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Research article

Multiple solutions to Kirchhoff-Schrödinger equations involving the $p(\cdot)$-Laplace- type operator

Yun-Ho Kim*

Department of Mathematics Education, Sangmyung University, Seoul 03016, Republic of Korea

* **Correspondence:** Email: kyh1213@smu.ac.kr.

Abstract: This paper is devoted to deriving several multiplicity results of nontrivial weak solutions to Kirchhoff-Schrödinger equations involving the $p(\cdot)$-Laplace-type operator. The aims of this paper are stated as follows. First, under some conditions on a nonlinear term, we show that our problem has a sequence of infinitely many large energy solutions. Second, we obtain the existence of a sequence of infinitely many small energy solutions to the problem on a new class of nonlinear term. The primary tools to obtain such multiplicity results are the fountain theorem and the dual fountain theorem, respectively.

Keywords: $p(x)$-Laplace type; variable exponent Lebesgue-Sobolev spaces; weak solution; fountain theorem; dual fountain theorem
Mathematics Subject Classification: 35D30, 35J20, 35J60, 35J92, 47J30

1. Introduction

In recent years, the variational problems with nonstandard growth conditions have been extensively studied by many researchers. The interest in variational problems with variable exponents is based in their popularity in diverse fields of mathematical physics, such as electrorheological fluid dynamics, elastic mechanics and image processing. We refer the readers to [2, 8, 9, 25, 28, 40, 43].

Set

$$C_+(\mathbb{R}^N) = \left\{ l \in C(\mathbb{R}^N) : \inf_{x \in \mathbb{R}^N} l(x) > 1 \right\}.$$

For any $m \in C_+(\mathbb{R}^N)$, we define

$$m^+ = \sup_{x \in \mathbb{R}^N} m(x) \quad \text{and} \quad m^- = \inf_{x \in \mathbb{R}^N} m(x).$$

In the present paper, we establish several multiplicity results of nontrivial weak solutions to the $p(\cdot)$-

Laplacian-like equations of Kirchhoff-Schrödinger type:

$$-M\left(\int_{\mathbb{R}^N}\Phi_0(x,\nabla y)\,dx\right)\operatorname{div}(\varphi(x,\nabla y))+\mathfrak{V}(x)\,|y|^{p(x)-2}\,y=\lambda\varrho(x)|y|^{t(x)-2}y+g(x,y)\text{ in }\mathbb{R}^N, \qquad (P_\lambda)$$

where $p\in C_+(\mathbb{R}^N)$ is Lipschitz continuous with $1<p^-\le p^+<N$, $p,q,t\in C_+(\mathbb{R}^N)$ and $1<t^-\le t^+<p^-\le p^+<q^-\le q^+<p^*(x)$ for all $x\in\mathbb{R}^N$, $\varphi(x,\xi)=\nabla_\xi\Phi_0(x,\xi)$ with $\Phi_0:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ being continuously differentiable with respect to the second variable; $M\in C(\mathbb{R}^+)$ is a real function, the function $\varphi(x,\xi)$ is of type $|\xi|^{p(x)-2}\,\xi$ with $p\in C_+(\mathbb{R}^N)$, $g:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and $\mathfrak{V}:\mathbb{R}^N\to(0,\infty)$ is a potential function with

(V) $\mathfrak{V}\in L^1_{loc}(\mathbb{R}^N)$, $\operatorname{ess\,inf}_{x\in\mathbb{R}^N}\mathfrak{V}(x)>0$, and $\lim_{|x|\to\infty}\mathfrak{V}(x)=+\infty$.

Also the Kirchhoff function $M:[0,\infty)\to\mathbb{R}^+$ fulfills the conditions as follows:

(K1) $M\in C(\mathbb{R}^+)$ fulfills $\inf_{t\in\mathbb{R}^+}M(t)\ge m_0>0$, where m_0 is a constant.

(K2) There exists $\vartheta\in[1,\frac{N}{N-p_+})$ such that $\vartheta\mathcal{M}(t)=\vartheta\int_0^t M(\tau)d\tau\ge M(t)t$ for any $t\ge 0$.

Regarding the nonlocal Kirchhoff term, it was first provided by Kirchhoff [29] to study an extension of the classical D'Alembert's wave equation by taking into account the changes in the length of the strings during the vibrations. Elliptic problems of Kirchhoff type have a strong background in diverse applications in physics and have been intensively investigated by many researchers in recent years; see for example, [3,4,6,7,12,15,16,19,22,23,26,31,35,36,39,42,48,49] and the references therein.

Suppose that φ, Φ_0 and g satisfy the assumptions as follows:

(Φ1) The equality

$$\Phi_0(x,\mathbf{0})=0$$

holds for almost all $x\in\mathbb{R}^N$.

(Φ2) There is a constant $b>0$ such that

$$|\varphi(x,\xi)|\le b|\xi|^{p(x)-1}$$

holds for almost all $x\in\mathbb{R}^N$ and for all $\xi\in\mathbb{R}^N$.

(Φ3) There is a positive constant d such that the relations

$$d\,|\xi|^{p(x)}\le\varphi(x,\xi)\cdot\xi \quad\text{and}\quad d|\xi|^{p(x)}\le p^+\Phi_0(x,\xi)$$

hold for all $x\in\mathbb{R}^N$ and $\xi\in\mathbb{R}^N$.

(Φ4) $\Phi_0(x,\cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x\in\mathbb{R}^N$.

(Φ5) The relation

$$r(x)\Phi_0(x,\xi)-\varphi(x,\xi)\cdot\xi\ge 0$$

holds for all $\xi\in\mathbb{R}^N$, where $r\in C_+(\mathbb{R}^N)$ is Lipschitz continuous with $p(\cdot)\le r(\cdot)<p^*(\cdot)$.

(H1) $0\le\varrho\in L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}\left(\mathbb{R}^N\right)\cap L^\infty\left(\mathbb{R}^N\right)$ with $\operatorname{meas}\left\{x\in\mathbb{R}^N:\varrho(x)\ne 0\right\}>0$.

(Ψ1) $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist a positive constant b_1 and a nonnegative function $\sigma_0 \in L^{q'(\cdot)}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ such that

$$|g(x,s)| \le \sigma_0(x) + b_1\,|s|^{q(x)-1}$$

for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$.

(Ψ2) There are $\mu > \vartheta r^+$, $T > 0$ and a function ω with $0 \le \omega \in L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\Lambda_1)$ on $\Lambda_1 := \{x \in \mathbb{R}^N : p(x) > p^-\}$ and $\omega(x) \equiv$ positive constant $\tilde{\omega}$ on $\Lambda_2 := \{x \in \mathbb{R}^N : p(x) = p^-\}$ such that $\text{meas}\{x \in \mathbb{R}^N : \omega(x) > 0\} \neq 0$ and

$$\mu G(x,s) \le s g(x,s) + \omega(x)\,|s|^{p^-}$$

for all $x \in \mathbb{R}^N$ and $|s| \ge T$, where $G(x,s) = \int_0^s g(x,t)\,dt$.

(Ψ3) There exist $C > 0$, $1 < \kappa^- \le \kappa^+ < p^- \le p^+$, $\tau(x) > 1$ with $p(x) \le \tau'(x)\kappa(x) \le p^*(x)$ for all $x \in \mathbb{R}^N$ and a positive function $\eta \in L^{\tau(\cdot)}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ such that

$$\liminf_{|s|\to 0} \frac{g(x,s)}{\eta(x)\,|s|^{\kappa(x)-2}\,s} \ge C$$

uniformly for almost all $x \in \mathbb{R}^N$.

The main goal of this paper is deriving several existence results of multiple solutions to the $p(\cdot)$-Laplacian-like equations of Kirchhoff-Schrödinger type with concave-convex nonlinearities. The first one is to discuss that the problem (P_λ) has a sequence of infinitely many large energy solutions. The other one is to establish the existence of a sequence of infinitely many small energy solutions to the problem (P_λ). The primary tools to obtain such multiplicity results are the fountain theorem and the dual fountain theorem, respectively. Such existence results of multiple solutions to nonlinear elliptic problems are particularly motivated by the contributions in recent studies [1, 5, 17, 18, 20, 24, 28, 31, 32, 34, 36–38, 44, 46], and the references therein. In particular, Alves and Liu [1] obtained the existence and multiplicity results to the superlinear $p(x)$-Laplacian problems:

$$-\text{div}(|\nabla y|^{p(x)-2}\nabla y) + \mathfrak{V}(x)|y|^{p(x)-2}y = g(x,y) \text{ in } \mathbb{R}^N.$$

Here, the potential function $\mathfrak{V} \in C(\mathbb{R}^N)$ satisfies the appropriate conditions and the Carathéodory function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

$(f1)$ $G(x,\ell) = o(|\ell|^{p(x)})$ as $\ell \to 0$ uniformly for all $x \in \mathbb{R}^N$.

$(f2)$ There is a constant $\theta \ge 1$ such that

$$\theta \mathcal{G}(x,\ell) \ge \mathcal{G}(x,s\ell)$$

for $(x,\ell) \in \mathbb{R}^N \times \mathbb{R}$ and $s \in [0,1]$, where $\mathcal{G}(x,\ell) = g(x,\ell)\ell - p^+ G(x,\ell)$.

The condition $(f2)$ is initially provided by the works of Jeanjean [21]. In the last few decades, there were substantial studies dealing with the p-Laplacian problem by assuming $(f2)$; see [37, 38]; see also [26, 45] for the case of variable exponents $p(\cdot)$. Recently, Lin and Tang [34] established various theorems on the existence of solutions of p-Laplacian equations with mild conditions for the superlinear term f, which is deeply different from those investigated in [21, 34, 37, 38]. Also, the authors of [20] obtained the existence results of infinitely many weak solutions to quasilinear elliptic equations with variable exponents under the following condition:

(f3) There exists a constant $C > 0$ such that

$$\ell g(x,\ell) - p^+ G(x,\ell) \le \varsigma g(x,\varsigma) - p^+ G(x,\varsigma) + C$$

for any $x \in \Omega$ and $0 < \ell < \varsigma$ or $\varsigma < \ell < 0$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary,

which was first provided by Miyagaki and Souto [41]. Let us consider the function

$$g(x,\ell) = \sigma(x)\left(|\ell|^{q(x)-2}\,\ell \ln(1+|\ell|) + \frac{|\ell|^{q(x)-1}\,\ell}{1+|\ell|}\right)$$

with its primitive function

$$G(x,\ell) = \frac{\sigma(x)}{q(x)}\,|\ell|^{q(x)}\ln(1+|\ell|)$$

for all $\ell \in \mathbb{R}$ and $q \in C_+(\mathbb{R}^N)$, where $p^+ < q(x)$ for all $x \in \mathbb{R}^N$ and $\sigma \in C(\mathbb{R}^N,\mathbb{R})$ with $0 < \inf_{x\in\mathbb{R}^N}\sigma(x) \le \sup_{x\in\mathbb{R}^N}\sigma(x) < \infty$. Then, this example satisfies the assumptions (f1)–(f3), but not (Ψ3).

Remark 1.1. *If we consider the function*

$$g(x,s) = \sigma(x)\left(\eta(x)\,|s|^{\kappa(x)-2}\,s + |s|^{p^- -2}\,s + \frac{2}{p^-}\sin s\right)$$

with its primitive function

$$G(x,s) = \sigma(x)\left(\frac{\eta(x)}{\kappa(x)}\,|s|^{\kappa(x)} + \frac{1}{p^-}\,|s|^{p^-} - \frac{2}{p^-}\cos s + \frac{2}{p^-}\right),$$

where $\sigma \in C(\mathbb{R}^N,\mathbb{R})$ *with* $0 < \inf_{x\in\mathbb{R}^N}\sigma(x) \le \sup_{x\in\mathbb{R}^N}\sigma(x) < \infty$, *and* κ *and* η *are given in* (Ψ3), *then it is clear that this example satisfies the conditions* (Ψ1)–(Ψ3), *but not* (f1)–(f3).

In this direction, regarding a new class of nonlinear term g which is different from the previous related works, we give the existence results of a sequence of infinitely many energy solutions by employing variational methods. However, our proof of the existence of multiple small energy solutions is slightly different from those of the previous related works [5,17,20,32,36,46,47]. Roughly speaking, in view of [5, 17, 20, 32], the condition (f1) plays an important role in ensuring all assumptions in the dual fountain theorem; however, we verify them when (f1) is changed into (Ψ3).

The outline of this paper is as follows. We present some necessary preliminary knowledge of function spaces which we will use throughout the paper. Next, we provide the variational framework related to the problem (P_λ) and then obtain various existence results of infinitely many nontrivial solutions to the $p(\cdot)$-Laplacian-like equations with concave-convex-type nonlinearities under suitable conditions on g.

2. Preliminaries

In this section, we briefly demonstrate some definitions and essential properties of Lebesgue-Sobolev spaces with a variable exponent in $\mathbb{R}^N$, which are main analysis tools for our work. For a deeper treatment on these spaces, we refer the reader to [8, 9, 14].

For any $l \in C_+(\mathbb{R}^N)$, we introduce the variable exponent Lebesgue space

$$L^{l(\cdot)}(\mathbb{R}^N) := \left\{ y : y \text{ is a measurable real-valued function, } \int_{\mathbb{R}^N} |y(x)|^{l(x)}\, dx < \infty \right\},$$

endowed with the Luxemburg norm

$$|y|_{L^{l(\cdot)}(\mathbb{R}^N)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left| \frac{y(x)}{\lambda} \right|^{l(x)} dx \le 1 \right\}.$$

The dual space of $L^{l(\cdot)}(\mathbb{R}^N)$ is $L^{l'(\cdot)}(\mathbb{R}^N)$, where $1/l(x) + 1/l'(x) = 1$.

The variable exponent Sobolev space $W^{1,l(\cdot)}(\mathbb{R}^N)$ is defined by

$$W^{1,l(\cdot)}(\mathbb{R}^N) = \left\{ y \in L^{l(\cdot)}(\mathbb{R}^N) : |\nabla y| \in L^{l(\cdot)}(\mathbb{R}^N) \right\},$$

with the norm

$$|y|_{W^{1,l(\cdot)}(\mathbb{R}^N)} = |\nabla y|_{L^{l(\cdot)}(\mathbb{R}^N)} + |y|_{L^{l(\cdot)}(\mathbb{R}^N)}. \tag{2.1}$$

We list some well-known results.

Lemma 2.1. *[14] The space $L^{l(\cdot)}(\mathbb{R}^N)$ is a uniformly convex and separable Banach space. For any $y \in L^{l(\cdot)}(\mathbb{R}^N)$ and $z \in L^{l'(\cdot)}(\mathbb{R}^N)$, we have*

$$\left| \int_{\mathbb{R}^N} yz\, dx \right| \le \left(\frac{1}{l^-} + \frac{1}{(l')_-} \right) |y|_{L^{l(\cdot)}(\mathbb{R}^N)} |z|_{L^{l'(\cdot)}(\mathbb{R}^N)} \le 2 |y|_{L^{l(\cdot)}(\mathbb{R}^N)} |z|_{L^{l'(\cdot)}(\mathbb{R}^N)}.$$

Lemma 2.2. ([14]) *If $1/l(x) + 1/m(x) + 1/n(x) = 1$, then, for any $y \in L^{l(\cdot)}(\mathbb{R}^N), z \in L^{m(\cdot)}(\mathbb{R}^N)$ and $w \in L^{n(\cdot)}(\mathbb{R}^N)$,*

$$\begin{aligned} \left| \int_{\mathbb{R}^N} yzw\, dx \right| &\le \left(\frac{1}{l^-} + \frac{1}{m^-} + \frac{1}{n^-} \right) |y|_{L^{l(\cdot)}(\mathbb{R}^N)} |z|_{L^{m(\cdot)}(\mathbb{R}^N)} |w|_{L^{n(\cdot)}(\mathbb{R}^N)} \\ &\le 3 |y|_{L^{l(\cdot)}(\mathbb{R}^N)} |z|_{L^{m(\cdot)}(\mathbb{R}^N)} |w|_{L^{n(\cdot)}(\mathbb{R}^N)}. \end{aligned}$$

Lemma 2.3. ([14]) *Denote*

$$\rho(y) = \int_{\mathbb{R}^N} |y|^{l(x)}\, dx \quad \text{for all } y \in L^{l(\cdot)}(\mathbb{R}^N).$$

Then,

(1) $\rho(y) > 1$ $(= 1;\ < 1)$ if and only if $|y|_{L^{p(\cdot)}(\mathbb{R}^N)} > 1$ $(= 1;\ < 1)$, respectively;

(2) if $|y|_{L^{l(\cdot)}(\mathbb{R}^N)} > 1$, then $|y|^{l^-}_{L^{l(\cdot)}(\mathbb{R}^N)} \le \rho(y) \le |y|^{l^+}_{L^{l(\cdot)}(\mathbb{R}^N)}$;

(3) if $|y|_{L^{l(\cdot)}(\mathbb{R}^N)} < 1$, then $|y|^{l^+}_{L^{l(\cdot)}(\mathbb{R}^N)} \le \rho(y) \le |y|^{l^-}_{L^{l(\cdot)}(\mathbb{R}^N)}$.

Lemma 2.4. ([10]) *Let $l \in C_+(\mathbb{R}^N)$ and $n \in L^\infty(\mathbb{R}^N)$ be such that $1 \le l(x)n(x) \le \infty$ for almost all $x \in \mathbb{R}^N$. If $y \in L^{l(\cdot)n(\cdot)}(\mathbb{R}^N)$ with $y \ne 0$, then the following is true:*

(1) *if* $|y|_{L^{l(\cdot)n(\cdot)}(\mathbb{R}^N)} > 1$, *then* $|y|^{n^-}_{L^{l(\cdot)n(\cdot)}(\mathbb{R}^N)} \leq \big|\,|y|^{n(x)}\,\big|_{L^{l(\cdot)}(\mathbb{R}^N)} \leq |y|^{n^+}_{L^{l(\cdot)n(\cdot)}(\mathbb{R}^N)}$;

(2) *if* $|y|_{L^{l(\cdot)n(\cdot)}(\mathbb{R}^N)} < 1$, *then* $|y|^{n^+}_{L^{l(\cdot)n(\cdot)}(\mathbb{R}^N)} \leq \big|\,|y|^{n(x)}\,\big|_{L^{l(\cdot)}(\mathbb{R}^N)} \leq |y|^{n^-}_{L^{l(\cdot)n(\cdot)}(\mathbb{R}^N)}$.

Lemma 2.5. ([14]) *Assume that* $h : \mathbb{R}^N \to \mathbb{R}$ *is Lipschitz continuous with* $1 < h^- \leq h^+ < N$. *Let* $n \in L^\infty(\mathbb{R}^N)$ *and* $h(x) \leq n(x) \leq h^*(x)$ *for almost all* $x \in \mathbb{R}^N$. *Then, we have a continuous embedding* $W^{1,h(\cdot)}(\mathbb{R}^N) \hookrightarrow L^{n(\cdot)}(\mathbb{R}^N)$.

When $p \in C_+(\mathbb{R}^N)$ and the potential function $\mathfrak{V}$ satisfies (V), let us define the linear subspace

$$X = \left\{ y \in W^{1,p(\cdot)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|\nabla y|^{p(x)} + \mathfrak{V}(x)\, |y|^{p(x)} \right) dx < +\infty \right\}$$

with the norm

$$|y|_X = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left(\left| \frac{\nabla y}{\lambda} \right|^{p(x)} + \mathfrak{V}(x) \left| \frac{y}{\lambda} \right|^{p(x)} \right) dx \leq 1 \right\},$$

which is equivalent to the norm (2.1).

Remark 2.6. ([13]) *Denote*

$$\rho(y) = \int_{\mathbb{R}^N} \left(|\nabla y|^{p(x)} + \mathfrak{V}(x)|y|^{p(x)} \right) dx \quad \text{for all } y \in X.$$

If the assumption (V) *is satisfied, then*

(1) $\rho(y) > 1$ $(= 1;\ < 1)$ *if and only if* $|y|_X > 1$ $(= 1;\ < 1)$, *respectively;*

(2) *if* $|y|_X > 1$, *then* $|y|^{p^-}_X \leq \rho(y) \leq |y|^{p^+}_X$;

(3) *if* $|y|_X < 1$, *then* $|y|^{p^+}_X \leq \rho(y) \leq |y|^{p^-}_X$.

Lemma 2.7. ([1]) *If the assumption* (V) *is satisfied, then*

(1) *we have a compact embedding* $X \hookrightarrow L^{p(\cdot)}(\mathbb{R}^N)$;

(2) *for any measurable function* $q : \mathbb{R}^N \to \mathbb{R}$ *with* $p(x) < q(x)$ *for all* $x \in \mathbb{R}^N$, *there exists a compact embedding* $X \hookrightarrow L^{q(\cdot)}(\mathbb{R}^N)$ *if* $\inf_{x\in\mathbb{R}^N}(p^*(x) - q(x)) > 0$.

Throughout this paper, let $p \in C_+(\mathbb{R}^N)$ be Lipschitz continuous with $1 < p^- \leq p^+ < N$ and the potential $\mathfrak{V}$ satisfy the condition (V) . Furthermore, $\langle \cdot, \cdot \rangle$ denotes the pairing of X and its dual X^*.

3. Existence of infinitely many solutions

In this section, we present the existence of infinitely many nontrivial solutions to the problem (P_λ) by utilizing the fountain theorem and the dual fountain theorem as the primary tools.

Definition 3.1. *By a solution of the problem* (P_λ), *we mean a function* $y \in X$ *such that*

$$M\left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla y)\, dx \right) \int_{\mathbb{R}^N} \varphi(x, \nabla y) \cdot \nabla z\, dx + \int_{\mathbb{R}^N} \mathfrak{V}(x)\, |y|^{p(x)-2}\, yz\, dx$$
$$= \lambda \int_{\mathbb{R}^N} \varrho(x)|y|^{r(x)-2} yz\, dx + \int_{\mathbb{R}^N} g(x, y) z\, dx$$

for all $z \in X$.

Let us define the functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(y) = \mathcal{M}\left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla y)\, dx\right) + \int_{\mathbb{R}^N} \frac{\mathfrak{V}(x)}{p(x)} |y|^{p(x)}\, dx.$$

Under the conditions (Φ1)–(Φ3), we have, by Lemma 3.2 in [33], that Φ is well defined on X, $\Phi \in C^1(X, \mathbb{R})$ and its Fréchet derivative is given by

$$\langle \Phi'(y), z\rangle = M\left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla y)\, dx\right) \int_{\mathbb{R}^N} \varphi(x, \nabla y) \cdot \nabla z\, dx + \int_{\mathbb{R}^N} \mathfrak{V}(x)\, |y|^{p(x)-2}\, yz\, dx.$$

According to the analogous arguments in [30, 33], the following assertion is easily verified, so we omit the proof.

Lemma 3.2. *Suppose that* (K1)*,* (K2) *and* (Φ1)–(Φ4) *are fulfilled. Then,* $\Phi : X \to \mathbb{R}$ *is weakly lower semicontinuous and convex on X. In addition,* Φ' *is a mapping of type* (S_+)*, i.e., if* $y_n \rightharpoonup y$ *in X and* $\limsup_{n\to\infty} \langle \Phi'(y_n) - \Phi'(y), y_n - y\rangle \le 0$*, then* $y_n \to y$ *in X as* $n \to \infty$*.*

Define the functional $\Psi : X \to \mathbb{R}$ by

$$\Psi(y) = \lambda \int_{\mathbb{R}^N} \frac{\varrho(x)}{t(x)} |y|^{t(x)} dx + \int_{\mathbb{R}^N} G(x, y)\, dx.$$

Then, $\Psi \in C^1(X, \mathbb{R})$ and its Fréchet derivative is

$$\langle \Psi'(y), z\rangle \ := \lambda \int_{\mathbb{R}^N} \varrho(x)|y|^{t(x)-2} yz\, dx + \int_{\mathbb{R}^N} g(x, y) z\, dx$$

for any $y, z \in X$. Next, the functional $I_\lambda : X \to \mathbb{R}$ is defined by

$$I_\lambda(y) = \Phi(y) - \Psi(y).$$

Then it is clear that $I_\lambda \in C^1(X, \mathbb{R}^N)$ and its Fréchet derivative is

$$\begin{aligned}\langle I_\lambda'(y), z\rangle = M\left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla y)\, dx\right) \int_{\mathbb{R}^N} \varphi(x, \nabla y) \cdot \nabla z\, dx + \int_{\mathbb{R}^N} \mathfrak{V}(x)\, |y|^{p(x)-2}\, yz\, dx \\ - \int_{\mathbb{R}^N} g(x, y) z\, dx - \lambda \int_{\mathbb{R}^N} \varrho(x)|y|^{t(x)-2} yz\, dx \end{aligned} \tag{3.1}$$

for any $y, z \in X$.

Proceeding the same arguments as in [13, Lemma 3.2], we have that the functionals Ψ and Ψ' are compact operators on X.

Lemma 3.3. *Assume that* (H1) *and* (Ψ1) *hold. Then,* Ψ *and* Ψ' *are compact operators on X.*

With the help of Lemmas 3.2 and 3.3, we show that the energy functional I_λ ensures the Cerami condition ((C)-condition for short), i.e., any sequence $\{y_n\} \subset X$ such that $\{I_\lambda(y_n)\}$ is bounded and $\|I_\lambda'(y_n)\|_{X^*}(1 + \|y_n\|_X) \to 0$ as $n \to \infty$ has a convergent subsequence. This plays a key role in obtaining the existence of nontrivial weak solutions for the given problem. The basic idea of proofs of these consequences follows the analogous arguments in [24]; see also [27].

Lemma 3.4. *Assume that* (H1), (K1), (K2), (Φ1)–(Φ5), (Ψ1) *and* (Ψ2) *hold. Furthermore, we assume that*

(Ψ4) $\lim_{|s|\to\infty} \frac{G(x,s)}{|s|^{\vartheta p^+}} = \infty$ *uniformly for almost all* $x \in \mathbb{R}^N$.

Then, the functional I_λ *satisfies the* (C)*-condition for any* $\lambda > 0$.

Proof. Let $\{y_n\}$ be a (C)-sequence in X, i.e.,

$$\sup_{n\in\mathbb{N}} |I_\lambda(y_n)| \le \Re_1 \text{ and } \langle I'_\lambda(y_n), y_n\rangle = o(1) \to 0, \text{ as } n \to \infty, \tag{3.2}$$

where $\Re_1$ is a positive constant. By virtue of Lemmas 3.2 and 3.3, we have that Φ' is a mapping of type (S_+) and Ψ' is a compact operator on X. Thus, because I'_λ is of type (S_+) and X is reflexive, it is enough to ensure that the sequence $\{y_n\}$ is bounded in X. To this end, suppose, on the contrary, that $\|y_n\|_X > 1$ and $\|y_n\|_X \to \infty$ as $n \to \infty$. Let $z_n = y_n/\|y_n\|_X$. Then, $\|z_n\|_X = 1$. Passing to a subsequence, we may assume that $z_n \rightharpoonup z$ as $n \to \infty$ in X; then, according to Lemma 2.7,

$$z_n \to z \text{ in } L^{s(\cdot)}(\mathbb{R}^N), \quad p(x) \le s(x) < p^*(x) \text{ and } z_n(x) \to z(x) \quad \text{a.e. in } \mathbb{R}^N. \tag{3.3}$$

Denote $\{a \le |y_n| \le b\}| = \{x \in \mathbb{R}^N : a \le |y_n(x)| \le b\}$ for any real number a and b. Since $\mathfrak{V}(x) \to +\infty$ as $|x| \to \infty$, we have

$$\begin{aligned}\left(\frac{1}{\vartheta p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \mathfrak{V}(x)|y_n|^{p(x)}dx - C_1\int_{\{|y_n|\le T\}} \left(|y_n|^{p(x)} + \sigma_0(x)|y_n| + b_1|y_n|^{q(x)}\right)dx \\ \ge \frac{1}{2}\left(\frac{1}{\vartheta p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \mathfrak{V}(x)|y_n|^{p(x)}dx - \Re_0\end{aligned}$$

for any positive constant C_1 and some positive constant $\Re_0$. In fact, by Young's inequality, we know that

$$\begin{aligned}&\left(\frac{1}{\vartheta p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \mathfrak{V}(x)|y_n|^{p(x)}dx - C_1\int_{\{|y_n|\le T\}} \left(|y_n|^{p(x)} + \sigma_0(x)|y_n| + b_1|y_n|^{q(x)}\right) dx \\ &\ge \left(\frac{1}{\vartheta p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \mathfrak{V}(x)|y_n|^{p(x)}dx - C_1\int_{\{|y_n|\le T\}} \left(|y_n|^{p(x)} + \sigma_0^{q'(x)}(x) + |y_n|^{q(x)} + b_1|y_n|^{q(x)}\right) dx \\ &\ge \frac{1}{2}\left(\frac{1}{\vartheta p^+} - \frac{1}{\mu}\right)\left[\int_{\mathbb{R}^N} \mathfrak{V}(x)|y_n|^{p(x)}\, dx + \int_{\{|y_n|\le T\}} \mathfrak{V}(x)|y_n|^{p(x)}\, dx\right] \\ &- C_1\int_{\{|y_n|\le 1\}} \left(|y_n|^{p(x)} + |y_n|^{q(x)} + b_1|y_n|^{q(x)}\right) dx \\ &- C_1\int_{\{1<|y_n|\le T\}} \left(|y_n|^{p(x)} + |y_n|^{q(x)} + b_1|y_n|^{q(x)}\right) dx - C_1(1 + \|\sigma_0\|_{L^{q'(\cdot)}(\mathbb{R}^N)}^{(q')^+}) \\ &\ge \frac{1}{2}\left(\frac{1}{\vartheta p^+} - \frac{1}{\mu}\right)\left[\int_{\mathbb{R}^N} \mathfrak{V}(x)|y_n|^{p(x)}\, dx + \int_{\{|y_n|\le T\}} \mathfrak{V}(x)|y_n|^{p(x)}\, dx\right] \\ &- C_1\,(2 + b_1)\int_{\{|y_n|\le 1\}} |y_n|^{p(x)}\, dx - C_1 T^{q^+-p^-}\,(2 + b_1)\int_{\{1<|y_n|\le T\}} |y_n|^{p(x)}\, dx - \widetilde{C}_1 \\ &\ge \frac{1}{2}\left(\frac{1}{\vartheta p^+} - \frac{1}{\mu}\right)\left[\int_{\mathbb{R}^N} \mathfrak{V}(x)|y_n|^{p(x)}\, dx + \int_{\{|y_n|\le T\}} \mathfrak{V}(x)|y_n|^{p(x)}\, dx\right]\end{aligned}$$

$$- \widetilde{C}_0 \int_{\{|y_n|\le T\}} |y_n|^{p(x)}\,dx - \widetilde{C}_1, \tag{3.4}$$

where $\widetilde{C}_0 := C_1 T^{q^+-p^-}(2+b_1)$ and $\widetilde{C}_1$ is a positive constant. Since $\mathfrak{V}(x)\to+\infty$ as $|x|\to\infty$, there is $\gamma_0>0$ such that $|x|\ge\gamma_0$ implies that $\mathfrak{V}(x)\ge \frac{2\vartheta p^+\mu\widetilde{C}_0}{\mu-\vartheta p^+}$. Then, we know that

$$\mathfrak{V}(x)|y_n|^{p(x)} \ge \frac{2\vartheta p^+\mu\widetilde{C}_0}{\mu-\vartheta p^+}|y_n|^{p(x)} \tag{3.5}$$

for $|x|\ge\gamma_0$. Set $B_\gamma := \{x\in\mathbb{R}^N : |x|<\gamma\}$. Then, since $\mathfrak{V}\in L^1_{loc}(\mathbb{R}^N)$, we infer

$$\int_{\{|y_n|\le T\}\cap B_{\gamma_0}} \mathfrak{V}(x)|y_n|^{p(x)}\,dx \le \widetilde{C}_2 \quad\text{and}\quad \int_{\{|y_n|\le T\}\cap B_{\gamma_0}} |y_n|^{p(x)}\,dx \le \widetilde{C}_3$$

for some positive constants $\widetilde{C}_2$ and $\widetilde{C}_3$. This, together with (3.4) and (3.5), yields

$$\begin{aligned}
&\left(\frac{1}{\vartheta p^+}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}\mathfrak{V}(x)|y_n|^{p(x)}\,dx - C_1\int_{\{|y_n|\le T\}}\Big(|y_n|^{p(x)}+\sigma_0(x)|y_n|+b_1|y_n|^{q(x)}\Big)\,dx\\
&\ge \frac12\left(\frac{1}{\vartheta p^+}-\frac{1}{\mu}\right)\left[\int_{\mathbb{R}^N}\mathfrak{V}(x)|y_n|^{p(x)}\,dx+\int_{\{|y_n|\le T\}\cap B^c_{\gamma_0}}\mathfrak{V}(x)|y_n|^{p(x)}\,dx+\int_{\{|y_n|\le T\}\cap B_{\gamma_0}}\mathfrak{V}(x)|y_n|^{p(x)}\,dx\right]\\
&- \widetilde{C}_0\left[\int_{\{|y_n|\le T\}\cap B^c_{\gamma_0}}|y_n|^{p(x)}\,dx+\int_{\{|y_n|\le T\}\cap B_{\gamma_0}}|y_n|^{p(x)}\,dx\right]-\widetilde{C}_1\\
&\ge \frac12\left(\frac{1}{\vartheta p^+}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}\mathfrak{V}(x)|y_n|^{p(x)}\,dx+\int_{\{|y_n|\le T\}\cap B^c_{\gamma_0}}\frac12\left(\left(\frac{1}{\vartheta p^+}-\frac{1}{\mu}\right)\mathfrak{V}(x)-\widetilde{C}_0\right)|y_n|^{p(x)}\,dx-\mathfrak{K}_0\\
&\ge \frac12\Big(\frac{1}{\vartheta p^+}-\frac{1}{\mu}\Big)\int_{\mathbb{R}^N}\mathfrak{V}(x)|y_n|^{p(x)}\,dx-\mathfrak{K}_0,
\end{aligned}$$

where $\mathfrak{K}_0 := \widetilde{C}_0\widetilde{C}_3+\widetilde{C}_1$, as claimed. Combining this with (K1), (K2), (Φ3), (Φ5), (Ψ1) and (Ψ2), one has

$$\begin{aligned}
\mathfrak{K}_1+1 &\ge I_\lambda(y_n)-\frac{1}{\mu}\langle I'_\lambda(y_n),y_n\rangle\\
&= \mathcal{M}\left(\int_{\mathbb{R}^N}\Phi_0(x,\nabla y_n)\,dx\right)+\int_{\mathbb{R}^N}\frac{\mathfrak{V}(x)}{p(x)}|y_n|^{p(x)}\,dx-\lambda\int_{\mathbb{R}^N}\frac{\varrho(x)}{t(x)}|y_n|^{t(x)}\,dx\\
&\quad-\int_{\mathbb{R}^N}G(x,y_n)\,dx-\frac{1}{\mu}M\left(\int_{\mathbb{R}^N}\Phi_0(x,\nabla y_n)\,dx\right)\int_{\mathbb{R}^N}\varphi(x,\nabla y_n)\cdot\nabla y_n\,dx\\
&\quad-\frac{1}{\mu}\int_{\mathbb{R}^N}\mathfrak{V}(x)\,|y_n|^{p(x)}\,dx+\frac{\lambda}{\mu}\int_{\mathbb{R}^N}\varrho(x)\,|y_n|^{t(x)}\,dx+\frac{1}{\mu}\int_{\mathbb{R}^N}g(x,y_n)y_n\,dx\\
&\ge \frac{1}{\vartheta r^+}M\left(\int_{\mathbb{R}^N}\Phi_0(x,\nabla y_n)\,dx\right)\int_{\mathbb{R}^N}\varphi(x,\nabla y_n)\cdot\nabla y_n\,dx+\int_{\mathbb{R}^N}\frac{\mathfrak{V}(x)}{p(x)}|y_n|^{p(x)}\,dx\\
&\quad-\lambda\int_{\mathbb{R}^N}\frac{\varrho(x)}{t(x)}|y_n|^{t(x)}\,dx-\int_{\mathbb{R}^N}G(x,y_n)\,dx\\
&\quad-\frac{1}{\mu}M\left(\int_{\mathbb{R}^N}\Phi_0(x,\nabla y_n)\,dx\right)\int_{\mathbb{R}^N}\varphi(x,\nabla y_n)\cdot\nabla y_n\,dx
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\mu}\int_{\mathbb{R}^N} \mathfrak{V}(x)\,|y_n|^{p(x)}\,dx + \frac{\lambda}{\mu}\int_{\mathbb{R}^N} \varrho(x)\,|y_n|^{t(x)}\,dx + \frac{1}{\mu}\int_{\mathbb{R}^N} g(x,y_n)y_n\,dx\\
\geq\; & d\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right) M\left(\int_{\mathbb{R}^N} \Phi_0(x,\nabla y_n)\,dx\right)\int_{\mathbb{R}^N} |\nabla y_n|^{p(x)}\,dx\\
&+ \left(\frac{1}{\vartheta p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \mathfrak{V}(x)\,|y_n|^{p(x)}\,dx + \frac{1}{\mu}\int_{\mathbb{R}^N} (g(x,y_n)y_n - \mu G(x,y_n))\,dx\\
&- \lambda\left(\frac{1}{t^-} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \varrho(x)\,|y_n|^{t(x)}\,dx\\
\geq\; & dm_0\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} |\nabla y_n|^{p(x)}\,dx + \left(\frac{1}{\vartheta p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \mathfrak{V}(x)\,|y_n|^{p(x)}\,dx\\
&+ \frac{1}{\mu}\int_{\{|y_n|\leq T\}} (g(x,y_n)y_n - \mu G(x,y_n))\,dx + \frac{1}{\mu}\int_{\{|y_n|\geq T\}} (g(x,y_n)y_n - \mu G(x,y_n))\,dx\\
&+ \lambda\left(\frac{1}{\mu} - \frac{1}{t^-}\right)\int_{\mathbb{R}^N} \varrho(x)\,|y_n|^{t(x)}\,dx\\
\geq\; & dm_0\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} |\nabla y_n|^{p(x)}\,dx + \left(\frac{1}{\vartheta p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \mathfrak{V}(x)\,|y_n|^{p(x)}\,dx\\
&- C_1\int_{\{|y_n|\leq T\}} \left(|y_n|^{p(x)} + \sigma_0(x)|y_n| + b_1|y_n|^{q(x)}\right)dx - \frac{1}{\mu}\int_{\{|y_n|\geq T\}} \omega(x)\,|y_n|^{p^-}\,dx\\
&- \lambda\left(\frac{1}{t^-} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \varrho(x)\,|y_n|^{t(x)}\,dx\\
\geq\; & dm_0\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} |\nabla y_n|^{p(x)}\,dx + \frac{1}{2}\left(\frac{1}{\vartheta p^+} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \mathfrak{V}(x)\,|y_n|^{p(x)}\,dx\\
&\quad - \frac{1}{\mu}\int_{\mathbb{R}^N} \omega(x)\,|y_n|^{p^-}\,dx - \lambda\left(\frac{1}{t^-} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \varrho(x)\,|y_n|^{t(x)}\,dx - \mathfrak{K}_0\\
\geq\; & \frac{\min\{dm_0, 1\}}{2}\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right)\left(\int_{\mathbb{R}^N} |\nabla y_n|^{p(x)}\,dx + \int_{\mathbb{R}^N} \mathfrak{V}(x)\,|y_n|^{p(x)}\,dx\right)\\
&\quad - \frac{1}{\mu}\left(\int_{\Lambda_1} \omega(x)\,|y_n|^{p^-}\,dx + \int_{\Lambda_2} \omega(x)\,|y_n|^{p^-}\,dx\right)\\
&\quad - 2\lambda\left(\frac{1}{t^-} - \frac{1}{\mu}\right)|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} \max\left\{|y_n|^{t^-}_{L^{p(\cdot)}(\mathbb{R}^N)}, |y_n|^{t^+}_{L^{p(\cdot)}(\mathbb{R}^N)}\right\} - \mathfrak{K}_0\\
\geq\; & \frac{\min\{dm_0, 1\}}{2}\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right)|y_n|^{p^-}_X - \frac{1}{\mu}\left(2|\omega|_{L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\Lambda_1)}|y_n|^{p^-}_{L^{p(\cdot)}(\Lambda_1)} + \tilde{\omega}|y_n|^{p^-}_{L^{p(\cdot)}(\Lambda_2)}\right)\\
&\quad - 2\lambda\left(\frac{1}{t^-} - \frac{1}{\mu}\right)|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} \max\left\{|y_n|^{t^-}_{L^{p(\cdot)}(\mathbb{R}^N)}, |y_n|^{t^+}_{L^{p(\cdot)}(\mathbb{R}^N)}\right\} - \mathfrak{K}_0\\
\geq\; & \frac{\min\{dm_0, 1\}}{2}\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right)|y_n|^{p^-}_X - \frac{1}{\mu}\left(2|\omega|_{L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\Lambda_1)} + \tilde{\omega}\right)|y_n|^{p^-}_{L^{p(\cdot)}(\mathbb{R}^N)}\\
&\quad - 2\lambda\left(\frac{1}{t^-} - \frac{1}{\mu}\right)|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} \max\left\{|y_n|^{t^-}_{L^{p(\cdot)}(\mathbb{R}^N)}, |y_n|^{t^+}_{L^{p(\cdot)}(\mathbb{R}^N)}\right\} - \mathfrak{K}_0
\end{aligned}
$$

$$\geq \frac{\min\{dm_0, 1\}}{2}\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right)|y_n|_X^{p^-} - \frac{1}{\mu}\left(2|\omega|_{L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\Lambda_1)}|y_n|^{p^-}_{L^{p(\cdot)}(\Lambda_1)} + \tilde{\omega}|y_n|^{p^-}_{L^{p(\cdot)}(\Lambda_2)}\right)$$
$$- 2\lambda\left(\frac{1}{t^-} - \frac{1}{\mu}\right)|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} \max\left\{|y_n|^{t^-}_{L^{p(\cdot)}(\mathbb{R}^N)}, |y_n|^{t^+}_{L^{p(\cdot)}(\mathbb{R}^N)}\right\} - \Re_0$$
$$\geq \frac{\min\{dm_0, 1\}}{2}\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right)|y_n|_X^{p^-} - \frac{1}{\mu}\left(2|\omega|_{L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\Lambda_1)} + \tilde{\omega}\right)|y_n|^{p^-}_{L^{p(\cdot)}(\mathbb{R}^N)}$$
$$- 2\lambda\left(\frac{1}{t^-} - \frac{1}{\mu}\right)|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)}(C_2^{t^+} + C_2^{t^-})|y_n|_X^{t^+} - \Re_0,$$

where C_2 is an embedding constant of $X \hookrightarrow L^{p(\cdot)}(\mathbb{R}^N)$. Hence, we know that

$$\Re_1 + 1 + \frac{1}{\mu}\left(2|\omega|_{L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\Lambda_1)} + \tilde{\omega}\right)|y_n|^{p^-}_{L^{p(\cdot)}(\mathbb{R}^N)}$$
$$+ 2\lambda\left(\frac{1}{t^-} - \frac{1}{\mu}\right)|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)}(C_2^{t^+} + C_2^{t^-})|y_n|_X^{t^+} + \Re_0 \geq \frac{\min\{dm_0, 1\}}{2}\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right)|y_n|_X^{p^-}.$$

Let us divide this by $\frac{\min\{dm_0,1\}}{2}\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right)|y_n|_X^{p^-}$ and then take the limit supremum of this inequality as $n \to \infty$. Then, this together with the relation (3.3) yields that

$$1 \leq \frac{2\left(2|\omega|_{L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\Lambda_1)} + \tilde{\omega}\right)}{\mu \min\{dm_0, 1\}(\frac{1}{\vartheta r^+} - \frac{1}{\mu})}\limsup_{n\to\infty}|z_n|^{p^-}_{L^{p(\cdot)}(\mathbb{R}^N)} = \frac{2\left(2|\omega|_{L^{\frac{p(\cdot)}{p(\cdot)-p^-}}(\Lambda_1)} + \tilde{\omega}\right)}{\mu \min\{dm_0, 1\}(\frac{1}{\vartheta r^+} - \frac{1}{\mu})}|z|^{p^-}_{L^{p(\cdot)}(\mathbb{R}^N)}.$$

Hence, it follows that $z \neq 0$. Due to Remark 2.6, (K1), (K2), (Φ3) and the relation (3.2), we have that

$$I_\lambda(y_n) \geq \mathcal{M}\left(\int_{\mathbb{R}^N}\Phi_0(x, \nabla y_n)\,dx\right) + \int_{\mathbb{R}^N}\frac{\mathfrak{V}(x)}{p(x)}|y_n|^{p(x)}\,dx$$
$$- \int_{\mathbb{R}^N}G(x, y_n)\,dx - \lambda\int_{\mathbb{R}^N}\frac{\varrho(x)}{t(x)}|y_n|^{t(x)}\,dx$$
$$\geq \frac{m_0}{\vartheta}\int_{\mathbb{R}^N}\Phi_0(x, \nabla y_n)dx + \int_{\mathbb{R}^N}\frac{\mathfrak{V}(x)}{p(x)}|y_n|^{p(x)}dx$$
$$- \int_{\mathbb{R}^N}G(x, y_n)dx - \lambda\int_{\mathbb{R}^N}\frac{\varrho(x)}{t(x)}|y_n|^{t(x)}dx$$
$$\geq \frac{dm_0}{\vartheta p^+}\int_{\mathbb{R}^N}|\nabla y_n|^{p(x)}dx + \frac{1}{p^+}\int_{\mathbb{R}^N}\mathfrak{V}(x)|y_n|^{p(x)}dx$$
$$- \int_{\mathbb{R}^N}G(x, y_n)dx - \lambda\int_{\mathbb{R}^N}\frac{\varrho(x)}{t(x)}|y_n|^{t(x)}dx$$
$$\geq \frac{\min\{dm_0, \vartheta\}}{\vartheta p^+}\int_{\mathbb{R}^N}|\nabla y_n|^{p(x)} + \mathfrak{V}(x)|y_n|^{p(x)}dx$$
$$- \int_{\mathbb{R}^N}G(x, y_n)dx - \lambda\int_{\mathbb{R}^N}\frac{\varrho(x)}{t(x)}|y_n|^{t(x)}dx$$
$$\geq \frac{\min\{dm_0, \vartheta\}}{\vartheta p^+}|y_n|_X^{p^-} - \int_{\mathbb{R}^N}G(x, y_n)\,dx$$
$$- \frac{2\lambda}{t^-}|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)}\max\left\{C_2^{t^-}|y_n|_X^{t^-}, C_2^{t^+}|y_n|_X^{t^+}\right\} + o(1). \tag{3.6}$$

Since $\|y_n\|_X \to \infty$ as $n \to \infty$, we assert by (3.6) that

$$\int_{\mathbb{R}^N} G(x, y_n)\, dx \geq \frac{\min\{dm_0, \vartheta\}}{\vartheta p^+} \|y_n\|_X^{p^-} - \frac{2\lambda}{t^-} \|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} (C_2^{t^+} + C_2^{t^-}) \|y_n\|_X^{t^+} - I_\lambda(y_n) \to \infty \quad \text{as} \quad n \to \infty. \tag{3.7}$$

In addition, Lemma 2.1 and the assumptions (K2) and (Φ2) imply that

$$\begin{aligned} I_\lambda(y_n) &= \mathcal{M}\left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla y_n) dx\right) + \int_{\mathbb{R}^N} \frac{\mathfrak{V}(x)}{p(x)} |y_n|^{p(x)}\, dx \\ &\qquad - \int_{\mathbb{R}^N} G(x, y_n) dx - \lambda \int_{\mathbb{R}^N} \frac{\varrho(x)}{t(x)} |y_n|^{t(x)}\, dx \\ &\leq \mathcal{M}\left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla y_n)\, dx\right) + \int_{\mathbb{R}^N} \frac{\mathfrak{V}(x)}{p(x)} |y_n|^{p(x)}\, dx - \int_{\mathbb{R}^N} G(x, y_n) dx \\ &\leq \mathcal{M}\left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla y_n)\, dx\right) + \frac{1}{p^-} \int_{\mathbb{R}^N} \mathfrak{V}(x)\, |y_n|^{p(x)}\, dx - \int_{\mathbb{R}^N} G(x, y_n) dx \\ &\leq \mathcal{M}(1)\left(1 + \left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla y_n) dx\right)^{\vartheta}\right) + \frac{1}{p^-} \int_{\mathbb{R}^N} \mathfrak{V}(x)\, |y_n|^{p(x)}\, dx - \int_{\mathbb{R}^N} G(x, y_n) dx \\ &\leq C_3 \max\{\mathcal{M}(1), \frac{1}{p_-}\}\left(1 + \int_{\mathbb{R}^N} \Phi_0(x, \nabla y_n) dx + \int_{\mathbb{R}^N} \mathfrak{V}(x)\, |y_n|^{p(x)}\, dx\right)^{\vartheta} - \int_{\mathbb{R}^N} G(x, y_n) dx \\ &\leq C_4 \left(1 + b \int_{\mathbb{R}^N} |\nabla y_n|^{p(x)}\, dx + \int_{\mathbb{R}^N} \mathfrak{V}(x)\, |y_n|^{p(x)}\, dx\right)^{\vartheta} - \int_{\mathbb{R}^N} G(x, y_n) dx \\ &\leq 2^{\vartheta} C_5 \|y_n\|_X^{\vartheta p^+} - \int_{\mathbb{R}^N} G(x, y_n) dx \end{aligned} \tag{3.8}$$

for some positive constants C_3, C_4 and C_5, where $\mathcal{M}(\tau) \leq \mathcal{M}(1)\left(1 + \tau^{\vartheta}\right)$ for all $\tau \in \mathbb{R}^+$, because, if $0 \leq \tau < 1$, then $\mathcal{M}(\tau) = \int_0^\tau M(s)\, ds \leq \mathcal{M}(1)$; also, if $\tau > 1$, then $\mathcal{M}(\tau) \leq \mathcal{M}(1)\tau^{\vartheta}$. Then, we obtain by the relation (3.8) that

$$2^{\vartheta} C_5 \geq \frac{1}{\|y_n\|_X^{\vartheta p^+}} \left(\int_{\mathbb{R}^N} G(x, y_n)\, dx + I_\lambda(y_n)\right). \tag{3.9}$$

From (Ψ4), we can choose $s_0 > 1$ such that $G(x, s) > |s|^{\vartheta p^+}$ for all $x \in \mathbb{R}^N$ and $|s| > s_0$. Using (Ψ1), there exists a positive constant $\mathcal{K}$ such that $|G(x, s)| \leq \mathcal{K}$ for all $(x, s) \in \mathbb{R}^N \times [-s_0, s_0]$. Hence, there exists a real number $\mathcal{K}_0$ such that $G(x, s) \geq \mathcal{K}_0$ for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$; thus

$$\frac{G(x, y_n) - \mathcal{K}_0}{\|y_n\|_X^{\vartheta p^+}} \geq 0 \tag{3.10}$$

for all $x \in \mathbb{R}^N$ and all $n \in \mathbb{N}$. Set $\Gamma_1 = \left\{x \in \mathbb{R}^N : z(x) \neq 0\right\}$. Suppose that $\text{meas}(\Gamma_1) \neq 0$. By the convergence (3.3), we infer that $|y_n(x)| = |z_n(x)|\, \|y_n\|_X \to \infty$ as $n \to \infty$ for all $x \in \Gamma_1$. Furthermore, owing to (Ψ4), one has

$$\lim_{n\to\infty} \frac{G(x, y_n)}{\|y_n\|_X^{\vartheta p^+}} = \lim_{n\to\infty} \frac{G(x, y_n)}{|y_n|^{\vartheta p^+}}\, |z_n|^{\vartheta p^+} = \infty \tag{3.11}$$

for all $x \in \Gamma_1$. According to (3.7)–(3.11) and the Fatou lemma, we deduce that

$$\begin{aligned}
2^{\vartheta} C_5 &= \liminf_{n\to\infty} \frac{2^{\vartheta} C_5 \int_{\mathbb{R}^N} G(x,y_n)\,dx}{\int_{\mathbb{R}^N} G(x,y_n)\,dx + I_\lambda(y_n)} \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{G(x,y_n)}{\|y_n\|_X^{\vartheta p^+}}\,dx \\
&= \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{G(x,y_n)}{\|y_n\|_X^{\vartheta p^+}}\,dx - \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{\mathcal{K}_0}{\|y_n\|_X^{\vartheta p^+}}\,dx \\
&\geq \liminf_{n\to\infty} \int_{\Gamma_1} \frac{G(x,y_n) - \mathcal{K}_0}{\|y_n\|_X^{\vartheta p^+}}\,dx \\
&\geq \int_{\Gamma_1} \liminf_{n\to\infty} \frac{G(x,y_n) - \mathcal{K}_0}{\|y_n\|_X^{\vartheta p^+}}\,dx \\
&= \int_{\Gamma_1} \liminf_{n\to\infty} \frac{G(x,y_n)}{|y_n|^{\vartheta p^+}} |z_n|^{\vartheta p^+}\,dx - \int_{\Gamma_1} \limsup_{n\to\infty} \frac{\mathcal{K}_0}{\|y_n\|_X^{\vartheta p^+}}\,dx = \infty,
\end{aligned}$$

which is a contradiction. Hence, we have that $\mathrm{meas}(\Gamma_1) = 0$; thus, $z(x) = 0$ for almost all $x \in \mathbb{R}^N$. Consequently, this leads to a contradiction; thus, $\{y_n\}$ is bounded in X. □

Let $\mathfrak{W}$ be a separable and reflexive Banach space. Then, it is known (see [11, 50]) that there are $\{e_n\} \subseteq \mathfrak{W}$ and $\{h_n^*\} \subseteq \mathfrak{W}^*$ such that

$$\mathfrak{W} = \overline{\mathrm{span}\{e_n : n = 1, 2, \cdots\}}, \quad \mathfrak{W}^* = \overline{\mathrm{span}\{h_n^* : n = 1, 2, \cdots\}}$$

and

$$\left\langle h_i^*, e_j \right\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let us denote $\mathfrak{W}_n = \mathrm{span}\{e_n\}$, $\mathfrak{Y}_k = \bigoplus_{n=1}^{k} \mathfrak{W}_n$ and $\mathfrak{Z}_k = \overline{\bigoplus_{n=k}^{\infty} \mathfrak{W}_n}$.

Lemma 3.5. ([1, 47]) *Suppose that* $(\mathfrak{E}, \|\cdot\|)$ *is a Banach space, the functional* $\mathcal{F} \in C^1(\mathfrak{E}, \mathbb{R})$ *ensures the* $(C)_c$*-condition for any* $c > 0$ *and* $\mathcal{F}$ *is even. If, for each large enough* $k \in \mathbb{N}$, *there are* $\beta_k > \alpha_k > 0$ *such that*

(1) $b_k := \inf\{\mathcal{F}(y) : \|y\|_{\mathfrak{E}} = \alpha_k, y \in \mathfrak{Z}_k\} \to \infty$ *as* $k \to \infty$

(2) $a_k := \max\{\mathcal{F}(y) : \|y\|_{\mathfrak{E}} = \beta_k, y \in \mathfrak{Y}_k\} \leq 0$,

then, $\mathcal{F}$ *has an unbounded sequence of critical values, i.e., there is a sequence* $\{y_n\} \subset \mathfrak{E}$ *such that* $\mathcal{F}'(y_n) = 0$ *and* $\mathcal{F}(y_n) \to +\infty$ *as* $n \to +\infty$.

Theorem 3.6. *Assume that* (H1), (K1), (K2), (Φ1)–(Φ5), (Ψ1), (Ψ2) *and* (Ψ4) *hold. If* $\Phi_0(x, -\xi) = \Phi_0(x, \xi)$ *holds for all* $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$ *and* $g(x, -s) = -g(x, s)$ *holds for all* $(x, s) \in \mathbb{R}^N \times \mathbb{R}$, *then for any* $\lambda > 0$, *the problem* (P_λ) *possesses an unbounded sequence of nontrivial weak solutions* $\{y_n\}$ *in* X *such that* $I_\lambda(y_n) \to \infty$ *as* $n \to \infty$.

Proof. Immediately, I_λ is an even functional and ensures the (C)-condition by Lemma 3.4. It suffices to verify that there exist $\beta_k > \alpha_k > 0$ such that

(1) $b_k := \inf\{I_\lambda(y) : \|y\|_X = \alpha_k, y \in \mathfrak{Z}_k\} \to \infty \quad \text{as} \quad k \to \infty$

(2) $a_k := \max\{I_\lambda(y) : \|y\|_X = \beta_k, y \in \mathfrak{Y}_k\} \leq 0$

for k large enough. For convenience, we denote

$$\nu_{1,k} = \sup_{\|y\|_X=1, y\in\mathfrak{Z}_k} \|y\|_{L^{p(\cdot)}(\mathbb{R}^N)}, \quad \nu_{2,k} = \sup_{\|y\|_X=1, y\in\mathfrak{Z}_k} \|y\|_{L^{q(\cdot)}(\mathbb{R}^N)}.$$

Then, it is easy to ensure that $\nu_{1,k} \to 0$ and $\nu_{2,k} \to 0$ as $k \to \infty$ (see [20]). Denote $\nu_k = \max\{\nu_{1,k}, \nu_{2,k}\}$. Then, we derive that $\nu_k < 1$ for k large enough. For any $y \in \mathfrak{Z}_k$, assume that $\|y\|_X > 1$. With an analogous argument to that in (3.6), it follows from the assumption (Φ4), Lemmas 2.1 and 2.5 and Remark 2.6 that, for k large enough,

$$\begin{aligned}
I_\lambda(y) &= \mathcal{M}\left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla y)\, dx\right) + \int_{\mathbb{R}^N} \frac{\mathfrak{V}(x)}{p(x)} |y|^{p(x)}\, dx - \lambda \int_{\mathbb{R}^N} \frac{\varrho(x)}{t(x)} |y|^{t(x)} - \int_{\mathbb{R}^N} G(x, y)\, dx \\
&\geq \frac{\min\{dm_0, \vartheta\}}{\vartheta p^+} \|y\|_X^{p^-} - \lambda \int_{\mathbb{R}^N} \frac{\varrho(x)}{t(x)} |y|^{t(x)}\, dx - \int_{\mathbb{R}^N} G(x, y)\, dx \\
&\geq \frac{\min\{dm_0, \vartheta\}}{\vartheta p^+} \|y\|_X^{p^-} - \frac{2\lambda}{t^-} \|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} \max\left\{\|y\|^{t^-}_{L^{p(\cdot)}(\mathbb{R}^N)}, \|y\|^{t^+}_{L^{p(\cdot)}(\mathbb{R}^N)}\right\} \\
&\quad - 2\|\sigma_0\|_{L^{p'(\cdot)}(\mathbb{R}^N)} \nu_k \|y\|_X - \frac{2b_1}{q^-} \int_{\mathbb{R}^N} |y|^{q(x)}\, dx \\
&\geq \frac{\min\{dm_0, \vartheta\}}{\vartheta p^+} \|y\|_X^{p^-} - \frac{2\lambda}{t^-} \|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} \nu_k^{t^-} \|y\|_X^{t^+} \\
&\quad - 2\|\sigma_0\|_{L^{p'(\cdot)}(\mathbb{R}^N)} \nu_k \|y\|_X - \frac{2b_1}{q^-} \nu_k^{q^-} \|y\|_X^{q^+}.
\end{aligned} \tag{3.12}$$

Choose

$$\alpha_k = \left(\frac{4\vartheta p^+ \nu_k^{q^-} b_1}{q^- \min\{dm_0, \vartheta\}}\right)^{\frac{1}{p^- - q^+}}.$$

Since $p^- < q^+$ and $\nu_k \to 0$ as $k \to \infty$, we assert that $\alpha_k \to \infty$ as $k \to \infty$. Hence, if $y \in \mathfrak{Z}_k$ and $\|y\|_X = \alpha_k$, then we deduce that

$$I_\lambda(y) \geq \frac{\min\{dm_0, \vartheta\}}{2\vartheta p^+} \alpha_k^{p^-} - \frac{2\lambda}{t^-} \|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} \nu_k^{t^-} \alpha_k^{t^+} - 2\|\sigma_0\|_{L^{p'(\cdot)}(\mathbb{R}^N)} \nu_k \alpha_k \to \infty \quad \text{as} \quad k \to \infty,$$

which implies (1).

The proof of the condition (2) is carried out in a similar fashion as that of Theorem 1.3 of [1] (see also [5]). For the convenience of readers, we give the proof. Suppose that the condition (2) does not hold for some k. Then, there exists a sequence $\{y_n\}$ in $\mathfrak{Y}_k$ such that

$$\|y_n\|_X \to \infty \text{ as } n \to \infty \quad \text{and} \quad I_\lambda(y_n) \geq 0. \tag{3.13}$$

Let $z_n = y_n/\|y_n\|_X$. Then, it is immediate that $\|z_n\|_X = 1$. Since $\dim \mathfrak{Y}_k < \infty$, we can choose $z \in \mathfrak{Y}_k \setminus \{0\}$ such that, up to a subsequence,

$$\|z_n - z\|_X \to 0 \quad \text{and} \quad z_n(x) \to z(x) \text{ for a.e. } x \in \mathbb{R}^N \text{as } n \to \infty.$$

We claim that $z(x) = 0$ for almost all $x \in \mathbb{R}^N$. If $z(x) \neq 0$, then $|y_n(x)| \to \infty$ for all $x \in \mathbb{R}^N$ as $n \to \infty$. In accordance with (Ψ4), it follows that

$$\lim_{n\to\infty} \frac{G(x, y_n)}{\|y_n\|_X^{\vartheta p^+}} = \lim_{n\to\infty} \frac{G(x, y_n)}{|y_n|^{\vartheta p^+}} |z_n|^{\vartheta p^+} = \infty \tag{3.14}$$

for all $x \in \Gamma_2 := \left\{x \in \mathbb{R}^N : z(x) \neq 0\right\}$. The analogous arguments to that in Lemma 3.4 yield that we choose a $\mathcal{K} \in \mathbb{R}$ such that $G(x, s) \geq \mathcal{K}$ for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}^N$; thus

$$\frac{G(x, y_n) - \mathcal{K}}{\|y_n\|_X^{\vartheta p^+}} \geq 0$$

for all $x \in \mathbb{R}^N$ and all $n \in \mathbb{N}$. Using (3.14) and the Fatou lemma, one has

$$\begin{aligned}
\liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{G(x, y_n)}{\|y_n\|_X^{\vartheta p^+}} dx &= \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{G(x, y_n)}{\|y_n\|_X^{\vartheta p^+}} dx - \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{\mathcal{K}}{\|y_n\|_X^{\vartheta p^+}} dx \\
&\geq \liminf_{n\to\infty} \int_{\Gamma_2} \frac{G(x, y_n) - \mathcal{K}}{\|y_n\|_X^{\vartheta p^+}} dx \\
&\geq \int_{\Gamma_2} \liminf_{n\to\infty} \frac{G(x, y_n) - \mathcal{K}}{\|y_n\|_X^{\vartheta p^+}} dx \\
&= \int_{\Gamma_2} \liminf_{n\to\infty} \frac{G(x, y_n)}{\|y_n\|_X^{\vartheta p^+}} dx - \int_{\Gamma_2} \limsup_{n\to\infty} \frac{\mathcal{K}}{\|y_n\|_X^{\vartheta p^+}} dx = \infty.
\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^N} \frac{G(x, y_n)}{\|y_n\|_X^{\vartheta p^+}} dx \to \infty \quad \text{as } n \to \infty.$$

We may suppose that $\|y_n\|_X > 1$. Using the relation (3.8), we have

$$\begin{aligned}
I_\lambda(y_n) &\leq 2^\vartheta C_5 \|y_n\|_X^{\vartheta p^+} - \int_{\mathbb{R}^N} G(x, y_n)\, dx \\
&= \|y_n\|_X^{\vartheta p^+} \left(2^\vartheta C_5 - \int_{\mathbb{R}^N} \frac{G(x, y_n)}{\|y_n\|_X^{\vartheta p^+}} dx\right) \to -\infty \quad \text{as } n \to \infty,
\end{aligned}$$

which is a contradiction to (3.13). This completes the proof. □

Definition 3.7. Suppose that $(E, \|\cdot\|)$ is a real reflexive and separable Banach space, $\mathcal{F} \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. We say that $\mathcal{F}$ fulfills the $(C)_c^*$-condition (with respect to $\mathfrak{Y}_n$) if any sequence $\{v_n\}_{n\in\mathbb{N}} \subset E$ for which $v_n \in \mathfrak{Y}_n$, for any $n \in \mathbb{N}$,

$$\mathcal{F}(v_n) \to c \quad \text{and} \quad \|(\mathcal{F}|_{\mathfrak{Y}_n})'(v_n)\|_{E^*}(1 + \|v_n\|) \to 0, \text{ as } n \to \infty,$$

has a subsequence converging to a critical point of $\mathcal{F}$.

Proposition 3.8. ([20]) *Suppose that* $(E, \|\cdot\|)$ *is a Banach space and* $\mathcal{F} \in C^1(E, \mathbb{R})$ *is an even functional. If there is* $k_0 > 0$ *so that, for each* $k \geq k_0$*, there exist* $\beta_k > \alpha_k > 0$ *such that*

(D1) $\inf\{\mathcal{F}(v) : \|v\|_E = \beta_k, v \in \mathfrak{Z}_k\} \geq 0$*;*

(D2) $b_k := \max\{\mathcal{F}(v) : \|v\|_E = \alpha_k, v \in \mathfrak{Y}_k\} < 0$*;*

(D3) $c_k := \inf\{\mathcal{F}(v) : \|v\|_E \leq \beta_k, v \in \mathfrak{Z}_k\} \to 0$ *as* $k \to \infty$*;*

(D4) $\mathcal{F}$ *satisfies the* $(C)_c^*$*-condition for every* $c \in [c_{k_0}, 0)$*,*

then $\mathcal{F}$ *admits a sequence of negative critical values* $c_n < 0$ *satisfying* $c_n \to 0$ *as* $n \to \infty$*.*

Lemma 3.9. *Suppose that* (H1)*,* (K1)*,* (K2)*,* (Φ1)–(Φ5) *and* (Ψ1)–(Ψ4) *are satisfied. Then,* I_λ *satisfies the* $(C)_c^*$*-condition.*

Proof. According to Lemmas 3.2 and 3.3, Φ' is mapping of type (S_+) and Ψ' is a compact operator on X. Because X is a reflexive Banach space, the idea of the proof is essentially the same as that in [20, Lemma 3.12]. □

Theorem 3.10. *Suppose that* (H1)*,* (K1)*,* (K2)*,* (Φ1)–(Φ5) *and* (Ψ1)–(Ψ4) *are satisfied. If* $\Phi_0(x, -\xi) = \Phi_0(x, \xi)$ *holds for all* $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$ *and* $g(x, -s) = -g(x, s)$ *holds for all* $(x, s) \in \mathbb{R}^N \times \mathbb{R}$*, then the problem* (P_λ) *admits a sequence of nontrivial solutions* $\{y_n\}$ *in* X *such that* $I_\lambda(y_n) \to 0$ *as* $n \to \infty$ *for all* $\lambda > 0$*.*

Proof. Due to Lemma 3.9, we note that the functional I_λ is even and satisfies the $(C)_c^*$-condition for every $c \in [d_{k_0}, 0)$. Now, we ensure the properties (D_1)–(D_3) in Proposition 3.8. To do this, let $\nu_k < 1$ for k large enough, where ν_k is given in Theorem 3.6.

(D1): From (Ψ2), the definition of ν_k and an analogous argument to that for (3.6) and (3.12), it follows that

$$\begin{aligned}
I_\lambda(y) \geq\ & \frac{\min\{dm_0, \vartheta\}}{\vartheta p^+}\|y\|_X^{p^-} - \frac{2\lambda}{t^-}\|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} \max\left\{\|y\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{t^-}, \|y\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{t^+}\right\} \\
& - 2\|\sigma_0\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\|y\|_{L^{p'(\cdot)}(\mathbb{R}^N)} - \frac{b_1}{q^-}\max\{\|y\|_{L^{q(\cdot)}(\mathbb{R}^N)}^{q^-}, \|y\|_{L^{q(\cdot)}(\mathbb{R}^N)}^{q^+}\} \\
\geq\ & \frac{\min\{dm_0, \vartheta\}}{\vartheta p^+}\|y\|_X^{p^-} - \frac{2\lambda}{t^-}\|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} \nu_k^{t^-}\|y\|_X^{t^+} \\
& - 2\|\sigma_0\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\nu_k\|y\|_X - \frac{b_1}{q^-}\nu_k^{q^-}\|y\|_X^{q^+} \\
\geq\ & \frac{\min\{dm_0, \vartheta\}}{\vartheta p^+}\|y\|_X^{p^-} - \left(\frac{2\lambda}{t^-}\|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} + \frac{b_1}{q^-}\right)\nu_k^{t^-}\|y\|_X^{q^+} - 2\|\sigma_0\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\nu_k\|y\|_X
\end{aligned}$$

for a sufficiently large k and $\|y\|_X \geq 1$. Choose

$$\beta_k = \left[\left(\frac{4\lambda}{t^-}\|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} + \frac{2b_1}{q^-}\right)\frac{\nu_k^{t^-}}{C_0}\right]^{\frac{1}{p^- - 2q^+}}, \tag{3.15}$$

where

$$C_0 := \frac{\min\{dm_0, \vartheta\}}{\vartheta p^+}.$$

Let $y \in \mathfrak{Z}_k$ with $\|y\|_X = \beta_k > 1$ for k large enough. Then, we choose a $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} I_\lambda(y) &\geq C_0\|y\|_X^{p^-} - \left(\frac{2\lambda}{t^-}\|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} + \frac{b_1}{q^-}\right)\nu_k^{t^-}\|y\|_X^{2q^+} - 2\|\sigma_0\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\nu_k\|y\|_X \\ &\geq \frac{C_0}{2}\beta_k^{p^-} - 2\|\sigma_0\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\left[\left(\frac{4\lambda}{t^-}\|\varrho\|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} + \frac{2b_1}{q^-}\right)C_0^{-1}\right]^{\frac{1}{p^--2q^+}}\nu_k^{\frac{t^-+p^--2q^+}{p^--2q^+}} \\ &\geq 0 \end{aligned}$$

for all $k \in \mathbb{N}$ with $k \geq k_0$, since $\lim_{k\to\infty}\beta_k = \infty$. Therefore,

$$\inf\{I_\lambda(y) : y \in \mathfrak{Z}_k, \|y\|_X = \beta_k\} \geq 0.$$

(D2): Since $\mathfrak{Y}_k$ is finite-dimensional, all norms are equivalent. Then, we find positive constants $\varsigma_{1,k}$ and $\varsigma_{2,k}$ such that

$$\varsigma_{1,k}\|y\|_X \leq \|y\|_{L^{\kappa(\cdot)}(\eta,\mathbb{R}^N)} \text{ and } \|y\|_{L^{q(\cdot)}(\mathbb{R}^N)} \leq \varsigma_{2,k}\|y\|_X$$

for any $y \in \mathfrak{Y}_k$. Let $y \in \mathfrak{Y}_k$ with $\|y\|_X \leq 1$. From (Ψ1) and (Ψ3), there are $\mathfrak{C}_1, \mathfrak{C}_2 > 0$ such that

$$G(x, s) \geq \mathfrak{C}_1\eta(x)|s|^{\kappa(x)} - \mathfrak{C}_2|s|^{q(x)}$$

for almost all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$. From ($\Phi$2), we get

$$\int_{\mathbb{R}^N}\Phi_0(x, \nabla y)\,dx \leq \mathfrak{C}_3$$

for some positive constant $\mathfrak{C}_3$. Then, we have

$$\begin{aligned} I_\lambda(y) &= \mathcal{M}\left(\int_{\mathbb{R}^N}\Phi_0(x, \nabla y)\,dx\right) + \int_{\mathbb{R}^N}\frac{\mathfrak{V}(x)}{p(x)}|y|^{p(x)}dx - \lambda\int_{\mathbb{R}^N}\frac{\varrho(x)}{t(x)}|y|^{t(x)}dx - \int_{\mathbb{R}^N}G(x, y)\,dx \\ &\leq \left(\sup_{0\leq\xi\leq\mathfrak{C}_3}M(\xi)\right)\int_{\mathbb{R}^N}\Phi_0(x, \nabla y)\,dx + \int_{\mathbb{R}^N}\frac{\mathfrak{V}(x)}{p(x)}|y|^{p(x)}dx \\ &\qquad - \mathfrak{C}_1\int_{\mathbb{R}^N}\eta(x)|y|^{\kappa(x)}dx + \mathfrak{C}_2\int_{\mathbb{R}^N}|y|^{q(x)}\,dx \\ &\leq C_6\|y\|_X^{p^-} - \mathfrak{C}_1\min\{\|y\|^{\kappa^+}_{L^{\kappa(\cdot)}(\eta,\mathbb{R}^N)}, \|y\|^{\kappa^-}_{L^{\kappa(\cdot)}(\eta,\mathbb{R}^N)}\} + \mathfrak{C}_2\max\{\|y\|^{q^-}_{L^{q(\cdot)}(\mathbb{R}^N)}, \|y\|^{q^+}_{L^{q(\cdot)}(\mathbb{R}^N)}\} \\ &\leq C_6\|y\|_X^{p^-} - \mathfrak{C}_1\min\{\varsigma_{1,k}^{\kappa^-}, \varsigma_{1,k}^{\kappa^+}\}\|y\|_X^{\kappa^+} + \mathfrak{C}_2\max\{\varsigma_{2,k}^{q-}, \varsigma_{2,k}^{q+}\}\|y\|_X^{q^-} \end{aligned}$$

for some positive constant C_6. Let $f(s) = C_6 s^{p^-} - \mathfrak{C}_1\min\{\varsigma_{1,k}^{\kappa^-}, \varsigma_{1,k}^{\kappa^+}\}s^{\kappa^+} + \mathfrak{C}_2\max\{\varsigma_{2,k}^{q-}, \varsigma_{2,k}^{q+}\}s^{q^-}$. Since $\kappa^+ < p^- < q^-$, we infer that $f(s) < 0$ for all $s \in (0, s_0)$ for a sufficiently small $s_0 \in (0, 1)$. Hence, we can find $\alpha_k > 0$ such that $I_\lambda(y) < 0$ for all $y \in \mathfrak{Y}_k$ with $\|y\|_X = \alpha_k < s_0$ for k large enough. If necessary, we can change k_0 to a large value so that $\beta_k > \alpha_k > 0$ and

$$b_k := \max\{I_\lambda(y) : y \in \mathfrak{Y}_k, \|y\|_X = \alpha_k\} < 0.$$

(D3): Because $\mathfrak{Y}_k \cap \mathfrak{Z}_k \neq \phi$ and $0 < \alpha_k < \beta_k$, we have that $c_k \leq b_k < 0$ for all $k \geq k_0$. For any $y \in \mathfrak{Z}_k$ with $\|y\|_X = 1$ and $0 < \tau < \beta_k$, one has

$$I_\lambda(\tau y) = \mathcal{M}\left(\int_{\mathbb{R}^N}\Phi_0(x, \nabla\tau y)\,dx\right) + \int_{\mathbb{R}^N}\frac{\mathfrak{V}(x)}{p(x)}|\tau y|^{p(x)}dx - \lambda\int_{\mathbb{R}^N}\frac{\varrho(x)}{t(x)}|\tau y|^{t(x)}dx - \int_{\mathbb{R}^N}G(x, \tau y)\,dx$$

$$\geq -\frac{2\lambda}{t^-}|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} \max\left\{|\tau y|^{t^-}_{L^{p(\cdot)}(\mathbb{R}^N)}, |\tau y|^{t^+}_{L^{p(\cdot)}(\mathbb{R}^N)}\right\}$$
$$- 2|\sigma_0|_{L^{p'(\cdot)}(\mathbb{R}^N)}|\tau y|_{L^{p(\cdot)}(\mathbb{R}^N)} - \frac{b_1}{q^-}\max\{|\tau y|^{q^-}_{L^{q(\cdot)}(\mathbb{R}^N)}, |\tau y|^{q^+}_{L^{q(\cdot)}(\mathbb{R}^N)}\}$$
$$\geq -\frac{2\lambda}{t^-}|\varrho|_{L^{\frac{q(\cdot)}{q(\cdot)-t(\cdot)}}(\mathbb{R}^N)}\beta_k^{t^+}\nu_k^{t^-} - 2|\sigma_0|_{L^{p'(\cdot)}(\mathbb{R}^N)}\beta_k\nu_k - \frac{b_1}{q^-}\beta_k^{q^+}\nu_k^{q^-}$$

for a sufficiently large k. Hence, from the definition of β_k, we infer

$$\begin{aligned} 0 > c_k &\geq -\frac{2\lambda}{t^-}|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)}\beta_k^{t^+}\nu_k^{t^-} - 2|\sigma_0|_{L^{p'(\cdot)}(\mathbb{R}^N)}\beta_k\nu_k - \frac{b_1}{q^-}\beta_k^{q^+}\nu_k^{q^-} \\ &= -\frac{2\lambda}{t^-}|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)}\left[\left(\frac{4\lambda}{t^-}|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} + \frac{2b_1}{q^-}\right)C_0^{-1}\right]^{\frac{t^+}{p^--2q^+}}\nu_k^{\frac{t^-(t^++p^--2q^+)}{p^--2q^+}} \\ &\quad - 2|\sigma_0|_{L^{p'(\cdot)}(\mathbb{R}^N)}\left[\left(\frac{4\lambda}{t^-}|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} + \frac{2b_1}{q^-}\right)C_0^{-1}\right]^{\frac{1}{p^--2q^+}}\nu_k^{\frac{t^-+p^--2q^+}{p^--2q^+}} \\ &\quad - \frac{b_1}{q^-}\left[\left(\frac{4\lambda}{t^-}|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} + \frac{2b_1}{q^-}\right)C_0^{-1}\right]^{\frac{q^+}{p^--2q^+}}\nu_k^{\frac{t^-q^++q^-(p^--2q^+)}{p^--2q^+}}. \end{aligned} \tag{3.16}$$

Because $p^- < q^+$, $t^+ + p^- < 2q^+$, $t^-q^+ + q^-p^- < 2q^-q^+$ and $\nu_k \to 0$ as $k \to \infty$, we conclude that $\lim_{k\to\infty} c_k = 0$.

Therefore, all properties of Proposition 3.8 are fulfilled, and we assert that the problem (P_λ) admits a sequence of nontrivial solutions $\{y_n\}$ in X such that $I_\lambda(y_n) \to 0$ as $n \to \infty$ for all $\lambda > 0$. □

Remark 3.11. *From the viewpoint of [5, 17, 20, 32], the assumptions* $(\Psi4)$ *and* $(f1)$ *are essential in obtaining Theorem 3.10. Under these two assumptions, the existence of two sequences* $0 < \alpha_k < \beta_k$ *sufficiently large is established in the papers [5, 17, 20, 32]. Regrettably, as a result of utilizing an analogous argument to that in [17, 20], we cannot show the property (D3) in Theorem 3.10. More precisely, if we replace* β_k *in* (3.15) *with*

$$\tilde{\beta}_k = \left[\left(\frac{4\lambda}{t^-}|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} + \frac{2b_1}{q^-}\right)\frac{\nu_k^{t^-}}{C_0}\right]^{\frac{1}{p^--q^+}}$$

and $t^+ + p^- > q^+$, *then, in the relation* (3.16),

$$\tilde{\beta}_k^{t^+}\nu_k^{t^-} = \left[\left(\frac{4\lambda}{t^-}|\varrho|_{L^{\frac{p(\cdot)}{p(\cdot)-t(\cdot)}}(\mathbb{R}^N)} + \frac{2b_1}{q^-}\right)C_0^{-1}\right]^{\frac{t^+}{p^--q^+}}\nu_k^{\frac{t^-(t^++p^--q^+)}{p^--q^+}} \to \infty \text{ as } k \to \infty;$$

thus, we cannot obtain the property (D3) in $\tilde{\beta}_k$. *However, the authors of [5, 27, 32] overcame the difficulty resulting from this new setting for* β_k. *In contrast, the existence of two sequences* $0 < \alpha_k < \beta_k \to 0$ *as* $k \to \infty$ *is obtained in [36, 46, 47] when* $(f1)$ *is satisfied. On the other hand, we get Theorem 3.10 when* $(\Psi4)$ *is not assumed and* $(f1)$ *is changed into* $(\Psi3)$. *In this direction, the proof of Theorem 3.10 is different from that in the recent works [5, 17, 20, 32, 36, 46, 47].*

4. Conclusions

In the present paper, on a new class of nonlinear term g, we present the existence results of a sequence of infinitely many solutions by utilizing the fountain theorem and the dual fountain theorem as the main tools. In particular, when we ensure assumptions in the dual fountain theorem, the conditions on the nonlinear term g near zero and at infinity are crucial, however, we obtain the existence of infinitely many small solutions without assuming them. This is a novelty of the present paper. Additionally, a new research direction in strong relationship with several related applications is the study of critical Kirchhoff-type equations:

$$-M\left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla y)\, dx\right) \operatorname{div}(\varphi(x, \nabla y)) + \mathfrak{V}(x)\, |y|^{p(x)-2}\, y = \lambda \varrho(x) |y|^{t(x)-2} y + g(x, y) \text{ in } \mathbb{R}^N,$$

where $p(x) < t(x)$ for all $x \in \mathbb{R}^N$ and $\{x \in \mathbb{R}^N : t(x) = p^*(x)\} \neq \emptyset$.

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Conflict of interest

The author declares that there are no competing interests.

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