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Research article

On symmetry of the product of two higher-order quasi-differential operators

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Abstract: The symmetric realizations of the product of two higher-order quasi-differential expressions in Hilbert space are investigated. By means of the construction theory of symmetric operators, we characterize symmetric domains determined by two-point boundary conditions for product of two symmetric differential expressions with regular or limit-circle singular endpoints. The presented result contains the characterization of self-adjoint domains as a special case. Several examples of singular symmetric product operators are given.

Keywords: product of differential operators; symmetric boundary conditions; singular operators
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1. Introduction

Consider the equation

$$My = \lambda wy \quad on\ J = (a, b),\ -\infty \leq a < b \leq \infty, \tag{1.1}$$

where M is a symmetric differential expression of order n, and w is a positive weight function.

In 1995, Möller-Zettl [12] first characterized the symmetric realizations of (1.1) for regular problems. Motivated by the method used by Sun [10] for self-adjoint operators and by the method of Möller-Zettl [12], Wang-Zettl [3, 4] characterized the domains of the symmetric realizations of (1.1) for singular problems with any deficiency index.

Based on the self-adjoint GKN theorem and von Neumann's formula for the adjoint of a symmetric operator in Hilbert space, for classical expression M, Sun [10] gave a decomposition of the maximal domain using certain solutions for non-real value of the spectral parameter λ, and then characterized all the self-adjoint realizations. Sun's work is an important contribution to the study of self-adjoint domains. Wang et al. [6] established a new representation in terms of certain solutions for real λ. This leads to a classification of solutions as limit-point(LP) or limit-circle(LC). The LC solutions contribute to the singular boundary conditions, but the LP solutions do not. In 2012, Hao et al. [18] extended this result to the case when both endpoints are singular. This real λ decomposition of the maximal domain and the construction of LC and LP solutions also play a critical role in the investigation of the spectrum of self-adjoint operators, the classification of self-adjoint boundary conditions, and the characterization of symmetric domains.

In [1, 4, 5], Wang-Zettl characterized the symmetric operators in $H = L^2(J,w)$ and proved a symmetric GKN-Type theorem. The result contains the self-adjoint GKN theorem as a special case. It is well known that the self-adjoin GKN theorem is widely used to study self-adjoint operators, difference operators, Hamiltonian systems, multi-interval operators, etc. The symmetric GKN-Type theorem maybe will have similar extensions for symmetric problems.

For products and powers of differential expressions, in [2, 15, 16], the deficiency indices of powers of classical expressions and of quasi-differential expressions were discussed. In [20], the self-adjointness of the product of two second-order differential operators was obtained. Based on [10, 20], An-Sun [8] characterized the self-adjointness of product of two nth-order real classical differential expressions with two regular endpoints and extended to problems with one regular endpoint and one singular endpoint [9]. In recent years, there are some works on the self-adjointness of products of differential operators, see [7, 11, 13, 14, 17, 19].

In this paper, we study the symmetric domain characterization for product of two quasi-differential expressions of order n, even or odd, with complex coefficients. We consider the cases when each endpoint is either regular or LC singular. The self-adjoint characterization for product of two differential expressions is a special case.

The organization of this paper is as follows. Following this introduction, the quasi-differential expressions, Lagrange identity, maximal and minimal operators, and powers of differential expressions are given in Section 2. Section 3 is devoted to the symmetric domain characterization of product of two differential operators. In subsection 3.1, we consider the case when one endpoint is regular and the other LC singular. In subsection 3.2, we consider the case when both endpoints are singular. Some examples are given in Section 4.

2. Preliminaries

We first repeat some definitions and basic properties of quasi-differential expressions. See the book [1] for more details.

Definition 1. *For $n > 1$, let*

$$\begin{aligned} Z_n(J) := \Big\{&Q = (q_{rs})_{r,s=1}^n \in M_n(L_{loc}(J)),\\ &q_{r,r+1} \neq 0 \text{ a.e. on } J, q_{r,r+1}^{-1} \in L_{loc}(J), 1 \le r \le n-1,\\ &q_{rs} = 0 \text{ a.e. on } J, 2 \le r+1 < s \le n; q_{rs} \in L_{loc}(J), s \neq r+1, 1 \le r \le n-1\Big\}. \end{aligned} \tag{2.1}$$

For $Q \in Z_n(J)$ we define

$$V_0 := \{y : J \to \mathbb{C}, y \text{ is measurable}\}$$

and $y^{[0]} = y\,(y \in V_0)$. Inductively, for $r = 1, \dots, n$, we define

$$V_r = \left\{y \in V_{r-1} : y^{[r-1]} \in (AC_{loc}(J))\right\},$$

$$y^{[r]} = q_{r,r+1}^{-1}\left\{y^{[r-1]'} - \sum_{s=1}^{r} q_{rs}y^{[s-1]}\right\} \quad (y \in V_r)\,, \tag{2.2}$$

where $q_{n,n+1} := 1$. *Finally we set*

$$My = M_Q y = i^n y^{[n]} \quad (y \in V_n)\,. \tag{2.3}$$

The expression $M = M_Q$ *is called the quasi-differential expression generated by* Q. *For* V_n *we also use the notations* $D(Q)$ *and* $D(M)$.

Definition 2. *Let* $Q \in Z_n(J)$, $J = (a, b)$. *The expression* $M = M_Q$ *is said to be regular at* a *or we say* a *is a regular endpoint, if for some* $c, a < c < b$, *we have*

$$\begin{aligned} q_{r,r+1}^{-1} &\in L(a,c), \quad r = 1, \cdots, n-1, \\ q_{rs} &\in L(a,c), \quad 1 \le r, s \le n, \quad s \ne r+1. \end{aligned}$$

Similarly the endpoint b *is regular if for some* $c, a < c < b$, *we have*

$$\begin{aligned} q_{r,r+1}^{-1} &\in L(c,b), \quad r = 1, \cdots, n-1, \\ q_{rs} &\in L(c,b), \quad 1 \le r, s \le n, \quad s \ne r+1. \end{aligned}$$

Note that from the definition of $Q \in Z_n(J)$ *it follows that if the above hold for some* $c \in J$, *then they hold for any* $c \in J$. *We say that* M *is regular on* J, *if* M *is regular at both endpoints. An endpoint is singular if it is not regular.*

Definition 3. *Let* $\mathbb{N}_2 = \{2, 3, 4, \cdots, \}$. *For* $k \in \mathbb{N}_2$, *we define the matrix* E_k *as follows:*

$$E_k = ((-1)^r \delta_{r,k+1-s})_{r,s=1}^{k}\,, \tag{2.4}$$

where $\delta_{i,j}$ *is the Kronecker* δ. *Note that*

$$E_k^* = E_k^{-1} = (-1)^{k+1} E_k. \tag{2.5}$$

Lemma 1 (Lagrange Identity). *Let* $Q \in Z_n(J), P = -E^{-1}Q^*E$ *where* $E = E_n$ *is defined in Definition 3. Then* $P \in Z_n(J)$ *and for any* $y \in D(M_Q), z \in D(M_P)$, *we have*

$$\bar{z} M_Q y - y\overline{M_P z} = [y, z]', \tag{2.6}$$

where

$$[y, z] = i^n \sum_{r=0}^{n} (-1)^{n+1-r} \bar{z}_P^{[n-r-1]} y_Q^{[r]} = -i^n Z^* E Y. \tag{2.7}$$

Here we call $[y, z]$ *or just* $[\cdot, \cdot]$ *a Lagrange bracket.*

Corollary 1. *If* $My = \lambda wy$ *and* $Mz = \bar{\lambda} wz$, *then* $[y, z]$ *is constant on* J. *In particular, if* λ *is real and* $My = \lambda wy$, $Mz = \lambda wz$, *then* $[y, z]$ *is constant on* J.

The above symplectic matrix E_k and the Lagrange Identity play an important role in the study of general symmetric differential expressions and the characterization of domains of symmetric and self-adjoint boundary conditions.

Definition 4. *Let $Q \in Z_n(J)$ and suppose that Q satisfies*

$$Q = -E^{-1}Q^*E, \quad \text{where } E = E_n = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n, \text{ i.e.}$$
$$q_{rs} = (-1)^{r+s-1}\bar{q}_{n+1-s,n+1-r}, \quad 1 \le r, s \le n.$$

Then Q is called a Lagrange symmetric matrix and the expression $M = M_Q$ is called a Lagrange symmetric, or just a symmetric, differential expression.

Definition 5. *Let $Q \in Z_n(J)$, $Q = -E^{-1}Q^*E$, let $M = M_Q$, $H = L^2(J, w)$, and let w be a weight function. The maximal operator $S_{\max} = S_{\max}(Q, J)$ with domain $D_{\max} = D_{\max}(Q, J)$ is defined by:*

$$D_{\max} = \left\{ y \in H : y \in D(M), w^{-1}My \in H \right\},$$
$$S_{\max}y = w^{-1}My, \quad y \in D_{\max}.$$

$D_{\max}(Q, J)$ is dense in H. The minimal operator $S_{\min} = S_{\min}(Q, J)$ with domain $D_{\min} = D_{\min}(Q, J)$ is defined as $S_{\min} = S^*_{\max}$. $S_{\min}$ is a closed symmetric operator in H with dense domain and $S^*_{\min} = S_{\max}$.

The following result is immediate from the Lagrange Identity and integration.

Corollary 2. *For any $y, z \in D(M)$, $J = (a, b)$, the limits $\lim_{t\to b^-}[y, z](t), \lim_{t\to a^+}[y, z](t)$ exist and are finite. And then we have:*

$$\int_a^b \{\bar{z}My - y\overline{Mz}\} = [y, z](b) - [y, z](a). \tag{2.8}$$

Lemma 2. *Let $a_1 < \cdots < a_k \in J$, where a_1 and a_k can also be regular endpoints. Let $\alpha_{jr} \in \mathbb{C}(j = 1, \ldots, k; r = 0, \ldots, n-1)$. Then there is a $y \in D_{\max}$ such that*

$$y^{[r]}(a_j) = \alpha_{jr} \quad (j = 1, \ldots, k; r = 0, \ldots, n-1).$$

Lemma 3. *Let $Q \in Z_n(J)$, $Q = -E^{-1}Q^*E$. Then*

$$D(S_{min}) = \{y \in D_{max} : [y, z](a) = 0 = [y, z](b), \text{for all } z \in D_{max}\}. \tag{2.9}$$

Given a Lagrange symmetric matrix $Q \in Z_n(J)$ and its associated symmetric expression $M = M_Q$ the construction and properties of powers M^s ($s \in \mathbb{N}_2$) of quasi-differential expression M was given in [1, 2].

Lemma 4. *Assume that $Q \in Z_n(J)$ is a Lagrange symmetric matrix. Let $M = M_Q$ and define M^2 by $M^2y = M(My), \cdots, M^sy = M(M^{s-1}y)$. Let $Q^{[1]} = Q$ and for $s \in \mathbb{N}_2$, let $Q^{[s]}$ denote the block diagonal matrix*

$$Q^{[s]} = \begin{bmatrix} Q & & \\ & \ddots & \\ & & Q \end{bmatrix}, \tag{2.10}$$

where there are s matrices Q on the diagonal and all other entries in this $sn \times sn$ matrix are zero except for the entries in positions $(n, n+1), (2n, 2n+1), \cdots, ((s-1)n, (s-1)n+1)$, these are all equal to 1. Then, for any positive integer s, the matrices $Q^{[s]}$ are in $Z_{sn}(J)$, and are Lagrange symmetric and the symmetric differential expression M^S is given by

$$M^s = M_{Q^{[s]}}. \tag{2.11}$$

Lemma 5. *Let $Q \in Z_n(J)$ be Lagrange symmetric and let $M = M_Q$ be the associated symmetric expression. If all solutions of $My = \lambda wy$ are in $L^2(J, w)$ for some $\lambda \in \mathbb{C}$, then this is true for all solutions of $M^sy = \lambda wy$ for every $\lambda \in \mathbb{C}$ and every $s \in \mathbb{N}$.*

3. Symmetric domain characterization of product of two differential operators

Wang-Zettl [1] characterized the domains of symmetric realizations S of the equation

$$My = \lambda wy \quad J = (a, b), -\infty \leq a < b \leq +\infty, \tag{3.1}$$

in the Hilbert space $H = L^2(J, w)$, where $M = M_Q$ is Lagrange symmetric, w is a positive weighted function. Based on the work of [1], we now study the symmetric realizations of product of two quasi-differential expressions in the case that each endpoint is either regular or limit-circle(LC) singular.

In the following we always let $Q \in Z_n(J), J = (a, b), n \in \mathbb{N}_2$, be a Lagrange symmetric matrix and $M = M_Q$ the corresponding symmetric differential expression of order n, even or odd, with real or complex coefficients.

3.1. The case when a is regular and b is LC singular

In this subsection, we consider the case when one endpoint of $J = (a, b)$ is regular and the other LC singular. By the Patching Lemma 2 and the decomposition of the maximal domain given by Theorem 4.4.3 in [1], we have the following result.

Lemma 6. *Let the endpoint a be regular, b LC singular, and let $a < c < b$. Then the deficiency index of M is n and there exist n linearly independent solutions $u_1(t), \ldots, u_n(t)$ of $My = 0$ in $L^2(J)$ such that*

$$u_i^{[j-1]}(a) = \delta_{ij} \quad (i, j = 1, 2, \ldots, n). \tag{3.2}$$

And then we have the decomposition of $D_{\max}$:

$$D_{\max} = D_{\min} \dotplus \operatorname{span}\{z_1, z_2, \cdots, z_n\} \dotplus \operatorname{span}\{u_1, u_2, \cdots u_n\}, \tag{3.3}$$

where $z_i \in D_{\max}, i = 1, \cdots n$ such that $z_i(t) = 0$ for $t \geq c$ and $z_i^{[j-1]}(a) = \delta_{ij}, i, j = 1, \cdots, n$, and δ_{ij} is the Kronecker δ.

By the Lagrange Identity and Corollary 1, we have

$$[u_i, u_j]_n(b) = [u_i, u_j]_n(a), \quad i, j = 1, 2, \cdots, n, \tag{3.4}$$

where $[\cdot, \cdot]_n$ denotes the Lagrange bracket of differential expression M.

The following theorem can be found in Chapter 6 of [1].

Theorem 1. *Let $M = M_Q, Q \in Z_n(J), J = (a, b), -\infty \leq a < b \leq +\infty$ be Lagrange symmetric, w a weight function, and let a be regular and b LC singular. Let the composed matrix $U = (A : B)$ be a boundary condition matrix with $\operatorname{rank}(U) = l, 0 \leq l \leq 2n$, where $A_{l,n}, B_{l,n}$ are complex matrices. Define the operator $S(U)$ in $L^2(J, w)$ by*

$$D(S(U)) = \{y \in D_{\max} : UY_{a,b} = 0\},$$
$$S(U)y = S_{\max}y \quad \text{for } y \in D(S(U)),$$

where

$$Y_{a,b} = \begin{pmatrix} Y_a \\ Y_b \end{pmatrix}, \quad Y_a = \begin{pmatrix} y^{[0]}(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix}, \quad Y_b = \begin{pmatrix} [y, u_1]_n(b) \\ \vdots \\ [y, u_n]_n(b) \end{pmatrix}, \tag{3.5}$$

functions $u_1, \cdots, u_n$ *are given by* (3.2)*. Let* $C = AE_nA^* - BE_{m_b}(b)B^*$ *and* $r = rank(C)$*, where* $E_n = ((-1)^r\delta_{r,n+1-s})_{r,s=1}^n$*. Then we have:*
(i) If $l < n$*, then* $S(U)$ *is not symmetric;*
(ii) If $l = n$*, then* $S(U)$ *is self-adjoint (and hence symmetric) if and only if* $r = 0$.
(iii) Let $l = n + s, 0 < s \leq n$*. Then* $S(U)$ *is symmetric and not self-adjoint if and only if* $r = 2s$.

Proof. This follows directly from Theorem 6.3.3 of [1]. Note that here we choose the solutions $u_1(t), \ldots, u_n(t)$ of $My = 0$. In terms of (3.4) and (3.2), for the case when a is regular and b is LC singular(then $m_b = n$), the matrix $E_{m_b}(b)$ given by Theorem 6.3.3 [1] can be written as

$$E_{m_b}(b) = E_n(b) = \begin{bmatrix} [u_1, u_1]_n(b) & \cdots & [u_n, u_1]_n(b) \\ \vdots & \ddots & \vdots \\ [u_1, u_n]_n(b) & \cdots & [u_n, u_n]_n(b) \end{bmatrix} = \begin{bmatrix} [u_1, u_1]_n(a) & \cdots & [u_n, u_1]_n(a) \\ \vdots & \ddots & \vdots \\ [u_1, u_n]_n(a) & \cdots & [u_n, u_n]_n(a) \end{bmatrix} = -i^n E_n, \tag{3.6}$$

where $i = \sqrt{-1}$. This concludes the proof. □

For $M^2y = M(My)$, we obviously have $D_{\max}(M^2) \subset D_{\max}(M)$ and $D_{\min}(M^2) \subset D_{\min}(M)$.

Lemma 7. *For any* $y, z \in D_{\max}(M^2)$*, we have*

$$[y, z]_{2n}(t) = [My, z]_n(t) + [y, Mz]_n(t), \tag{3.7}$$

where $[\cdot, \cdot]_{2n}$ *denotes the Lagrange bracket of differential expression* M^2.

Proof. By the Lagrange Identity (2.6), we have

$$[y, z]_{2n} = \int \bar{z}M^2y\mathrm{d}t - \int y\overline{M^2z}\mathrm{d}t,$$

$$[y, Mz]_n = \int \overline{Mz}My\mathrm{d}t - \int y\overline{M^2z}\mathrm{d}t,$$

$$[My, z]_n = \int \bar{z}M^2y\mathrm{d}t - \int My\overline{Mz}\mathrm{d}t = \int \bar{z}M^2y\mathrm{d}t - [y, Mz]_n - \int y\overline{M^2z}\mathrm{d}t.$$

Therefore

$$[y, z]_{2n} = [My, z]_n + [y, Mz]_n.$$

□

Now we study the symmetric domain characterizations of product of two nth-order differential operators L_1 and L_2 which are generated by the same symmetric differential expression M (may be with same or different boundary conditions). Let

$$L_i(y) : \begin{cases} L_i(y) = My, \quad \forall y \in D_i, \\ D_i = \{y \in D_{\max}(M) : U_iY_{a,b} = 0\}, \end{cases} \quad i = 1, 2, \tag{3.8}$$

where $U_i = (A_i : B_i)$ is a composed matrix with rank $(U_i) = n + s\,(0 \leq s \leq n)$, A_i, B_i are $(n + s) \times n$ complex matrices, and $Y_{a,b}$ is given by (3.5).

Set $L(y) = (L_2 \circ L_1)(y) = L_2(L_1(y))$, $y \in D_1, My \in D_2$. By (3.8), we have

$$L : \begin{cases} L(y) = M^2 y, \\ U_1 Y_{a,b} = 0, \\ U_2 (MY)_{a,b} = 0. \end{cases} \tag{3.9}$$

where

$$(MY)_{a,b} = \begin{pmatrix} (MY)_a \\ (MY)_b \end{pmatrix}, \quad (MY)_a = \begin{pmatrix} (My)^{[0]}(a) \\ \vdots \\ (My)^{[n-1]}(a) \end{pmatrix}, \quad (MY)_b = \begin{pmatrix} [My, u_1]_n(b) \\ \vdots \\ [My, u_n]_n(b) \end{pmatrix}, \tag{3.10}$$

and the functions $u_1, u_2, \cdots, u_n$ are defined in Lemma 6.

Theorem 2. *The quasi-derivative* $(My)^{[m]} = i^n y^{[n+m]}$, $0 \le m \le n$.

Proof. From Lemma 4, we have

$$Q^{[2]} = \begin{bmatrix} Q & F \\ 0 & Q \end{bmatrix}, \text{ where } F = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}.$$

When $n = 2$, we have

$$\begin{aligned} y^{[1]} &= q_{12}^{-1}(y' - q_{11}y), \\ y^{[2]} &= (y^{[1]})' - q_{21}y - q_{22}y^{[1]}, \\ y^{[3]} &= q_{12}^{-1}(y^{[2]\prime} - q_{11}y^{[2]}), \\ y^{[4]} &= y^{[3]\prime} - q_{21}y^{[2]} - q_{22}y^{[3]}. \end{aligned}$$

From $My = i^2 y^{[2]}$, it follows that

$$\begin{aligned} (My)^{[1]} &= q_{12}^{-1}((My)' - q_{11}My) \\ &= i^2 q_{12}^{-1}(y^{[2]\prime} - q_{11}y^{[2]}). \end{aligned}$$

Therefore $(My)^{[1]} = i^2 y^{[3]}$.

When $n > 2$, we have

$$\begin{aligned} y^{[1]} &= q_{12}^{-1}(y' - q_{11}y), \\ y^{[2]} &= q_{23}^{-1}\{(y^{[1]})' - q_{21}y - q_{22}y^{[1]}\}, \\ &\cdots\cdots \\ y^{[z]} &= q_{z,z+1}^{-1}\{y^{[z-1]\prime} - \sum_{h=1}^{z} q_{zh}y^{[h-1]}\}, \end{aligned}$$

where $z = 1, 2, \cdots, 2n$. Since

$$\begin{aligned} &My = i^n y^{[n]}, \\ &y^{[n+1]} = q_{12}^{-1}\{y^{[n]\prime} - q_{11}y^{[n]}\}, \\ &(My)^{[1]} = q_{12}^{-1}((My)' - q_{11}My) = i^n q_{12}^{-1}(y^{[n]\prime} - q_{11}y^{[n]}), \end{aligned}$$

it follows that $(My)^{[1]} = i^n y^{[n+1]}$. Assume that for any given $m = k-1 (2 \le k < n)$, $(My)^{[k-1]} = i^n y^{[n+k-1]}$ holds. Consider the case when $m = k$, then we have

$$\begin{aligned} y^{[n+k]} &= q^{-1}_{n+k,n+k+1}\{y^{[n+k-1]'} - \sum_{h=1}^{n+k} q_{n+k,h} y^{[h-1]}\} \\ &= q^{-1}_{k,k+1}\{y^{[n+k-1]'} - \sum_{h=n+1}^{n+k} q_{n+k,h} y^{[h-1]}\} \\ &= q^{-1}_{k,k+1}\{y^{[n+k-1]'} - \sum_{h=1}^{k} q_{kh} y^{[n+h-1]}\} \end{aligned}$$

and

$$\begin{aligned} (My)^{[k]} &= q^{-1}_{k,k+1}\{My^{[k-1]'} - \sum_{h=1}^{k} q_{kh} My^{[h-1]}\} \\ &= i^n q^{-1}_{k,k+1}\{y^{[n+k-1]'} - \sum_{h=1}^{k} q_{kh} y^{[n+h-1]}\}. \end{aligned}$$

Hence $(My)^{[k]} = i^n y^{[n+k]}$ holds. This proof is completed by mathematical induction. □

Corollary 3.

$$\begin{pmatrix} (My)^{[0]}(t) \\ \vdots \\ (My)^{[n-1]}(t) \end{pmatrix} = i^n \begin{pmatrix} (y^{[n]})^{[0]}(t) \\ \vdots \\ (y^{[n]})^{[n-1]}(t) \end{pmatrix} = i^n \, (O : I_n) \begin{pmatrix} y^{[0]}(t) \\ \vdots \\ y^{[2n-1]}(t) \end{pmatrix}, \tag{3.11}$$

where O is the $n \times n$ zero matrix, I_n is the $n \times n$ identity matrix, and $(O : I_n)$ is a composed matrix.

Proof. This result follows directly from Theorem 2. □

In the following, we will rewrite the operator defined in (3.9) in a clear form. We first give the maximal domain decomposition of the differential expression M^2.

It is obvious that $u_1, \cdots, u_n$ defined in Lemma 6 are linearly independent solutions of $M^2(y) = 0$. Let $\psi_1, \psi_2, \cdots, \psi_n$ be n solutions of $M^2(y) = 0$ which satisfy

$$\psi_i^{[j-1]}(a) = 0, \ \psi_i^{[n+j-1]}(a) = \delta_{ij}, \ \ i, j = 1, 2, \cdots, n. \tag{3.12}$$

Combining the conditions of (3.2), we can obtain that $u_1, \cdots, u_n, \psi_1, \cdots, \psi_n$ are $2n$ linearly independent solutions of $M^2 y = 0$. Similar to Lemma 6, we have the following decomposition.

Corollary 4. *The maximal domain $D_{\max}(M^2)$ can has the representation:*

$$D_{\max}(M^2) = D_{\min}(M^2) \dotplus \mathrm{span}\{z_1, \cdots, z_n, z_{n+1}, \cdots, z_{2n}\} \dotplus \mathrm{span}\{u_1, u_2, \cdots u_n, \psi_1, \cdots, \psi_n\}, \tag{3.13}$$

where $z_i \in D_{\max}(M^2), i = 1, \cdots 2n$ such that $z_i(t) = 0$ for $t \ge c$ and $z_i^{[j-1]}(a) = \delta_{ij}, i, j = 1, \cdots, 2n$.

Proof. This proof can be completed by using the same method as the proof of Theorem 4.4.3 in [1]. □

Since $\psi_i \in D_{\max}(M^2)$ and $D_{\max}(M^2) \subset D_{\max}(M)$, by Lemma 6, each ψ_i has a unique representation:

$$\psi_i = y_{i0} + \sum_{j=1}^{n} d_{ij} z_j + \sum_{j=1}^{n} a_{ij} u_j, \quad i = 1, 2, \cdots, n, \tag{3.14}$$

where $y_{i0} \in D_{\min}(M)$, $d_{ij}, a_{ij} \in \mathbb{C}$. Set

$$N = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \tag{3.15}$$

where the entries a_{ij} $(i, j = 1, 2, \cdots, n)$ of N are given in (3.14).

Theorem 3. *The operator L defined in* (3.9) *can be rewritten as*

$$L : \begin{cases} L(y) = M^2 y, \\ UY_{a,b} = 0, \end{cases} \tag{3.16}$$

where $U = (A : B)$ is a composed matrix of matrices

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & i^n A_2 \end{pmatrix}, \tag{3.17}$$

$$B = \begin{pmatrix} -i^n B_1 E_n N^T E_n & i^n B_1 \\ B_2 & 0 \end{pmatrix}, \tag{3.18}$$

and matrices A_1, A_2, B_1, B_2 are given in (3.8).

Proof. By (3.14) and Corollary 4, every $y \in D_{\max}(M^2)$ can be uniquely written as

$$\begin{aligned} y &= y_0 + \sum_{i=1}^{2n} d_i z_i + \sum_{i=1}^{n} c_i u_i + \sum_{i=1}^{n} c_i^* \psi_i \\ &= y_0 + \sum_{i=1}^{2n} d_i z_i + \sum_{i=1}^{n} c_i u_i + \sum_{i=1}^{n} c_i^* (y_{i0} + \sum_{j=1}^{n} d_{ij} z_j + \sum_{j=1}^{n} a_{ij} u_j), \end{aligned} \tag{3.19}$$

where $y_0 \in D_{\min}(M^2)$, $d_i, c_i, c_i^* \in \mathbb{C}$. It follows from Lemma 6 and Lemma 7 that

$$[u_i, u_j]_{2n}(a) = [M(u_i), u_j]_n(a) + [u_i, M(u_j)]_n(a) = 0, \quad i, j = 1, 2, \cdots, n.$$

Let

$$V(a) = \begin{pmatrix} u_1(a) & \cdots & u_n(a) & \psi_1(a) & \cdots & \psi_n(a) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_1^{[2n-1]}(a) & \cdots & u_n^{[2n-1]}(a) & \psi_1^{[2n-1]}(a) & \cdots & \psi_n^{[2n-1]}(a) \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ D & I_n \end{pmatrix}, \tag{3.20}$$

$$W(t)=\begin{pmatrix} [u_1,u_1]_{2n} & \cdots & [u_n,u_1]_{2n} & [\psi_1,u_1]_{2n} & \cdots & [\psi_n,u_1]_{2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ [u_1,u_n]_{2n} & \cdots & [u_n,u_n]_{2n} & [\psi_1,u_n]_{2n} & \cdots & [\psi_n,u_n]_{2n} \\ [u_1,\psi_1]_{2n} & \cdots & [u_n,\psi_1]_{2n} & [\psi_1,\psi_1]_{2n} & \cdots & [\psi_n,\psi_1]_{2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ [u_1,\psi_n]_{2n} & \cdots & [u_n,\psi_n]_{2n} & [\psi_1,\psi_n]_{2n} & \cdots & [\psi_n,\psi_n]_{2n} \end{pmatrix}. \tag{3.21}$$

From (2.7), we have

$$\begin{aligned} W(a) &= -i^{2n}V^*(a)E_{2n}V(a) \\ &= (-1)^{n+1}\begin{pmatrix} I_n & D^* \\ 0 & I_n \end{pmatrix}\begin{pmatrix} 0 & E_n \\ (-1)^nE_n & 0 \end{pmatrix}\begin{pmatrix} I_n & 0 \\ D & I_n \end{pmatrix} \\ &= (-1)^{n+1}\begin{pmatrix} (-1)^nD^*E_n+E_nD & E_n \\ (-1)^nE_n & 0 \end{pmatrix}. \end{aligned} \tag{3.22}$$

Moreover

$$W(a)=(-1)^{n+1}\begin{pmatrix} 0 & E_n \\ (-1)^nE_n & 0 \end{pmatrix}. \tag{3.23}$$

Therefore $(-1)^nD^*E_n+E_nD=0$. Note that u_j,ψ_j, $j=1,2,\cdots,n$ are solutions of M^2y. By the Lagrange identity, we know $W(t)$ is constant on $J=(a,b)$. By (2.9), (3.2) and (3.19), we have

$$\begin{pmatrix} [y,u_1]_{2n}(b) \\ \vdots \\ [y,u_n]_{2n}(b) \\ [y,\psi_1]_{2n}(b) \\ \vdots \\ [y,\psi_n]_{2n}(b) \end{pmatrix} = W(b)\begin{pmatrix} c_1 \\ \vdots \\ c_n \\ c_1^* \\ \vdots \\ c_n^* \end{pmatrix} = W(a)\begin{pmatrix} c_1 \\ \vdots \\ c_n \\ c_1^* \\ \vdots \\ c_n^* \end{pmatrix}. \tag{3.24}$$

Hence

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \\ c_1^* \\ \vdots \\ c_n^* \end{pmatrix} = (W(a))^{-1}\begin{pmatrix} [y,u_1]_{2n}(b) \\ \vdots \\ [y,u_n]_{2n}(b) \\ [y,\psi_1]_{2n}(b) \\ \vdots \\ [y,\psi_n]_{2n}(b) \end{pmatrix} = (-1)^{n+1}\begin{pmatrix} 0 & (-1)^nE_n^{-1} \\ E_n^{-1} & 0 \end{pmatrix}\begin{pmatrix} [y,u_1]_{2n}(b) \\ \vdots \\ [y,u_n]_{2n}(b) \\ [y,\psi_1]_{2n}(b) \\ \vdots \\ [y,\psi_n]_{2n}(b) \end{pmatrix}. \tag{3.25}$$

From the decomposition (3.19), we have

$$\begin{pmatrix} [y,u_1]_n(b) \\ \vdots \\ [y,u_n]_n(b) \end{pmatrix} = \begin{pmatrix} -i^nE_n & -i^nE_nN^T \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_n \\ c_1^* \\ \vdots \\ c_n^* \end{pmatrix}$$

$$
= (-i)^n \begin{pmatrix} E_n & E_n N^T \end{pmatrix} \begin{pmatrix} 0 & (-1)^n E_n^{-1} \\ E_n^{-1} & 0 \end{pmatrix} \begin{pmatrix} [y,u_1]_{2n}(b) \\ \vdots \\ [y,u_n]_{2n}(b) \\ [y,\psi_1]_{2n}(b) \\ \vdots \\ [y,\psi_n]_{2n}(b) \end{pmatrix}
$$

$$
= (-i)^n \begin{pmatrix} E_n N^T E_n^{-1} & (-1)^n I_n \end{pmatrix} \begin{pmatrix} [y,u_1]_{2n}(b) \\ \vdots \\ [y,u_n]_{2n}(b) \\ [y,\psi_1]_{2n}(b) \\ \vdots \\ [y,\psi_n]_{2n}(b) \end{pmatrix}. \tag{3.26}
$$

For any $y \in D_{\max}(M^2)$, it follows from Lemma 7 and $Mu_i = 0$ that

$$
[y,u_i]_{2n} = [My,u_i]_n + [y,Mu_i]_n = [My,u_i]_n, \quad i = 1,2,\cdots,n. \tag{3.27}
$$

Namely

$$
[y,u_i]_{2n} = [My,u_i]_n, \quad i = 1,2,\cdots,n. \tag{3.28}
$$

The proof of Theorem 3 is now immediate from the above discussion and (3.9). □

Furthermore, we have the following result.

Theorem 4. *The relationship $\overline{N}E_n + (-1)^n E_n N^T = 0$ holds.*

Proof. By Lemmas 7 and Lemma 3, we have

$$
\begin{aligned}
[\psi_i,\psi_j]_{2n}(b) &= [M\psi_i,\psi_j]_n(b) + [\psi_i, M\psi_j]_n(b) \\
&= [M\psi_i, \sum_{k=1}^n a_{jk}u_k]_n(b) + [\sum_{k=1}^n a_{ik}u_k, M\psi_j]_n(b) \\
&= \sum_{k=1}^n \overline{a}_{jk}[M\psi_i,u_k]_n(b) + \sum_{k=1}^n a_{ik}[u_k, M\psi_j]_n(b) \\
&= \sum_{k=1}^n \overline{a}_{jk}[\psi_i,u_k]_{2n}(b) + \sum_{k=1}^n a_{ik}[u_k,\psi_j]_{2n}(b) \\
&= \sum_{k=1}^n \overline{a}_{jk}[\psi_i,u_k]_{2n}(a) + \sum_{k=1}^n a_{ik}[u_k,\psi_j]_{2n}(a) \\
&= 0.
\end{aligned}
$$

Let $\overline{N}E_n + (-1)^n E_n N^T = (b_{ij})_{1\le i,j\le n}$. Then by (3.23), we have

$$
b_{ij} = \sum_{k=1}^n (-1)^{n+1}\overline{a}_{jk}[\psi_i,u_k]_{2n}(a) + \sum_{k=1}^n (-1)^{n+1} a_{ik}[u_k,\psi_j]_{2n}(a) = 0.
$$

This completes the proof. □

Based on the above lemmas and theorems, we now obtain our main result: the symmetric characterization of product of two differential operators.

Theorem 5. *Let the hypothesis and notations of Theorem 3 hold. Then the product operator $L = L_2 \circ L_1$ is symmetric if and only if*

$$\operatorname{rank}(A_1 E_n A_2^* - B_1 E_n B_2^*) = 2s, \tag{3.29}$$

where $0 \leq s \leq n$.

Proof. Since $\operatorname{rank}(A_i : B_i) = n + s \, (i = 1, 2)$ and

$$\begin{aligned} \operatorname{rank}(U) &= \operatorname{rank}\begin{pmatrix} A_1 & 0 & -i^n B_1 E_n N^T E_n & i^n B_1 \\ 0 & i^n A_2 & B_2 & 0 \end{pmatrix} \\ &= \operatorname{rank}\begin{pmatrix} A_1 & i^n B_1 & 0 & -i^n B_1 E_n N^T E_n \\ 0 & 0 & i^n A_2 & B_2 \end{pmatrix}, \end{aligned}$$

we can obtain $\operatorname{rank}(U) = 2n + 2s$. By computation, we have

$$AE_{2n}A^* = \begin{pmatrix} 0 & i^n A_1 E_n A_2^* \\ i^n A_2 E_n A_1^* & 0 \end{pmatrix},$$

$$\begin{aligned} BW(b)B^* &= (-1)^{n+1} \begin{pmatrix} -i^n B_1 E_n N^T E_n & i^n B_1 \\ B_2 & 0 \end{pmatrix} \\ &\begin{pmatrix} 0 & E_n \\ (-1)^n E_n & 0 \end{pmatrix} \begin{pmatrix} (-1)^{n+1} i^n E_n \overline{N} E_n B_1^* & B_2^* \\ (-i)^n B_1^* & 0 \end{pmatrix} \\ &= (-1)^{n+1} \begin{pmatrix} B_1(\overline{N} E_n + (-1)^n E_n N^T) B_1^* & (-1)^{n+1} i^n B_1 E_n B_2^* \\ (-1)^{n+1} i^n B_2 E_n B_1^* & 0 \end{pmatrix}. \end{aligned}$$

Now we will use the basic Theorem 1 to prove our result. Note that here the matrix $E_{m_b}(b)$ given in Theorem 1 is $W(b)$. Then, combining with Theorem 4, we have

$$\begin{aligned} C &= AE_n A^* - BW(b)B^* \\ &= \begin{pmatrix} 0 & i^n (A_1 E_n A_2^* - B_1 E_n B_2^*) \\ i^n (A_2 E_n A_1^* - B_2 E_n B_1^*) & 0 \end{pmatrix}. \end{aligned}$$

Therefore, by Theorem 1, the operator L is symmetric if and only if

$$\operatorname{rank}(C) = 4s$$

which is equal to

$$\operatorname{rank}(A_1 E_n A_2^* - B_1 E_n B_2^*) = 2s.$$

Thus the proof is completed. □

Remark 1. *For theorem 5, if $s = 0$, then L is self-adjoint if and only if $A_1 E_n A_2^* = B_1 E_n B_2^*$.*

3.2. The case when both endpoints are LC singular

We will consider the symmetric characterization of product of two differential expressions for the case when both endpoints are LC singular.

In the following we always let $Q \in Z_n(J), J = (a,b), n \in \mathbb{N}_2$, be Lagrange symmetric, w a weight function, the two endpoints a and b LC singular, and let $My = M_Q y = \lambda wy$ be the corresponding symmetric differential equation.

By Lemma 5, we have the next lemma.

Lemma 8. *If the expression $M = M_Q$ is LC singular at both endpoints of $J = (a,b)$, then M^2 is LC singular at both endpoints.*

We first reduce the decomposition Theorem 4.4.4 of [1] to the case when both endpoints are LC singular.

Lemma 9. *Let M be Lagrange symmetric, a and b LC singular, and let $c \in (a,b)$. Then $Q \in Z_n((a,c))$, $Q \in Z_n((c,b))$, and the deficiency indices of $My = \lambda wy$ on $(a,c),(c,b)$ and (a,b) are all n. Then*
(1) There exist n linearly independent solutions $p_1,\cdots,p_n$ of $My = 0$ on (a,c) such that $p_i^{[j-1]}(c) = \delta_{ij}$ $(i,j = 1,2,\ldots,n)$. For $1 \le i,j \le n$, we have

$$[p_i,p_j]_n(a) = [p_i,p_j]_n(c). \tag{3.30}$$

The solutions $p_1,\cdots,p_n$ can be extended to (a,b) such that the extended functions, also denoted by $p_1,\cdots,p_n$, are in $D_{\max}(a,b)$ and are identically 0 near b.
(2) There exist n linearly independent solutions $v_1,\cdots,v_n$ of $My = 0$ on (c,b) such that $v_i^{[j-1]}(c) = \delta_{ij}$ $(i,j = 1,2,\ldots,n)$. For $1 \le i,j \le n$, we have

$$[v_i,v_j]_n(b) = [v_i,v_j]_n(c). \tag{3.31}$$

The solutions $v_1,\cdots,v_n$ can be extended to (a,b) such that the extended functions, also denoted by $v_1,\cdots,v_n$, are in $D_{\max}(a,b)$ and are identically 0 near a.
(3) By the extended functions p_i and v_i, the maximal domain has the following decomposition:

$$D_{\max}(a,b) = D_{\min}(a,b) \dot{+} \operatorname{span}\{p_1,\cdots,p_n\} \dot{+} \operatorname{span}\{v_1,\cdots,v_n\}. \tag{3.32}$$

Proof. This lemma can be directly obtained from Theorem 4.4.4 of [1]. Since p_i and v_i are respectively the solutions of $My = 0$ on (a,c) and (c,b), together with the Lagrange identity, we can obtain (3.30) and (3.31) hold. □

Let

$$\widehat{E}(a) = \begin{pmatrix} [p_1,p_1]_n(a) & \cdots & [p_n,p_1]_n(a) \\ \vdots & \ddots & \vdots \\ [p_1,p_n]_n(a) & \cdots & [p_n,p_n]_n(a) \end{pmatrix}, \quad \widehat{E}(b) = \begin{pmatrix} [v_1,v_1]_n(b) & \cdots & [v_n,v_1]_n(b) \\ \vdots & \ddots & \vdots \\ [v_1,v_n]_n(b) & \cdots & [v_n,v_n]_n(b) \end{pmatrix}. \tag{3.33}$$

By (1) and (2) of Lemma 9, we have $\widehat{E}(a) = \widehat{E}(b) = -i^n E_n$.

Let

$$\widehat{Y}_{a,b} = \begin{pmatrix} \widehat{Y}_a \\ \widehat{Y}_b \end{pmatrix}, \quad \widehat{Y}_a = \begin{pmatrix} [y,p_1]_n(a) \\ \vdots \\ [y,p_n]_n(a) \end{pmatrix}, \quad \widehat{Y}_b = \begin{pmatrix} [y,v_1]_n(b) \\ \vdots \\ [y,v_n]_n(b) \end{pmatrix}. \tag{3.34}$$

The following is a minor modification of Theorem 6.3.3 of [1].

Lemma 10. *Let the hypothesis and notations of Lemma 9 hold. Suppose U is a boundary matrix with* $\operatorname{rank}(U) = l, 0 \le l \le 2n$. *Let $U = (A : B)$, where $A_{l\times n}$ and $B_{l\times n}$ are complex matrices. Define the operator $S(U)$ in $L^2(J, w)$ by*

$$D(S(U)) = \{y \in D_{\max} : U\widehat{Y}_{a,b} = 0\},$$
$$S(U)y = S_{\max}y \quad \text{for } y \in D(S(U)).$$

Let $\operatorname{rank}(A\widehat{E}(a)A^* - B\widehat{E}(b)B^*) = r$, *where $\widehat{E}(a)$ and $\widehat{E}(b)$ are given in* (3.33). *Then we have:*
(i) If $l < n$, then $S(U)$ is not symmetric;
(ii) Let $l = n + s, 0 \le s \le n$. Then $S(U)$ is symmetric if and only if $r = 2s$.

Let the operators $\widehat{L_1}$ and $\widehat{L_2}$ be generated by the same LC symmetric differential expression M :

$$\widehat{L_i} : \begin{cases} \widehat{L_i}(y) = My, \quad \forall y \in \widehat{D_i}, \\ \widehat{D_i} = \{y \in D_{\max} : U_i\widehat{Y}_{a,b} = 0\}, \end{cases} \quad i = 1, 2, \tag{3.35}$$

where $U_i = (A_i : B_i)$, $\operatorname{rank}(U_i) = n + s$, $0 \le s \le n$, A_i, B_i are $(n + s) \times n$ matrices, and $\widehat{Y}_{a,b}$ is given in (3.34).

Let $\widehat{L} = \widehat{L_2}\circ\widehat{L_1}$, $y \in \widehat{D_1}$, $My \in \widehat{D_2}$, then we have:

$$\widehat{L} : \begin{cases} \widehat{L}(y) = M^2y, \\ U_1\widehat{Y}_{a,b} = 0, \\ U_2(\widehat{MY})_{a,b} = 0, \end{cases} \tag{3.36}$$

where

$$(\widehat{MY})_{a,b} = \begin{pmatrix} (\widehat{MY})_a \\ (\widehat{MY})_b \end{pmatrix}, \ (\widehat{MY})_a = \begin{pmatrix} [My, p_1]_n(a) \\ \vdots \\ [My, p_n]_n(a) \end{pmatrix}, \ (\widehat{MY})_b = \begin{pmatrix} [My, v_1]_n(b) \\ \vdots \\ [My, v_n]_n(b) \end{pmatrix}. \tag{3.37}$$

It is obvious that $p_1, \cdots, p_n$ are solutions of $M^2(y) = 0$ on (a, c), and $v_1, \cdots, v_n$ are solutions of $M^2(y) = 0$ on (c, b). Let $\theta_1, \theta_2, \cdots, \theta_n$ be solutions of $M^2(y) = 0$ on (a, c) , and let $\beta_1, \beta_2, \cdots, \beta_n$ be solutions of $M^2(y) = 0$ on (c, b), which satisfy

$$\theta_i^{[j-1]}(c) = 0, \ \theta_i^{[n-1+j]}(c) = \delta_{ij}, \quad i, j = 1, 2, \cdots, n, \tag{3.38}$$

$$\beta_i^{[j-1]}(c) = 0, \ \beta_i^{[n-1+j]}(c) = \delta_{ij}, \quad i, j = 1, 2, \cdots, n. \tag{3.39}$$

It is clear that $p_1, \cdots, p_n, \theta_1, \cdots, \theta_n$ are $2n$ linearly independent solutions of $M^2y = 0$ on (a, c) and $v_1, \cdots, v_n, \beta_1, \cdots, \beta_n$ are $2n$ linearly independent solutions of $M^2y = 0$ on (c, b). By Naimark Patching Lemma 2, $\theta_1, \cdots, \theta_n$ can be extended to (a, b) such that the extended functions, also denoted by $\theta_1, \cdots, \theta_n$, are in $D_{\max}(a, b)$ and are identically 0 near b. Similarly, $\beta_1, \cdots, \beta_n$ can be extended to (a, b) such that the extended functions, also denoted by $\beta_1, \cdots, \beta_n$, are in $D_{\max}(a, b)$ and are identically 0 near a. Similar to Lemma 9 , we have the following decomposition.

Corollary 5. *The maximal domain $D_{\max}(M^2) = D_{\max}(M^2, (a, b))$ has the representation:*

$$D_{\max}(M^2) = D_{\min}(M^2) \dotplus \operatorname{span}\{p_1, \cdots, p_n, \theta_1, \cdots, \theta_n\} \dotplus \operatorname{span}\{v_1, v_2, \cdots v_n, \beta_1, \cdots, \beta_n\}. \tag{3.40}$$

Since $\theta_i, \beta_i \in D_{\max}(M^2)$ and $D_{\max}(M^2) \subset D_{\max}(M)$, by (3.32), each θ_i and β_i has a unique representation:

$$\theta_i = y_{i0} + \sum_{j=1}^{n} a_{ij} p_j + \sum_{j=1}^{n} b_{ij} v_j, \quad i = 1, 2, \cdots, n, \tag{3.41}$$

$$\beta_i = y_{0i} + \sum_{j=1}^{n} c_{ij} p_j + \sum_{j=1}^{n} d_{ij} v_j, \quad i = 1, 2, \cdots, n, \tag{3.42}$$

where $y_{i0}, y_{0i} \in D_{\min}(M)$, $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbb{C}$.

Set

$$N_1 = (a_{ij})_{n\times n}, \quad N_2 = (d_{ij})_{n\times n}, \tag{3.43}$$

where the entries a_{ij}, d_{ij} are given in (3.41) and (3.42).

Theorem 6. *The operator $\widehat{L}$ defined in (3.36) can be rewritten as*

$$\widehat{L} : \begin{cases} \widehat{L}(y) = M^2 y, \\ U\widehat{Y}_{a,b} = 0, \end{cases} \tag{3.44}$$

where $U = (A : B)$,

$$A = \begin{pmatrix} -i^n A_1 E_n N_1^T E_n & i^n A_1 \\ A_2 & 0 \end{pmatrix}, \tag{3.45}$$

$$B = \begin{pmatrix} -i^n B_1 E_n N_2^T E_n & i^n B_1 \\ B_2 & 0 \end{pmatrix}. \tag{3.46}$$

and matrices A_1, A_2, B_1, B_2 are given in (3.35).

Proof. By Corollary 5, for every $y \in D_{\max}(M^2)$, we have

$$y = y_0 + \sum_{i=1}^{n} a_i p_i + \sum_{i=1}^{n} c_i \theta_i + \sum_{i=1}^{n} b_i v_i + \sum_{i=1}^{n} d_i \beta_i, \tag{3.47}$$

where $y_0 \in D_{\min}(M^2), a_i, b_i, c_i, d_i \in \mathbb{C}$.

Let

$$\widehat{U}(c) = \begin{pmatrix} p_1(c) & \cdots & p_n(c) & \theta_1(c) & \cdots & \theta_n(c) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ p_1^{[2n-1]}(c) & \cdots & p_n^{[2n-1]}(c) & \theta_1^{[2n-1]}(c) & \cdots & \theta_n^{[2n-1]}(c) \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ D_1 & I_n \end{pmatrix}, \tag{3.48}$$

$$\widehat{V}(c) = \begin{pmatrix} v_1(c) & \cdots & v_n(c) & \beta_1(c) & \cdots & \beta_n(c) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_1^{[2n-1]}(c) & \cdots & v_n^{[2n-1]}(c) & \beta_1^{[2n-1]}(c) & \cdots & \beta_n^{[2n-1]}(c) \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ D_2 & I_n \end{pmatrix}, \tag{3.49}$$

$$\widehat{W}(t) = \begin{pmatrix} [p_1, p_1]_{2n} & \cdots & [p_n, p_1]_{2n} & [\theta_1, p_1]_{2n} & \cdots & [\theta_n, p_1]_{2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ [p_1, p_n]_{2n} & \cdots & [p_n, p_n]_{2n} & [\theta_1, p_n]_{2n} & \cdots & [\theta_n, p_n]_{2n} \\ [p_1, \theta_1]_{2n} & \cdots & [p_n, \theta_1]_{2n} & [\theta_1, \theta_1]_{2n} & \cdots & [\theta_n, \theta_1]_{2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ [p_1, \theta_n]_{2n} & \cdots & [p_n, \theta_n]_{2n} & [\theta_1, \theta_n]_{2n} & \cdots & [\theta_n, \theta_n]_{2n} \end{pmatrix}, \tag{3.50}$$

$$\widehat{Z}(t)=\begin{pmatrix} [v_1,v_1]_{2n} & \cdots & [v_n,v_1]_{2n} & [\beta_1,v_1]_{2n} & \cdots & [\beta_n,v_1]_{2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ [v_1,v_n]_{2n} & \cdots & [v_n,v_n]_{2n} & [\beta_1,v_n]_{2n} & \cdots & [\beta_n,v_n]_{2n} \\ [v_1,\beta_1]_{2n} & \cdots & [v_n,\beta_1]_{2n} & [\beta_1,\beta_1]_{2n} & \cdots & [\beta_n,\beta_1]_{2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ [v_1,\beta_n]_{2n} & \cdots & [v_n,\beta_n]_{2n} & [\beta_1,\beta_n]_{2n} & \cdots & [\beta_n,\beta_n]_{2n} \end{pmatrix}. \tag{3.51}$$

From (2.7), we have

$$\begin{aligned}\widehat{W}(c)&=-i^{2n}\widehat{U}^*(c)E_{2n}\widehat{U}(c)\\ &=(-1)^{n+1}\begin{pmatrix} I_n & D_1^* \\ 0 & I_n\end{pmatrix}\begin{pmatrix} 0 & E_n \\ (-1)^nE_n & 0\end{pmatrix}\begin{pmatrix} I_n & 0 \\ D_1 & I_n\end{pmatrix}\\ &=(-1)^{n+1}\begin{pmatrix} (-1)^nD_1^*E_n+E_nD_1 & E_n \\ (-1)^nE_n & 0\end{pmatrix},\end{aligned}$$

$$\begin{aligned}\widehat{Z}(c)&=-i^{2n}\widehat{V}^*(c)E_{2n}\widehat{V}(c)\\ &=(-1)^{n+1}\begin{pmatrix} I_n & D_2^* \\ 0 & I_n\end{pmatrix}\begin{pmatrix} 0 & E_n \\ (-1)^nE_n & 0\end{pmatrix}\begin{pmatrix} I_n & 0 \\ D_2 & I_n\end{pmatrix}\\ &=(-1)^{n+1}\begin{pmatrix} (-1)^nD_2^*E_n+E_nD_2 & E_n \\ (-1)^nE_n & 0\end{pmatrix}.\end{aligned}$$

By Lemma 7, we have

$$[p_i,p_j]_{2n}(c)=[Mp_i,p_j]_n(c)+[p_i,Mp_j]_n(c)=0,\quad i,j=1,2,\cdots,n,$$

$$[v_i,v_j]_{2n}(c)=[Mv_i,v_j]_n(c)+[v_i,Mv_j]_n(c)=0,\quad i,j=1,2,\cdots,n.$$

Therefore

$$\widehat{W}(c)=\widehat{Z}(c)=(-1)^{n+1}\begin{pmatrix} 0 & E_n \\ (-1)^nE_n & 0\end{pmatrix}.$$

At the same time, we obtain that $(-1)^nD_i^*E_n+E_nD_i=0$, $i=1,2$. Since p_j,θ_j $(j=1,2,\cdots,n)$ are solutions of $M^2y=0$ on (a,c), v_j,β_j $(j=1,2,\cdots,n)$ are solutions of $M^2y=0$ on (c,b). It follows from the Lagrange identity that $\widehat{W}(t)$, $\widehat{Z}(t)$ are constant on (a,c) and (c,b), respectively. Note that p_i,θ_i are identically 0 near b and v_i,β_i are identically 0 near a. Then, for $i,j=1,2\cdots,n$, we have

$$[p_i,v_j]_n(b)=[p_i,\beta_j]_n(b)=[\theta_i,v_j]_n(b)=[\theta_i,\beta_j]_n(b)=0, \tag{3.52}$$

$$[v_i,p_j]_n(a)=[v_i,\theta_j]_n(a)=[\beta_i,p_jt]_n(a)=[\beta_i,\theta_j]_n(a)=0, \tag{3.53}$$

$$[p_i,v_j]_{2n}(b)=[p_i,\beta_j]_{2n}(b)=[\theta_i,v_j]_{2n}(b)=[\theta_i,\beta_j]_{2n}(b)=0, \tag{3.54}$$

$$[v_i,p_j]_{2n}(a)=[v_i,\theta_j]_{2n}(a)=[\beta_i,p_j]_{2n}(a)=[\beta_i,\theta_j]_{2n}(a)=0. \tag{3.55}$$

By (2.9), (3.47), (3.54) and (3.55), we have

$$\begin{pmatrix} [y, p_1]_{2n}(a) \\ \vdots \\ [y, p_n]_{2n}(a) \\ [y, \theta_1]_{2n}(a) \\ \vdots \\ [y, \theta_n]_{2n}(a) \end{pmatrix} = \widehat{W}(a) \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \widehat{W}(c) \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ c_1 \\ \vdots \\ c_n \end{pmatrix}, \tag{3.56}$$

$$\begin{pmatrix} [y, v_1]_{2n}(b) \\ \vdots \\ [y, v_n]_{2n}(b) \\ [y, \beta_1]_{2n}(b) \\ \vdots \\ [y, \beta_n]_{2n}(b) \end{pmatrix} = \widehat{Z}(b) \begin{pmatrix} b_1 \\ \vdots \\ b_n \\ d_1 \\ \vdots \\ d_n \end{pmatrix} = \widehat{Z}(c) \begin{pmatrix} b_1 \\ \vdots \\ b_n \\ d_1 \\ \vdots \\ d_n \end{pmatrix}. \tag{3.57}$$

Hence

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = (\widehat{W}(c))^{-1} \begin{pmatrix} [y, p_1]_{2n}(a) \\ \vdots \\ [y, p_n]_{2n}(a) \\ [y, \theta_1]_{2n}(a) \\ \vdots \\ [y, \theta_n]_{2n}(a) \end{pmatrix} = (-1)^{n+1} \begin{pmatrix} 0 & (-1)^n E_n^{-1} \\ E_n^{-1} & 0 \end{pmatrix} \begin{pmatrix} [y, p_1]_{2n}(a) \\ \vdots \\ [y, p_n]_{2n}(a) \\ [y, \theta_1]_{2n}(a) \\ \vdots \\ [y, \theta_n]_{2n}(a) \end{pmatrix}, \tag{3.58}$$

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \\ d_1 \\ \vdots \\ d_n \end{pmatrix} = (\widehat{Z}(c))^{-1} \begin{pmatrix} [y, v_1]_{2n}(b) \\ \vdots \\ [y, v_n]_{2n}(b) \\ [y, \beta_1]_{2n}(b) \\ \vdots \\ [y, \beta_n]_{2n}(b) \end{pmatrix} = (-1)^{n+1} \begin{pmatrix} 0 & (-1)^n E_n^{-1} \\ E_n^{-1} & 0 \end{pmatrix} \begin{pmatrix} [y, v_1]_{2n}(b) \\ \vdots \\ [y, v_n]_{2n}(b) \\ [y, \beta_1]_{2n}(b) \\ \vdots \\ [y, \beta_n]_{2n}(b) \end{pmatrix}. \tag{3.59}$$

By (3.41), (3.42), (3.47), (3.52) and (3.53), we have

$$\begin{aligned}
\begin{pmatrix} [y,p_1]_n(a) \\ \vdots \\ [y,p_n]_n(a) \end{pmatrix} &= \begin{pmatrix} -i^n E_n & -i^n E_n N_1^T \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ c_1 \\ \vdots \\ c_n \end{pmatrix} \\
&= (-i)^n \begin{pmatrix} E_n & E_n N_1^T \end{pmatrix} \begin{pmatrix} 0 & (-1)^n E_n^{-1} \\ E_n^{-1} & 0 \end{pmatrix} \begin{pmatrix} [y,p_1]_{2n}(a) \\ \vdots \\ [y,p_n]_{2n}(a) \\ [y,\theta_1]_{2n}(a) \\ \vdots \\ [y,\theta_n]_{2n}(a) \end{pmatrix} \\
&= (-i)^n \begin{pmatrix} E_n N_1^T E_n^{-1} & (-1)^n I_n \end{pmatrix} \begin{pmatrix} [y,p_1]_{2n}(a) \\ \vdots \\ [y,p_n]_{2n}(a) \\ [y,\theta_1]_{2n}(a) \\ \vdots \\ [y,\theta_n]_{2n}(a) \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\begin{pmatrix} [y,v_1]_n(b) \\ \vdots \\ [y,v_n]_n(b) \end{pmatrix} &= \begin{pmatrix} -i^n E_n & -i^n E_n N_2^T \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \\ d_1 \\ \vdots \\ d_n \end{pmatrix} \\
&= (-i)^n \begin{pmatrix} E_n & E_n N_2^T \end{pmatrix} \begin{pmatrix} 0 & (-1)^n E_n^{-1} \\ E_n^{-1} & 0 \end{pmatrix} \begin{pmatrix} [y,v_1]_{2n}(b) \\ \vdots \\ [y,v_n]_{2n}(b) \\ [y,\beta_1]_{2n}(b) \\ \vdots \\ [y,\beta_n]_{2n}(b) \end{pmatrix} \\
&= (-i)^n \begin{pmatrix} E_n N_2^T E_n^{-1} & (-1)^n I_n \end{pmatrix} \begin{pmatrix} [y,v_1]_{2n}(b) \\ \vdots \\ [y,v_n]_{2n}(b) \\ [y,\beta_1]_{2n}(b) \\ \vdots \\ [y,\beta_n]_{2n}(b) \end{pmatrix}.
\end{aligned}$$

For any $y \in D_{\max}(M^2)$, it follows from Lemma 7, together with $Mp_i = 0, Mv_i = 0$, that

$$[y, p_i]_{2n} = [My, p_i]_n + [y, Mp_i]_n = [My, p_i]_n, \quad i = 1, 2, \cdots, n; \tag{3.60}$$

$$[y, v_i]_{2n} = [My, v_i]_n + [y, Mv_i]_n = [My, v_i]_n, \quad i = 1, 2, \cdots, n. \tag{3.61}$$

Thus, in terms of above discussion, the proof for Theorem 6 is completed. □

Theorem 7. *The relationships* $\overline{N_1}E_n + (-1)^n E_n N_1^T = 0, \overline{N_2}E_n + (-1)^n E_n N_2^T = 0$ *hold.*

Proof. Similar to the proof of Theorem 4, we have

$$\begin{aligned}[\theta_i, \theta_j]_{2n}(a) &= [M\theta_i, \theta_j]_n(a) + [\theta_i, M\theta_j]_n(a) \\ &= \sum_{k=1}^{n} \overline{a}_{jk}[\theta_i, p_k]_{2n}(a) + \sum_{k=1}^{n} a_{ik}[p_k, \theta_j]_{2n}(a) = 0\end{aligned}$$

and

$$\begin{aligned}[\beta_i, \beta_j]_{2n}(b) &= [M\beta_i, \beta_j]_n(b) + [\beta_i, M\beta_j]_n(b) \\ &= \sum_{k=1}^{n} \overline{d}_{jk}[\beta_i, v_k]_{2n}(b) + \sum_{k=1}^{n} d_{ik}[v_k, \beta_j]_{2n}(b) = 0.\end{aligned}$$

Then we can easily obtain $\overline{N}_1 E_n + (-1)^n E_n N_1^T = 0$ and $\overline{N}_2 E_n + (-1)^n E_n N_2^T = 0$. □

Based on above discussion, we present the main result:

Theorem 8. *Let the hypothesis and notations of Theorem 6 hold. Then the product operator* $\widehat{L} = \widehat{L_2} \circ \widehat{L_1}$ *is symmetric if and only if*

$$\operatorname{rank}(A_1 E_n A_2^* - B_1 E_n B_2^*) = 2s, \tag{3.62}$$

where $0 \leq s \leq n$.

Proof. Since

$$\operatorname{rank}(U) = \operatorname{rank}\begin{pmatrix} -i^n A_1 E_n N_1^T E_n & i^n A_1 & -i^n B_1 E_n N_2^T E_n & i^n B_1 \\ A_2 & 0 & B_2 & 0 \end{pmatrix}$$

and $\operatorname{rank}(A_i : B_i) = n + s (i = 1, 2)$, we have $\operatorname{rank}(U) = 2n + 2s$.

By computation, we get

$$\begin{aligned} &A\widehat{W}(a)A^* = (-1)^{n+1}\begin{pmatrix} -i^n A_1 E_n N_1^T E_n & i^n A_1 \\ A_2 & 0 \end{pmatrix} \\ &\begin{pmatrix} 0 & (-1)^n E_n^{-1} \\ E_n^{-1} & 0 \end{pmatrix}\begin{pmatrix} (-1)^{n+1} i^n E_n \overline{N}_1 E_n A_1^* & A_2^* \\ (-i)^n A_1^* & 0 \end{pmatrix} \\ &= (-1)^{n+1}\begin{pmatrix} A_1(\overline{N}_1 E_n + (-1)^n E_n N_1^T) A_1^* & (-1)^{n+1} i^n A_1 E_n A_2^* \\ (-1)^{n+1} i^n A_2 E_n A_1^* & 0 \end{pmatrix} \end{aligned}$$

and

$$B\widehat{Z}(b)B^* = (-1)^{n+1}\begin{pmatrix} -i^n B_1(\overline{N}_2 E_n + (-1)^n E_n N_2^T)B_1^* & (-1)^{n+1} i^n B_1 E_n B_2^* \\ (-1)^{n+1} i^n B_2 E_n B_1^* & 0 \end{pmatrix}.$$

To prove the furthermore part we note that here the matrices $\widehat{E}(a)$, $\widehat{E}(b)$ given in Lemma 10 are $\widehat{W}(a)$, $\widehat{Z}(b)$, respectively. Thus, by Lemma 10 and Theorem 7, we have

$$\begin{aligned} &A\widehat{W}(a)A^* - B\widehat{Z}(b)B^* \\ &= (-1)^{n+1}\begin{pmatrix} 0 & (-1)^{n+1} i^n (A_1 E_n A_2^* - B_1 E_n B_2^*) \\ i^n (A_1 E_n A_2^* - B_1 E_n B_2^*)^* & 0 \end{pmatrix}. \end{aligned}$$

It follows from Lemma 10 that L is symmetric if and only if

$$\operatorname{rank}(A\widehat{W}(a)A^* - B\widehat{Z}(b)B^*) = 4s$$

which is equal to

$$\operatorname{rank}(A_1 E_n A_2^* - B_1 E_n B_2^*) = 2s.$$

□

The next corollary is the self-adjoint special case of Theorem 8.

Corollary 6. *Let the hypothesis and notations of Theorem 6 hold. Then the product operator $\widehat{L} = \widehat{L_2} \circ \widehat{L_1}$ is self-adjoint if and only if*

$$A_1 E_n A_2^* - B_1 E_n B_2^* = 0.$$

Proof. This is the special case $s = 0$ of Theorem 8. □

Remark 2. *Recall that the Lagrange brackets $[y, z]$ are well defined at each singular endpoint. These brackets can be used to replace the quasi-derivatives. Our symmetric domain characterization Theorem 8 can be adapted to the maximal deficiency case which occurs when each endpoint is either regular or LC singular. Namely, our result can be used for the four cases for the endpoints: R/R, R/LC, LC/R, LC/LC.*

4. Examples

Example 1. *Consider $My = -(py')' + qy$ on $J = (a, b), -\infty \le a < b \le +\infty$, where $p^{-1}, q \in L_{loc}(J, \mathbb{R})$. Let a and b be LC singular. Set*

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & i \\ 0 & 0 \end{pmatrix}, \ B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & 1 \\ i & 0 \\ 0 & 0 \end{pmatrix}, \ B_2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

The operator $L_1 y = My$ is determined by the boundary conditions:

$$[y, p_1](a) = 0, \ \ i[y, p_2](a) + [y, v_2](b) = 0, \ \ [y, v_1](b) = 0.$$

The operator $L_2y = My$ is determined by the boundary conditions:

$$[y, p_2](a) = 0, \quad i[y, p_1](a) = [y, v_2](b), \quad [y, v_1](b) + [y, v_2](b) = 0.$$

Since here $s = 1$ and $\operatorname{rank}(A_1E_2A_2^* - B_1E_2B_2^*) = 2s = 2$, by Theorem 8, the product operator $L = L_2^\circ L_1$ is symmetric.

Example 2. *Consider $My = -(py')' + qy$ on $J = (a, b), -\infty \le a < b \le +\infty$, where $p^{-1}, q \in L_{loc}(J, \mathbb{R})$. Let a and b be LC singular. Set*

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The operator $L_1y = My$ is determined by the boundary conditions:

$$[y, p_1](a) = 0, \quad [y, p_2](a) = 0, \quad [y, v_1](b) = 0.$$

The operator $L_2y = My$ is determined by the boundary conditions:

$$[y, p_1](a) = 0, \quad [y, p_2](a) = 0, \quad [y, v_2](b) = 0.$$

Since here $s = 1$ and $\operatorname{rank}(A_1E_2A_2^* - B_1E_2B_2^*) = 3$, by Theorem 8, the product operator $L = L_2^\circ L_1$ is not symmetric.

Remark 3. *For the above examples, when the endpoint a is regular for this M, we can simply replace $[y, p_1](a), [y, p_2](a)$ with $y(a)$, $y^{[1]}(a)$, respectively. Similarly for a regular endpoint b, we can replace $[y, v_1](b), [y, v_2](b)$ with $y(b)$, $y^{[1]}(b)$, respectively.*

Example 3. *Let $Q \in Z_n(J), n \in \mathbb{N}_2$, $J = (a, b), -\infty \le a < b \le +\infty$, $M = M_Q$ be the symmetric expression, and let $N_{s\times n}$ $(0 \le s \le n)$ be of full row rank, I_n the $n \times n$ identity matrix. Assume that a and b are LC singular. Choose*

$$A_1 = \begin{pmatrix} N_{s\times n} \\ 0_{n\times n} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0_{s\times n} \\ I_n \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0_{s\times n} \\ I_n \end{pmatrix}, \quad B_2 = \begin{pmatrix} N_{s\times n} \\ 0_{n\times n} \end{pmatrix}.$$

Let

$$U_1 = (A_1 : B_1) = \begin{pmatrix} N & 0 \\ 0 & I_n \end{pmatrix}, \quad U_2 = (A_2 : B_2) = \begin{pmatrix} 0 & N \\ I_n & 0 \end{pmatrix}.$$

We define the operators L_1, L_2 as (3.35). By computation, we have

$$A_1E_nA_2^* - B_1E_nB_2^* = \begin{pmatrix} 0 & NE_n \\ -E_nN^* & 0 \end{pmatrix}.$$

Obviously, $\operatorname{rank}(U_1) = \operatorname{rank}(U_2) = n + s$, $\operatorname{rank}(A_1E_nA_2^* - B_1E_nB_2^*) = 2s$. *Then, by Theorem 8, we know $L = L_2^\circ L_1$ is a symmetric operator.*

Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

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