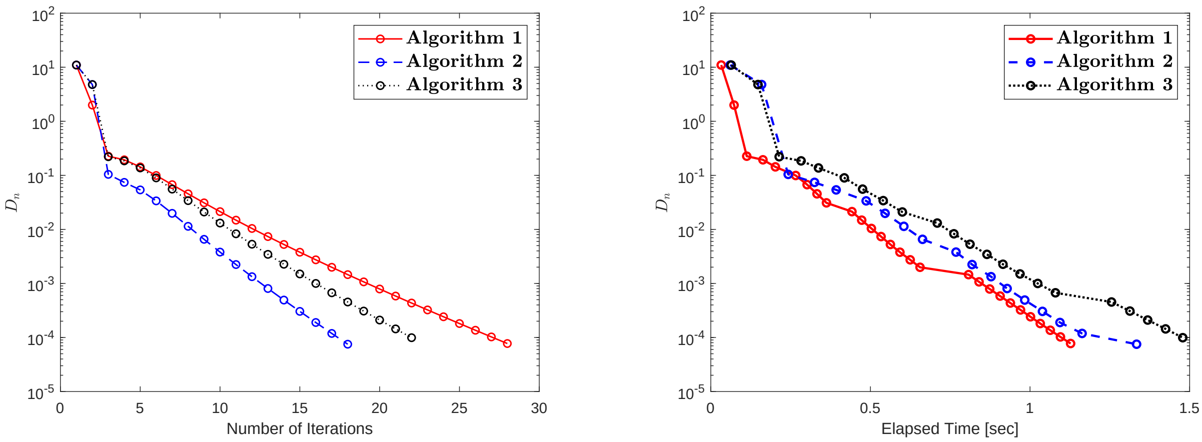

Figure 1. Numerical illustrations of all proposed algorithms when $\vartheta_i = \frac{1}{(i+2)}$.

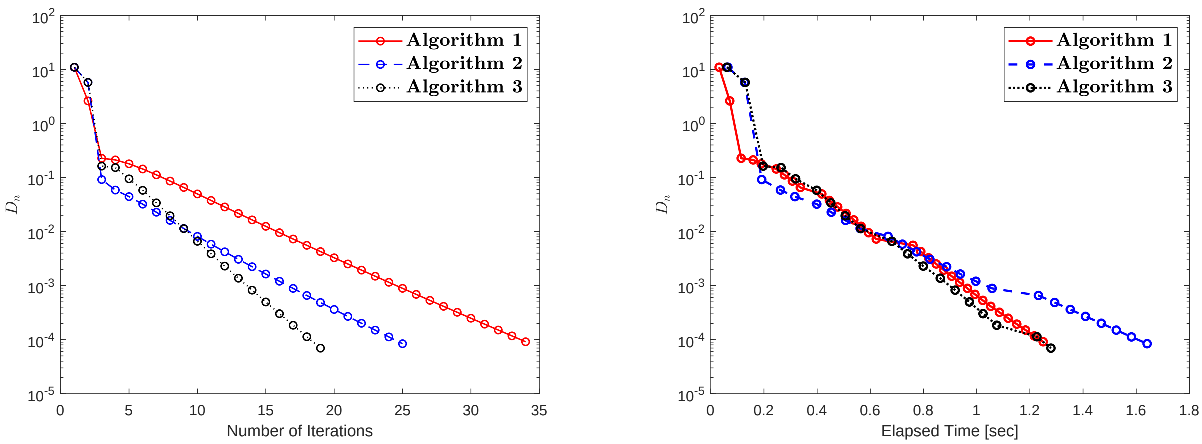

Figure 2. Numerical illustrations of all proposed algorithms when $\vartheta_i = \frac{1}{2(i+2)}$.

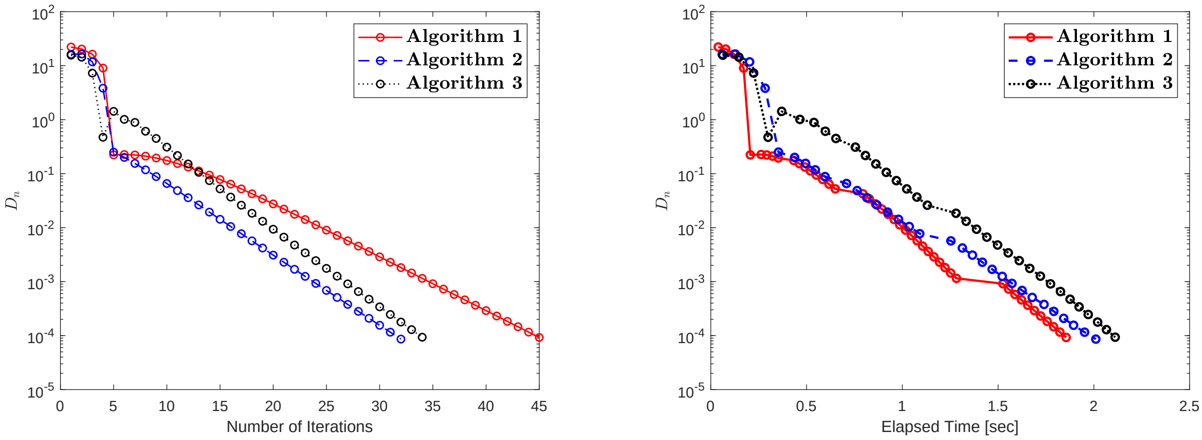

Figure 3. Numerical illustrations of all proposed algorithms when $\vartheta_i = \frac{1}{5(i+2)}$.

The numerical findings are shown in Table 2 and Figures 4–6. The control conditions are considered in the following way: (1) Algorithm 1: $\varkappa = \frac{0.7}{L}, \theta = 0.66, \epsilon_i = \frac{1}{(i+1)^2}, \vartheta_i = \frac{1}{(i+2)}, D_i = \|w_i - v_i\|$; (2) Algorithm 2: $\varkappa_1 = 0.45, \chi = 0.44, \theta = 0.66, \vartheta_i = \frac{1}{(i+2)}, \epsilon_i = \frac{1}{(i+1)^2}, D_i = \|w_i - v_i\|$; (3) Algorithm 3: $\varkappa_1 = 0.45, \chi = 0.44, \theta = 0.66, \varphi_i = \frac{100}{(i+1)^2}, \vartheta_i = \frac{1}{(i+2)}, \epsilon_i = \frac{1}{(i+1)^2}, D_i = \|w_i - v_i\|$.

Table 2. The obtained from the numerical calculations for Example 4.1.

	Number of iterations			Execution time in seconds		
u_1	Algorithm 1	Algorithm 2	Algorithm 3	Algorithm 1	Algorithm 2	Algorithm 3
$(2, 2, \cdots, 2_{50000}, 0, 0, \cdots)$	34	28	26	1.30	1.78	1.70
$(1, 2, \cdots, 50000, 0, 0, \cdots)$	43	36	30	1.79	2.54	1.90
$(10, 10, \cdots, 10_{50000}, 0, 0, \cdots)$	56	43	37	2.39	2.86	2.38

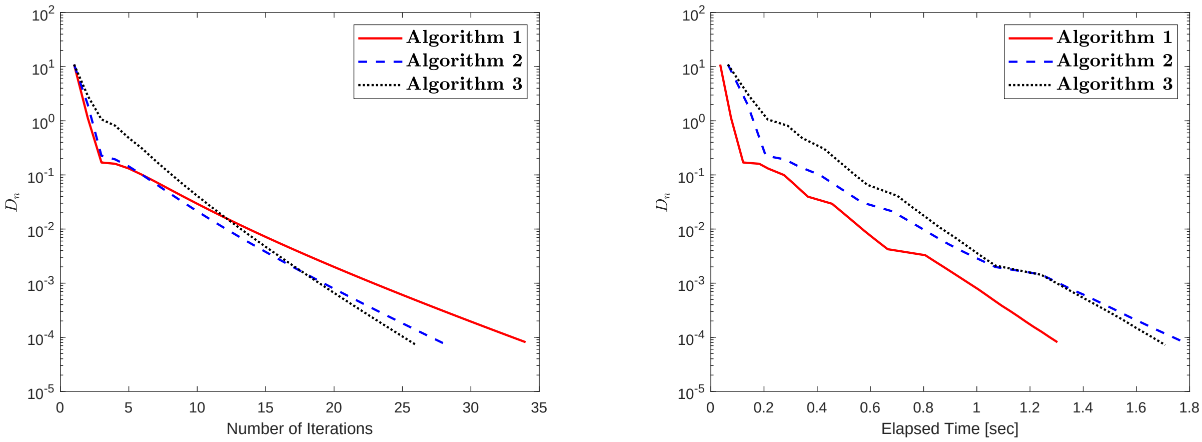

Figure 4. Numerical illustrations of all proposed algorithms when $u_1 = (2, 2, \cdots, 2_{50000}, 0, 0, \cdots)$.

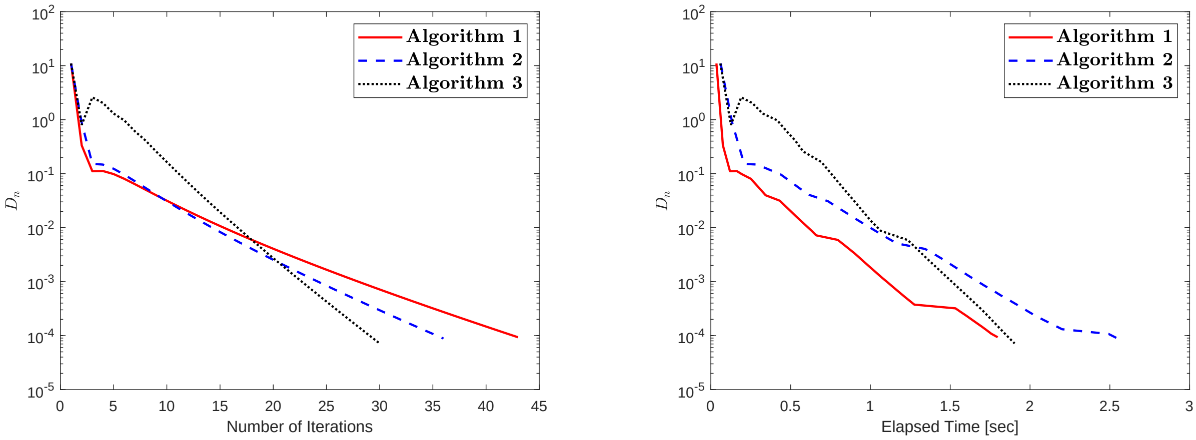

Figure 5. Numerical illustrations of all proposed algorithms when $u_1 = (1, 2, \cdots, 50000, 0, 0, \cdots)$.

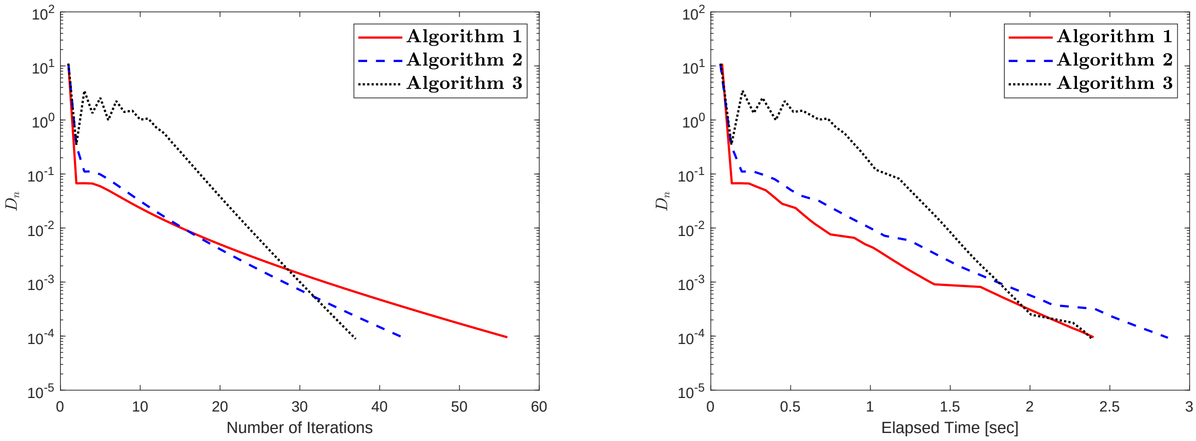

Figure 6. Numerical illustrations of all proposed algorithms when $u_1 = (10, 10, \cdots, 10_{50000}, 0, 0, \cdots)$.

5. Conclusions

This research focused on various types of strong convergence results for variational inequalities problems involving quasi-monotone and semistrictly quasi-monotone operators in the setting of real Hilbert spaces. We modify the two forms of extragradient methods by employing an inertial scheme and two types of variable stepsize rules. The numerical results are presented to evaluate the effectiveness of the proposed algorithms in terms of their stepsize rule. These numerical experiments have shown that the variable step size has an impact on the effectiveness of the iterative sequence in this context.

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Conflict of interest

All authors declare that they have no conflicts of interest.

References

1. A. Antipin, On a method for convex programs using a symmetrical modification of the lagrange function, *Ekonomika i Matematicheskie Metody*, **12** (1976), 1164–1173.
2. T. Bantaojai, N. Pakkaranang, H. ur Rehman, P. Kumam, W. Kumam, Convergence analysis of self-adaptive inertial extra-gradient method for solving a family of pseudomonotone equilibrium problems with application, *Symmetry*, **12** (2020), 1332. http://dx.doi.org/10.3390/sym12081332
3. L. Ceng, A. Petruşel, X. Qin, J. Yao, A modified inertial subgradient extragradient method for solving pseudomonotone variational inequalities and common fixed point problems, *Fixed Point Theory*, **21** (2020), 93–108. http://dx.doi.org/10.24193/fpt-ro.2020.1.07
4. L. Ceng, A. Petruşel, C. Wen, J. Yao, Inertial-like subgradient extragradient methods for variational inequalities and fixed points of asymptotically nonexpansive and strictly pseudocontractive mappings, *Mathematics*, **7** (2019), 860. http://dx.doi.org/10.3390/math7090860
5. L. Ceng, A. Petruşel, J. Yao, On mann viscosity subgradient extragradient algorithms for fixed point problems of finitely many strict pseudocontractions and variational inequalities, *Mathematics*, **7** (2019), 925. http://dx.doi.org/10.3390/math7100925
6. L. Ceng, M. Shang, Composite extragradient implicit rule for solving a hierarch variational inequality with constraints of variational inclusion and fixed point problems, *J. Inequal. Appl.*, **2020** (2020), 33. http://dx.doi.org/10.1186/s13660-020-2306-1
7. L. Ceng, C. Wen, Y. Liou, J. Yao, A general class of differential hemivariational inequalities systems in reflexive banach spaces, *Mathematics*, **9** (2021), 3173. http://dx.doi.org/10.3390/math9243173
8. Y. Censor, A. Gibali, S. Reich, The subgradient extragradient method for solving variational inequalities in hilbert space, *J. Optim. Theory Appl.*, **148** (2011), 318–335. http://dx.doi.org/10.1007/s10957-010-9757-3

9. Y. Censor, A. Gibali, S. Reich, Strong convergence of subgradient extragradient methods for the variational inequality problem in hilbert space, *Optim. Method. Softw.*, **26** (2011), 827–845. http://dx.doi.org/10.1080/10556788.2010.551536

10. Y. Censor, A. Gibali, S. Reich, Extensions of korpelevich extragradient method for the variational inequality problem in euclidean space, *Optimization*, **61** (2012), 1119–1132. http://dx.doi.org/10.1080/02331934.2010.539689

11. S. Chang, Salahuddin, L. Wang, M. Liu, On the weak convergence for solving semistrictly quasi-monotone variational inequality problems, *J. Inequal. Appl.*, **2019** (2019), 74. http://dx.doi.org/10.1186/s13660-019-2032-8

12. C. Elliott, Variational and quasivariational inequalities applications to free-boundary problems, *SIAM Review*, **29** (1987), 314–315. http://dx.doi.org/10.1137/1029059

13. G. Grillo, G. Stampacchia, Formes bilinéaires coercitives sur les ensembles convexes, *Académie des Sciences de Paris*, **258** (1964), 4413–4416.

14. N. Hadjisavvas, S. Schaible, On strong pseudomonotonicity and (semi)strict quasimonotonicity, *J. Optim. Theory Appl.*, **79** (1993), 139–155. http://dx.doi.org/10.1007/BF00941891

15. L. He, Y. Cui, L. Ceng, T. Zhao, D. Wang, H. Hu, Strong convergence for monotone bilevel equilibria with constraints of variational inequalities and fixed points using subgradient extragradient implicit rule, *J. Inequal. Appl.*, **2021** (2021), 146. http://dx.doi.org/10.1186/s13660-021-02683-y

16. A. Iusem, B. Svaiter, A variant of korpelevich's method for variational inequalities with a new search strategy, *Optimization*, **42** (1997), 309–321. http://dx.doi.org/10.1080/02331939708844365

17. G. Kassay, J. Kolumbán, Z. Páles, On nash stationary points, *Publ. Math. Debrecen*, **54** (1999), 267–279. http://dx.doi.org/10.5486/pmd.1999.1902

18. G. Kassay, J. Kolumbán, Z. Páles, Factorization of minty and stampacchia variational inequality systems, *Eur. J. Oper. Res.*, **143** (2002), 377–389. http://dx.doi.org/10.1016/S0377-2217(02)00290-4

19. D. Kinderlehrer, G. Stampacchia, *An introduction to variational inequalities and their applications*, New York: Society for Industrial and Applied Mathematics, 2000. http://dx.doi.org/10.1137/1.9780898719451

20. I. Konnov, *Equilibrium models and variational inequalities*, New York: Elsevier, 2007.

21. G. Korpelevich, The extragradient method for finding saddle points and other problems, *Matecon*, **12** (1976), 747–756.

22. H. Liu, J. Yang, Weak convergence of iterative methods for solving quasimonotone variational inequalities, *Comput. Optim. Appl.*, **77** (2020), 491–508. http://dx.doi.org/10.1007/s10589-020-00217-8

23. L. Liu, Ishikawa and mann iterative process with errors for nonlinear strongly accretive mappings in banach spaces, *J. Math. Anal. Appl.*, **194** (1995), 114–125. http://dx.doi.org/10.1006/jmaa.1995.1289

24. L. Liu, S. Cho, J. Yao, Convergence analysis of an inertial tseng's extragradient algorithm for solving pseudomonotone variational inequalities and applications, *J. Nonlinear Var. Anal.*, **5** (2021), 627–644. http://dx.doi.org/10.23952/jnva.5.2021.4.09

25. P. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, *Set-Valued Anal.*, **16** (2008), 899–912. http://dx.doi.org/10.1007/s11228-008-0102-z

26. A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.*, **241** (2000), 46–55. http://dx.doi.org/10.1006/jmaa.1999.6615

27. A. Nagurney, *Network economics: a variational inequality approach*, New York: Springer, 1999. http://dx.doi.org/10.1007/978-1-4757-3005-0

28. M. Noor, Some iterative methods for nonconvex variational inequalities, *Comput. Math. Model.*, **21** (2010), 97–108. http://dx.doi.org/10.1007/s10598-010-9057-7

29. P. Peeyada, W. Cholamjiak, D. Yambangwai, A hybrid inertial parallel subgradient extragradient-line algorithm for variational inequalities with an application to image recovery, *J. Nonlinear Funct. Anal.*, **2022** (2022), 9. http://dx.doi.org/ 10.23952/jnfa.2022.9

30. S. Regmi, *Optimized iterative methods with applications in diverse disciplines*, New York: Nova Science Publishers, 2020.

31. Salahuddin, The extragradient method for quasi-monotone variational inequalities, *Optimization*, **71** (2022), 2519–2528. http://dx.doi.org/10.1080/02331934.2020.1860979

32. B. Tan, S. Cho, J. Yao, Accelerated inertial subgradient extragradient algorithms with non-monotonic step sizes for equilibrium problems and fixed point problems, *J. Nonlinear Var. Anal.*, **6** (2022), 89–122. http://dx.doi.org/10.23952/jnva.6.2022.1.06

33. B. Tan, S. Li, Revisiting projection and contraction algorithms for solving variational inequalities and applications, *Applied Set-Valued Analysis and Optimization*, **4** (2022), 167–183. http://dx.doi.org/10.23952/asvao.4.2022.2.03

34. B. Tan, X. Qin, S. Cho, Revisiting subgradient extragradient methods for solving variational inequalities, *Numer. Algor.*, **90** (2022), 1593–1615. http://dx.doi.org/10.1007/s11075-021-01243-1

35. B. Tan, X. Qin, J. Yao, Strong convergence of inertial projection and contraction methods for pseudomonotone variational inequalities with applications to optimal control problems, *J. Glob. Optim.*, **82** (2022), 523–557. http://dx.doi.org/10.1007/s10898-021-01095-y

36. B. Tan, X. Qin, J. Yao, Strong convergence of self-adaptive inertial algorithms for solving split variational inclusion problems with applications, *J. Sci. Comput.*, **87** (2021), 20. http://dx.doi.org/10.1007/s10915-021-01428-9

37. P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, *SIAM J. Control Optim.*, **38** (2000), 431–446. http://dx.doi.org/10.1137/S0363012998338806

38. H. ur Rehman, M. Özdemir, I. Karahan, N. Wairojjana, The Tseng's extragradient method for semistrictly quasimonotone variational inequalities, *J. Appl. Numer. Optim.*, **4** (2022), 203–214.

39. H. ur Rehman, A. Gibali, P. Kumam, K. Sitthithakerngkiet, Two new extragradient methods for solving equilibrium problems, *RACSAM*, **115** (2021), 75. http://dx.doi.org/10.1007/s13398-021-01017-3

40. H. ur Rehman, P. Kumam, Y. Cho, P. Yordsorn, Weak convergence of explicit extragradient algorithms for solving equilibirum problems, *J. Inequal. Appl.*, **2019** (2019), 282. http://dx.doi.org/10.1186/s13660-019-2233-1

41. H. ur Rehman, P. Kumam, A. Gibali, W. Kumam, Convergence analysis of a general inertial projection-type method for solving pseudomonotone equilibrium problems with applications, *J. Inequal. Appl.*, **2021** (2021), 63. http://dx.doi.org/10.1186/s13660-021-02591-1

42. H. ur Rehman, P. Kumam, Y. Cho, Y. Suleiman, W. Kumam, Modified popov's explicit iterative algorithms for solving pseudomonotone equilibrium problems, *Optim. Method. Soft.*, **36** (2021), 82–113. http://dx.doi.org/10.1080/10556788.2020.1734805

43. H. ur Rehman, W. Kumam, P. Kumam, M. Shutaywi, A new weak convergence non-monotonic self-adaptive iterative scheme for solving equilibrium problems, *AIMS Mathematics*, **6** (2021), 5612–5638. http://dx.doi.org/10.3934/math.2021332

44. H. ur Rehman, N. Pakkaranang, A. Hussain, N. Wairojjana, A modified extra-gradient method for a family of strongly pseudomonotone equilibrium problems in real Hilbert spaces, *J. Math. Comput. Sci.*, **22** (2021), 38–48. http://dx.doi.org/10.22436/jmcs.022.01.04

45. N. Wairojjana, H. ur Rehman, I. Argyros, N. Pakkaranang, An accelerated extragradient method for solving pseudomonotone equilibrium problems with applications, *Axioms*, **9** (2020), 99. http://dx.doi.org/10.3390/axioms9030099

46. Y. Wang, T. Xu, J. Yao, B. Jiang, Self-adaptive method and inertial modification for solving the split feasibility problem and fixed-point problem of quasi-nonexpansive mapping, *Mathematics*, **10** (2022), 1612. http://dx.doi.org/10.3390/math10091612

47. J. Yang, H. Liu, A modified projected gradient method for monotone variational inequalities, *J. Optim. Theory Appl.*, **179** (2018), 197–211. http://dx.doi.org/10.1007/s10957-018-1351-0

48. J. Yang, H. Liu, Z. Liu, Modified subgradient extragradient algorithms for solving monotone variational inequalities, *Optimization*, **67** (2018), 2247–2258. http://dx.doi.org/10.1080/02331934.2018.1523404

49. L. Zhang, C. Fang, S. Chen, An inertial subgradient-type method for solving single-valued variational inequalities and fixed point problems, *Numer. Algor.*, **79** (2018), 941–956. http://dx.doi.org/10.1007/s11075-017-0468-9

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