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Research article

Entropy of credibility distribution for intuitionistic fuzzy variable

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Abstract: This paper handles the new information entropy measure and divergence measure associated with intuitionistic fuzzy variables (IFVs). Based on credibility distribution and credibility measure of intuitionistic fuzzy variable, the credibility entropy formulas of discrete and continuous IFVs are proposed and some of their properties are investigated. The cross-entropy of intuitionistic fuzzy variable and its relationship with credibility entropy are then discussed. Finally, some numerical examples are given to illustrate the practicability of the presented credibility entropy and cross-entropy of intuitionistic fuzzy variable. Also, we make some comparative analysis on the credibility cross-entropy measure and some existing distance measures of IFVs in the pattern recognition problem.

Keywords: intuitionistic fuzzy variable; credibility distribution; credibility degree; credibility entropy; cross-entropy
Mathematics Subject Classification: 03E72

1. Introduction

As a generalization of fuzzy set, intuitionistic fuzzy set (IFS) is more flexible in dealing with uncertainty since it can present both membership degree and non-membership degree. In real-life application fields, the preference value of an alternative on criteria and the weight of criteria generally take intuitionistic fuzzy variable rather than fuzzy variable. Intuitionistic fuzzy variables (IFVs) or intuitionistic fuzzy numbers (IFNs) have been widely applied to multi-criteria decision-making [12,13,29,30], portfolio selection [31] and fault diagnosis problems [32].

Intuitionistic fuzzy entropy, a term used to quantify the degree of uncertainty of IFS, plays an important role in the intuitionistic fuzzy application area. Szmidt [23] firstly defined the entropy of intuitionistic fuzzy set by extending De Luca's axiom of fuzzy entropy [4]. Then, Vlachos [25], Chen [3], Zhang [33], Bhardwaj [2], Garg [6], Joshi [11], Shangguan [24] proposed some entropy formulas for IFS and applied them to pattern recognition, multi-criteria decision making, supplier selection and image processing. After that, Hung [10], Xia [28] presented some intuitionistic fuzzy cross-entropy formulas (or divergence measures) and applied them to decision-making [14,26], edge detection [1], performance evaluation [19] under intuitionistic fuzzy environment. However, the above-mentioned intuitionistic fuzzy entropy formulas are only applicable to discrete IFSs and not to continuous IFSs. In this paper, we introduce a new type of entropy and cross-entropy of continuous intuitionistic fuzzy variable based on credibility theory. In 2008, Li [15] defined a credibility distribution for fuzzy variable and introduced a credibility measure for quantifying the occurrence chance of fuzzy event. Also, Liu [16] studied the credibility expectation of fuzzy variable. Li [17] studied the credibility mean, variance and skewness for fuzzy variable. Then, Mehlawat [20], Qin [21], Zhang [34] utilized credibilistic mean, entropy and cross-entropy of fuzzy variables to handle portfolio decision problems. More recently, many scholars have studied possibility, necessity and credibility measures of intuitionistic fuzzy number and applied them to multi-product manufacturing system [7], inventory optimization problem [8,9], group decision-making [27]. However, there are few literatures on the credibility entropy of intuitionistic fuzzy variable. Although Rahimi [22] proposed an entropy formula for IFV based on credibility measure, it was very complex in computing because it employed both possibility and determinacy of IFV. Moreover, IFV defined in [22] does not fulfill the normalization condition which is generally satisfied by fuzzy variable.

Motivated by the above related studies, we will develop the new credibility measure and the credibility entropy theory for IFV based on credibility distribution of IFV with normalization condition. First, we propose the credibility entropy for IFV based on both the Shannon's entropy function and the credibility degrees of all the intuitionistic fuzzy values. Then, by employing credibility measure and credibility distribution of IFVs, we also present some credibilistic cross-entropy formulas for IFVs, which can represent the divergence measures between two IFVs and are widely applied to pattern recognition, medical diagnosis and clustering analysis. From information theoretical view, the proposed credibility entropy of IFV represents the extent of uncertainty associated with IFV resulted from information deficiency. However, the traditional intuitionistic fuzzy entropy formulas only represent the uncertainty degree in determining whether an element belongs to an IFS or not. Hence, the proposed credibility entropy of IFV has more meaning than the existing intuitionistic fuzzy entropy of IFV.

This paper aims to introduce a new credibility entropy based on credibility distribution of IFV to measure the information deficiency associated with IFV. The presented credibility entropy and cross-entropy of IFV are defined directly by the credibility degree of focal elements and only require that the intuitionistic fuzzy distribution has normalization. The remainder of this paper is organized as follows. In Section 2, we introduce some basic concepts and notations of intuitionistic fuzzy set and IFV. In Section 3, the credibility entropy of discrete credibility distribution of IFV is proposed and some of its properties are discussed in details. In Section 4, we present the credibility entropy and the credibility cross-entropy for credibility distribution of continuous IFV. Then, the close relationship between the proposed credibility entropy and credibility cross-entropy of IFVs is also investigated. In Section 5, we deduce the credibility entropy formulas for trapezoidal and triangular IFVs, which are widely used in application fields. In Section 6, we give some numerical examples to demonstrate the applicability of the proposed credibility entropy and cross-entropy of IFVs. Especially in example 3 we make some comparison analysis of the proposed credibility cross-entropy and the existing distance measures of IFVs in pattern recognition application area. The recognition results illustrate that the proposed credibility cross-entropy is more powerful for discriminating two intuitionistic fuzzy patterns than the existing distance measures of IFVs.

2. Preliminary

In this section, we give some notations, concepts of intuitionistic fuzzy set and intuitionistic fuzzy variable.

Definition 1. [23] An intuitionistic fuzzy set (IFS) $\tilde{A}$ over a universe X is defined by a function $\tilde{A}: X \to \langle I \rangle$, where $\tilde{A}(x) = \langle \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \rangle \in \langle I \rangle = \{\langle u, v \rangle \mid 0 \le u + v \le 1\}$, and $\mu_{\tilde{A}}(x)$, $\nu_{\tilde{A}}(x)$ represent the membership degree and nonmembership degree of x in IFS $\tilde{A}$, respectively.

We let $IF(X)$ denote all the intuitionistic fuzzy sets on universe X.

Definition 2. An intuitionistic fuzzy variable $\tilde{\xi}$ is defined by a function from sample space to intuitionistic fuzzy set space on universe $X \subset R$, that is, $\tilde{\xi}: \Omega \to IF(X)$ with an intuitionistic fuzzy restriction $\langle \mu_{\tilde{\xi}(\omega)}(x), \nu_{\tilde{\xi}(\omega)}(x) \rangle \in \langle I \rangle$ on each value x.

In this paper we always assume $\tilde{\xi}$ is a normal IFV with membership function $\mu_{\tilde{\xi}}(x)$ and non-membership $\nu_{\tilde{\xi}}(x)$, satisfying $\sup_{x \in X} \tilde{\xi}(x) = \langle \sup_{x \in X} \mu_{\tilde{\xi}}(x), \inf_{x \in X} \nu_{\tilde{\xi}}(x) \rangle = \langle 1, 0 \rangle$.

Definition 3. For any crisp set $A \subset R$, the possibility and necessity measures for the intuitionistic fuzzy event (IFE) $\{\tilde{\xi} \in A\}$ are respectively defined by

$$Pos\{\tilde{\xi} \in A\} = \sup_{x \in A} \tfrac{\mu_{\tilde{\xi}}(x) + 1 - \nu_{\tilde{\xi}}(x)}{2}. \tag{1}$$

$$Nec\{\tilde{\xi} \in A\} = 1 - \sup_{x \notin A} \tfrac{\mu_{\tilde{\xi}}(x) + 1 - \nu_{\tilde{\xi}}(x)}{2}. \tag{2}$$

Since the possibility measure only provides non-probabilistic information on the occurrence of the intuitionistic fuzzy event $\{\tilde{\xi} \in A\}$ while the necessity measure quantifies the impossibility of its opposite event, we give the following credibility measure to quantify the chance of occurrence of intuitionistic fuzzy event (IFE) $\{\tilde{\xi} \in A\}$.

Definition 4. For any crisp set $A \subset R$, the credibility measure of IFE $\{\tilde{\xi} \in A\}$ is defined as

$$Cr\{\tilde{\xi} \in A\} = \tfrac{1}{2}[\sup_{x \in A} \tfrac{\mu_{\tilde{\xi}}(x) + 1 - \nu_{\tilde{\xi}}(x)}{2} + 1 - \sup_{x \notin A} \tfrac{\mu_{\tilde{\xi}}(x) + 1 - \nu_{\tilde{\xi}}(x)}{2}], \tag{3}$$

which indicates all the non-probabilistic information about intuitionistic fuzzy event $\{\tilde{\xi} \in A\}$.

Notably, $Cr\{\tilde{\xi} \in A\} \in [0,1]$, and it is self-dual, i.e., $Cr\{\tilde{\xi} \in A\} + Cr\{\tilde{\xi} \in A^c\} = 1$, which make credibility measure useful and convenient in real-life applications.

Let $\mathrm{Supp}(\tilde{\xi})=\{x\in X/\mu_{\tilde{\xi}}(x)\geq 0,\nu_{\tilde{\xi}}(x)<1\}$ denote the support of $\tilde{\xi}$, and its elements are called focal elements. For each $x_i\in Supp\{\tilde{\xi}\}$, intuitionistic fuzzy event $\{\tilde{\xi}=x_i\}$ is called a basic IFE and $pos\{\tilde{\xi}=x_i\}=\frac{1}{2}[\mu_{\tilde{\xi}}(x_i)+1-\nu_{\tilde{\xi}}(x_i)]$ is called the possibility degree of $\tilde{\xi}$ at x_i, where $\mu_{\tilde{\xi}}(x_i)=\mu^i_{\tilde{\xi}}$, $\nu_{\tilde{\xi}}(x_i)=\nu^i_{\tilde{\xi}}$ denote the true and false possibility distribution of $\tilde{\xi}$ on x_i, respectively. $\langle\mu_{\tilde{\xi}}(x_i),\nu_{\tilde{\xi}}(x_i)\rangle$ denotes the possibility distribution of $\tilde{\xi}$ on x_i. With formulas (1)–(3) we deduce the credibility distribution function of $\tilde{\xi}$ on x_i as

$$Cr\{\tilde{\xi}=x_i\}=\frac{1}{2}\left[\frac{\mu_{\tilde{\xi}}(x_i)+1-\nu_{\tilde{\xi}}(x_i)}{2}+1-\sup_{j\neq i}\frac{\mu_{\tilde{\xi}}(x_j)+1-\nu_{\tilde{\xi}}(x_j)}{2}\right] \tag{4}$$

3. Entropy of credibility distribution for discrete intuitionistic fuzzy variable

In this section, we give the definition of credibility entropy of discrete intuitionistic fuzzy variable. Six basic properties and one theorem about the entropy of credibility distribution for discrete intuitionistic fuzzy variable are then verified.

Definition 5. Let $\tilde{\xi}$ be an intuitionistic fuzzy variable (IFV) taking values in discrete universe $X=\{x_1,x_2,\cdots\}$, and $Cr\{\tilde{\xi}=x_i\}$ is the credibility distribution of $\tilde{\xi}$ on x_i, then the entropy of its credibility distribution is defined by $H(\tilde{\xi})=\sum_{i=1}^{\infty}S(Cr(\tilde{\xi}=x_i))$.

Especially, when $X=\{x_1,x_2,\cdots x_n\}$, the credibility entropy of IFV $\tilde{\xi}$ reduces to

$$H(\tilde{\xi})=\sum_{i=1}^{n}S(Cr(\tilde{\xi}=x_i))=\sum_{i=1}^{n}S\left(\frac{1}{2}\left[\frac{\mu_{\tilde{\xi}}(x_i)+1-\nu_{\tilde{\xi}}(x_i)}{2}+1-\sup_{x\neq x_i}\frac{\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)}{2}\right]\right), \tag{5}$$

where $S(t)=-t\ln t-(1-t)\ln(1-t),\forall t\in[0,1]$, is the well-known Shannon's entropy function.

Theorem 1. Let $\tilde{\xi}$ be an IFV taking values in $\{x_1,x_2,\cdots,x_n\}$ with true and false possibility distribution $\{\langle\mu^1_{\tilde{\xi}},\nu^1_{\tilde{\xi}}\rangle,\langle\mu^2_{\tilde{\xi}},\nu^2_{\tilde{\xi}}\rangle,\cdots\langle\mu^n_{\tilde{\xi}},\nu^n_{\tilde{\xi}}\rangle\}$. If there exists an index k such that $Cr(\tilde{\xi}=x_k)=1$, then $Cr(\tilde{\xi}=x_i)=0,\forall i\neq k$.

Proof. If $Cr(\tilde{\xi}=x_k)$ $=\frac{1}{2}\left[\frac{\mu_{\tilde{\xi}}(x_k)+1-\nu_{\tilde{\xi}}(x_k)}{2}+1-\sup_{i\neq k}\frac{\mu_{\tilde{\xi}}(x_i)+1-\nu_{\tilde{\xi}}(x_i)}{2}\right]$ $=1$, then we deduce that $\frac{\mu_{\tilde{\xi}}(x_k)+1-\nu_{\tilde{\xi}}(x_k)}{2}=1$, and $\sup_{i\neq k}\frac{\mu_{\tilde{\xi}}(x_i)+1-\nu_{\tilde{\xi}}(x_i)}{2}=0$.

Hence, we get $\mu_{\tilde{\xi}}(x_k)=1$, $\nu_{\tilde{\xi}}(x_k)=0$, and $\mu_{\tilde{\xi}}(x_i)=0,\nu_{\tilde{\xi}}(x_i)=1,\forall i\neq k$.

It yields to $Cr(\tilde{\xi}=x_i)=\frac{1}{2}\left[\frac{\mu_{\tilde{\xi}}(x_i)+1-\nu_{\tilde{\xi}}(x_i)}{2}+1-\sup_{j\neq i}\frac{\mu_{\tilde{\xi}}(x_j)+1-\nu_{\tilde{\xi}}(x_j)}{2}\right]=0$, $\forall i\neq k$.

From Theorem 1 one can see that there are not two basic IFEs $\{\tilde{\xi}=x_k\}$ and $\{\tilde{\xi}=x_i\}$ such that $Cr\{\tilde{\xi}=x_k\}=1$ and $Cr\{\tilde{\xi}=x_i\}>0$ simultaneously.

Property 1. Let $\tilde{\xi},\tilde{\eta}$ be two IFVs taking values in $\mathrm{X}=\{x_1,x_2,\cdots,x_n\}$ with possibility distributions $\{\langle\mu_{\tilde{\xi}}^1,\nu_{\tilde{\xi}}^1\rangle,\langle\mu_{\tilde{\xi}}^2,\nu_{\tilde{\xi}}^2\rangle,\cdots\langle\mu_{\tilde{\xi}}^n,\nu_{\tilde{\xi}}^n\rangle\}$, and $\{\langle\mu_{\tilde{\eta}}^1,\nu_{\tilde{\eta}}^1\rangle,\langle\mu_{\tilde{\eta}}^2,\nu_{\tilde{\eta}}^2\rangle,\cdots\langle\mu_{\tilde{\eta}}^n,\nu_{\tilde{\eta}}^n\rangle\}$, respectively. If $\{\langle\mu_{\tilde{\eta}}^i,\nu_{\tilde{\eta}}^i\rangle\}$ is a rearrangement of the sequence $\{\langle\mu_{\tilde{\xi}}^i,\nu_{\tilde{\xi}}^i\rangle\}$, then we have $H(\tilde{\xi})=H(\tilde{\eta})$.

Proof. The property follows immediately from Definition 5 of the entropy of credibility distribution of IFV. This symmetry property shows that the credibility entropy is invariant with respect to permutations of a given possibility or credibility distribution.

Property 2. Let $\tilde{\xi}$ be an IFV that takes value in $X=\{x_1,x_2,\cdots\}$ with possibility distribution $\{\langle\mu_{\xi}^1,\nu_{\xi}^1\rangle,\langle\mu_{\xi}^2,\nu_{\xi}^2\rangle,\cdots\}$. Then $H\{\tilde{\xi}\}\ge 0$ and the equality holds iff $\tilde{\xi}$ is a crisp number.

Proof. Since $Cr(\tilde{\xi}=x_i)\in[0,1]$ and $S(t)\ge 0,\forall t\in[0,1]$, we immediately deduce that

$$H(\tilde{\xi})=\sum_{i=1}^{\infty}S(Cr(\tilde{\xi}=x_i))\ge 0.$$

(1) If $H(\tilde{\xi})=\sum_{i=1}^{\infty}S(Cr(\tilde{\xi}=x_i))=0$, we have $Cr(\tilde{\xi}=x_i)=0$ or 1, $\forall x_i\in X$.

It follows from Theorem 1 that there exists only one index k with $Cr\{\tilde{\xi}=x_k\}=1$ such that the true and false possibility distribution of IFV $\tilde{\xi}$ is

$$\prod_{\tilde{\xi}}(x)=\begin{cases}\langle 1,0\rangle, x=x_k\\ \langle 0,1\rangle, x\ne x_k\end{cases}.$$

That is to say, $\tilde{\xi}$ is a crisp number.

(2) On the other hand, if $\tilde{\xi}$ is a crisp number, then IFV $\tilde{\xi}$ has the possibility distribution $\prod_{\tilde{\xi}}$ as the above form. With Definition 5 one can easily verify that the entropy $H(\tilde{\xi})=0$ in this case.

Property 3. Let $\tilde{\xi}$ be an IFV that takes value in $X=\{x_1,x_2,\cdots,x_n\}$ with possibility distribution $\{\langle\mu_{\xi}^1,\nu_{\xi}^1\rangle,\langle\mu_{\xi}^2,\nu_{\xi}^2\rangle,\cdots\langle\mu_{\xi}^n,\nu_{\xi}^n\rangle\}$. Then $H(\tilde{\xi})\le n\ln 2$ and the equality holds iff the possibility distribution of IFV $\tilde{\xi}$ is $\langle\mu_{\tilde{\xi}}(x_i),\nu_{\tilde{\xi}}(x_i)\rangle=\langle 1,0\rangle$, for all $i=1,2,\cdots,n$.

Proof. Since the Shannon's entropy function $S(t)$ reaches its maximum $\ln 2$ at $t=\frac{1}{2}$, we have

$$H(\tilde{\xi})=\sum_{i=1}^{n}S(Cr(\tilde{\xi}=x_i))\le n\ln 2.$$

And the above equality holds iff

$$Cr(\tilde{\xi}=x_i)=\tfrac{1}{2}[\tfrac{\mu_{\tilde{\xi}}(x_i)+1-\nu_{\tilde{\xi}}(x_i)}{2}+1-\max_{j\ne i}\tfrac{\mu_{\tilde{\xi}}(x_j)+1-\nu_{\tilde{\xi}}(x_j)}{2}]=\tfrac{1}{2},(\forall i=1,2,\cdots,n),$$

which yields to $\langle\mu_{\tilde{\xi}}(x_i),\nu_{\tilde{\xi}}(x_i)\rangle=\langle 1,0\rangle$ for any $i=1,2,\cdots,n$.

Property 4. Let $\tilde{\xi}$ be an IFV that takes value in $X=\{x_1,x_2,\cdots,x_n\}$ with possibility distribution $\{\langle\mu^1,\nu^1\rangle,\langle\mu^2,\nu^2\rangle,\cdots\langle\mu^n,\nu^n\rangle\}$, and $\tilde{\eta}$ be an IFV that takes value in $X=\{x_1,x_2,\cdots,x_n,x_{n+1}\}$ with possibility distribution $\{\langle\mu^1,\nu^1\rangle,\langle\mu^2,\nu^2\rangle,\cdots\langle\mu^n,\nu^n\rangle,\langle\mu^{n+1},\nu^{n+1}\rangle\}$. Then $H(\tilde{\xi})\le H(\tilde{\eta})$ and the equality holds iff $\langle\mu^{n+1},\nu^{n+1}\rangle=\langle 0,1\rangle$.

Proof. For $i=1,2,\cdots,n,$ we deduce the following result.

If $\langle\mu^i,\nu^i\rangle\ne\langle 1,0\rangle,$ *then* $Cr(\tilde{\xi}=x_i)=\frac{1}{2}[\frac{\mu^i+1-\nu^i}{2}+1-\max_{k\ne i,k\in\{1,\Lambda\ n\}}\frac{\mu^k+1-\nu^k}{2}]=\frac{1}{2}[\frac{\mu^i+1-\nu^i}{2}]$;

$$Cr(\tilde{\eta}=x_i)=\frac{1}{2}[\frac{\mu^i+1-\nu^i}{2}+1-\max_{k\ne i,k\in\{1,\Lambda\ n+1\},}\frac{\mu^k+1-\nu^k}{2}]=\frac{1}{2}[\frac{\mu^i+1-\nu^i}{2}].$$

We have $S(Cr(\tilde{\xi}=x_i))=S(Cr(\tilde{\eta}=x_i)).$

If $\langle\mu^i,\nu^i\rangle=\langle 1,0\rangle,$ *then* $Cr(\tilde{\xi}=x_i)=\frac{1}{2}[\frac{\mu^i+1-\nu^i}{2}+1-\max_{k\ne i,k\in\{1,\Lambda\ n\}}\frac{\mu^k+1-\nu^k}{2}]$

$$\ge\frac{1}{2}[\frac{\mu^i+1-\nu^i}{2}+1-\max_{k\ne i,k\in\{1,\Lambda\ n+1\},}\frac{\mu^k+1-\nu^k}{2}]=Cr(\tilde{\eta}=x_i)\in[\tfrac{1}{2},1].$$

We have $S(Cr(\tilde{\xi}=x_i))\le S(Cr(\tilde{\eta}=x_i))$ due to the decreasing property of function $S(t)$ in $[\frac{1}{2},1]$.

So, we get $H(\tilde{\xi})=\sum_{i=1}^{n}S(Cr(\tilde{\xi}=x_i))\le\sum_{i=1}^{n}S(Cr(\tilde{\eta}=x_i))=H(\tilde{\eta})$.

Also, from the normalization condition of IFV $\tilde{\xi}$ we know

$$Cr(\tilde{\eta}=x_{n+1})=\frac{1}{2}[\frac{\mu^{n+1}+1-\nu^{n+1}}{2}+1-\max_{1\le i\le n}\frac{\mu^i+1-\nu^i}{2}]=\frac{1}{2}[\frac{\mu^{n+1}+1-\nu^{n+1}}{2}+1-1]=\frac{\mu^{n+1}+1-\nu^{n+1}}{4}\ge 0.$$

Thus, we get $H(\tilde{\xi})=\sum_{i=1}^{n}S(Cr(\tilde{\xi}=x_i))\le\sum_{i=1}^{n+1}S(Cr(\tilde{\eta}=x_i))=H(\tilde{\eta})$.

Obviously, the above equality holds iff $\mu^{n+1}=0,\ \ \nu^{n+1}=1$.

The above property indicates that the credibility entropy of IFV only relates to the true and false possibilities of all the focal elements of IFV.

Property 5. Let $\tilde{\xi},\tilde{\eta}$ be two IFVs taking values in $X=\{x_1,x_2,\cdots,x_n\}$ with possibility distributions $\{\langle\mu_{\tilde{\xi}}^1,\nu_{\tilde{\xi}}^1\rangle,\langle\mu_{\tilde{\xi}}^2,\nu_{\tilde{\xi}}^2\rangle,\cdots\langle\mu_{\tilde{\xi}}^n,\nu_{\tilde{\xi}}^n\rangle\}$ and $\{\langle\mu_{\tilde{\eta}}^1,\nu_{\tilde{\eta}}^1\rangle,\langle\mu_{\tilde{\eta}}^2,\nu_{\tilde{\eta}}^2\rangle,\cdots\langle\mu_{\tilde{\eta}}^n,\nu_{\tilde{\eta}}^n\rangle\}$, respectively. If $\langle\mu_{\tilde{\xi}}^i,\nu_{\tilde{\xi}}^i\rangle\le\langle\mu_{\tilde{\eta}}^i,\nu_{\tilde{\eta}}^i\rangle$, for all $i=1,2,\cdots,n,$ then $H(\tilde{\xi})\le H(\tilde{\eta})$.

Proof. **Case (1):** If $\langle\mu_{\tilde{\xi}}^i,\nu_{\tilde{\xi}}^i\rangle=\langle 1,0\rangle$ with $\langle\mu_{\tilde{\xi}}^i,\nu_{\tilde{\xi}}^i\rangle\le\langle\mu_{\tilde{\eta}}^i,\nu_{\tilde{\eta}}^i\rangle,$ *i.e.,* $\mu_{\tilde{\xi}}^i\le\mu_{\tilde{\eta}}^i,\nu_{\tilde{\xi}}^i\ge\nu_{\tilde{\eta}}^i,\forall i=1,\cdots,n,$

then we infer that $\langle\mu_{\tilde{\eta}}^i,\nu_{\tilde{\eta}}^i\rangle=\langle 1,0\rangle$ and $\sup_{j\ne i}\frac{\mu_{\tilde{\xi}}^j+1-\nu_{\tilde{\xi}}^j}{2}\le\sup_{j\ne i}\frac{\mu_{\tilde{\eta}}^j+1-\nu_{\tilde{\eta}}^j}{2}$.

Thus, we have

$$Cr(\tilde{\xi}=x_i)=\frac{1}{2}(\frac{\mu_{\tilde{\xi}}^i+1-\nu_{\tilde{\xi}}^i}{2}+1-\sup_{j\ne i}\frac{\mu_{\tilde{\xi}}^j+1-\nu_{\tilde{\xi}}^j}{2})=\frac{1}{2}(\frac{1+1-0}{2}+1-\sup_{j\ne i}\frac{\mu_{\tilde{\xi}}^j+1-\nu_{\tilde{\xi}}^j}{2})$$

$$\ge\frac{1}{2}(\frac{1+1-0}{2}+1-\sup_{j\ne i}\frac{\mu_{\tilde{\eta}}^j+1-\nu_{\tilde{\eta}}^j}{2})=\frac{1}{2}(\frac{\mu_{\tilde{\eta}}^i+1-\nu_{\tilde{\eta}}^i}{2}+1-\sup_{j\ne i}\frac{\mu_{\tilde{\eta}}^j+1-\nu_{\tilde{\eta}}^j}{2})$$

$$= Cr(\tilde{\eta} = x_i) \geq \tfrac{1}{2}.$$

So, it follows from the monotonically decreasing property of function $S(t)$ in interval $[\frac{1}{2},1]$ that

$$H(\tilde{\xi}) = \sum_{i=1}^{n} S(Cr(\tilde{\xi} = x_i)) \leq \sum_{i=1}^{n} S(Cr(\tilde{\eta} = x_i)) = H(\tilde{\eta}).$$

Case (2): If $\langle \mu_{\tilde{\xi}}^i, \nu_{\tilde{\xi}}^i \rangle \neq \langle 1,0 \rangle$, there must exist an index k such that $\langle \mu_{\tilde{\xi}}^k, \nu_{\tilde{\xi}}^k \rangle = \langle 1,0 \rangle$ and $\langle \mu_{\tilde{\eta}}^k, \nu_{\tilde{\eta}}^k \rangle = \langle 1,0 \rangle$ due to the normalization condition of IFV and $\langle \mu_{\tilde{\xi}}^i, \nu_{\tilde{\xi}}^i \rangle \leq \langle \mu_{\tilde{\eta}}^i, \nu_{\tilde{\eta}}^i \rangle, i.e., \mu_{\tilde{\xi}}^i \leq \mu_{\tilde{\eta}}^i, \nu_{\tilde{\xi}}^i \geq \nu_{\tilde{\eta}}^i, \forall i = 1, \cdots, n.$

Then, we get $\sup_{k\neq i} \frac{\mu_{\tilde{\xi}}^k + 1 - \nu_{\tilde{\xi}}^k}{2} = \sup_{k\neq i} \frac{\mu_{\tilde{\eta}}^k + 1 - \nu_{\tilde{\eta}}^k}{2} = \frac{1+1-0}{2} = 1$. And

$$Cr(\tilde{\xi} = x_i) = \tfrac{1}{2}\left(\frac{\mu_{\tilde{\xi}}^i + 1 - \nu_{\tilde{\xi}}^i}{2} + 1 - \sup_{k\neq i} \frac{\mu_{\tilde{\xi}}^k + 1 - \nu_{\tilde{\xi}}^k}{2}\right) = \tfrac{1}{2}\left(\frac{\mu_{\tilde{\xi}}^i + 1 - \nu_{\tilde{\xi}}^i}{2} + 1 - 1\right) = \left(\frac{\mu_{\tilde{\xi}}^i + 1 - \nu_{\tilde{\xi}}^i}{4}\right);$$

$$Cr(\tilde{\eta} = x_i) = \tfrac{1}{2}\left(\frac{\mu_{\tilde{\eta}}^i + 1 - \nu_{\tilde{\eta}}^i}{2} + 1 - \sup_{k\neq i} \frac{\mu_{\tilde{\eta}}^k + 1 - \nu_{\tilde{\eta}}^k}{2}\right) = \frac{\mu_{\tilde{\eta}}^i + 1 - \nu_{\tilde{\eta}}^i}{4}.$$

From $\langle \mu_{\tilde{\xi}}^i, \nu_{\tilde{\xi}}^i \rangle \leq \langle \mu_{\tilde{\eta}}^i, \nu_{\tilde{\eta}}^i \rangle$, we know $\mu_{\tilde{\xi}}^i \leq \mu_{\tilde{\eta}}^i,\ \nu_{\tilde{\xi}}^i \geq \nu_{\tilde{\eta}}^i, (\forall i = 1, \cdots, n)$, which gives rise to

$$0 \leq Cr(\tilde{\xi} = x_i) = \left(\frac{\mu_{\tilde{\xi}}^i + 1 - \nu_{\tilde{\xi}}^i}{4}\right) \leq \frac{\mu_{\tilde{\eta}}^i + 1 - \nu_{\tilde{\eta}}^i}{4} = Cr(\tilde{\eta} = x_i) < \tfrac{1}{2}.$$

It follows from the monotonically increasing property of function $S(t)$ in interval $[0, \frac{1}{2}]$ that

$$H(\tilde{\xi}) = \sum_{i=1}^{n} S(Cr(\tilde{\xi} = x_i)) \leq \sum_{i=1}^{n} S(Cr(\tilde{\eta} = x_i)) = H(\tilde{\eta}).$$

Property 6. Let $\tilde{\xi}, \tilde{\eta}$ be two IFVs taking value in $\mathrm{X} = \{x_1, x_2, \cdots, x_n\}$ with possibility distributions $\{\langle \mu_{\tilde{\xi}}^1, \nu_{\tilde{\xi}}^1 \rangle, \langle \mu_{\tilde{\xi}}^2, \nu_{\tilde{\xi}}^2 \rangle, \cdots \langle \mu_{\tilde{\xi}}^n, \nu_{\tilde{\xi}}^n \rangle\}$ and $\{\langle \mu_{\tilde{\eta}}^1, \nu_{\tilde{\eta}}^1 \rangle, \langle \mu_{\tilde{\eta}}^2, \nu_{\tilde{\eta}}^2 \rangle, \cdots \langle \mu_{\tilde{\eta}}^n, \nu_{\tilde{\eta}}^n \rangle\}$, respectively. Then we have $\sum_{i=1}^{n} S(\theta Cr(\tilde{\xi} = x_i) + (1-\theta) Cr(\tilde{\eta} = x_i)) \geq \theta H(\tilde{\xi}) + (1-\theta) H(\tilde{\eta})$, for any $\theta \in [0,1]$.

Proof. Due to the concavity of Shannon's function $S(t)$ in interval $[0,1]$, we infer that

$$S(\theta Cr(\tilde{\xi} = x_i) + (1-\theta) Cr(\tilde{\eta} = x_i)) \geq \theta S(Cr(\tilde{\xi} = x_i)) + (1-\theta) S(Cr(\tilde{\eta} = x_i)), \quad \forall i = 1, \cdots, n.$$

By summing both sides of the above inequalities we immediately obtain the result of Property 6.

Example 1. Suppose that "the age of Jack is about 30" can be modeled by the following IFV:

$$\tilde{\xi} = \{(25, \langle 0.3, 0.5 \rangle), (27, \langle 0.6, 0.3 \rangle), (28, \langle 0.7, 0.2 \rangle), (30, \langle 1, 0 \rangle), (31, \langle 0.8, 0.1 \rangle), (32, \langle 0.75, 0.2 \rangle), (35, \langle 0.2, 0.7 \rangle)\},$$

where $(28, \langle 0.7, 0.2 \rangle)$ represents $\tilde{\xi}$ taking value 28 with true possibility 0.7 and false possibility 0.2.

By using the credibility formula (4) we can compute the credibility distribution of IFV $\tilde{\xi}$ as follows.

$$Cr(\tilde{\xi} = 25) = 0.2, \quad Cr(\tilde{\xi} = 27) = 0.325, \quad Cr(\tilde{\xi} = 28) = 0.375, \quad Cr(\tilde{\xi} = 30) = 0.575,$$

$$Cr(\tilde{\xi}=31)=0.425,\quad Cr(\tilde{\xi}=32)=0.3875,\quad Cr(\tilde{\xi}=35)=0.125.$$

Thus, with formula (5) we get credibility entropy of IFV $\tilde{\xi}$ as $H(\tilde{\xi})=\sum_{i=1}^{7}S(Cr(\tilde{\xi}=x_i))=4.2008.$ The corresponding normalized credibility entropy is $\overline{H}(\tilde{\xi})=\frac{1}{7\ln 2}\sum_{i=1}^{7}S(Cr(\tilde{\xi}=x_i))=0.8658,$ which is greater than the intuitionisitc fuzzy entropy $E(\tilde{\xi})=0.8178$ introduced in [33].

4. Entropy and cross-entropy of credibility distribution for continuous IFV

In this section, we give the definitions of credibility entropy and cross-entropy of continuous IFVs. Some important property theorems about the credibility entropy for continuous IFVs are verified. Further, we disclose the close relationship between credibility entropy and cross-entropy of continuous IFVs.

$\tilde{\xi}$ is called a continuous IFV if the true possibility function $\mu_{\tilde{\xi}}$ and the false possibility function $\nu_{\tilde{\xi}}$ are all continuous on real number set R. For any $x\in R$, the credibility distribution of basic intuitionistic fuzzy event $\{\tilde{\xi}=x\}$ can be calculated by

$$Cr\{\tilde{\xi}=x\}=\tfrac{1}{2}[\tfrac{\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)}{2}+1-\sup_{y\neq x}\tfrac{\mu_{\tilde{\xi}}(y)+1-\nu_{\tilde{\xi}}(y)}{2}]=\tfrac{1}{2}[\tfrac{\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)}{2}+1-\tfrac{1+1-0}{2}]=\tfrac{\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)}{4}. \quad (6)$$

Definition 6. Let $\tilde{\xi}$ be a continuous intuitionistic fuzzy variable taking values in $[a,b]\subset R$. Then, the entropy of its credibility distribution is defined by

$$H(\tilde{\xi})=\int_a^b S(Cr(\tilde{\xi}=x)dx$$
$$=-\int_a^b(\tfrac{u_{\tilde{\xi}(x)}+1-\nu_{\tilde{\xi}(x)}}{4})\ln(\tfrac{u_{\tilde{\xi}(x)}+1-\nu_{\tilde{\xi}(x)}}{4})dx-\int_a^b(1-\tfrac{u_{\tilde{\xi}(x)}+1-\nu_{\tilde{\xi}(x)}}{4})\ln(1-\tfrac{u_{\tilde{\xi}(x)}+1-\nu_{\tilde{\xi}(x)}}{4})dx \quad (7)$$

where $S(t)=-t\ln t-(1-t)\ln(1-t)$ is the well-known Shannon's entropy function.

Theorem 2. Let $\tilde{\xi}$ be a continuous IFV taking values in $[a,b]\subset R$. Then, we have

$$0\le H(\xi)=\int_a^b S(Cr(\tilde{\xi}=x)dx\le(b-a)\ln 2.$$

Proof. Since $0\le\frac{u_{\tilde{\xi}(x)}+1-\nu_{\tilde{\xi}(x)}}{4}\le\frac{1}{2}$, we easily know $0=S(0)\le S(\frac{u_{\tilde{\xi}(x)}+1-\nu_{\tilde{\xi}(x)}}{4})\le S(\frac{1}{2})=\ln 2$ by the monotonically increasing of Shannon function in $[0,\frac{1}{2}]$. Further, we have

$$0\le H(\xi)=\int_a^b S(Cr(\tilde{\xi}=x)dx=\int_a^b S(\tfrac{u_{\tilde{\xi}(x)}+1-\nu_{\tilde{\xi}(x)}}{4})dx\le\int_a^b S(\tfrac{1}{2})dx\le(b-a)\ln 2.$$

Especially, when $u_{\tilde{\xi}(x)}=1,\nu_{\tilde{\xi}(x)}=0,\forall x\in(a,b)$, credibility entropy $H(\tilde{\xi})$ reaches its maximum $(b-a)\ln 2;$ when $u_{\tilde{\xi}(x)}=0,\nu_{\tilde{\xi}(x)}=1,\forall x\in[a,b]$, credibility entropy $H(\tilde{\xi})$ reaches its minimum 0.

Theorem 3. Let $\tilde{\xi},\tilde{\eta}$ be two continuous IFVs with possibility distributions $\langle\mu_{\xi}(x),\nu_{\xi}(x)\rangle$ and $\langle\mu_{\tilde{\eta}}(x),\nu_{\tilde{\eta}}(x)\rangle,$ respectively. If $\langle\mu_{\xi}(x),\nu_{\xi}(x)\rangle\le\langle\mu_{\tilde{\eta}}(x),\nu_{\tilde{\eta}}(x)\rangle$, $\forall x\in R$, then $H(\tilde{\xi})\le H(\tilde{\eta})$.

Proof. Since $\langle \mu_{\xi}(x), \nu_{\xi}(x)\rangle \le \langle \mu_{\tilde\eta}(x), \nu_{\tilde\eta}(x)\rangle$; *i.e.*, $\mu_{\tilde\xi}(x) \le \mu_{\tilde\eta}(x), \nu_{\tilde\xi}(x) \ge \nu_{\tilde\eta}(x)$, $\forall x \in R$,

$$Cr(\tilde\xi = x) = \tfrac{1}{2}\left(\frac{\mu_{\tilde\xi}^{i}+1-\nu_{\tilde\xi}^{i}}{2} + 1\text{-}\sup_{j\ne i}\frac{\mu_{\tilde\xi}^{j}+1-\nu_{\tilde\xi}^{j}}{2}\right) = \frac{\mu_{\tilde\xi}(x)+1-\nu_{\tilde\xi}(x)}{4} \in [0, \tfrac{1}{2}];$$

$$Cr(\tilde\eta = x) = \tfrac{1}{2}\left(\frac{\mu_{\tilde\eta}^{i}+1-\nu_{\tilde\eta}^{i}}{2} + 1\text{-}\sup_{j\ne i}\frac{\mu_{\tilde\eta}^{j}+1-\nu_{\tilde\eta}^{j}}{2}\right) = \frac{\mu_{\tilde\eta}(x)+1-\nu_{\tilde\eta}(x)}{4} \in [0, \tfrac{1}{2}].$$

Thus, we have $0 \le Cr(\tilde\xi = x) \le Cr(\tilde\eta = x) \le \frac{1}{2}$.

Hence, $H(\xi) = \int_a^b S(Cr(\tilde\xi = x)dx \le \int_a^b S(Cr(\tilde\eta = x)dx = H(\tilde\eta)$.

Theorem 4. Let $\tilde\xi$ be a continuous IFV taking values in $[a,b]$ with true possibility function $u_{\tilde\xi}$ and false possibility function $v_{\tilde\xi}$, for any real number k and λ, then $H(\lambda\tilde\xi + k) = |\lambda| H(\tilde\xi)$.

Proof. Since $\tilde\xi$ is a continuous IFV taking values in $[a,b]$, then we know $\lambda\tilde\xi + k$ is also a continuous IFV taking values in $[\lambda a + k, \lambda b + k]$ if $\lambda \ge 0$. Thus, we have

$$H(\lambda\tilde\xi + k) = \int_{\lambda a+k}^{\lambda b+k} S(Cr(\lambda\tilde\xi + k = x))dx$$

$$= \int_a^b S(Cr(\tilde\xi = t)d(\lambda t + k) \quad = \lambda\int_a^b S(Cr(\tilde\xi = t)dt = |\lambda| H(\tilde\xi).$$

On the other hand, if $\lambda < 0$, we have

$$H(\lambda\tilde\xi + k) = \int_{\lambda b+k}^{\lambda a+k} S(Cr(\lambda\tilde\xi + k) = x)dx$$

$$= \int_b^a S(Cr(\tilde\xi = t)d(\lambda t + k) \quad = -\lambda\int_a^b S(Cr(\tilde\xi = t)dt = |\lambda| H(\tilde\xi).$$

Definition 7. Let $\tilde\xi$ and $\tilde\eta$ be two continuous IFVs with credibility distributions on R. The credibility cross-entropy of $\tilde\xi$ from $\tilde\eta$ is defined by

$$D(\tilde\xi, \tilde\eta) = \int_{-\infty}^{+\infty} \{Cr(\tilde\xi = x)\ln\frac{Cr(\tilde\xi = x)}{\frac{1}{2}[Cr(\tilde\xi = x) + Cr(\tilde\eta = x)]} + (1 - Cr(\tilde\xi = x))\ln\frac{1 - Cr(\tilde\xi = x)}{\frac{1}{2}[1 - Cr(\tilde\xi = x) + 1 - Cr(\tilde\eta = x)]}\}dx,$$

which is more rational than the definition given in [21]. It shows the discrimination information in favor of $\tilde\xi$ against $\tilde\eta$ or divergence degree between IFVs $\tilde\xi$ and $\tilde\eta$.

Remark 1. If $\tilde\xi$ and $\tilde\eta$ are continuous IFVs taking values in $[a,b]$, substituting formula $Cr\{\tilde\xi = x\} = \frac{\mu_{\tilde\xi}(x)+1-\nu_{\tilde\xi}(x)}{4}$, $Cr\{\tilde\eta = x\} = \frac{\mu_{\tilde\eta}(x)+1-\nu_{\tilde\eta}(x)}{4}$ into Definition 7 we get the following cross-entropy:

$$D(\tilde\xi, \tilde\eta) = \int_a^b \frac{\mu_{\tilde\xi}(x)+1-\nu_{\tilde\xi}(x)}{4}\ln\frac{(\mu_{\tilde\xi}(x)+1-\nu_{\tilde\xi}(x))}{\frac{1}{2}[\mu_{\tilde\xi}(x)+1-\nu_{\tilde\xi}(x)+\mu_{\tilde\eta}(x)+1-\nu_{\tilde\eta}(x)]}dx + \int_a^b \left(\frac{3\text{-}\mu_{\tilde\xi}(x)+\nu_{\tilde\xi}(x)}{4}\right)\ln\frac{\frac{3\text{-}\mu_{\tilde\xi}(x)+\nu_{\tilde\xi}(x)}{4}}{\frac{1}{2}[\frac{3-\mu_{\tilde\xi}(x)+\nu_{\tilde\xi}(x)}{4} + \frac{3-\mu_{\tilde\eta}(x)+\nu_{\tilde\eta}(x)}{4}]}dx. \quad (8)$$

According to Shannon's inequality (see [18]), one can easily prove that $D(\tilde\xi, \tilde\eta) \ge 0$; and

$D(\tilde{\xi},\tilde{\eta})=0$ if $\mu_{\tilde{\xi}}(x)=\mu_{\tilde{\eta}}(x), \nu_{\tilde{\xi}}(x)=\nu_{\tilde{\eta}}(x), i.e., \tilde{\xi}=\tilde{\eta}$; However, $D(\tilde{\xi},\tilde{\eta})\neq D(\tilde{\eta},\tilde{\xi})$, i.e., $D(\tilde{\xi},\tilde{\eta})$ is not symmetric. So, it should be modified to a symmetric form of credibility cross-entropy for IFVs as below.

Definition 8. For two IFVs $\tilde{\xi},\tilde{\eta}$, the symmetric credibility cross-entropy of IFVs is defined as

$$D^*(\tilde{\xi},\tilde{\eta})=D(\tilde{\xi},\tilde{\eta})+D(\tilde{\eta},\tilde{\xi}). \tag{9}$$

From the above formula, one can easily verify that
(1) $D^*(\tilde{\xi},\tilde{\eta})\geq 0$;
(2) If $\tilde{\xi}=\tilde{\eta}$, then $D^*(\tilde{\xi},\tilde{\eta})=0$;
(3) $D^*(\tilde{\xi},\tilde{\eta})=D^*(\tilde{\eta},\tilde{\xi})$, which is consistent with our intuition of discrimination between two IFVs. So, the symmetric credibility cross-entropy is more appropriate to show the discrimination information of IFVs $\tilde{\xi}$ from $\tilde{\eta}$.

Below, we give a theorem to discover the relationship between the proposed credibility entropy and the symmetric credibility cross-entropy form of IFVs.

Theorem 5. For any continuous IFV $\tilde{\xi}$ taking values in interval $[a,b]$ with true possibility function $\mu_{\tilde{\xi}}$ and false possibility function $\nu_{\tilde{\xi}}$, then we get

$$D^*(\tilde{\xi},\tilde{\xi}^c)=-[H(\tilde{\xi})+H(\tilde{\xi}^c)]+2S(\tfrac{1}{4})(b-a),$$

where $S(t)=-t\ln t-(1-t)\ln(1-t)$ is the well-known Shannon's entropy function.

Proof. Since $\tilde{\xi}^c$ is the complement of IFV $\tilde{\xi}$, we have $\mu_{\tilde{\xi}^c}(x)=\nu_{\tilde{\xi}}(x), \nu_{\tilde{\xi}^c}(x)=\mu_{\tilde{\xi}}(x)$ and $Cr(\tilde{\xi}^c=x)=\frac{1}{2}[\frac{\mu_{\tilde{\xi}^c}(x)+1-\nu_{\tilde{\xi}^c}(x)}{2}+1-\sup_{x'\neq x}\frac{\mu_{\tilde{\xi}^c}(x')+1-\nu_{\tilde{\xi}^c}(x')}{2}]=\frac{\nu_{\tilde{\xi}}(x)+1-\mu_{\tilde{\xi}}(x)}{4}$.
Then it follows from formulas (6)–(8) that

$$\begin{aligned}
D(\tilde{\xi},\tilde{\xi}^c)&=\int_a^b Cr(\tilde{\xi}=x)\ln\frac{Cr(\tilde{\xi}=x)}{\frac{1}{2}[Cr(\tilde{\xi}=x)+Cr(\tilde{\xi}^c=x)]}+(1-Cr(\tilde{\xi}=x))\ln\frac{1-Cr(\tilde{\xi}=x)}{1-\frac{1}{2}[Cr(\tilde{\xi}=x)+Cr(\tilde{\xi}^c=x)]})dx\\
&=\int_a^b\frac{\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)}{4}\ln\frac{\frac{(\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x))}{4}}{\frac{1}{2}[\frac{\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)}{4}+\frac{\nu_{\tilde{\xi}}(x)+1-\mu_{\tilde{\xi}}(x)}{4}]}dx+\int_a^b(\frac{3-\mu_{\tilde{\xi}}(x)+\nu_{\tilde{\xi}}(x)}{4})\ln\frac{\frac{3-\mu_{\tilde{\xi}}(x)+\nu_{\tilde{\xi}}(x)}{4}}{\frac{1}{2}[\frac{3-\mu_{\tilde{\xi}}(x)+\nu_{\tilde{\xi}}(x)}{4}+\frac{3-\nu_{\tilde{\xi}}(x)+\mu_{\tilde{\xi}}(x)}{4}]}dx\\
&=\int_a^b\frac{\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)}{4}\ln(4\times\frac{(\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x))}{4})dx+\int_a^b(\frac{3-\mu_{\tilde{\xi}}(x)+\nu_{\tilde{\xi}}(x)}{4})\ln(\frac{4}{3}\times\frac{3-\mu_{\tilde{\xi}}(x)+\nu_{\tilde{\xi}}(x)}{4})dx\\
&=\int_a^b\frac{\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)}{4}\ln(\frac{(\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x))}{4})dx+\int_a^b\frac{\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)}{4}\ln(4)dx\\
&+\int_a^b(\frac{3-\mu_{\tilde{\xi}}(x)+\nu_{\tilde{\xi}}(x)}{4})\ln(\frac{3-\mu_{\tilde{\xi}}(x)+\nu_{\tilde{\xi}}(x)}{4})dx+\int_a^b(\frac{3-\mu_{\tilde{\xi}}(x)+\nu_{\tilde{\xi}}(x)}{4})\ln(\tfrac{4}{3})dx.
\end{aligned}$$

Similarly, we conclude that

$$D(\tilde{\xi}^c,\tilde{\xi})=\int_a^b Cr(\tilde{\xi}^c=x)\ln\frac{Cr(\tilde{\xi}^c=x)}{\frac{1}{2}[Cr(\tilde{\xi}^c=x)+Cr(\tilde{\xi}=x)]}+(1-Cr(\tilde{\xi}^c=x))\ln\frac{1-Cr(\tilde{\xi}^c=x)}{1-\frac{1}{2}[Cr(\tilde{\xi}^c=x)+Cr(\tilde{\xi}=x)]})dx$$

$$
= \int_a^b \frac{\mu_{\tilde{\xi}^c}(x)+1-\nu_{\tilde{\xi}^c}(x)}{4} \ln \frac{(\mu_{\tilde{\xi}^c}(x)+1-\nu_{\tilde{\xi}^c}(x))}{\frac{1}{2}[\mu_{\tilde{\xi}^c}(x)+1-\nu_{\tilde{\xi}^c}(x)+\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)]} dx + \int_a^b \left(\frac{3-\mu_{\tilde{\xi}^c}(x)+\nu_{\tilde{\xi}^c}(x)}{4}\right) \ln \frac{\frac{3-\mu_{\tilde{\xi}^c}(x)+\nu_{\tilde{\xi}^c}(x)}{4}}{\frac{1}{2}[\frac{3-\mu_{\tilde{\xi}^c}(x)+\nu_{\tilde{\xi}^c}(x)}{4}+\frac{3-\mu_{\tilde{\xi}}(x)+\nu_{\tilde{\xi}}(x)}{4}]} dx
$$

$$
= \int_a^b \frac{\nu_{\tilde{\xi}}(x)+1-\mu_{\tilde{\xi}}(x)}{4} \ln \frac{\frac{\nu_{\tilde{\xi}}(x)+1-\mu_{\tilde{\xi}}(x)}{4}}{\frac{1}{2}[\frac{\nu_{\tilde{\xi}}(x)+1-\mu_{\tilde{\xi}}(x)}{4}+\frac{\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)}{4}]} dx + \int_a^b \left(\frac{3-\nu_{\tilde{\xi}}(x)+\mu_{\tilde{\xi}}(x)}{4}\right) \ln \frac{\frac{3-\nu_{\tilde{\xi}}(x)+\mu_{\tilde{\xi}}(x)}{4}}{\frac{1}{2}[\frac{3-\nu_{\tilde{\xi}}(x)+\mu_{\tilde{\xi}}(x)}{4}+\frac{3-\mu_{\tilde{\xi}}(x)+\nu_{\tilde{\xi}}(x)}{4}]} dx
$$

$$
= \int_a^b \frac{\nu_{\tilde{\xi}}(x)+1-\mu_{\tilde{\xi}}(x)}{4} \ln\left(\frac{\nu_{\tilde{\xi}}(x)+1-\mu_{\tilde{\xi}}(x)}{4}\right) dx + \int_a^b \frac{\nu_{\tilde{\xi}}(x)+1-\mu_{\tilde{\xi}}(x)}{4} \ln 4dx
$$

$$
+ \int_{-\infty}^{+\infty} \left(\frac{3-\nu_{\tilde{\xi}}(x)+\mu_{\tilde{\xi}}(x)}{4}\right) \ln\left(\frac{3-\nu_{\tilde{\xi}}(x)+\mu_{\tilde{\xi}}(x)}{4}\right) dx + \int_{-\infty}^{+\infty} \left(\frac{3-\nu_{\tilde{\xi}}(x)+\mu_{\tilde{\xi}}(x)}{4}\right) \ln\left(\frac{4}{3}\right) dx .
$$

Thus, with formula (9) we have

$$
D^*(\tilde{\xi}, \tilde{\xi}^c) = D(\tilde{\xi}, \tilde{\xi}^c) + D(\tilde{\xi}^c, \tilde{\xi})
$$

$$
= -H(\tilde{\xi}) + \int_a^b \frac{\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)}{4} \ln 4dx + \int_a^b \left(\frac{3-\mu_{\tilde{\xi}}(x)+\nu_{\tilde{\xi}}(x)}{4}\right) \ln(\tfrac{4}{3}) dx
$$

$$
+ (-H(\tilde{\xi}^c) + \int_a^b \frac{\nu_{\tilde{\xi}}(x)+1-\mu_{\tilde{\xi}}(x)}{4} \ln(4)\ dx + \int_a^b \left(\frac{3-\nu_{\tilde{\xi}}(x)+\mu_{\tilde{\xi}}(x)}{4}\right) \ln(\tfrac{4}{3})\ dx
$$

$$
= -[H(\tilde{\xi}) + H(\tilde{\xi}^c)] + [\int_a^b \frac{\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)}{4} \ln 4dx + \int_a^b \frac{\nu_{\tilde{\xi}}(x)+1-\mu_{\tilde{\xi}}(x)}{4} \ln 4dx]
$$

$$
+ [\int_a^b \left(\frac{3-\mu_{\tilde{\xi}}(x)+\nu_{\tilde{\xi}}(x)}{4}\right) \ln(\tfrac{4}{3}) dx + \int_a^b \left(\frac{3-\nu_{\tilde{\xi}}(x)+\mu_{\tilde{\xi}}(x)}{4}\right) \ln(\tfrac{4}{3})\ dx]
$$

$$
= -[H(\tilde{\xi}) + H(\tilde{\xi}^c)] + 2\int_a^b (-\tfrac{1}{4} \ln \tfrac{1}{4} - \tfrac{3}{4} \ln \tfrac{3}{4}) dx
$$

$$
= -[H(\tilde{\xi}) + H(\tilde{\xi}^c)] + 2S(\tfrac{1}{4})(b-a) .
$$

It is noticed that $H(\tilde{\xi}^c) \neq H(\tilde{\xi})$, which is not consistent with the axiomatic property of traditional intuitionistic fuzzy entropy in [23]. So, the proposed credibility entropy $H(\tilde{\xi})$ of IFV $\tilde{\xi}$ in this paper is different from the existing entropy $E(\tilde{\xi})$ of IFV $\tilde{\xi}$ stated in [25,28,33].

5. Credibility entropy of trapezoidal and triangular intuitionistic fuzzy variable

In this part, we give the definitions of trapezoidal intuitionistic fuzzy variable (TIFV) and triangular intuitionistic fuzzy variable (TrIFV). Then, we deduce the corresponding credibility entropy formulas for TIFV and TrIFV which are frequently used in real-life applications.

Definiton 9. [30] A trapezoidal intuitionistic fuzzy variable (TIFV) is determined by a multi-tuple $\tilde{\xi} = (a_1', a_1, a_2, a_3, a_4, a_4')$ of crisp numbers with $-\infty < a_1' \le a_1 \le a_2 \le a_3 \le a_4 \le a_4' < +\infty$, whose true possibility function and false possibility function are respectively defined as

$$\mu_{\tilde{\xi}}(x)=\begin{cases}(x-a_1)/(a_2-a_1), if \quad a_1\le x\le a_2;\\ 1, if \quad a_2\le x\le a_3;\\ (a_4-x)/(a_4-a_3), if \quad a_3\le x\le a_4;\\ 0, otherwise.\end{cases} \quad \nu_{\tilde{\xi}}(x)=\begin{cases}(a_2-x)/(a_2-a_1'), if \quad a_1'\le x\le a_2;\\ 0, if \quad a_2\le x\le a_3;\\ (x-a_3)/(a_4'-a_3), if \quad a_3\le x\le a_4';\\ 1, otherwise.\end{cases}$$

Theorem 6. Let $\tilde{\xi}=(a_1',a_1,a_2,a_3,a_4,a_4')$ be a TIFV taking values in $[a_1',a_4']$. Then, the entropy $H(\tilde{\xi})$ of its credibility distribution is

$$\begin{aligned}\mathrm{H}(\tilde{\xi})&=2(a_2-a_1')[(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2\ln\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')}-(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2\ln(\tfrac{a_1-a_1'}{4(a_2-a_1')})+\tfrac{1}{2}(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2-\tfrac{1}{2}(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2+\tfrac{1}{2}]\\&+\tfrac{2(a_2-a_1)(a_2-a_1')}{(2a_2-a_1-a_1')}[(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2\ln\tfrac{a_1-a_1'}{4(a_2-a_1')}-(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2\ln\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')}-\tfrac{1}{2}(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2+\tfrac{1}{2}(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2]+(a_3-a_2)\ln 2\\&+\tfrac{2(a_4-a_3)((a_4'-a_3)}{a_4'+a_4-2a_3}[(\tfrac{a_4'-a_4}{4(a_4'-a_3)})^2\ln(\tfrac{a_4'-a_4}{4(a_4'-a_3)})-(\tfrac{3a_4'+a_4-4a_3}{4(a_4'-a_3)})^2\ln(\tfrac{3a_4'+a_4-4a_3}{4(a_4'-a_3)})-\tfrac{1}{2}(\tfrac{a_4'-a_4}{4(a_4'-a_3)})^2+\tfrac{1}{2}(\tfrac{3a_4'+a_4-4a_3}{4(a_4'-a_3)})^2]\\&+2(a_4'-a_3)[(\tfrac{3a_4'+a_4-4a_3}{4(a_4'-a_3)})^2\ln\tfrac{3a_4'+a_4-4a_3}{4(a_4'-a_3)}-(\tfrac{a_4'-a_4}{4(a_4'-a_3)})^2\ln(\tfrac{a_4'-a_4}{4(a_4'-a_3)})+\tfrac{1}{2}(\tfrac{a_4'-a_4}{4(a_4'-a_3)})^2-\tfrac{1}{2}(\tfrac{3a_4'+a_4-4a_3}{4(a_4'-a_3)})^2+\tfrac{1}{2}],\end{aligned}\tag{10}$$

Proof. According to credibility entropy formula (7), we get the credibility entropy of TIFV $\tilde{\xi}$ by

$$H(\tilde{\xi})=\int_{a_1}^{a_1'}S(Cr(\tilde{\xi}=x)dx+\int_{a_1}^{a_2}S(Cr(\tilde{\xi}=x)dx+\int_{a_2}^{a_3}S(Cr(\tilde{\xi}=x)dx+\int_{a_3}^{a_4}S(Cr(\tilde{\xi}=x)dx+\int_{a_4}^{a_4'}S(Cr(\tilde{\xi}=x)dx.$$

So, the credibility entropy of TIFV $\tilde{\xi}$ comprises the following five integrals:

(1) $\int_{a_1'}^{a_1}S(Cr(\tilde{\xi}=x))dx$

$$=\int_{a_1'}^{a_1}S(\tfrac{1}{2}(\tfrac{0+1-\nu_{\tilde{\xi}}(x)}{2}+1-\sup_{y\ne x}\tfrac{\mu_{\tilde{\xi}}(y)+1-\nu_{\tilde{\xi}}(y)}{2})dx$$

$$=\int_{a_1'}^{a_1}S(\tfrac{1-\nu_{\tilde{\xi}}(x)}{4})dx$$

$$=-\int_{a_1'}^{a_1}(\tfrac{1}{4}(1-\tfrac{a_2-x}{a_2-a_1'}))\ln(\tfrac{1}{4}(1-\tfrac{a_2-x}{a_2-a_1'}))dx-\int_{a_1'}^{a_1}(1-\tfrac{1}{4}(1-\tfrac{a_2-x}{a_2-a_1'}))\ln(1-\tfrac{1}{4}(1-\tfrac{a_2-x}{a_2-a_1'}))dx$$

$$=2(a_2-a_1')[(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2\ln\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')}-(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2\ln(\tfrac{a_1-a_1'}{4(a_2-a_1')})+\tfrac{1}{2}(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2-\tfrac{1}{2}(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2+\tfrac{1}{2}].$$

(2) $\int_{a_1}^{a_2}S(Cr(\tilde{\xi}=x))dx$

$$=\int_{a_1}^{a_2}S(\tfrac{\mu_{\tilde{\xi}}(x)+1-\nu_{\tilde{\xi}}(x)}{4})dx$$

$$=-\int_{a_1}^{a_2}(\tfrac{1}{4}(\tfrac{x-a_1}{a_2-a_1}+1-\tfrac{a_2-x}{a_2-a_1'})\ln(\tfrac{1}{4}(\tfrac{x-a_1}{a_2-a_1}+1-\tfrac{a_2-x}{a_2-a_1'}))dx-\int_{a_1}^{a_2}(1-(\tfrac{1}{4}(\tfrac{x-a_1}{a_2-a_1}+1-\tfrac{a_2-x}{a_2-a_1'}))\ln(1-(\tfrac{1}{4}(\tfrac{x-a_1}{a_2-a_1}+1-\tfrac{a_2-x}{a_2-a_1'}))dx$$

$$=\frac{2(a_2-a_1)(a_2-a_1')}{(2a_2-a_1-a_1')}[(\frac{a_1-a_1'}{4(a_2-a_1')})^2\ln\frac{a_1-a_1'}{4(a_2-a_1')}-(\frac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2\ln\frac{4a_2-a_1-3a_1'}{4(a_2-a_1')}-\frac{1}{2}(\frac{a_1-a_1'}{4(a_2-a_1')})^2+\frac{1}{2}(\frac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2].$$

(3) $\int_{a_2}^{a_3}S(Cr(\tilde{\xi}=x))dx$

$$=\int_{a_2}^{a_3}S(\frac{1}{2}(\frac{\mu_{\tilde{\xi}}(x)+1-v_{\tilde{\xi}}(x)}{2}+1-\sup_{y\neq x}\frac{\mu_{\tilde{\xi}}(y)+1-v_{\tilde{\xi}}(y)}{2})dx$$

$$=\int_{a_2}^{a_3}S(\frac{1}{2}(\frac{1+1-0}{2}+1-\frac{1+1-0}{2})dx$$

$$=(a_3-a_2)\ln 2.$$

(4) $\int_{a_3}^{a_4}S(Cr(\tilde{\xi}=x))dx$

$$=\int_{a_3}^{a_4}S(\frac{\mu_{\tilde{\xi}}(x)+1-v_{\tilde{\xi}}(x)}{4})dx$$

$$=-\int_{a_3}^{a_4}(\frac{1}{4}(\frac{a_4-x}{a_4-a_3}+1-\frac{x-a_3}{a_4'-a_3})\ln(\frac{1}{4}(\frac{a_4-x}{a_4-a_3}+1-\frac{x-a_3}{a_4'-a_3})dx-\int_{a_3}^{a_4}(1-\frac{1}{4}(\frac{a_4-x}{a_4-a_3}+1-\frac{x-a_3}{a_4'-a_3})\ln(1-\frac{1}{4}(\frac{a_4-x}{a_4-a_3}+1-\frac{x-a_3}{a_4'-a_3})dx$$

$$=\frac{2(a_4-a_3)((a_4'-a_3)}{a_4'+a_4-2a_3}[(\frac{a_4'-a_4}{4(a_4'-a_3)})^2\ln(\frac{a_4'-a_4}{4(a_4'-a_3)})-(\frac{3a_4'+a_4-4a_3}{4(a_4'-a_3)})^2\ln(\frac{3a_4'+a_4-4a_3}{4(a_4'-a_3)})-\frac{1}{2}(\frac{a_4'-a_4}{4(a_4'-a_3)})^2+\frac{1}{2}(\frac{3a_4'+a_4-4a_3}{4(a_4'-a_3)})^2].$$

(5) $\int_{a_4}^{a_4'}S(Cr(\tilde{\xi}=x))dx$

$$=\int_{a_4}^{a_4'}S(\frac{1}{2}(\frac{0+1-v_{\tilde{\xi}}(x)}{2}+1-\sup_{y\neq x}\frac{\mu_{\tilde{\xi}}(y)+1-v_{\tilde{\xi}}(y)}{2})dx$$

$$=\int_{a_4}^{a_4'}S(\frac{1-v_{\tilde{\xi}}(x)}{4})dx$$

$$=-\int_{a_4}^{a_4'}(\frac{1}{4}(1-\frac{x-a_3}{a_4'-a_3}))\ln(\frac{1}{4}(1-\frac{x-a_3}{a_4'-a_3}))dx-\int_{a_4}^{a_4'}(1-\frac{1}{4}(1-\frac{x-a_3}{a_4'-a_3}))\ln(1-\frac{1}{4}(1-\frac{x-a_3}{a_4'-a_3}))dx$$

$$=2(a_4'-a_3)[(\frac{3a_4'+a_4-4a_3}{4(a_4'-a_3)})^2\ln\frac{3a_4'+a_4-4a_3}{4(a_4'-a_3)}-(\frac{a_4'-a_4}{4(a_4'-a_3)})^2\ln(\frac{a_4'-a_4}{4(a_4'-a_3)})+\frac{1}{2}(\frac{a_4'-a_4}{4(a_4'-a_3)})^2-\frac{1}{2}(\frac{3a_4'+a_4-4a_3}{4(a_4'-a_3)})^2+\frac{1}{2}].$$

Thus, by summing the above five parts we immediately obtain the credibility entropy $H(\tilde{\xi})$.

Remark 2. If $a_1'=a_1, a_4'=a_4$, then TIFV $\tilde{\xi}=(a_1',a_1,a_2,a_3,a_4,a_4')$ reduces to ordinary trapezoidal fuzzy variable $\tilde{\xi}=(a_1,a_2,a_3,a_4)$, and the credibility entropy formula (10) of TIFV $\tilde{\xi}$ reduces to

$$H(\tilde{\xi})=\frac{1}{2}(a_4-a_1)+(a_3-a_2)(\ln 2-\frac{1}{2}),$$

which is consistent with that of fuzzy variable obtained in [15]. However, the deduced entropy value of trapezoidal fuzzy variable in [22] is not in agreement with this entropy result.

Definition 10. [13,32] A triangular intuitionistic fuzzy variable (TrIFV) can be determined by a quintuple $\tilde{\zeta}=(a_1',a_1,a_2,a_3,a_3')$ with $-\infty<a_1'\le a_1\le a_2\le a_3\le a_3'<+\infty$, whose true possibility and false possibility function are defined as

$$\mu_{\tilde{\zeta}}(x)=\begin{cases}(x-a_1)/(a_2-a_1), if \quad a_1\le x\le a_2;\\ 1, if \quad x=a_2;\\ (a_3-x)/(a_3-a_2), if \quad a_2\le x\le a_3;\\ 0, otherwise.\end{cases} \quad \nu_{\tilde{\zeta}}(x)=\begin{cases}(a_2-x)/(a_2-a_1'), if \quad a_1'\le x\le a_2;\\ 0, if \quad x=a_2;\\ (x-a_2)/(a_3'-a_2), if \quad a_2\le x\le a_3';\\ 1, otherwise.\end{cases}$$

Theorem 7. Let $\tilde{\zeta}=(a_1',a_1,a_2,a_3,a_3')$ be a TrIFV taking values in $[a_1',a_3']$, the credibility entropy of its credibility distribution is

$$\begin{aligned}H(\tilde{\zeta})&=2(a_2-a_1')[(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2\ln\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')}-(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2\ln(\tfrac{a_1-a_1'}{4(a_2-a_1')})+\tfrac{1}{2}(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2-\tfrac{1}{2}(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2+\tfrac{1}{2}]\\&+\tfrac{2(a_2-a_1)(a_2-a_1')}{(2a_2-a_1-a_1')}\{(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2\ln\tfrac{a_1-a_1'}{4(a_2-a_1')}-(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2\ln\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')}-\tfrac{1}{2}(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2+\tfrac{1}{2}(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2\\&+\tfrac{2(a_3-a_2)((a_3'-a_2)}{a_3'+a_3-2a_2}[(\tfrac{a_3'-a_3}{4(a_3'-a_2)})^2\ln(\tfrac{a_3'-a_3}{4(a_3'-a_2)})-(\tfrac{3a_3'+a_3-4a_2}{4(a_3'-a_2)})^2\ln(\tfrac{3a_3'+a_3-4a_2}{4(a_3'-a_2)})-\tfrac{1}{2}(\tfrac{a_3'-a_3}{4(a_3'-a_2)})^2+\tfrac{1}{2}(\tfrac{3a_3'+a_3-4a_2}{4(a_3'-a_2)})^2]\\&+2(a_3'-a_2)[(\tfrac{3a_3'+a_3-4a_2}{4(a_3'-a_2)})^2\ln\tfrac{3a_3'+a_3-4a_2}{4(a_3'-a_2)}-(\tfrac{a_3'-a_3}{4(a_3'-a_2)})^2\ln(\tfrac{a_3'-a_3}{4(a_3'-a_2)})+\tfrac{1}{2}(\tfrac{a_3'-a_3}{4(a_3'-a_2)})^2-\tfrac{1}{2}(\tfrac{3a_3'+a_3-4a_2}{4(a_3'-a_2)})^2+\tfrac{1}{2}]. \quad (11)\end{aligned}$$

Proof. With Definition 6, we can calculate the credibility entropy of TrIFV $\tilde{\zeta}$ as

$$H(\tilde{\zeta})=\int_{a_1'}^{a_1}S(Cr(\tilde{\zeta}=x)dx+\int_{a_1}^{a_2}S(Cr(\tilde{\zeta}=x)dx+\int_{a_2}^{a_3}S(Cr(\tilde{\zeta}=x)dx+\int_{a_3}^{a_3'}S(Cr(\tilde{\zeta}=x)dx.$$

So, the credibility entropy of TrIFV $\tilde{\xi}$ contains the following four integrals.

(1) $\int_{a_1'}^{a_1}S(Cr(\tilde{\zeta}=x))dx$

$$=\int_{a_1'}^{a_1}S(\tfrac{1-\nu_{\tilde{\zeta}(x)}}{4})dx$$

$$=-\int_{a_1'}^{a_1}(\tfrac{1}{4}(1-\tfrac{a_2-x}{a_2-a_1'}))\ln(\tfrac{1}{4}(1-\tfrac{a_2-x}{a_2-a_1'}))dx-\int_{a_1'}^{a_1}(1-\tfrac{1}{4}(1-\tfrac{a_2-x}{a_2-a_1'}))\ln(1-\tfrac{1}{4}(1-\tfrac{a_2-x}{a_2-a_1'}))dx$$

$$=2(a_2-a_1')[(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2\ln\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')}-(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2\ln(\tfrac{a_1-a_1'}{4(a_2-a_1')})+\tfrac{1}{2}(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2-\tfrac{1}{2}(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2+\tfrac{1}{2}].$$

(2) $\int_{a_1}^{a_2}S(Cr(\tilde{\zeta}=x))dx$

$$=\int_{a_1}^{a_2}S(\tfrac{\mu_{\tilde{\zeta}(x)}+1-\nu_{\tilde{\zeta}(x)}}{4})dx$$

$$=-\int_{a_1}^{a_2}(\tfrac{1}{4}(\tfrac{x-a_1}{a_2-a_1}+1-\tfrac{a_2-x}{a_2-a_1'})\ln(\tfrac{1}{4}(\tfrac{x-a_1}{a_2-a_1}+1-\tfrac{a_2-x}{a_2-a_1'}))dx-\int_{a_1}^{a_2}(1-(\tfrac{1}{4}(\tfrac{x-a_1}{a_2-a_1}+1-\tfrac{a_2-x}{a_2-a_1'}))\ln(1-(\tfrac{1}{4}(\tfrac{x-a_1}{a_2-a_1}+1-\tfrac{a_2-x}{a_2-a_1'}))dx$$

$$=\tfrac{2(a_2-a_1)(a_2-a_1')}{(2a_2-a_1-a_1')}[(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2\ln\tfrac{a_1-a_1'}{4(a_2-a_1')}-(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2\ln\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')}-\tfrac{1}{2}(\tfrac{a_1-a_1'}{4(a_2-a_1')})^2+\tfrac{1}{2}(\tfrac{4a_2-a_1-3a_1'}{4(a_2-a_1')})^2].$$

(3) $\int_{a_2}^{a_3}S(Cr(\tilde{\zeta}=x))dx$

$$=\int_{a_2}^{a_3} S(\tfrac{1}{2}(\tfrac{\mu_{\tilde{\zeta}(x)}+1-v_{\tilde{\zeta}(x)}}{2}+1-\sup_{y\neq x}\tfrac{\mu_{\tilde{\zeta}}(y)+1-v_{\tilde{\zeta}}(y)}{2})dx$$

$$=\int_{a_2}^{a_3} S(\tfrac{\mu_{\tilde{\zeta}(x)}+1-v_{\tilde{\zeta}(x)}}{4})dx$$

$$=-\int_{a_2}^{a_3}(\tfrac{1}{4}(\tfrac{a_3-x}{a_3-a_2}+1-\tfrac{x-a_2}{a_3'-a_2})\ln(\tfrac{1}{4}(\tfrac{a_3-x}{a_3-a_2}+1-\tfrac{x-a_2}{a_3'-a_2}))dx-\int_{a_2}^{a_3}(1-\tfrac{1}{4}(\tfrac{a_3-x}{a_3-a_2}+1-\tfrac{x-a_2}{a_3'-a_2})\ln(1-\tfrac{1}{4}(\tfrac{a_3-x}{a_3-a_2}+1-\tfrac{x-a_2}{a_3'-a_2}))dx$$

$$=\tfrac{2(a_3-a_2)(a_3'-a_2)}{a_3'+a_3-2a_2}[(\tfrac{a_3'-a_3}{4(a_3'-a_2)})^2\ln(\tfrac{a_3'-a_3}{4(a_3'-a_2)})-(\tfrac{3a_3'+a_3-4a_2}{4(a_3'-a_2)})^2\ln(\tfrac{3a_3'+a_3-4a_2}{4(a_3'-a_2)})-\tfrac{1}{2}(\tfrac{a_3'-a_3}{4(a_3'-a_2)})^2+\tfrac{1}{2}(\tfrac{3a_3'+a_3-4a_2}{4(a_3'-a_2)})^2].$$

(4) $\int_{a_3}^{a_3'} S(Cr(\tilde{\zeta}=x))dx$

$$=\int_{a_3}^{a_3'} S(\tfrac{1-v_{\tilde{\zeta}(x)}}{4})dx$$

$$=-\int_{a_3}^{a_3'}(\tfrac{1}{4}(1-\tfrac{x-a_2}{a_3'-a_2}))\ln(\tfrac{1}{4}(1-\tfrac{x-a_2}{a_3'-a_2}))dx-\int_{a_3}^{a_3'}(1-\tfrac{1}{4}(1-\tfrac{x-a_2}{a_3'-a_2}))\ln(1-\tfrac{1}{4}(1-\tfrac{x-a_2}{a_3'-a_2}))dx$$

$$=2(a_3'-a_2)[(\tfrac{3a_3'+a_3-4a_2}{4(a_3'-a_2)})^2\ln\tfrac{3a_3'+a_3-4a_2}{4(a_3'-a_2)}-(\tfrac{a_3'-a_3}{4(a_3'-a_2)})^2\ln(\tfrac{a_3'-a_3}{4(a_3'-a_2)})+\tfrac{1}{2}(\tfrac{a_3'-a_3}{4(a_3'-a_2)})^2-\tfrac{1}{2}(\tfrac{3a_3'+a_3-4a_2}{4(a_3'-a_2)})^2+\tfrac{1}{2}].$$

Thus, by summing the above four parts of integrals we complete the proof.

Remark 3. If $a_1'=a_1, a_3'=a_3$, then TrIFV $\tilde{\zeta}=(a_1',a_1,a_2,a_3,a_3')$ reduces to ordinary triangular fuzzy variable $\tilde{\zeta}=(a_1,a_2,a_3)$, and the credibility entropy formula (11) of TrIFV $\tilde{\zeta}$ degenerates to

$$H(\tilde{\zeta})=\int_{a_1}^{a_2} S(Cr(\tilde{\zeta}=x)dx+\int_{a_2}^{a_3} S(Cr(\tilde{\zeta}=x)dx \quad =\tfrac{1}{2}[(a_3-a_1)],$$

which is in agreement with the credibility entropy value of triangular fuzzy variable stated in [15]. Therefore, from the above Remark 3 and formula (11) one can see that the presented entropy of credibility distribution of IFV is a generalization of that of fuzzy variable [15,16]. The credibility entropy of fuzzy variable in [15] is just the special case of that of IFV. So, the credibility entropy of IFV proposed in this paper is more powerful for dealing with uncertainty.

6. Comparison analysis for credibility entropy of intuitionistic fuzzy variable

In fact, intuitionistic fuzzy variables are also regarded as linguistic values and widely used to model humanistic systems such as artificial intelligence, pattern recognition, medical diagnosis and decision support system. The applications of entropy measure, cross-entropy measure and similarity measure of IFVs can be refereed to literatures [1,2,5,6,10,11,14,26,28]. In this section, two comparative examples are given to illustrate the applicability of the proposed credibility entropy and cross-entropy of IFVs.

Example 2. Let TIFV $\tilde{\xi}=(0.5,0.6,0.7,0.8,0.9,1.0)$ represent a linguistic value "big" in uncertain humanistic system. According to the multiplication operator of IFVs given in [30], we can calculate the power n of TIFV $\tilde{A}$ as $\tilde{\xi}^n=(0.5^n,0.6^n,0.7^n,0.8^n,0.9^n,1^n)$; $\forall n>0$.

By taking different values of $n=\tfrac{1}{2},2,3,4$, we can compute

$\tilde{\xi}^{1/2} = (0.7071, 0.7746, 0.8367, 0.8944, 0.9487, 1.0)$, which denotes "more or less big";

$\tilde{\xi}^{2} = (0.25, 0.36, 0.49, 0.64, 0.81, 1.0)$, which denotes "very big";

$\tilde{\xi}^{3} = (0.125, 0.216, 0.343, 0.512, 0.729, 1.0)$, which denotes "quite very big";

$\tilde{\xi}^{4} = (0.0625, 0.1296, 0.2401, 0.4096, 0.6561, 1.0)$, which denotes "very very big".

By employing the proposed credibility entropy formula (10), we obtain

$$H(\tilde{\xi}^{1/2}) = 0.1356 < H(\tilde{\xi}) = 0.2321 < H(\tilde{\xi}^{2}) = 0.3482 < H(\tilde{\xi}^{3}) = 0.4034 < H(\tilde{\xi}^{4}) = 0.4266,$$

which is consistent with our intuition. The smaller is n, the smaller is the credibility entropy of IFV.

$$\tilde{\xi}^{1/20} = (0.9659, 0.9748, 0.9823, 0.9889, 0.9947, 1.0); \qquad H(\tilde{\xi}^{1/20}) = 0.0157;$$

$$\tilde{\xi}^{1/100} = (0.9931, 0.9949, 0.9964, 0.9978, 0.9989, 1.0); \qquad H(\tilde{\xi}^{1/100}) = 0.0032.$$

And

$$\lim_{n\to 0} \tilde{\xi}^{n} = (0.5^n, 0.6^n, 0.7^n, 0.8^n, 0.9^n, 1^n) = (1,1,1,1,1,1), \qquad H(\lim_{n\to 0} \tilde{\xi}^{n}) = 0.$$

When n becomes zero, the IFV $\tilde{\xi}^{n}$ reduces to a crisp real number 1 with true possibility 1 and false possibility 0, its credibility entropy arrives at the minimum value 0, which is in accordance with the previous credibility entropy result of Property 2. It also complies with the information theoretical view that the uncertainty degree reaches minimum when the intuitionistic fuzzy variable always takes a unique value. And the credibility entropy of IFV becomes greater when the IFV takes more values with possibility 1.

On the other hand, by using the following ordinary intuitionistic fuzzy entropy formula of IFV $\tilde{A}$,

$$\mathrm{E}(\tilde{A}) = -\int_{-\infty}^{+\infty} [\tfrac{\mu_{\tilde{A}}(x)+1-\nu_{\tilde{A}}(x)}{2} \ln \tfrac{\mu_{\tilde{A}}(x)+1-\nu_{\tilde{A}}(x)}{2} + (1-\tfrac{\mu_{\tilde{A}}(x)+1-\nu_{\tilde{A}}(x)}{2}) \ln(1-\tfrac{\mu_{\tilde{A}}(x)+1-\nu_{\tilde{A}}(x)}{2})]dx,$$

we can also get

$$E(\tilde{\xi}^{1/2}) = 0.1058 < E(\tilde{\xi}) = 0.18 < E(\tilde{\xi}^{2}) = 0.27 < E(\tilde{\xi}^{3}) = 0.31825 < E(\tilde{\xi}^{4}) = 0.347.$$

Clearly, the ascending rank obtained by the proposed credibility entropy of intuitionistic fuzzy variable is also consistent with that of intuitionistic fuzzy entropy stated in [33]. However, the proposed credibility entropy formula in [22] can only deal with the kind of TIFV $\tilde{\xi} = (a, b, c, d; \varphi, \omega)$, it cannot handle the credibility entropy of this kind of trapezoidal intuitionistic fuzzy variable $\tilde{\xi} = (a', a, b, c, d, d') = (0.5, 0.6, 0.7, 0.8, 0.9, 1.0)$ with normalization condition.

Example 3. Let us consider a pattern recognition problem. Assume there are three given patterns which correspond to three classifications, respectively. The three known patterns are represented by the following triangular intuitionistic fuzzy variables which are adapted from example in [5].

$$\tilde{P}_1 = (0,2,4,5,6) = ((2,4,5),(0,4,6));$$

$$P_2 = (0,1,4,5,6) = ((1,4,5),(0,4,6));$$

$$P_3 = (0,1,3,4,6) = ((1,3,4),(0,3,6)).$$

Provided that we obtain an unknown pattern $Q = (0,1,3,5,6) = ((1,3,5),(0,3,6))$, where $(1,3,5)$ denotes the true fuzzy possibility function, $(0,3,6)$ denotes the false fuzzy possibility function. In order to determine the proper classification the unknown pattern $\tilde{Q}$ belongs to, we can utilize the proposed credibility cross-entropy formulas (8) and (9) to calculate the divergence measure between the unknown pattern $\tilde{Q}$ and each given pattern $\tilde{P}_i$ as below.

$$D^*(\tilde{Q},\tilde{P}_1) = D(\tilde{Q},\tilde{P}_1) + D(\tilde{P}_1,\tilde{Q}) = 0.097;$$

$$D^*(\tilde{Q},\tilde{P}_2) = D(\tilde{Q},\tilde{P}_2) + D(\tilde{P}_2,\tilde{Q}) = 0.0575;$$

$$D^*(\tilde{Q},\tilde{P}_3) = D(\tilde{Q},\tilde{P}_3) + D(\tilde{P}_3,\tilde{Q}) = 0.015.$$

Hence, $D^*(\tilde{Q},\tilde{P}_3) = \min_{1\le i\le 3} D^*(\tilde{Q},\tilde{P}_i)$, i.e., the discrimination information between pattern $\tilde{Q}$ and pattern $\tilde{P}_3$ is the smallest. So, the unknown pattern $\tilde{Q}$ should be assigned to classification $\tilde{P}_3$, which also coincides with our intuition.

For comparison, we employ Jaccard similarity, Dice similarity and Cosine similarity measure of IFNs introduced in [30] to compute the distance measures between unknown pattern $\tilde{Q}$ and each given pattern $\tilde{P}_i$ $(i = 1,2,3)$ by the following formulas.

$$D_{Jaccard}(\tilde{Q},\tilde{P}_i) = 1 - S_{Jaccard}(\tilde{Q},\tilde{P}_i),\quad D_{Dice}(\tilde{Q},\tilde{P}_i) = 1 - S_{Dice}(\tilde{Q},\tilde{P}_i),\quad D_{Cosine}(\tilde{Q},\tilde{P}_i) = 1 - S_{Cosine}(\tilde{Q},\tilde{P}_i).$$

The calculated distance measures and credibility cross-entropy results between the unknown pattern $\tilde{Q}$ and the given pattern $\tilde{P}_i$ $(i = 1,2,3)$ are displayed in the following Table 1.

Table 1. The ranking and recognition result based on cross-entropy and distance between $\tilde{Q}$ and $\tilde{P}_i$.

$\tilde{Q}$	$\tilde{P}_1$	$\tilde{P}_2$	$\tilde{P}_3$	Ranking	Recognition result
Cross-entropy	0.097	0.0575	0.015	$D^*(\tilde{Q},\tilde{P}_3) < D^*(\tilde{Q},\tilde{P}_2) < D^*(\tilde{Q},\tilde{P}_1)$	$\tilde{P}_3$
Jaccard distance	0.0333	0.0227	0.0132	$D_J(\tilde{Q},\tilde{P}_3) < D_J(\tilde{Q},\tilde{P}_2) < D_J(\tilde{Q},\tilde{P}_1)$	$\tilde{P}_3$
Dice distance	0.0169	0.0115	0.0066	$D_D(\tilde{Q},\tilde{P}_3) < D_D(\tilde{Q},\tilde{P}_2) < D_D(\tilde{Q},\tilde{P}_1)$	$\tilde{P}_3$
Cosine distance	0.0124	0.0083	0.0049	$D_C(\tilde{Q},\tilde{P}_3) < D_C(\tilde{Q},\tilde{P}_2) < D_C(\tilde{Q},\tilde{P}_1)$	$\tilde{P}_3$

From Table 1 one can see that no matter which distance is used, we can obtain $D(\tilde{Q},\tilde{P}_3) = \min_{1\le i\le 3} D(\tilde{Q},\tilde{P}_i)$, which shows that the distance between pattern $\tilde{Q}$ and pattern $\tilde{P}_3$ is the least. Thus, the unknown pattern $\tilde{Q}$ should also be classified to $\tilde{P}_3$.

By comparing the results in Table 1, one can see that the same ranking order and pattern recognition classification are obtained whether we use the proposed credibility cross-entropy or the distance measures. So, the proposed credibility cross-entropy is very efficient in dealing with the uncertain pattern recognition problem because the credibility cross-entropy not only can determine the discrimination information between two patterns, but also reflect the divergence degree or distance measure of two patterns.

To illustrate the advantage of the proposed credibility cross-entropy, we also display the following example. Suppose there is the fourth triangular intuitionistic fuzzy pattern $\tilde{P}_4$ as

$$\tilde{P}_4 = (0, 1.6849, 3, 4.5084, 5.4622) = ((1.6849, 3, 4.5084), (0, 3, 5.4622)) .$$

If we employ the above Jaccard, Dice and Cosine distance measures of IFVs, we can compute

$$D_{Jaccard}(\tilde{Q}, \tilde{P}_4) = D_{Jaccard}(\tilde{Q}, \tilde{P}_3) = 0.0132,$$

$$D_{Dice}(\tilde{Q}, \tilde{P}_4) = D_{Dice}(\tilde{Q}, \tilde{P}_3) = 0.0066,$$

$$D_{Cosine}(\tilde{Q}, \tilde{P}_4) = D_{Cosine}(\tilde{Q}, \tilde{P}_3) = 0.0049.$$

Obviously, it is difficult to assign $\tilde{Q}$ to pattern $\tilde{P}_3$ or $\tilde{P}_4$ by means of the above-mentioned distance measures. However, using our proposed cross-entropy formulas (8) and (9) we can get

$$D^*(\tilde{Q}, \tilde{P}_4) = D(\tilde{Q}, \tilde{P}_4) + D(\tilde{P}_4, \tilde{Q}) = 0.0135 + 0.01947 = 0.03297 > D^*(\tilde{Q}, \tilde{P}_3) = 0.015.$$

So, compared with $\tilde{P}_4$, $\tilde{P}_3$ is closer to $\tilde{Q}$, which is consistent with people's intuition. Then, we can identify the unknown pattern $\tilde{Q}$ as category $\tilde{P}_3$. Therefore, the advantage of our presented credibility cross-entropy is that it is more powerful and flexible to discriminate two IFVs than the previous distance measures $D_{Jaccard}, D_{Dice}, D_{\text{cosine}}$ stated in [30].

The disadvantage of the presented credibility cross-entropy is that the calculation is a litter more complicated. But our proposed credibility cross-entropy adds the meaning of intuitionistic fuzzy information entropy and can be used to measure the uncertain discrimination information between two intuitionistic fuzzy patterns. Thus, it can better reflect the difference between two IFVs than the traditional intuitionistic fuzzy distance formulas.

7. Conclusions

In this manuscript, a new kind of entropy based on credibility distribution of IFV is introduced and some important properties of the credibility entropy of IFV are investigated. The proposed credibility entropy is different from the existing entropy measures of intuitionisitc fuzzy set [23,33], which can better measure the information deficiency of IFV in the framework of credibility degree rather than intuitionistic fuzzy uncertainty. The presented credibility entropy of intuitionistic fuzzy variable can be employed to determine the rational weight of intuitionistic fuzzy attribute in the multi-attribute decision-making problem. Also, by extending the fuzzy cross-entropy formula [21] we present an enhanced version of credibility cross-entropy of IFVs, which is different from the existing cross-entropy of intuitionisitc fuzzy sets [25,10,28] and can better measure the divergence degree between two IFVs. To illustrate that our proposed credibility cross-entropy is well suited in pattern recognition problem, we borrow the intuitionisitc fuzzy patterns from [5]. The numerical example given in section 6 indicates that our proposed credibility cross-entropy is good in intuitionistic fuzzy pattern recognition problem. In some special intuitionistic fuzzy environment, one can see that the recognition methods based on the previous distance measures cannot identify the unknown pattern, but we can get the accurate identification result by using the proposed credibility cross-entropy. Thus, the presented credibility cross-entropy has more advantages in the application field of pattern recognition than the existing distance measures. In the future, we may extend the presented credibility entropy and cross-entropy to other domains such as intuitionistic fuzzy clustering, intuitionistic fuzzy group decision and intuitionistic fuzzy portfolio selection problems.

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Conflict of interest

There is no conflict of interest.

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