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Research article

The distribution of exterior transmission eigenvalues for spherically stratified media

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Abstract: The exterior transmission eigenvalues corresponding to spherical symmetry media and spherically symmetric eigenfunctions are considered. Under various coefficient conditions, we give the number and the asymptotic distribution (described by the subscript numbers) of these eigenvalues in the complex plane.

Keywords: inverse scattering; exterior transmission eigenvalues; number; distribution; spectral theory
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1. Introduction

The transmission eigenvalue problem appears in inverse scattering theory and has attracted wide attention recently (see, e.g., [1–16] and the references therein). It is non-self-adjoint and irregular in the Birkhoff (and even the Stone) sense [17]. According to the incident fields and the measuring positions, the transmission eigenvalue problem can be divided into the interior transmission eigenvalue problem and the exterior transmission eigenvalue problem. The subject matter of this paper is the latter problem which plays a central role in many important physical problems, such as the inverse scattering problems of determining the shape of underground reservoirs or nuclear reactors [18].

The exterior transmission eigenvalue problem for spherically stratified media in $\mathbb{R}^3$ (see [19, 20]) is to find functions $w, v \in C^2(\mathbb{R}^3\backslash\overline{D})$ such that

$$\Delta w + k^2 n(r) w = 0 \text{ in } \mathbb{R}^3\backslash\overline{D}, \tag{1.1}$$

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^3\backslash\overline{D}, \tag{1.2}$$

$$w = v \text{ on } \partial D, \tag{1.3}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \text{ on } \partial D, \tag{1.4}$$

$$\lim_{r\to\infty} r\left(\frac{\partial w}{\partial r} - ikw\right) = 0, \tag{1.5}$$

$$\lim_{r\to\infty} r\left(\frac{\partial v}{\partial r} - ikv\right) = 0, \tag{1.6}$$

where $r := |x|$, $x \in \mathbb{R}^3$, $D := \{x : |x| < a\}$, ν is the unit outward normal to ∂D, $n \in W_2^2(a, b)$, $n(r) > 0$ for $a < r < b$ and $n(r) = 1$ for $r > b$. Values of $k \in \mathbb{C}$ such that there exists a nontrivial solution to (1.1)–(1.6) are called *exterior transmission eigenvalues*.

In this paper, not all the exterior transmission eigenvalues but a subset consisting of those eigenvalues for which the corresponding eigenfunctions are spherically symmetric are considered. Let

$$w(r) = a_0\frac{y(r)}{r}, v(r) = b_0\frac{y_0(r)}{r},$$

then the relations (1.1)–(1.6) become the following ordinary differential equations

$$y'' + k^2 n(r)y = 0, \ r \in (a, \infty), \tag{1.7}$$

$$y_0'' + k^2 y_0 = 0, \ r \in (a, \infty), \tag{1.8}$$

$$a_0 y(a) = b_0 y_0(a), \tag{1.9}$$

$$a_0 y'(a) = b_0 y_0'(a), \tag{1.10}$$

and

$$y(r) = y_0(r) = e^{ikr}, \ r > b. \tag{1.11}$$

It is known that the transmission eigenvalues carry information about the refractive index and are used in sampling type methods for the reconstruction of the support of an inhomogeneous medium (for details, see [21, 22]). The interesting and important questions are the distribution of the transmission eigenvalues. For the interior transmission eigenvalue problems, Xu and his collaborators gave the asymptotics of non-real transmission eigenvalues in [23, 24]. For the exterior transmission eigenvalue problems, Chen [25] described a asymptotic eigenvalue density for each scattered angle. In [15], we showed that there are infinitely many non-real eigenvalues and gave the asymptotics of exterior transmission eigenvalues without the description of the subscript numbers. Recently, several intrinsic local and global geometric patterns of the transmission eigenfunctions have been revealed. The local geometric property was first discovered and investigated in [26, 27], and was further studied in [28–31] for different geometric and physical setups. The global geometric property was first discovered and investigated in [32], and was further studied in [33–35] for different geometric and physical setups.

In this paper, we continue to study the asymptotic behavior which is more accurate than that in [15]. Furthermore, the asymptotic behavior in this paper is described by the subscript numbers. The asymptotics of eigenvalues with the description of the subscript numbers is more difficult to investigate (due to the non-self-adjointness), but it has important applications in the inverse spectral problems, especially in the inverse spectral problem with the mixed spectral data and the stability of the inverse spectral problem [5, 14, 17, 36–38].

The paper is organized as follows. In Section 2, we study the asymptotics of the characteristic function and give the asymptotic solution of a transcendental equation. Then, we concern the counting results and the asymptotic behavior of exterior transmission eigenvalues for the case when $n(a) \neq 1$, $n(b) \neq 1$ in Section 3 and for the case when $n(a) = 1,\ n(b) \neq 1$ and the case when $n(a) \neq 1, n(b) = 1$ in Section 4.

2. Preliminaries

Firstly, we reduce the eigenvalue problem (1.7)–(1.11) to a problem of finding roots of a relative function, which is referred to as *the characteristic function* in the following. It appears in [15] and we give it without proof.

Lemma 2.1. *The exterior transmission eigenvalues coincide with the zeros of the function $D(k)$, where*

$$D(k) = ikY(a, k) - Y'(a, k), \tag{2.1}$$

and $Y(r, k)$ is the unique solution of the initial value problem

$$\begin{cases} Y'' + k^2 n(r)Y = 0,\ r \in (a, b), \\ Y(b, k) = 1, Y'(b, k) = ik. \end{cases} \tag{2.2}$$

In order to look for the properties of the characteristic function $D(k)$, we now give the behaviours of $Y(r, k)$ and $Y'(r, k)$. Make use of the modified Liouville transformation

$$\xi = \xi(r) := \int_r^b \sqrt{n(t)}\mathrm{d}t, \tag{2.3}$$

$$z(\xi) := n(r)^{\frac{1}{4}} Y(r, k), \tag{2.4}$$

and define the quantity

$$\gamma := \xi(a) = \int_a^b \sqrt{n(t)}\mathrm{d}t \tag{2.5}$$

which has a physical meaning as the travel time for a wave to move from $r = a$ to $r = b$ in the corresponding wave scattering problem. The problem (2.2) is transformed in the following form [15]

$$z'' + [k^2 - p(\xi)]z = 0,\ \xi \in (0, \gamma), \tag{2.6}$$

$$z(0) = n(b)^{\frac{1}{4}},\ z'(0) = -n(b)^{-\frac{1}{4}} \left[\frac{n'(b)}{4n(b)} + ik\right], \tag{2.7}$$

where

$$p(\xi) := \frac{1}{4}\frac{n''(r)}{n(r)^2} - \frac{5}{16}\frac{n'(r)^2}{n(r)^3}. \tag{2.8}$$

Let $z_1(\xi)$ and $z_2(\xi)$ be the solutions of (2.6) which satisfy $z_1(0) = 1,\ z_1'(0) = 0$ and $z_2(0) = 0$, $z_2'(0) = 1$. Then $z(\xi)$ can be represented in the form

$$z(\xi) = z(0)z_1(\xi) + z'(0)z_2(\xi)$$

$$= n(b)^{\frac{1}{4}} z_1(\xi) - n(b)^{-\frac{1}{4}} \left[\frac{n'(b)}{4n(b)} + ik\right] z_2(\xi). \tag{2.9}$$

By using (2.4) and (2.1), we calculate that

$$Y(a,k) = n(a)^{-\frac{1}{4}} z(\gamma) = \frac{1}{B} z_1(\gamma) - \frac{1}{A}\left(\frac{D}{4} + ik\right) z_2(\gamma), \tag{2.10}$$

$$\begin{aligned} Y'(a,k) &= -\frac{1}{4} n(a)^{-\frac{5}{4}} n'(a) z(\gamma) - n(a)^{\frac{1}{4}} z'(\gamma) \\ &= -\frac{C}{4B} z_1(\gamma) + \frac{C}{4A}\left(\frac{D}{4} + ik\right) z_2(\gamma) - A z_1'(\gamma) + B\left(\frac{D}{4} + ik\right) z_2'(\gamma), \end{aligned} \tag{2.11}$$

and then

$$\begin{aligned} D(k) = {} & \frac{1}{B}\left(ik + \frac{C}{4}\right) z_1(\gamma) - \frac{1}{A}\left(-k^2 + \frac{C+D}{4} ik + \frac{CD}{16}\right) z_2(\gamma) \\ & + A z_1'(\gamma) - B\left(ik + \frac{D}{4}\right) z_2'(\gamma), \end{aligned} \tag{2.12}$$

where

$$A = [n(a)n(b)]^{\frac{1}{4}}, B = \left[\frac{n(a)}{n(b)}\right]^{\frac{1}{4}}, C = \frac{n'(a)}{n(a)}, D = \frac{n'(b)}{n(b)}. \tag{2.13}$$

From the basic estimates in Chapter 1 of [39], we know that if $p \in L^2[0,\gamma]$,

$$\begin{aligned} z_1(\gamma) &= \cos(k\gamma) + O\left(\frac{e^{|\mathrm{Im}(k)|\gamma}}{k}\right), \\ z_2(\gamma) &= \frac{\sin(k\gamma)}{k} + O\left(\frac{e^{|\mathrm{Im}(k)|\gamma}}{k^2}\right), \end{aligned}$$

if $p \in W_2^1[0,\gamma]$,

$$z_1(\gamma) = \cos(k\gamma) + \frac{\sin(k\gamma)}{2k} Q + O\left(\frac{e^{|\mathrm{Im}(k)|\gamma}}{k^2}\right), \tag{2.14}$$

$$z_2(\xi) = \frac{\sin(k\gamma)}{k} - \frac{\cos(k\gamma)}{2k^2} Q + O\left(\frac{e^{|\mathrm{Im}(k)|\gamma}}{k^3}\right), \tag{2.15}$$

and if $p \in W_2^2[0,\gamma]$,

$$z_1(\gamma) = \cos(k\gamma) + \frac{\sin(k\gamma)}{2k} Q + \frac{\cos(k\gamma)}{4k^2}\left(p(\gamma) - p(0) - \frac{1}{2}Q^2\right) + O\left(\frac{e^{|\mathrm{Im}(k)|\gamma}}{k^3}\right), \tag{2.16}$$

$$z_2(\gamma) = \frac{\sin(k\gamma)}{k} - \frac{\cos(k\gamma)}{2k^2} Q + \frac{\sin(k\gamma)}{4k^3}\left(p(\gamma) + p(0) - \frac{1}{2}Q^2\right) + O\left(\frac{e^{|\mathrm{Im}(k)|\gamma}}{k^4}\right), \tag{2.17}$$

where

$$Q = \int_0^\gamma p(s)\mathrm{d}s.$$

A straightforward computation yields the following asymptotic expansions for $D(k)$,

$$\frac{D(k)}{k} = c_1 \sin k\gamma + c_2 i \cos k\gamma + O\left(\frac{e^{|\mathrm{Im}(k)|\gamma}}{k}\right), p \in L^2[0,\gamma], \tag{2.18}$$

$$\begin{aligned}\frac{D(k)}{k} &= c_1 \sin k\gamma + c_2 i \cos k\gamma + \frac{1}{k}\left(d_1 i \sin k\gamma + d_2 \cos k\gamma\right)\\ &+ O\left(\frac{e^{|\mathrm{Im}k|\gamma}}{k^2}\right), p \in W_2^1[0,\gamma],\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}\frac{D(k)}{k} &= c_1 \sin k\gamma + c_2 i \cos k\gamma + \frac{1}{k}\left(d_1 i \sin k\gamma + d_2 \cos k\gamma\right)\\ &+ \frac{1}{k^2}\left(e_1 \sin k\gamma + e_2 i \cos k\gamma\right) + O\left(\frac{e^{|\mathrm{Im}k|\gamma}}{k^3}\right), p \in W_2^2[0,\gamma],\end{aligned} \tag{2.20}$$

where

$$c_1 = \frac{1}{A} - A, \tag{2.21}$$

$$c_2 = \frac{1}{B} - B, \tag{2.22}$$

$$d_1 = \left(\frac{1}{B} - B\right)\frac{Q}{2} - \frac{C+D}{4A}, \tag{2.23}$$

$$d_2 = \left(A - \frac{1}{A}\right)\frac{Q}{2} + \frac{1}{4}\left(\frac{C}{B} - BD\right), \tag{2.24}$$

$$e_1 = \frac{p(\gamma)+p(0)}{4}\left(A + \frac{1}{A}\right) + \frac{Q^2}{8}\left(A - \frac{1}{A}\right) + \frac{Q}{8}\left(\frac{C}{B} - BD\right) - \frac{CD}{16A}, \tag{2.25}$$

$$e_2 = \frac{p(\gamma)-p(0)}{4}\left(\frac{1}{B} + B\right) + \frac{Q^2}{8}\left(B - \frac{1}{B}\right) + \frac{Q}{8A}\left(D + C\right). \tag{2.26}$$

Note that (2.20) can be rewritten as

$$\begin{aligned}\frac{D(k)}{k} &= i\left(\frac{c_1+c_2}{2}e^{-ik\gamma} - \frac{c_1-c_2}{2}e^{ik\gamma}\right) + \frac{1}{k}\left(\frac{d_1+d_2}{2}e^{ik\gamma} - \frac{d_1-d_2}{2}e^{-ik\gamma}\right)\\ &+ \frac{i}{k^2}\left(\frac{e_1+e_2}{2}e^{-ik\gamma} - \frac{e_1-e_2}{2}e^{ik\gamma}\right) + O\left(\frac{e^{|\mathrm{Im}k|\gamma}}{k^3}\right),\ k \to \infty.\end{aligned}$$

To get the asymptotics of the non-real exterior transmission eigenvalues, we introduce the following transcendental equation:

$$z - \kappa \ln z = w, \tag{2.27}$$

where κ is a constant in $\mathbb{C}$ and $\ln z = \ln|z| + i \arg z$ with $-\pi < \arg z \leq \pi$.

Lemma 2.2. *The transcendental Eq (2.27) has a unique solution*

$$z(w) = w + \kappa \ln w + \kappa^2 \frac{\ln w}{w} + O\left(\frac{\ln^2 |w|}{|w|^2}\right)$$

for any sufficiently large $|w|$.

Proof. The proof can be found in [24]. □

3. The distribution of the exterior transmission eigenvalues

Based on our previous results about the asymptotics of the exterior transmission eigenvalues [15], we continue to focus on more precise asymptotics with the description of the subscript numbers. When we count the exterior transmission eigenvalues, we always count the multiple zeros (if there exist) with their multiplicities. Below, the standard notation

$$\|p\| = \left(\int_0^\gamma |p(t)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}}$$

is used for the norm on $L^2[0,\gamma]$.

3.1. The case when $n(a) \neq 1$ and $n(b) \neq 1$

Define

$$g_1(k) := i\left(\frac{c_2 - c_1}{2} e^{ik\gamma} + \frac{c_1 + c_2}{2} e^{-ik\gamma}\right). \tag{3.1}$$

The conditions $n(a) \neq 1$ and $n(b) \neq 1$ imply that

$$\begin{aligned} c_1 + c_2 &= \left[n(a)^{-\frac{1}{4}} - n(a)^{\frac{1}{4}}\right]\left[n(b)^{-\frac{1}{4}} + n(b)^{\frac{1}{4}}\right] \\ &= [n(a)n(b)]^{-\frac{1}{4}}\left[1 - \sqrt{n(a)}\right]\left[1 + \sqrt{n(b)}\right] \neq 0, \\ c_1 - c_2 &= \left[n(a)^{-\frac{1}{4}} + n(a)^{\frac{1}{4}}\right]\left[n(b)^{-\frac{1}{4}} - n(b)^{\frac{1}{4}}\right] \\ &= [n(a)n(b)]^{-\frac{1}{4}}\left[1 + \sqrt{n(a)}\right]\left[1 - \sqrt{n(b)}\right] \neq 0, \end{aligned}$$

and

$$\begin{aligned} \frac{c_1 - c_2}{c_1 + c_2} &= \frac{\left[1 + \sqrt{n(a)}\right]\left[1 - \sqrt{n(b)}\right]}{\left[1 - \sqrt{n(a)}\right]\left[1 + \sqrt{n(b)}\right]} \\ &= \frac{n(b) - 1}{n(a) - 1}\left[\frac{\sqrt{n(a)} + 1}{\sqrt{n(b)} + 1}\right]^2. \end{aligned}$$

It follows that the function $g_1(k)$ has zeros in $\mathbb{C}$. Since $\frac{c_1 - c_2}{c_1 + c_2} > 0$ if $[n(a) - 1]\,[n(b) - 1] > 0$ and $\frac{c_1 - c_2}{c_1 + c_2} < 0$ if $[n(a) - 1]\,[n(b) - 1] < 0$, then the zeros of $g_1(k)$ are

$$k_j^0 = x_j^0 + iy^0,\ j \in \mathbb{Z}, \tag{3.2}$$

where

$$y^0 = \frac{1}{2\gamma} \ln\left|\frac{c_1 - c_2}{c_1 + c_2}\right|,$$

and

$$x_j^0 = \begin{cases} \frac{j\pi}{\gamma}, \text{ if } [n(a) - 1]\,[n(b) - 1] > 0, \\ \frac{j\pi}{\gamma} - \frac{\pi}{2\gamma}, \text{ if } [n(a) - 1]\,[n(b) - 1] < 0. \end{cases} \tag{3.3}$$

Note that as $k \to \infty$,

$$\frac{D(k)}{k} = g_1(k) + O\left(\frac{e^{|\mathrm{Im}(k)|\gamma}}{k}\right),\ \text{if } p \in L^2[a,b], \tag{3.4}$$

and

$$\frac{D(k)}{k} = g_1(k) + \frac{1}{k}(d_1 i \sin k\gamma + d_2 \cos k\gamma) + O\left(\frac{e^{|\mathrm{Im}k|\gamma}}{k^2}\right), \text{ if } p \in W_2^1[a,b]. \tag{3.5}$$

Lemma 3.1. *Suppose $n \in W_2^2(a,b)$ and $n(a) \neq 1$, $n(b) \neq 1$. Then*

$$\left|\frac{D(k)}{k} - g_1(k)\right| \leq \frac{K}{|k|} \exp\left(\|p\| \gamma^{\frac{3}{2}}\right) \exp\left(|\mathrm{Im}\, k| \gamma\right),$$

where A, B, C, D are defined as in (2.13), and

$$K = \frac{1}{\gamma}\left(A + B + \frac{1}{A} + \frac{1}{B}\right) + \frac{|C|}{4B} + \frac{|C+D|}{4A} + \frac{|CD|\gamma}{16A} + \frac{|BD|}{4}. \tag{3.6}$$

Proof. With the use of the Picard iteration method [39], we know that for $\xi \in [0, \gamma]$,

$$z_1(\xi) = \sum_{n=0}^{\infty} C_n(\xi, k, p), \tag{3.7}$$

$$z_2(\xi) = \sum_{n=0}^{\infty} S_n(\xi, k, p), \tag{3.8}$$

where

$$\begin{aligned} &C_0(\xi, k, p) = \cos k\xi, \\ &C_n(\xi, k, p) = \int_0^{\xi} \frac{\sin k(\xi - s)}{k} p(s) C_{n-1}(s) \mathrm{d}s, \ n = 1, 2, \ldots, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &S_0(\xi, k, p) = \frac{\sin k\xi}{k}, \\ &S_n(\xi, k, p) = \int_0^{\xi} \frac{\sin k(\xi - s)}{k} p(s) S_{n-1}(s) \mathrm{d}s, \ n = 1, 2, \ldots. \end{aligned} \tag{3.10}$$

Using (3.7) and (3.8) in (2.12) we obtain that

$$\begin{aligned} \frac{D(k)}{k} - g_1(k) = &\frac{i}{B} \sum_{n=1}^{\infty} C_n(\gamma, k, p) + \frac{k}{A} \sum_{n=1}^{\infty} S_n(\gamma, k, p) + \frac{A}{k} \sum_{n=1}^{\infty} C_n'(\gamma, k, p) \\ &- iB \sum_{n=1}^{\infty} S_n'(\gamma, k, p) + \frac{C}{4Bk} \sum_{n=0}^{\infty} C_n(\gamma, k, p) - \frac{C+D}{4A} i \sum_{n=0}^{\infty} S_n(\gamma, k, p) \\ &- \frac{CD}{16Ak} \sum_{n=0}^{\infty} S_n(\gamma, k, p) - \frac{BD}{4k} \sum_{n=0}^{\infty} S_n'(\gamma, k, p). \end{aligned} \tag{3.11}$$

Recall the elementary inequalities

$$|\cos k\gamma| = \frac{1}{2}\left|e^{ik\gamma} + e^{-ik\gamma}\right| \leq \exp\left(|\,\mathrm{Im}\, k\gamma|\right),$$

$$|\sin k\gamma| = \frac{1}{2}\left|e^{ik\gamma} - e^{-ik\gamma}\right| \leq \exp\left(|\operatorname{Im} k\gamma|\right),$$

$$\left|\frac{\sin k\gamma}{k}\right| = \left|\int_0^\gamma \cos kt\mathrm{d}t\right| \leq \int_0^\gamma \exp\left(|\operatorname{Im} kt|\right)\mathrm{d}t \leq \gamma \exp\left(|\operatorname{Im} k\gamma|\right), \tag{3.12}$$

and

$$\left|\frac{\sin k\gamma}{k}\right| \leq \frac{\exp\left(|\operatorname{Im} k\gamma|\right)}{|k|}. \tag{3.13}$$

Then the terms on the right-hand side of (3.11) can be majorized as follows,

$$\begin{aligned}
&\left|\frac{i}{B}\sum_{n=1}^{\infty} C_n(\gamma, k, p)\right| \\
&\leq \frac{1}{B}\sum_{n=1}^{\infty}\int_{0\leq t_1\leq\cdots\leq t_{n+1}=\gamma} |\cos kt_1| \prod_{j=1}^{n-1}\left|\frac{\sin k(t_{j+1}-t_j)}{k}p(t_j)\right|\left|\frac{\sin k(t_{n+1}-t_n)}{k}p(t_n)\right|\mathrm{d}t_1\cdots\mathrm{d}t_n \\
&\leq \frac{1}{B}\sum_{n=1}^{\infty}\frac{\exp\left(|\operatorname{Im} k|\,\gamma\right)\gamma^{n-1}}{|k|}\int_{0\leq t_1\leq\cdots\leq t_{n+1}=\gamma}\prod_{j=1}^{n}|p(t_j)|\mathrm{d}t_1\cdots\mathrm{d}t_n.
\end{aligned}$$

The value of the integral in the last line does not change under permutation of $t_1, \ldots, t_n$, and the union of all the permuted regions of integration is $[0, \gamma]^n$. It follows that

$$\begin{aligned}
\int_{0\leq t_1\leq\cdots\leq t_{n+1}=\gamma}\prod_{j=1}^{n}\left|p(t_j)\right|\mathrm{d}t_1\cdots\mathrm{d}t_n &= \frac{1}{n!}\int_{[0,\gamma]^n}\prod_{j=1}^{n}\left|p(t_j)\right|\mathrm{d}t_1\cdots\mathrm{d}t_n \\
&= \frac{1}{n!}\left[\int_0^\gamma |p(t)|\,\mathrm{d}t\right]^n \leq \frac{1}{n!}\left(\|p\|\,\sqrt{\gamma}\right)^n
\end{aligned}$$

by the Schwarz inequality. Thus,

$$\begin{aligned}
\left|\frac{i}{B}\sum_{n=1}^{\infty} C_n(\gamma, k, p)\right| &\leq \frac{1}{B}\frac{\exp\left(|\operatorname{Im} k|\,\gamma\right)}{|k|\gamma}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\|p\|\,\gamma^{\frac{3}{2}}\right)^n \\
&< \frac{1}{B\gamma|k|}\exp\left(|\operatorname{Im} k|\,\gamma + \|p\|\,\gamma^{\frac{3}{2}}\right).
\end{aligned}$$

The same reasoning applies to the other terms and yields, for $k \in \mathbb{C}$, $p \in L^2[0, \gamma]$,

$$\left|\frac{k}{A}\sum_{n=1}^{\infty} S_n(\gamma, k, p)\right| < \frac{1}{A\gamma|k|}\exp\left(|\operatorname{Im} k|\,\gamma + \|p\|\,\gamma^{\frac{3}{2}}\right),$$

$$\left|\frac{A}{k}\sum_{n=1}^{\infty} C_n'(\gamma, k, p)\right| < \frac{A}{\gamma\,|k|}\exp\left(|\operatorname{Im} k|\,\gamma + \|p\|\,\gamma^{\frac{3}{2}}\right),$$

$$\left|iB\sum_{n=1}^{\infty} S_n'(\gamma, k, p)\right| < \frac{B}{|k|\gamma}\exp\left(|\operatorname{Im} k\gamma| + \|p\|\,\gamma^{\frac{3}{2}}\right),$$

$$\left|\frac{C}{4Bk}\sum_{n=0}^{\infty}C_n(\gamma,k,p)\right| \le \frac{|C|}{4B\,|k|}\exp\left(|\operatorname{Im}k|\,\gamma+\|p\|\,\gamma^{\frac{3}{2}}\right),$$

$$\left|\frac{C+D}{4A}i\sum_{n=0}^{\infty}S_n(\gamma,k,p)\right| \le \frac{|C+D|}{4A\,|k|}\exp\left(|\operatorname{Im}k|\,\gamma+\|p\|\,\gamma^{\frac{3}{2}}\right),$$

$$\left|\frac{CD}{16Ak}\sum_{n=0}^{\infty}S_n(\gamma,k,p)\right| \le \frac{|CD|\,\gamma}{16A\,|k|}\exp\left(|\operatorname{Im}k|\,\gamma+\|p\|\,\gamma^{\frac{3}{2}}\right),$$

$$\left|\frac{BD}{4k}\sum_{n=0}^{\infty}S'_n(\gamma,k,p)\right| \le \frac{|BD|}{4\,|k|}\exp\left(|\operatorname{Im}k|\,\gamma+\|p\|\,\gamma^{\frac{3}{2}}\right).$$

Therefore,

$$\left|\frac{D(k)}{k}-g_1(k)\right| \le \left(\frac{A}{\gamma}+\frac{B}{\gamma}+\frac{1}{A\gamma}+\frac{1}{B\gamma}+\frac{|C|}{4B}+\frac{|C+D|}{4A}+\frac{|CD|\,\gamma}{16A}+\frac{|BD|}{4}\right) \times\frac{\exp\left(|\operatorname{Im}k|\,\gamma+\|p\|\,\gamma^{\frac{3}{2}}\right)}{|k|}.$$

This completes the proof. □

The following lemma can be obtained for arbitrary sine-type functions (see Lemma 1 on page 163 in [40]). In order to facilitate the readers, we show the proof.

Lemma 3.2. *Suppose $n \in W_2^2(a,b)$ and $n(a) \neq 1$, $n(b) \neq 1$. Let k_j^0, $j \in \mathbb{Z}$, be the zeros of $g_1(k)$ defined in (3.2). For any $\varepsilon > 0$, if $|k-k_j^0| \ge \varepsilon$ for all integers j, then there exists a number $M_\varepsilon > 0$, such that*

$$|g_1(k)| > M_\varepsilon \exp|\operatorname{Im}k\gamma|. \tag{3.14}$$

Proof. We only prove the inequality (3.14) in the case when $[n(a)-1]\,[n(b)-1] > 0$, i.e., $\frac{c_1-c_2}{c_1+c_2} > 0$, the proof in the case when $[n(a)-1]\,[n(b)-1] < 0$ can be completed similarly.

Denote $G_1(k) := |g_1(k)|^2 \exp(-2|\operatorname{Im}k|\gamma)$. Let $k = x+iy$. By a direct calculation, we have

$$|g_1(k)|^2 = \frac{(c_1-c_2)^2}{4}e^{-2\gamma y}\left[1+\left(\frac{c_1+c_2}{c_1-c_2}\right)^2 e^{4\gamma y}-2\frac{c_1+c_2}{c_1-c_2}e^{2\gamma y}\cos 2\gamma x\right]. \tag{3.15}$$

Then if $y \ge 0$,

$$G_1(k) = \frac{(c_1+c_2)^2}{4}\left[1+\left(\frac{c_1-c_2}{c_1+c_2}\right)^2 e^{-4\gamma y}-2\frac{c_1-c_2}{c_1+c_2}e^{-2\gamma y}\cos 2\gamma x\right], \tag{3.16}$$

and if $y \le 0$,

$$G_1(k) = \frac{(c_1-c_2)^2}{4}\left[1+\left(\frac{c_1+c_2}{c_1-c_2}\right)^2 e^{4\gamma y}-2\frac{c_1+c_2}{c_1-c_2}e^{2\gamma y}\cos 2\gamma x\right]. \tag{3.17}$$

By use of the periodicity of the cosine function, we just need to prove that $G_1(k)$ has a positive lower bound when $k \in D_\varepsilon$, where

$$D_\varepsilon := \left\{k = x+iy : x \in \left[-\frac{\pi}{2\gamma},\frac{\pi}{2\gamma}\right], |k-iy^0| \ge \varepsilon\right\}. \tag{3.18}$$

Choose $\delta_0 \in (0, \varepsilon)$ such that

$$1 + e^{-4\gamma\delta_0} - 2\cos 2\gamma \sqrt{\varepsilon^2 - \delta_0^2} > 0. \tag{3.19}$$

Firstly, let us consider the case when $0 < \left|\frac{c_1 - c_2}{c_1 + c_2}\right| < 1$, i.e., $y^0 < 0$.

(a) For $y \geq 0$, we get $e^{-2y\gamma} \leq 1$ and then

$$\begin{aligned} G_1(k) &\geq \frac{(c_1 + c_2)^2}{4}\left(1 - \left|\frac{c_1 - c_2}{c_1 + c_2}\right| e^{-2\gamma y}\right)^2 \geq \frac{(c_1 + c_2)^2}{4}\left(1 - \left|\frac{c_1 - c_2}{c_1 + c_2}\right|\right)^2 \\ &= \frac{1}{4}\left(|c_1 + c_2| - |c_1 - c_2|\right)^2 > 0. \end{aligned} \tag{3.20}$$

(b) For $y \in (-\infty, y^0 - \delta_0]$, we have $e^{2\gamma y} \leq e^{2\gamma y^0} e^{-2\gamma\delta_0} = \left|\frac{c_1 - c_2}{c_1 + c_2}\right| e^{-2\gamma\delta_0}$ and

$$G_1(k) \geq \frac{(c_1 - c_2)^2}{4}\left(1 - \left|\frac{c_1 + c_2}{c_1 - c_2}\right| e^{2\gamma y}\right)^2 \geq \frac{(c_1 - c_2)^2}{4}\left(1 - e^{-2\gamma\delta_0}\right)^2 > 0. \tag{3.21}$$

(c) For $y \in (y^0 - \delta_0, y^0]$, i.e., $\left|\frac{c_1 - c_2}{c_1 + c_2}\right| e^{-2\gamma\delta_0} < e^{2\gamma y} \leq \left|\frac{c_1 - c_2}{c_1 + c_2}\right|$, the assumption $|k - iy^0| \geq \varepsilon$ implies that

$$|x| \geq \sqrt{\varepsilon^2 - (y - y^0)^2} > \sqrt{\varepsilon^2 - \delta_0^2}.$$

Hence

$$G_1(k) > \frac{(c_1 - c_2)^2}{4}\left(1 + e^{-4\gamma\delta_0} - 2\cos 2\gamma \sqrt{\varepsilon^2 - \delta_0^2}\right) > 0. \tag{3.22}$$

(d) We now consider the remaining case $y \in \left(y^0, 0\right)$. If $\delta_0 < -y^0$, then $\left(y^0, 0\right) = (y^0, y^0 + \delta_0) \cup [y^0 + \delta_0, 0)$. For $y \in (y^0, y^0 + \delta_0) \subset \left(y^0, 0\right)$, we get $\left|\frac{c_1 - c_2}{c_1 + c_2}\right| < e^{2\gamma y} < 1$. Then

$$\begin{aligned} G_1(k) &> \frac{(c_1 - c_2)^2}{2}\left(1 - \left|\frac{c_1 + c_2}{c_1 - c_2}\right| \cos 2\gamma \sqrt{\varepsilon^2 - \delta_0^2}\right) \\ &> \frac{(c_1 - c_2)^2}{2}\left(1 - \cos 2\gamma \sqrt{\varepsilon^2 - \delta_0^2}\right) > 0. \end{aligned} \tag{3.23}$$

For $y \in [y^0 + \delta_0, 0)$, we obtain that $\left|\frac{c_1 - c_2}{c_1 + c_2}\right| e^{2\gamma\delta_0} \leq e^{2\gamma y} < 1$, and then

$$G_1(k) \geq \frac{(c_1 - c_2)^2}{4}\left(1 - e^{2\gamma\delta_0}\right)^2 > 0. \tag{3.24}$$

If $\delta_0 \geq -y^0$,then $y \in \left(y^0, 0\right) \subset (y^0, y^0 + \delta_0)$, $\left|\frac{c_1 - c_2}{c_1 + c_2}\right| < e^{2\gamma y} < 1$, and

$$\begin{aligned} G_1(k) &> \frac{(c_1 - c_2)^2}{2}\left(1 - \left|\frac{c_1 + c_2}{c_1 - c_2}\right| \cos 2\gamma \sqrt{\varepsilon^2 - \delta_0^2}\right) \\ &> \frac{(c_1 - c_2)^2}{2}\left(1 - \cos 2\gamma \sqrt{\varepsilon^2 - \delta_0^2}\right) > 0. \end{aligned} \tag{3.25}$$

Then, using the similar steps above, we can estimate the lower bound of $G_1(k)$ in the case when $\left|\frac{c_1-c_2}{c_1+c_2}\right| > 1$. More specifically,

$$
\begin{aligned}
&G_1(k) \geq \frac{1}{4}\left(|c_1-c_2|-|c_1+c_2|\right)^2 > 0,\ y\in(-\infty,0],\\
&G_1(k) > \frac{(c_1+c_2)^2}{4}\left(1+e^{-4\gamma\delta_0}-2\cos 2\gamma\sqrt{\varepsilon^2-\delta_0^2}\right) > 0, y\in[y^0, y^0+\delta_0),\\
&G_1(k) \geq \frac{(c_1+c_2)^2}{4}\left(1-e^{-2\gamma\delta_0}\right)^2 > 0,\ y\in[y^0+\delta_0,+\infty),
\end{aligned}
$$

and for the interval $(0, y^0)$,

$$
\begin{aligned}
&G_1(k) \geq \frac{(c_1+c_2)^2}{4}\left(1-e^{2\gamma\delta_0}\right)^2 > 0, y\in(0, y^0-\delta_0],\\
&G_1(k) > \frac{(c_1+c_2)^2}{2}\left(1-\cos 2\gamma\sqrt{\varepsilon^2-\delta_0^2}\right) > 0, y\in(y^0-\delta_0, y^0),
\end{aligned}
$$

or

$$
G_1(k) > \frac{(c_1+c_2)^2}{2}\left(1-\cos 2\gamma\sqrt{\varepsilon^2-\delta_0^2}\right) > 0, y\in(0, y^0).
$$

Lastly, for $\left|\frac{c_1-c_2}{c_1+c_2}\right| = 1$, we similarly conclude that

$$
\begin{aligned}
&G_1(k) \geq \frac{(c_1-c_2)^2}{4}\left(1-e^{-2\gamma\delta_0}\right)^2 > 0,\ y\in(-\infty,-\delta_0],\\
&G_1(k) \geq \frac{(c_1-c_2)^2}{4}\left(1+e^{-4\gamma\delta_0}-2\cos 2\gamma\sqrt{\varepsilon^2-\delta_0^2}\right) > 0, y\in(-\delta_0, 0),\\
&G_1(k) \geq \frac{(c_1+c_2)^2}{4}\left(1+e^{-4\gamma\delta_0}-2\cos 2\gamma\sqrt{\varepsilon^2-\delta_0^2}\right) > 0, y\in[0, \delta_0),\\
&G_1(k) \geq \frac{(c_1+c_2)^2}{4}\left(1-e^{-2\gamma\delta_0}\right)^2 > 0, y > [\delta_0, +\infty).
\end{aligned}
$$

To sum up, there exists a number $M_\varepsilon > 0$ such that $G_1(k) > M_\varepsilon^2$ for $k \in D_\varepsilon$, i.e.,

$$
|g_1(k)| > M_\varepsilon \exp\left(|\operatorname{Im} k|\gamma\right),\ \text{for } |k-k_j^0| \geq \varepsilon.
$$

This completes the proof. □

Remark 3.1. *If $\varepsilon = \frac{\pi}{4\gamma}$, we can take $\delta_0 = \frac{\pi}{8\gamma}$ to make the condition (3.19) true.*

In this case, $\left|1-e^{2\gamma\delta_0}\right| > \left|1-e^{-2\gamma\delta_0}\right| > \frac{1}{2}$, $\sqrt{1+e^{-4\gamma\delta_0}-2\cos 2\gamma\sqrt{\varepsilon^2-\delta_0^2}} > \frac{1}{2}$, and $\sqrt{1-\cos 2\gamma\sqrt{\varepsilon^2-\delta_0^2}} > \frac{1}{2}$. Then M_ε can be taken as follows:

$$
M = \begin{cases} \min\left\{|c_1|, |c_2|, \left|\frac{c_1-c_2}{4}\right|, \left|\frac{c_1+c_2}{4}\right|\right\}, & \text{if } \ |c_1-c_2| \neq |c_1+c_2|, \\ \frac{|c_1|}{4}, & \text{if } \ c_1-c_2 = c_1+c_2, \\ \frac{|c_2|}{4}, & \text{if } \ c_2-c_1 = c_1+c_2. \end{cases} \tag{3.26}
$$

For $j \in \mathbb{N}^+$, consider the rectangle contour $\Gamma_j^{(1)} = I_{1,j}^{(1)} \cup I_{2,j}^{(1)} \cup I_{3,j}^{(1)} \cup I_{4,j}^{(1)}$, where

$$I_{1,j}^{(1)} := \left\{ k : x = \frac{j\pi + \frac{\pi}{4}}{\gamma} \geq |y| \right\}, I_{2,j}^{(1)} := \left\{ k : |x| \leq \frac{j\pi + \frac{\pi}{4}}{\gamma} = y \right\},$$
$$I_{3,j}^{(1)} := \left\{ k : -x = \frac{j\pi + \frac{\pi}{4}}{\gamma} \geq |y| \right\}, I_{4,j}^{(1)} := \left\{ k : |x| \leq \frac{j\pi + \frac{\pi}{4}}{\gamma} = -y \right\}.$$

Fix $j > \frac{1}{2\pi} \left| \ln \left| \frac{c_1 - c_2}{c_1 + c_2} \right| \right|$, then for all $k \in \Gamma_j^{(1)}$, we have that $\left| k - k_j^0 \right| \geq \frac{\pi}{4\gamma}$, $j \in \mathbb{Z}$. By Lemma 3.2 and Remark 3.1, we get that, if $k \in \Gamma_j^{(1)}$, $j > \frac{1}{2\pi} \left| \ln \left| \frac{c_1 - c_2}{c_1 + c_2} \right| \right|$, then

$$|g_1(k)| > M \exp\left(|\operatorname{Im} k\gamma|\right). \tag{3.27}$$

Theorem 3.1. *Suppose $n \in W_2^2(a, b)$, $n(a) \neq 1$, $n(b) \neq 1$. The values γ, K, M, c_1, c_2 appear in (2.5), (3.6), (2.21), (2.22) and (3.26) respectively. Let*

$$N > \max \left\{ \frac{K\gamma \exp\left(\|p\| \gamma^{\frac{3}{2}}\right)}{M\pi}, \frac{1}{2\pi} \left| \ln \left| \frac{c_1 - c_2}{c_1 + c_2} \right| \right| \right\} \tag{3.28}$$

be an integer. Then, if $[n(a) - 1][n(b) - 1] > 0$, there are exactly $2N + 2$ exterior transmission eigenvalues, counted with multiplicities, inside $\Gamma_N^{(1)}$, if $[n(a) - 1][n(b) - 1] < 0$, there are exactly $2N + 1$ exterior transmission eigenvalues, counted with multiplicities, inside $\Gamma_N^{(1)}$, and for each $j > N$, exactly one simple root in the circular region

$$\left| k - k_j^0 \right| < \frac{\pi}{4\gamma},$$

where k_j^0 are defined as in (3.2). There are no other roots.

Proof. Fix N be an integer that satisfies (3.28), and let $L \geq N$ be an integer. Consider the contours $\Gamma_L^{(1)}$ and the contours

$$\left| k - k_j^0 \right| = \frac{\pi}{4\gamma}, j > N.$$

By Lemma 3.2 and Remark 3.1, the estimate $|g_1(k)| > M \exp\left(|\operatorname{Im} k\gamma|\right)$ holds on all of them. Therefore, by the estimate for $D(k)$ in Lemma 3.1,

$$\begin{aligned}
\left| \frac{D(k)}{k} - g_1(k) \right| &\leq \frac{K}{|k|} \exp\left(\|p\| \gamma^{\frac{3}{2}}\right) \exp\left(|\operatorname{Im} k| \gamma\right) \\
&< \frac{1}{|k|} \frac{K \exp\left(\|p\| \gamma^{\frac{3}{2}}\right)}{M} |g_1(k)| \\
&< \frac{N\pi}{|k\gamma|} |g_1(k)| \\
&< |g_1(k)|
\end{aligned}$$

also holds on them since $|k\gamma| \geq N\pi + \frac{\pi}{4}$ on them. Hence, by Rouché's theorem, $D(k)$ has as many roots, counted with multiplicities, as $kg_1(k)$ in each of the bounded regions. Since $g_1(k)$ has only the simple roots

$$k_j^0 = \begin{cases} \frac{j\pi}{\gamma} + \frac{i}{2\gamma} \ln \left| \frac{c_1 - c_2}{c_1 + c_2} \right|, & \text{if } [n(a) - 1][n(b) - 1] > 0, \\ \frac{j - \frac{1}{2}}{\gamma}\pi + \frac{i}{2\gamma} \ln \left| \frac{c_1 - c_2}{c_1 + c_2} \right|, & \text{if } [n(a) - 1][n(b) - 1] < 0, \end{cases}$$

and $L \geq N$ can be chosen arbitrarily large, the theorem follows. □

Since $[g_1(k)]^* = -g_1(-k^*)$, where k^* denotes the conjugate of k, then the distribution of the zeros of $g_1(k)$ is symmetrical with respect to the imaginary axes. Denote the zeros of $D(k)$ by $\left\{k_j\right\}_{j\geq 0} \cup \left\{-k_j^*\right\}_{j\geq 0}$ if $[n(a)-1]\,[n(b)-1] > 0$; by $\left\{k_j\right\}_{n\geq 0} \cup \left\{-k_j^*\right\}_{n\geq 1}$ if $[n(a)-1]\,[n(b)-1] < 0$. Moreover, it follows from Lemma 3.2 that

$$k_j = k_j^0 + \epsilon_j,\ \epsilon_j = o(1),\ j \to \infty. \tag{3.29}$$

Let us estimate ϵ_j, $j \geq 0$. Substituting (3.29) into (2.18), by a direct calculation, we have

$$0 = \frac{-2i}{c_2 - c_1}\frac{D(k_j)}{k_j e^{ik_j\gamma}} = \left(1 + \frac{c_1 + c_2}{c_2 - c_1}e^{-2ik_j\gamma}\right) + \alpha_j,$$

where $\alpha_j = O\left(\frac{1}{j}\right)$. Since $e^{-2ik_j\gamma} = \frac{c_1 - c_2}{c_1 + c_2}e^{-2i\epsilon_j\gamma}$, we have that $i\sin 2\epsilon_j\gamma + \alpha_j = 0$, and hence $\epsilon_j = O\left(\frac{1}{j}\right)$.

If $p \in W_2^1[0,\gamma]$, then we can obtain the more accurate expression of ϵ_j. Indeed, substituting (3.29) into (2.19), we obtain

$$\begin{aligned} 0 &= \frac{-2i}{c_2 - c_1}\frac{D(k_j)}{k_j e^{ik_j\gamma}} \\ &= \left(1 + \frac{c_1 + c_2}{c_2 - c_1}e^{-2ik_j\gamma}\right) - \frac{i}{k_j}\left(\frac{d_1 + d_2}{c_2 - c_1} + \frac{d_2 - d_1}{c_2 - c_1}e^{-2ik_j\gamma}\right) + \frac{\beta_j}{k_j}, \end{aligned}$$

and then

$$e^{-2i\epsilon_j\gamma} - 1 = \frac{i}{k_j^0 + \epsilon_j}\left(\frac{d_1 + d_2}{c_2 - c_1} + \frac{d_1 - d_2}{c_1 + c_2}e^{-2i\epsilon_j\gamma}\right) + \frac{\beta_j}{j},$$

where $\beta_j = O\left(\frac{1}{j}\right)$.

By use of the expansions

$$e^{-2i\epsilon_j\gamma} = 1 - 2i\epsilon_j\gamma + O\left(\frac{1}{j^2}\right),\ \frac{1}{k_j^0 + \epsilon_j} = \frac{\gamma}{j\pi}\left[1 + O\left(\frac{1}{j}\right)\right],$$

we get that

$$\epsilon_j = \frac{1}{2j\pi}\left(\frac{d_1 + d_2}{c_2 - c_1} + \frac{d_1 - d_2}{c_1 + c_2}\right) + O\left(\frac{1}{j^2}\right).$$

Then, under the conditions $n(a) \neq 1, n(b) \neq 1$, the exterior transmission eigenvalues can be numbered and estimated as follows.

Theorem 3.2. *Assume $n \in W_2^2(a,b)$, $n(a) \neq 1, n(b) \neq 1$. Denote the exterior transmission eigenvalues by $\left\{k_j\right\}_{j\geq 0} \cup \left\{-k_j^*\right\}_{j\geq 0}$ if $[n(a)-1]\,[n(b)-1] > 0$; by $\left\{k_j\right\}_{j\geq 0} \cup \left\{-k_j^*\right\}_{j\geq 1}$ if $[n(a)-1]\,[n(b)-1] < 0$. Then the sequence $\left\{k_j\right\}_{j\geq 0}$ has the following asymptotics:*

$$k_j = k_j^0 + O\left(\frac{1}{j}\right), j \to \infty,$$

where k_j^0, $j \in \mathbb{Z}$, is the sequence defined as in (3.2). If we further assume $p \in W_2^1[0,\gamma]$, then the sequence $\left\{k_j\right\}_{j\geq 0}$ has the following asymptotics:

$$k_j = k_j^0 + \frac{1}{2j\pi}\left(\frac{d_1 + d_2}{c_2 - c_1} + \frac{d_1 - d_2}{c_1 + c_2}\right) + O\left(\frac{1}{j^2}\right), j \to \infty,$$

where the numbers c_1, c_2, d_1, d_2 appear in (2.21)–(2.24).

3.2. *The case when $n(a) = 1, n(b) \neq 1$ or $n(a) \neq 1, n(b) = 1$*

In this subsection, we first study the distribution of zeros of the function $g_2(k)$ defined in (3.30) below. And then, we get a counting lemma and the asymptotics of the exterior transmission eigenvalues. Note that $c_1 + c_2 = 0, c_1 - c_2 \neq 0$ if $n(a) = 1, n(b) \neq 1$, and $c_1 + c_2 \neq 0, c_1 - c_2 = 0$ if $n(a) \neq 1, n(b) = 1$. In both cases, the function $g_1(k)$ has no zeros at all.

If $n(a) = 1, n(b) \neq 1$, then

$$\frac{d_1 - d_2}{c_1 - c_2} = \frac{1}{8}\frac{n'(a)}{n(b) - 1}\left[1 + \sqrt{n(b)}\right]^2.$$

Define

$$g_2(k) := i\frac{c_2 - c_1}{2}e^{ik\gamma} + \frac{1}{k}\left(\frac{d_1 + d_2}{2}e^{ik\gamma} + \frac{d_2 - d_1}{2}e^{-ik\gamma}\right). \tag{3.30}$$

Then as $k \to \infty$, if $p \in W_2^1[0, \gamma]$,

$$D(k) = kg_2(k) + O\left(\frac{e^{|\mathrm{Im}k|\gamma}}{k}\right), \tag{3.31}$$

and if $p \in W_2^1[0, \gamma]$,

$$D(k) = kg_2(k) + \frac{i}{k}\left(\frac{e_2 - e_1}{2}e^{ik\gamma} + \frac{e_1 + e_2}{2}e^{-ik\gamma}\right) + O\left(\frac{e^{|\mathrm{Im}k|\gamma}}{k^2}\right). \tag{3.32}$$

Since the function $g_2(k)$ has only one zero $k = -\frac{d_1+d_2}{c_1-c_2}i$ if $d_1 = d_2$, we further assume that $d_1 - d_2 \neq 0$, i.e., $n'(a) \neq 0$.

Lemma 3.3. *If $n(a) = 1, n(b) \neq 1$ and $n'(a) \neq 0$, then all zeros of $g_2(k)$, excepting one imaginary one, are simple algebraically.*

Proof. Assume k_0 be some zero of g_2 with multiplicities at least two. Then we have from $g_2(k_0) = 0$ and $g_2'(k_0) = 0$ that

$$k_0 = i\left(\frac{1}{2\gamma} - \frac{d_1 + d_2}{c_1 - c_2}\right)$$

which implies that the function $g_2(k)$ has only one multiple zero. The proof is complete. □

Now, let $z = -2ik\gamma$ in (3.30). Then the equation $\frac{2i}{c_1-c_2}ke^{ik\gamma}g_2(k) = 0$ is equivalent to

$$ze^{-z} = 2\gamma\frac{d_1 - d_2}{c_1 - c_2}\left(1 - \frac{d_1 + d_2}{d_1 - d_2}e^{-z}\right)$$

which implies

$$z - \ln z = w_j,\ w_j = -2j\pi i - \ln\left(2\gamma\frac{d_1 - d_2}{c_1 - c_2}\right) - \ln\left(1 - \frac{d_1 + d_2}{d_1 - d_2}e^{-z}\right),\ j \in \mathbb{Z}.$$

By Lemma 2.2, we have

$$z = w_j + \ln w_j + \frac{\ln w_j}{w_j} + O\left(\frac{\ln^2|w_j|}{|w_j|^2}\right), j \to \pm\infty. \tag{3.33}$$

It follows that $\operatorname{Re} z = \ln j + O(1)$, which implies

$$\ln\left(1 - \frac{d_1 + d_2}{d_1 - d_2} e^{-z}\right) = \begin{cases} O\left(\frac{1}{j}\right), \text{ if } d_1 + d_2 \neq 0, \\ 0, \text{ if } d_1 + d_2 = 0. \end{cases} \tag{3.34}$$

Since $g_2(-k^*) = -[g_2(k)]^*$, then the distribution of the zeros of $g_2(k)$ is symmetrical with respect to the imaginary axes. Let us consider the zeros of $g_2(k)$ with $\operatorname{Re} k > 0$, namely, assume $j > 0$. Going back to (3.33), we have

$$\begin{aligned} \ln w_j &= \ln\left\{-2j\pi i\left[1 + \frac{1}{2j\pi i}\ln\left(2\gamma\frac{d_1 - d_2}{c_1 - c_2}\right) + O\left(\frac{1}{j^2}\right)\right]\right\} \\ &= \ln(2j\pi) - \frac{\pi i}{2} - \frac{i}{2j\pi}\ln\left(2\gamma\frac{d_1 - d_2}{c_1 - c_2}\right) + O\left(\frac{1}{j^2}\right) \end{aligned} \tag{3.35}$$

and

$$\frac{\ln w_j}{w_j} = \frac{i\ln(2j\pi)}{2j\pi} + \frac{1}{4j} + O\left(\frac{\ln j}{j^2}\right). \tag{3.36}$$

Therefore, substituting (3.34)–(3.36) into (3.33), we have

$$z = -2j\pi i + \ln(2j\pi) - \frac{\pi i}{2} - \ln\left(2\gamma\frac{d_1 - d_2}{c_1 - c_2}\right) + \frac{i\ln(2j\pi)}{2j\pi} + O\left(\frac{1}{j}\right). \tag{3.37}$$

Hence

$$\ln\left(1 - \frac{d_1 + d_2}{d_1 - d_2}e^{-z}\right) = -\frac{d_1 + d_2}{c_1 - c_2}\frac{\gamma}{j\pi}\left[i + \frac{\ln(2j\pi)}{2j\pi}\right] + O\left(\frac{1}{j^2}\right),$$

and then

$$w_j = -2j\pi i - \ln\left(2\gamma\frac{d_1 - d_2}{c_1 - c_2}\right) + \frac{d_1 + d_2}{c_1 - c_2}\frac{\gamma i}{j\pi} + O\left(\frac{\ln j}{j^2}\right).$$

Hence we can obtain the expression of z which is more accurate than (3.37),

$$\begin{aligned} z &= w_j + \ln w_j + \frac{\ln w_j}{w_j} + O\left(\frac{\ln^2|w_j|}{|w_j|^2}\right) \\ &= \ln(2j\pi) - \ln\left(2\gamma\frac{d_1 - d_2}{c_1 - c_2}\right) + \frac{1}{4j} - \frac{i}{2j\pi}\ln\left(2\gamma\frac{d_1 - d_2}{c_1 - c_2}\right) \\ &\quad - 2j\pi i - \frac{\pi i}{2} + \frac{i\ln(2j\pi)}{2j\pi} + \frac{d_1 + d_2}{c_1 - c_2}\frac{\gamma i}{j\pi} + O\left(\frac{\ln^2 j}{j^2}\right). \end{aligned}$$

Denote the zeros of $g_2(k)$ by $\{\mu_j\}$. Let $\mu_j = \sigma_j + i\tau_j$. Since $z = -2ik\gamma$, we have that if $\frac{d_1 - d_2}{c_1 - c_2} > 0$,

$$\begin{cases} \sigma_j = \frac{j\pi}{\gamma} + \frac{\pi}{4\gamma} - \frac{\ln(2j\pi)}{4j\pi\gamma} - \frac{d_1+d_2}{c_1-c_2}\frac{1}{2j\pi} + \frac{1}{4j\pi\gamma}\ln\left(2\gamma\frac{d_1-d_2}{c_1-c_2}\right) + O\left(\frac{\ln^2 j}{j^2}\right), \\ \tau_j = \frac{1}{2\gamma}\left[\ln(2j\pi) - \ln\left(2\gamma\frac{d_1-d_2}{c_1-c_2}\right) + \frac{1}{4j}\right] + O\left(\frac{\ln^2 j}{j^2}\right); \end{cases} \tag{3.38}$$

and if $\frac{d_1 - d_2}{c_1 - c_2} < 0$,

$$\begin{cases} \sigma_j = \frac{j\pi}{\gamma} + \frac{3\pi}{4\gamma} - \frac{\ln(2j\pi)}{4j\pi\gamma} - \frac{d_1+d_2}{c_1-c_2}\frac{1}{2j\pi} + \frac{1}{4j\pi\gamma}\ln\left(-2\gamma\frac{d_1-d_2}{c_1-c_2}\right) + O\left(\frac{\ln^2 j}{j^2}\right), \\ \tau_j = \frac{1}{2\gamma}\left[\ln(2j\pi) - \ln\left(-2\gamma\frac{d_1-d_2}{c_1-c_2}\right) + \frac{3}{4j}\right] + O\left(\frac{\ln^2 j}{j^2}\right). \end{cases} \tag{3.39}$$

Let $k = x + iy$. Then

$$\frac{2e^{-ik\gamma}kg_2(k)}{c_1 - c_2} = y + \frac{d_1 + d_2}{c_1 - c_2} - \frac{d_1 - d_2}{c_1 - c_2}e^{2\gamma y}\cos 2\gamma x - i\left(x - \frac{d_1 - d_2}{c_1 - c_2}e^{2\gamma y}\sin 2\gamma x\right). \tag{3.40}$$

For sufficiently large $j \in \mathbb{N}^+$, consider the rectangle contour $\Gamma_j^{(2)} = I_{1,j}^{(2)} \cup I_{2,j}^{(2)} \cup I_{3,j}^{(2)} \cup I_{4,j}^{(2)}$, where

$$I_{1,j}^{(2)} := \{k : x = \frac{(j+1)\pi}{\gamma} \geq |y|\},\ I_{2,j}^{(2)} := \{k : |x| \leq \frac{(j+1)\pi}{\gamma} = y\},$$
$$I_{3,j}^{(2)} := \{k : -x = \frac{(j+1)\pi}{\gamma} \geq |y|\},\ I_{4,j}^{(2)} := \{k : |x| \leq \frac{(j+1)\pi}{\gamma} = -y\}.$$

Letting k start from the point $\left(\frac{(j+1)\pi}{\gamma}, -\frac{(j+1)\pi}{\gamma}i\right)$ and travel round $\Gamma_j^{(2)}$ by the counterclockwise, and using (3.40), one can easily obtain that, for large enough j, if $\frac{d_1-d_2}{c_1-c_2} > 0$, i.e., $\frac{n'(a)}{n(b)-1} > 0$, then the variations

$$\Delta_{I_{1,j}^{(2)}} \arg \frac{2e^{-ik\gamma}kg_2(k)}{c_1-c_2} = -\theta_1, \quad \Delta_{I_{2,j}^{(2)}} \arg \frac{2e^{-ik\gamma}kg_2(k)}{c_1-c_2} = 4(j+1)\pi - \theta_2,$$
$$\Delta_{I_{3,j}^{(2)}} \arg \frac{2e^{-ik\gamma}kg_2(k)}{c_1-c_2} = -\theta_3, \qquad \Delta_{I_{4,j}^{(2)}} \arg \frac{2e^{-ik\gamma}kg_2(k)}{c_1-c_2} = \theta_4,$$

where $\theta_1, \theta_2, \theta_3, \theta_4 \in (0, \pi)$, and $\theta_4 = \theta_2 + \theta_2 + \theta_3$; if $\frac{d_1-d_2}{c_1-c_2} < 0$, i.e., $\frac{n'(a)}{n(b)-1} < 0$, then the variations

$$\Delta_{I_{1,j}^{(2)}} \arg \frac{2e^{-ik\gamma}kg_2(k)}{c_1-c_2} = \theta_5, \quad \Delta_{I_{2,j}^{(2)}} \arg \frac{2e^{-ik\gamma}kg_2(k)}{c_1-c_2} = 4(j+1)\pi + \theta_6,$$
$$\Delta_{I_{3,j}^{(2)}} \arg \frac{2e^{-ik\gamma}kg_2(k)}{c_1-c_2} = \theta_7, \qquad \Delta_{I_{4,j}^{(2)}} \arg \frac{2e^{-ik\gamma}kg_2(k)}{c_1-c_2} = \theta_8,$$

where $\theta_5, \theta_6, \theta_7, \theta_8 \in (0, \pi)$, and $\theta_5 + \theta_6 + \theta_7 + \theta_8 = 2\pi$. Thus, $\Delta_{\Gamma_n^{(2)}} \arg \frac{2e^{-ik\gamma}kg_2(k)}{c_1-c_2} = 4(j+1)\pi$ if $\frac{n'(a)}{n(b)-1} > 0$; $\Delta_{\Gamma_n^{(2)}} \arg \frac{2e^{-ik\gamma}kg_2(k)}{c_1-c_2} = (4j+6)\pi$ if $\frac{n'(a)}{n(b)-1} < 0$. By the argument principal of entire functions, the number of zeros of the function $kg_2(k)$ inside $\Gamma_j^{(2)}$ equals $2j+2$ if $\frac{n'(a)}{n(b)-1} > 0$; equals $2j+3$ if $\frac{n'(a)}{n(b)-1} < 0$. Therefore, the zeros of $kg_2(k)$ can be numbered as follows.

Lemma 3.4. *If $\frac{n'(a)}{n(b)-1} > 0$, then the zeros of $kg_2(k)$, denoted by $\left\{\mu_j\right\}_{j\geq 0} \cup \left\{-\mu_j^*\right\}_{j\geq 0}$ with $\operatorname{Re}\mu_j \geq 0$, satisfy the asymptotic formula (3.38). If $\frac{n'(a)}{n(b)-1} < 0$, then the zeros of $kg_2(k)$, denoted by $\left\{\mu_j\right\}_{j\geq 0} \cup \left\{-\mu_j^*\right\}_{j\geq 1}$ with $\operatorname{Re}\mu_j \geq 0$, satisfy the asymptotic formula (3.39).*

Next, we study the asymptotics of exterior transmission eigenvalues under the assumption $p \in W_2^1[0, \gamma]$ or $p \in W_2^2[0, \gamma]$. In the latter case, we obtain the more accurate estimate for the asymptotics of exterior transmission eigenvalues. For arbitrary small $\varepsilon > 0$ and large j, consider the rectangle contour $\Delta_j := \Delta_{\sigma_j}^{\pm} \cup \Delta_{\tau_j}^{\pm}$, where

$$\Delta_{\sigma_j}^{\pm} := \left\{k : x = \sigma_j \pm \varepsilon, \left|y - \tau_j\right| \leq \varepsilon\right\},\ \Delta_{\tau_j}^{\pm} := \{k : y = \tau_n \pm \varepsilon, |x - \sigma_n| \leq \varepsilon\}.$$

Lemma 3.5. *If $n(a) = 1, n(b) \neq 1$ and $n'(a) \neq 0$, then for sufficiently large $j > 0$, and small $\varepsilon > 0$, the function $g_2(k)$ satisfies*

$$|kg_2(k)| \geq C_\varepsilon e^{|\operatorname{Im} k|\gamma}, k \in \Gamma_j^{(2)} \cup \Delta_j,$$

where $C_\varepsilon > 0$ does not depend on j.

Proof. Denote $G_2(k) := |kg_2(k)|^2\, e^{-2|\mathrm{Im}\, k|\gamma}$. Recall $k = x + iy$. By a direct calculation, we have

$$\begin{aligned}|kg_2(k)|^2 = &\frac{(c_1 - c_2)^2}{4} e^{-2\gamma y}\left(x^2 + y^2\right) + \frac{(d_1 + d_2)^2}{4} e^{-2\gamma y} + \frac{(d_1 - d_2)^2}{4} e^{2\gamma y}\\ &+ \frac{(c_1 - c_2)(d_1 + d_2)}{2} y e^{-2\gamma y} + \frac{(c_1 - c_2)(d_1 + d_2)}{2} x e^{-2\gamma y} \sin 2\gamma x\\ &- \frac{d_1^2 - d_2^2}{2} \cos 2\gamma x - \frac{(c_1 - c_2)(d_1 - d_2)}{2} (y \cos 2\gamma x + x \sin 2\gamma x).\end{aligned}$$

It follows that if $\mathrm{Im}\, k \geq 0$,

$$\begin{aligned}G_2(k) = &\frac{(c_1 - c_2)^2}{4} e^{-4\gamma y}\left(x^2 + y^2\right) + \frac{(d_1 + d_2)^2}{4} e^{-4\gamma y} + \frac{(d_1 - d_2)^2}{4}\\ &+ \frac{(c_1 - c_2)(d_1 + d_2)}{2} y e^{-4\gamma y} + \frac{(c_1 - c_2)(d_1 + d_2)}{2} x e^{-4\gamma y} \sin 2\gamma x\\ &- \frac{d_1^2 - d_2^2}{2} e^{-2\gamma y} \cos 2\gamma x - \frac{(c_1 - c_2)(d_1 - d_2)}{2} e^{-2\gamma y} (y \cos 2\gamma x + x \sin 2\gamma x),\end{aligned} \tag{3.41}$$

and if $\mathrm{Im}\, k < 0$,

$$\begin{aligned}G_2(k) = &\frac{(c_1 - c_2)^2}{4}\left(x^2 + y^2\right) + \frac{(c_1 - c_2)(d_1 + d_2)}{2} y + \frac{(c_1 - c_2)(d_1 + d_2)}{2} x \sin 2\gamma x\\ &+ \frac{(d_1 + d_2)^2}{4} + \frac{(d_1 - d_2)^2}{4} e^{4\gamma y} - \frac{(c_1 - c_2)(d_1 - d_2)}{2} e^{2\gamma y} (y \cos 2\gamma x + x \sin 2\gamma x)\\ &- \frac{d_1^2 - d_2^2}{2} e^{2y\gamma} \cos 2\gamma x.\end{aligned}$$

Let us first consider the case $k \in \Delta_j$. From (3.38) and (3.39), we have $\mathrm{Im}\, \mu_j > 0$, $\sigma_j e^{-2\tau_j \gamma} \to \left|\frac{d_1 - d_2}{c_1 - c_2}\right|$ as $j \to \infty$. It follows that when $k \in \Delta^{\pm}_{\tau_j}$,

$$x e^{-2\gamma y} = \left(\sigma_j + t\right) e^{-2\gamma\left(\tau_j \pm \varepsilon\right)} \to \left|\frac{d_1 - d_2}{c_1 - c_2}\right| e^{\mp 2\gamma\varepsilon},\ j \to \infty, \tag{3.42}$$

and

$$\sin 2\gamma x = \sin 2\gamma\left(\sigma_j + t\right) = \pm \cos 2\gamma t + o(1),\ \text{ when } \pm \frac{d_1 - d_2}{c_1 - c_2} > 0, \tag{3.43}$$

here $t \in [-\varepsilon, \varepsilon]$. Substituting (3.42) and (3.43) to (3.41), we have that

$$G_2(k) = \frac{(d_1 - d_2)^2}{4}\left(1 + e^{\mp 4\gamma\varepsilon} - 2e^{\mp 2\gamma\varepsilon} \cos 2\gamma t\right) + o(1),\ k \in \Delta^{\pm}_{\tau_j},\ j \to \infty. \tag{3.44}$$

Denote $q_1(t) := 1 + e^{\mp 4\gamma\varepsilon} - 2e^{\mp 2\gamma\varepsilon} \cos 2t\gamma$. It is easy to see that $q_1(t)$ has minimum value at $t = 0$. Thus, we have

$$G_2(k) \geq \frac{(d_1 - d_2)^2}{4}\left(1 - e^{\mp 2\gamma\varepsilon}\right)^2 + o(1),\ k \in \Delta^{\pm}_{\tau_j},\ j \to \infty. \tag{3.45}$$

Similar to (3.44), we can obtain that for $k \in \Delta^{\pm}_{\sigma_j}$,

$$G_2(k) = \frac{(d_1 - d_2)^2}{4}\left(1 + e^{-4\gamma t} - 2e^{-2\gamma t} \cos 2\gamma\varepsilon\right) + o(1),\ j \to \infty, \tag{3.46}$$

where $t \in [-\varepsilon, \varepsilon]$. Denote $q_2(t) := 1 + e^{-4\gamma t} - 2e^{-2\gamma t}\cos 2\gamma\varepsilon$. Easily, $q_2'(t) > 0$ for $\cos 2\gamma\varepsilon > e^{-2\gamma t}$ and $q_2'(t) < 0$ for $e^{-2\gamma t} > \cos 2\gamma\varepsilon$. Thus, it follows that

$$G_2(k) \geq \frac{(d_1 - d_2)^2}{4}\sin^2 2\gamma\varepsilon + o(1),\ k \in \Delta^{\pm}_{\sigma_j},\ j \to \infty. \tag{3.47}$$

Together with (3.45) and (3.47), we have proved that there exists $\eta_1 > 0$ that is dependent only on ε, such that $G_2(k) > \eta_1$ for $k \in \Delta_j$.

Now, let up pay attention to the case $k \in \Gamma_j^{(2)}$. Because $g_2(k)$ satisfies $g_2(-k^*) = -[g_2(k)]^*$, we only need to consider $k \in \Gamma_j^{(2)} \cap \{k : \operatorname{Re} k \geq 0\}$.

For $k \in \left\{k : 0 \leq x \leq \frac{j+1}{\gamma}\pi, y = \frac{j+1}{\gamma}\pi\right\}$, we have

$$G_2(k) = \frac{(d_1 - d_2)^2}{4} + o(1). \tag{3.48}$$

For $k \in \left\{k : x = \frac{j+1}{\gamma}\pi, \frac{1}{2\gamma}\ln j < y \leq \frac{j+1}{\gamma}\right\}$, i.e., $x = \sigma_j + \frac{j+1}{\gamma}\pi - \sigma_j$ and $y = \tau_j + t$ with $t \in \left[\frac{1}{2\gamma}\ln j - \tau_j, \frac{j+1}{\gamma} - \tau_j\right]$, similar to (3.46), and using (3.38) and (3.39), we have

$$G_2(k) = \frac{(d_1 - d_2)^2}{4}\left(1 + e^{-4\gamma t}\right) + o(1),\ j \to \infty. \tag{3.49}$$

For $k \in \left\{k : x = \frac{j+1}{\gamma}\pi, 0 \leq y \leq \frac{1}{2\gamma}\ln j\right\}$, we get

$$G_2(k) = \frac{(c_1 - c_2)^2 (j+1)^2 \pi^2}{4\gamma^2} e^{-4\tau\gamma}[1 + O(1)] \to \infty,\ j \to \infty. \tag{3.50}$$

For $k \in \left\{k : x = \frac{j+1}{\gamma}\pi, -\frac{j+1}{\gamma} \leq y < 0\right\} \cup \left\{k : 0 \leq x \leq \frac{j+1}{\gamma}\pi, y = -\frac{j+1}{\gamma}\pi\right\}$, we have

$$G_2(k) \geq \frac{(c_1 - c_2)^2 (j+1)^2 \pi^2}{4\gamma^2}[1 + o(1)] \to \infty,\ j \to \infty. \tag{3.51}$$

Together with (3.48)–(3.51), we obtain that there exists $\eta_2 > 0$ that is independent on j, such that $G_2(k) > \eta_2$ for $k \in \Gamma_j^{(2)}$. Taking $C_\varepsilon = \min\{\eta_1, \eta_2\}$, we complete the proof. □

Using Lemma 3.5 and (3.31), we get that $|kg_2(k)| > |D(k) - kg_2(k)|$ for $k \in \Gamma_j^{(2)} \cup \Delta_j$ for large j. By the Rouche's theorem, we conclude that the number of zeros of the function $D(k)$ coincides with the number of $kg_2(k)$ inside $\Gamma_j^{(2)}$ or Δ_j. It follows from Lemma 3.3 that all sufficiently large zeros of $D(k)$ are simple. By Lemma 3.4, the zeros of the function $D(k)$ can be numbered as follows: when $\frac{n'(a)}{n(b)-1} > 0$ denote the zeros of $D(k)$ by $\left\{k_j\right\}_{j\geq 0} \cup \left\{-k_j^*\right\}_{j\geq 0}$; when $\frac{n'(a)}{n(b)-1} < 0$ denote the zeros of $D(k)$ by $\left\{k_j\right\}_{j\geq 0} \cup \left\{-k_j^*\right\}_{j\geq 1}$. Moreover,

$$k_j = \mu_j + \epsilon_j, \epsilon_j = o(1), j \to \infty. \tag{3.52}$$

Furthermore, we estimate ϵ_j. Substituting (3.52) into (3.31), we get

$$0 = \frac{-2}{c_1 - c_2} e^{ik_j\gamma} D(k_j) = ik_j e^{2ik_j\gamma} - \frac{d_1 + d_2}{c_1 - c_2} e^{2ik_j\gamma} + \frac{d_1 - d_2}{c_1 - c_2} + \alpha_j,$$

where $\{\alpha_j\} \in l_2$. Noting $g_2(\mu_j) = 0$, i.e., $i\mu_j e^{2i\mu_j\gamma} - \frac{d_1+d_2}{c_1-c_2} e^{2i\mu_j\gamma} + \frac{d_1-d_2}{c_1-c_2} = 0$. Then

$$\left(1 - e^{2i\epsilon_j\gamma}\right)\frac{d_1 - d_2}{c_1 - c_2} + \alpha_j = 0.$$

Using the asymptotics of μ_j, we get $e^{2i\mu_j\gamma} = O\left(\frac{1}{j}\right)$. Then

$$e^{2i\epsilon_j\gamma} - 1 = \frac{c_1 - c_2}{d_1 - d_2} i\epsilon_j e^{2i\mu_j\gamma} e^{2i\epsilon_j\gamma} + \alpha_j = O\left(\frac{1}{j}\right), \tag{3.53}$$

and then $\epsilon_j = O\left(\frac{1}{j}\right)$.

If $p \in W_2^2[0, \gamma]$, then we can substitute (3.52) into (3.32) and obtain that

$$\begin{aligned} 0 = \frac{-2}{c_1 - c_2} D(k_j) e^{ik_j\gamma} = ik_j e^{2ik_j\gamma} - \frac{d_1 + d_2}{c_1 - c_2} e^{2ik_j\gamma} + \frac{d_2 - d_1}{c_1 - c_2} \\ - \frac{i}{k_j}\left(\frac{e_2 - e_1}{c_1 - c_2} e^{2ik_j\gamma} + \frac{e_1 + e_2}{c_1 - c_2}\right) + \frac{\beta_j}{k_j}, \end{aligned}$$

where $\beta_j = O\left(\frac{1}{j}\right)$. The fact $g_2(\mu_j) = 0$ implies that

$$\begin{aligned} 0 = \left(i\epsilon_j e^{2i\mu_j\gamma} - \frac{d_1 - d_2}{c_1 - c_2}\right) e^{2i\epsilon_j\gamma} + \frac{d_1 - d_2}{c_1 - c_2} \\ - \frac{i}{\mu_j + \epsilon_j}\left(\frac{e_2 - e_1}{c_1 - c_2} e^{2i\mu_j\gamma} e^{2i\epsilon_j\gamma} + \frac{e_1 + e_2}{c_1 - c_2}\right) + \frac{\beta_j}{\mu_j + \epsilon_j}, \end{aligned}$$

i.e.,

$$\begin{aligned} e^{2i\epsilon_j\gamma} - 1 = \frac{c_1 - c_2}{d_1 - d_2} i\epsilon_j e^{2i\mu_j\gamma} e^{2i\epsilon_j\gamma} \\ - \frac{i}{\mu_j + \epsilon_j}\left(\frac{e_2 - e_1}{d_1 - d_2} e^{2i\mu_j\gamma} e^{2i\epsilon_j\gamma} + \frac{e_1 + e_2}{d_1 - d_2}\right) + \frac{\beta_j}{\mu_j + \epsilon_j}. \end{aligned}$$

Since

$$\frac{1}{\mu_j + \epsilon_j} = \frac{\gamma}{j\pi}\left[1 - \frac{i\ln(2j\pi)}{2j\pi} + O\left(\frac{1}{j}\right)\right],$$

then

$$e^{2i\epsilon_j\gamma} - 1 = -\frac{e_1 + e_2}{d_1 - d_2}\frac{i}{\mu_j + \epsilon_j} + \frac{\beta_j}{j} = -\frac{e_1 + e_2}{d_1 - d_2}\frac{\gamma i}{j\pi}\left[1 - i\frac{\ln(2j\pi)}{2j\pi}\right] + \frac{\beta_j}{j},$$

hence,

$$\epsilon_j = -\frac{1}{2\pi}\frac{e_1 + e_2}{d_1 - d_2}\left[\frac{1}{j} - \frac{i}{2\pi}\frac{\ln(2j\pi)}{j^2}\right] + O\left(\frac{1}{j^2}\right).$$

Let us summarize what we have proved.

Theorem 3.3. *Assume $n(a) = 1$, $n(b) \neq 1$, $n'(a) \neq 0$, and $p \in W_2^1[0,\gamma]$. Denote the exterior transmission eigenvalues by $\left\{k_j\right\}_{j\geq 0} \cup \left\{-k_j^*\right\}_{j\geq 0}$ if $\frac{n'(a)}{n(b)-1} > 0$; by $\left\{k_j\right\}_{j\geq 0} \cup \left\{-k_j^*\right\}_{j\geq 1}$ if $\frac{n'(a)}{n(b)-1} < 0$. Then the sequence $\left\{k_j\right\}_{j\geq 0}$ has the following asymptotics:*

$$k_j = \mu_j + O\left(\frac{1}{j}\right), j \to \infty.$$

If we further assume $p \in W_2^2[0,\gamma]$, then the sequence $\left\{k_j\right\}_{j\geq 0}$ has the following asymptotics:

$$k_j = \mu_j - \frac{1}{2\pi}\frac{e_1 + e_2}{d_1 - d_2}\left[\frac{1}{j} - \frac{i}{2\pi}\frac{\ln(2j\pi)}{j^2}\right] + O\left(\frac{1}{j^2}\right), j \to \infty.$$

Here, the asymptotics of $\left\{\mu_j\right\}$ is given in (3.38) and (3.39), the numbers d_1, d_2, e_1, e_2 and the function p appear in (2.23)–(2.26) and (2.8).

When $n(a) \neq 1$, $n(b) = 1$, $n'(b) \neq 0$, the asymptotics of exterior transmission eigenvalues can be studied similarly. Let

$$\mu'_j = \sigma'_j + i\tau'_j, \tag{3.54}$$

where

$$\sigma'_j = \begin{cases} \frac{1}{2\gamma}\left[-2j\pi - \frac{3\pi}{2} + \frac{\gamma}{j\pi}\frac{d_1-d_2}{c_1+c_2} - \frac{1}{2j\pi}\ln\left(2\gamma\frac{d_1+d_2}{c_1+c_2}\right) + \frac{\ln(2j\pi)}{2j\pi}\right], & \text{if } \frac{n'(b)}{n(a)-1} > 0, \\ \frac{1}{2\gamma}\left[-2j\pi - \frac{\pi}{2} + \frac{\gamma}{j\pi}\frac{d_1-d_2}{c_1+c_2} - \frac{1}{2j\pi}\ln\left(-2\gamma\frac{d_1+d_2}{c_1+c_2}\right) + \frac{\ln(2j\pi)}{2j\pi}\right], & \text{if } \frac{n'(b)}{n(a)-1} < 0, \end{cases}$$

and

$$\tau'_j = \begin{cases} \frac{-1}{2\gamma}\left[\ln(2j\pi) - \ln\left(2\gamma\frac{d_1+d_2}{c_1+c_2}\right) + \frac{3}{4j}\right], & \text{if } \frac{n'(b)}{n(a)-1} > 0, \\ \frac{-1}{2\gamma}\left[\ln(2j\pi) - \ln\left(-2\gamma\frac{d_1+d_2}{c_1+c_2}\right) + \frac{1}{4j}\right], & \text{if } \frac{n'(b)}{n(a)-1} < 0. \end{cases}$$

We provide the following theorem without proving it.

Theorem 3.4. *Assume $n(a) \neq 1, n(b) = 1, n'(b) \neq 0$ and $p \in W_2^1[0,\gamma]$. Denote the exterior transmission eigenvalues by $\left\{k_j\right\}_{j\geq 0} \cup \left\{-k_j^*\right\}_{j\geq 2}$ if $\frac{n'(b)}{n(a)-1} > 0$; by $\left\{k_j\right\}_{j\geq 0} \cup \left\{-k_j^*\right\}_{j\geq 1}$ if $\frac{n'(b)}{n(a)-1} < 0$. Then the sequence $\left\{k_j\right\}_{j\geq 0}$ has the following asymptotics:*

$$k_j = \mu'_j + O\left(\frac{1}{j}\right), j \to \infty.$$

If we further assume $p \in W_2^2[0,\gamma]$, then the sequence $\left\{k_j\right\}_{j\geq 0}$ has the following asymptotics:

$$k_j = \mu'_j - \frac{1}{2\pi}\frac{e_1 - e_2}{d_1 + d_2}\left[\frac{1}{j} - \frac{i}{2\pi}\frac{\ln(2j\pi)}{j^2}\right] + O\left(\frac{1}{j^2}\right), j \to \infty,$$

where $\left\{\mu'_j\right\}$ is given in (3.54), the numbers d_1, d_2, e_1, e_2 and the function p appear in (2.23)–(2.26) and (2.8).

Remark 3.2. *When $n(a) = 1$ and $n(b) = 1$, the asymptotics of the exterior transmission eigenvalues can be studied similarly according to whether $n'(a)$ and $n'(b)$ are zeros.*

Example 3.1. *If* $n(r) = \frac{1}{r^4}, a = \frac{1}{2}, b = \frac{3}{2}$, *then* $n(a) = 2^4 \neq 1$, $n(b) = \left(\frac{2}{3}\right)^4 \neq 1$, $\gamma = \int_a^b \sqrt{n(t)}\mathrm{d}t = \frac{4}{3}$, $p(\xi) = \frac{1}{4}\frac{n''(r)}{n(r)^2} - \frac{5}{16}\frac{n'(r)^2}{n(r)^3} = 0$, *and*

$$D(k) = \left(-\frac{8}{3}ik + \frac{4}{3}\right)\cos\frac{4k}{3} - \left(\frac{7}{12}k - 2i + \frac{1}{k}\right)\sin\frac{4k}{3}.$$

By use of Theorem 3.2, we have

$$k_j = \frac{3\pi}{8}(2j-1) - \frac{1}{j\pi}\frac{176}{195} + \frac{3i}{8}\ln\frac{25}{39} + O\left(\frac{1}{j^2}\right), j \to \infty, \tag{3.55}$$

and they are near the line $z = \frac{3i}{8}\ln\frac{25}{39} \approx -0.166757i$ *for k large enough. Figure 1 shows the numerical distribution of the zeros of* $D(k)$, *which are the eigenvalues in (3.55).*

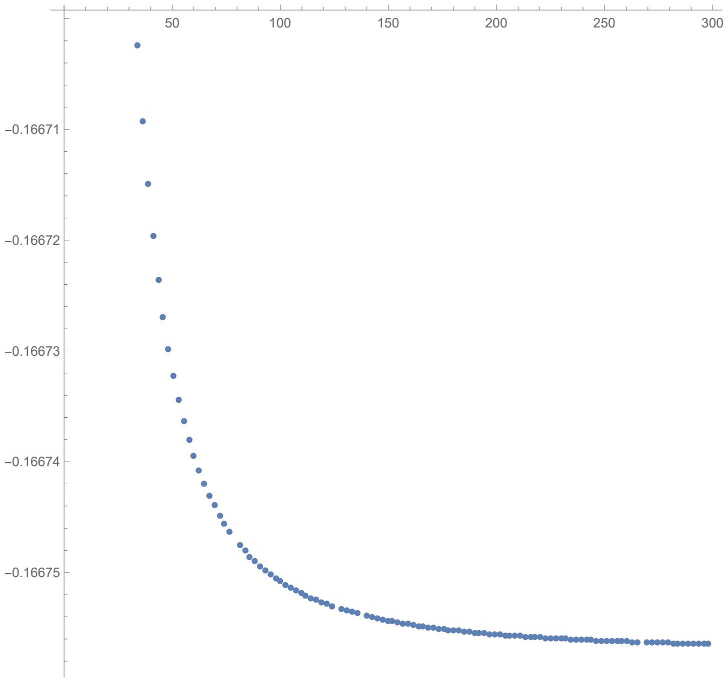

Figure 1. Example 3.1.

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Conflict of interest

The authors declare no conflict of interest.

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