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Research article

Cubic B-Spline method for the solution of the quadratic Riccati differential equation

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Abstract: The quadratic Riccati equations are first-order nonlinear differential equations with numerous applications in various applied science and engineering areas. Therefore, several numerical approaches have been derived to find their numerical solutions. This paper provided the approximate solution of the quadratic Riccati equation via the cubic b-spline method. The convergence analysis of the method is discussed. The efficiency and applicability of the proposed approach are verified through three numerical test problems. The obtained results are in good settlement with the exact solutions. Moreover, the numerical results indicate that the proposed cubic b-spline method attains a superior performance compared with some existing methods.

Keywords: Cubic b-spline method; nonlinear equations; numerical analysis; Riccati differential equation; first-order differential equations
Mathematics Subject Classification: 65D07, 65G99, 65L05

1. Introduction

Consider the differential equation

$$u'(x) = p(x) + q(x)u(x) + r(x)u^2(x), u(x_0) = \alpha. \tag{1}$$

Equation (1), which was developed by the Italian mathematician Jacopo Riccati and is known as the quadratic Riccati differential equation (QRDE) [1], arises in various applications in applied science and engineering including stochastic realization theory, random processes, diffusion problems, optimal control, and financial mathematics [2]. Therefore, the QRDE has gained attention and has been inspected by many studies. Moreover, finding exact solutions for such nonlinear equations is intolerable or impossible. Consequently, various techniques are developed to obtain its solution, for instance, Adomian's decomposition method [3–7], a piecewise variational iteration method [8], the classical Runge Kutta method of order four (RK4) [9], and the Bezier curves method [10], etc. [11,12].

An exciting property of the Riccati equation is that it can be revised as a second-order linear equation. To obtain a second-order linear equation, one can first use the transform $\delta(x) = r(x)u(x)$, to get

$$\delta'(x) = p(x)r(x) + \left(q(x) + \frac{r'(x)}{r(x)}\right)\delta(x) + \delta^2(x),$$

and then substituting $\delta(x) = -\frac{\phi'(x)}{\phi(x)}$ leads to

$$\phi''(x) - \left(q(x) + \frac{r'(x)}{r(x)}\right)\phi'(x) + r(x)p(x)\phi(x) = 0.$$

Moreover, if a particular solution u_p is found for (1), then a general solution can be acquired as

$$u = u_p + \frac{1}{\mu(x)},$$

where $\mu(x)$ is a solution of the associated Bernoulli equation

$$\mu'(x) = (q(x) + 2r(x)u_p)\mu(x) + r(x)\mu^2(x).$$

In the present work, our main motivation is to treat the QRDE numerically using the cubic B-spline method and also to establish error estimates of the method. The algorithm is developed, and the approximate solutions achieved by this algorithm are compared with some existing methods.

This article is structured as follows. Section 2 provides the cubic B-spline scheme for the solution of the QRDE. Section 3 discusses the convergence analysis of the presented method. Section 4 presents a comparison of two existing methods. Finally, Section 5 includes a brief conclusion.

2. Construction of cubic B-spline method

Suppose that the solution domain $[a, b]$ is divided up into n equal-length subintervals using the knots $x_i = a + ih,\ i = 0\ (1)\ n,$ where $h = (b\text{-}a)/n$. Let $B_i(x)$ denotes the cubic B-spline function, which is defined as

$$B_i(x)=\begin{cases}\frac{(x-x_i)^3}{6h^3}, & x\in[x_i,x_{i+1}]\\ \frac{(x-x_i)^3}{6h^3}-2\frac{(x-x_{i+1})^3}{3h^3}, & x\in[x_{i+1},x_{i+2}]\\ \frac{(x_{i+4}-x)^3}{6h^3}-2\frac{(x_{i+3}-x)^3}{3h^3}, & x\in[x_{i+2},x_{i+3}]\\ \frac{(x_{i+4}-x)^3}{6h^3}, & x\in[x_{i+3},x_{i+4}]\\ 0, & \text{else.}\end{cases}$$

Each spline basis function $B_i(x)$ is locally supported and nonnegative on $[x_i,x_{i+4}]$ [13]. Therefore, we introduce six additional points to both sides of the domain as $x_k=a+kh$ $(k=-3,-2,-1)$, and $x_k=a+kh$ $(k=1,2,3)$. In addition, the value of $B_i(x_j)$ and $B_i'(x_j)$ are listed in Table 1.

Table 1. Values of $B_i(x)$ and $B_i'(x)$ at the Knots.

	x_{i+1}	x_{i+2}	x_{i+3}	else
$B_i(x)$	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$	0
$B_i^{'}(x)$	$\frac{1}{2h}$	0	$-\frac{1}{2h}$	0

Our numerical treatment for solving the QRDE (1) utilizing the cubic B-spline method is to achieve an approximating solution $s(x)$ of the form

$$s(x)=\sum_{i=-3}^{n-1}c_i\,B_i(x). \tag{2}$$

By using (2) and Table 1, we get

$$u(x_j)=s(x_j)=\tfrac{1}{6}(c_{j-3}+4c_{j-2}+c_{j-1}), \tag{3}$$

$$u^{'}(x_j)=s^{'}(x_j)=\tfrac{1}{2h}(c_{j-1}-c_{j-3}). \tag{4}$$

Substituting (3) and (4) into the QRDE (1) produces

$$\tfrac{1}{2h}(c_{j-1}-c_{j-3})=p(x_i)+\tfrac{1}{6}q(x_i)(c_{j-1}+4c_{j-2}+c_{j-3})+\tfrac{1}{36}r(x_i)(c_{j-1}+4c_{j-2}+c_{j-3})^2. \tag{5}$$

Equation (5) provides (n+1) equations with (n+3) unknowns. Hence, two additional conditions must be added to solve the above system uniquely. Employing the initial condition gives

$$\tfrac{1}{6}(c_{-3}+4c_{-2}+c_{-1})=\alpha. \tag{6}$$

One more equation is still required. Differentiating (1) once again yields

$$u^{''}(x)=p^{'}(x)+q(x)u^{'}(x)+q^{'}(x)u(x)+2r(x)u(x)u^{'}(x)+r^{'}(x)u^2(x). \tag{7}$$

Moreover, in [14], we have

$$s''(x_0)=\frac{14c_{-3}-33c_{-2}+28c_{-1}-14c_0+6c_1-c_2}{12h^2}+O(h^4). \tag{8}$$

On substituting (3), (4), and (8) with ignoring the error term in (8) into (7), we get

$$\begin{aligned}&\frac{1}{12h^2}(14c_{-3}-33c_{-2}+28c_{-1}-14c_0+6c_1-c_2)\\&=p'(x_0)+\frac{1}{2h}q(x_0)(c_{-1}-c_{-3})+\frac{1}{6}q'(x_0)(c_{-3}+4c_{-2}+c_{-1})\\&+\frac{1}{6h}r(x_0)(c_{-3}+4c_{-2}+c_{-1})(c_{-1}-c_{-3})\\&+\frac{1}{36}r'(x_0)(c_{-3}+4c_{-2}+c_{-1})^2\end{aligned} \tag{9}$$

Once we solve the system (5), (6), and (9) for c_i's, the cubic B-spline is fully determined.

3. Convergence analysis

The convergence analysis of the proposed method is going to be demonstrated in this section. For this purpose, it is assumed that $u(x)\in C^5[a,b]$.

Using the shifting operator, $E(S(x_i))=S(x_{i+1})$ and (3), (4) can be expressed as [15]

$$\frac{h}{6}(E^{-1}+4+E)S'(x_i)=\frac{1}{2}(E-E^{-1})u(x_j). \tag{10}$$

As $E=e^{hD}$ and $D\equiv d/dx$, one can obtain

$$e^{hD}+e^{-hD}=2\sum_{k=0}^{\infty}\frac{(hD)^{2k}}{(2k)!}, \tag{11}$$

$$e^{hD}-e^{-hD}=2\sum_{k=0}^{\infty}\frac{(hD)^{2k+1}}{(2k+1)!}. \tag{12}$$

As a result, (10) can be expressed as [15]

$$\left[1+\frac{1}{3}\sum_{k=1}^{\infty}\frac{(hD)^{2k}}{(2k)!}\right]s'(x_j)=\left[\sum_{k=0}^{\infty}\frac{(hD)^{2k+1}}{(2k+1)!}\right]u(x_j). \tag{13}$$

Simplifying (13) gives

$$\begin{aligned}s'(x_j)&=\left[\sum_{k=0}^{\infty}\frac{(hD)^{2k+1}}{(2k+1)!}\right]\left(1+\frac{1}{3}\sum_{k=1}^{\infty}\frac{(hD)^{2k}}{(2k)!}\right)^{-1}u(x_j)\\&=(D+\frac{h^2D^3}{3!}+\frac{h^4D^5}{5!}+\cdots)\left[1-(\frac{h^2D^2}{6}+\frac{h^4D^4}{72}+\cdots)\right.\\&\left.+(\frac{h^2D^2}{6}+\frac{h^4D^4}{72}+\cdots)^2+\cdots\right]u(x_j)\\&=D(1-\frac{h^4D^4}{180}+\frac{h^6D^6}{1512}-\cdots)u(x_j).\end{aligned}$$

Therefore,

$$s'(x_j) = u'(x_j) - \frac{h^4}{180} u^{(5)}(x_j) + \cdots. \tag{14}$$

At this point, the error function $e(x)$ is defined as

$$e(x) = s(x) - u(x).$$

On substituting (14) into the Taylor expansion of $e(x_j + kh),\ 0 \le k \le 1,$ we get

$$e(x_j + kh) = -\frac{kh^4}{180} u^{(5)}(x_j) + O(h^6). \tag{15}$$

This proves that our approach to solving the QRDE (1) is of order $O(h^4)$.

4. Numerical experiments

In this section, three problems of the QRDE (1) are given to reveal the scheme's efficiency and support the theoretical discussion. All computations have been performed via *MATHEMATICA* 9 software.

Problem 4.1. Consider the following QRDE:

$$u'(x) = 16x^2 - 5 + 8xu(x) + u^2(x),\ u(0) = 1,\ 0 \le x \le 1.$$

The exact solution is $u(x) = 1 - 4x.$

In Problem 4.1, the numerical results are computed at specific points. The absolute errors are presented in Table 2. Figure 1 displays the graph between the exact and numerical solutions at the grid points. From Figure 1, our numerical results are in good agreement with the exact solution. In this problem, our numerical findings are more accurate than those of the RK4 in [9] and the Bezier curves method (BCM) in [10].

Table 2. Absolute errors for Problem 4.1.

x	The Presented Method	RK4 in [9]	BCM in [10]
0.1	1.1102×10^{-16}	0.0	2.336×10^{-4}
0.3	1.3878×10^{-16}	2.2204×10^{-16}	4.5422×10^{-4}
0.5	2.2205×10^{-16}	2.2204×10^{-16}	9.375×10^{-11}
0.7	6.6613×10^{-16}	2.2204×10^{-16}	4.5422×10^{-4}
0.9	4.4409×10^{-16}	4.4409×10^{-16}	2.336×10^{-4}
1	8.8818×10^{-16}	8.8818×10^{-16}	0.0

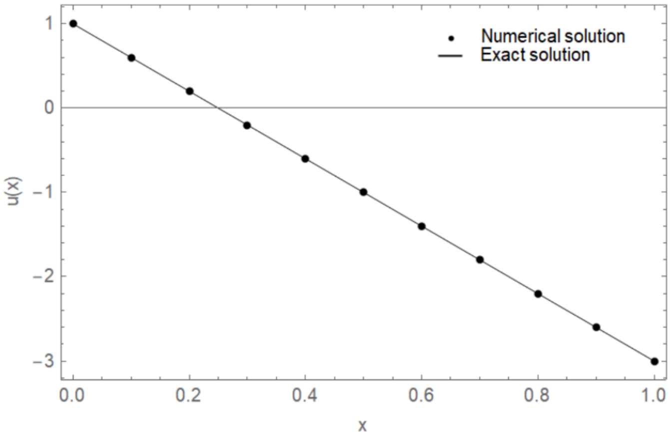

Figure 1. The graph of the numerical and exact solution for Problem 4.1 with n=10.

Problem 4.2. Consider the following QRDE

$$u'(x) = 1 + 2u(x) - u^2(x),\ u(0) = 0,\ 0 \leq x \leq 1.$$

The exact solution is $u(x) = 1 + \sqrt{2}\, tanh(\sqrt{2}x + \frac{1}{2} ln(\frac{\sqrt{2}-1}{\sqrt{2}+1}))$.

In Problem 4.2, the numerical results are computed at specific points. The absolute errors are presented in Table 3. Figure 2 displays the graph between the exact and numerical solutions at the grid points. From Figure 2, our numerical results are in good agreement with the exact solution. In this problem, our numerical findings are more accurate than those of the RK4 in [9] and the BCM in [10].

Table 3. Absolute errors for Problem 4.2.

x	The Presented Method	RK4 in [9]	BCM in [10]
0.1	3.3766×10^{-6}	2.2551×10^{-6}	2.4895×10^{-4}
0.3	1.092×10^{-5}	7.3083×10^{-6}	4.4482×10^{-4}
0.5	8.1363×10^{-6}	1.1301×10^{-5}	2.8944×10^{-10}
0.7	5.5194×10^{-6}	1.2940×10^{-5}	3.7412×10^{-4}
0.9	1.1756×10^{-5}	1.3141×10^{-5}	1.785×10^{-4}
1	9.2686×10^{-6}	1.3245×10^{-5}	3.2516×10^{-10}

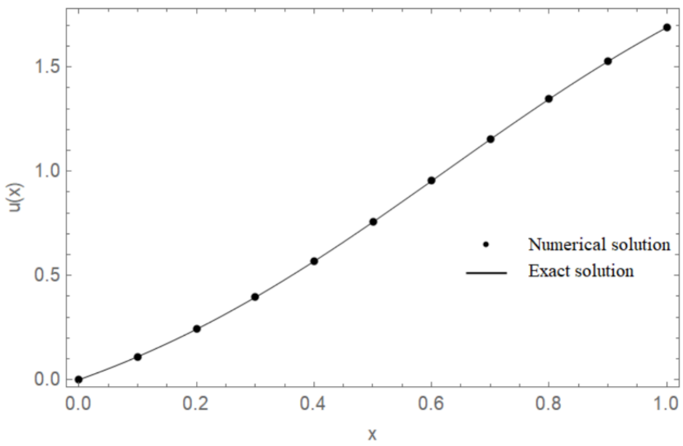

Figure 2. The graph of numerical and exact solutions for Problem 4.2 with *n*=10.

Problem 4.3. Consider the following QRDE

$$u'(x) = e^{x} - e^{3x} + 2e^{x}u(x) - e^{x}u^{2}(x),\ u(0) = 1,\ 0 \leq x \leq 1.$$

With the exact solution $u(x) = e^{x}$.

In Problem 4.3, the numerical results are computed at specific points. The absolute errors are presented in Table 4. Figure 3 displays the graph between the exact and numerical solutions at the grid points. From Figure 3, our numerical results are in good agreement with the exact solution. In this problem, our numerical findings are more accurate than those of the RK4 in [9] and the BCM in [10].

Table 4. Absolute errors for Problem 4.3.

x	The Presented Method	RK4 in [9]	BCM in [10]
0.1	6.5346×10^{-8}	1.1153×10^{-7}	3.4681×10^{-4}
0.3	2.0112×10^{-7}	4.6838×10^{-7}	6.7437×10^{-4}
0.5	3.6696×10^{-7}	1.1237×10^{-6}	3.8747×10^{-10}
0.7	5.6951×10^{-7}	2.3239×10^{-6}	6.7437×10^{-4}
0.9	8.1691×10^{-7}	4.5182×10^{-6}	3.4682×10^{-4}
1	9.5347×10^{-7}	6.2225×10^{-6}	0.0

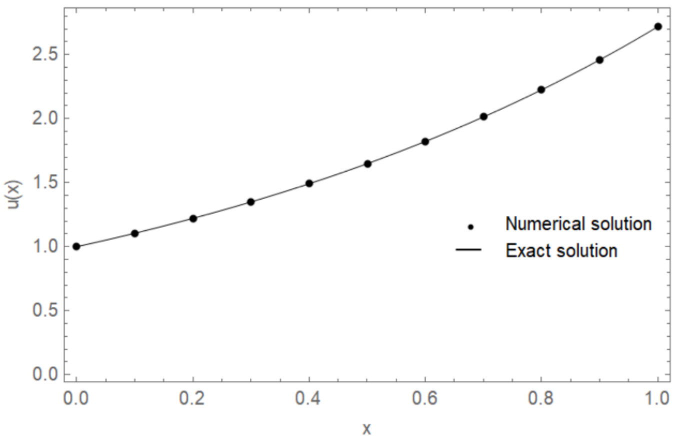

Figure 3. The graph of numerical and exact solutions for Problem 4.3 with n=10.

5. Conclusions

The cubic B-spline technique is developed for solving the QRDEs numerically. The convergence analysis of the cubic B-spline technique is analyzed. Three test examples have been considered to examine the efficiency of the developed algorithm. The comparisons of the absolute errors with those of the RK4 in [9] and the BCM in [10] seem to indicate the superiority of the proposed method over some existing methods in terms of error.

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Conflict of interest

The authors declare no conflict of interest.

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