Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 9443–9460.
DOI: 10.3934/math.2023476
Received: 10 December 2023
Revised: 28 January 2023
Accepted: 02 February 2023
Published: 17 February 2023

Research article

Fixed point results in b-metric spaces with applications to integral equations

Badriah Alamri and Jamshaid Ahmad*

University of Jeddah, College of Science, Department of Mathematics, Jeddah, Saudi Arabia;
baalamri@uj.edu.sa; jkhan@uj.edu.sa

* **Correspondence:** Email: jkhan@uj.edu.sa.

Abstract: The purpose of this article is to obtain common fixed point results in b-metric spaces for generalized rational contractions involving control functions of two variables. We provide an example to show the originality of our main result. As outcomes of our results, we derive certain fixed and common fixed point results for rational contractions presuming control functions of one variable and constants. As an application, we investigate the solution of an integral equation.

Keywords: common fixed point; b-metric space; control functions; generalized contraction; integral equation
Mathematics Subject Classification: 46S40, 47H10, 54H25

1. Introduction

Fixed point theory is one of the most celebrated and conventional theories in mathematics and has comprehensive applications in different fields. In this theory, the notion of metric space plays an important role, which was naturally accomplished by M. Frechet [1] in 1906. And the first and pioneer result in this theory is Banach contraction principle [2] in which the underlying space is the complete metric space. In this principle, the contractive mapping is necessarily continuous while it is not applicable in the case of discontinuity. The major drawback of this principle is how we apply this contractive mapping in case of discontinuity. This problem was overcome in the past by Kannan [3] where it proved a fixed point result without continuity. In 1972, Chaterjea [4] established a result which is independent from Banach contraction principle and Kannan fixed point theorem. Later on, Fisher [5] introduced rational inequality in fixed point theory and obtained a fixed point result in complete metric spaces. Motivated by the influence of the genuine concept of metric space and the Banach contraction principle on fixed point theory, various authors have undertaken several extensions of this concept and achieved this result in the past few years.

On the other hand, the conceptual framework of b-metric space have been done by Bakhtin [6] which was formally defined by Czerwik [7] in 1993 who discussed the convergence of measurable functions as a meaningful generalization of metric space and also established the Banach contraction principle in b-metric space. Koleva et al. [8] established fixed point results for Chaterjea type inequality in the background of b-metric spaces. Subsequently, Abbas et al. [9] obtained common fixed results for Fisher type inequality [5] in the setting of partial ordered b-metric spaces. Hammad et al. [10, 11] introduced the notions of cyclic η_s^q-rational contractions and $\beta_{q,\phi}^{s,\psi}$-contractions in b-metric-like spaces and investigated the solutions of integral equations. Ameer et al. [12] defined Ćirić type rational graphic (Y, Λ)-contraction in partial b-metric spaces endowed with a directed graph and obtained common fixed points of two self mappings. Furthermore, they presented some applications on electric circuit equations and fractional differential equations. Recently, Seddik et al. [13] established a common fixed point result for rational contraction in the setting of b-metric space.

In this research work, we define generalized rational contractive mappings by combining Banach's contraction [2], Chaterjea's contraction [4] and Fisher's contraction [5] and involve control functions of two variables and establish some common fixed point results in b-metric spaces. As outcomes of our results, we derive certain fixed and common fixed point results for rational contractions presuming control functions of one variable and constants. As an application, we investigate the solution of an integral equation.

2. Preliminaries

The well-known Banach contraction principle [2] is given in this way.

Theorem 1. *[2] Let $(\mathcal{F}, \varsigma)$ be a complete metric space and let $\Im : \mathcal{F} \to \mathcal{F}$. If there exists a nonnegative constant $\rho \in [0, 1)$ such that*

$$\varsigma(\Im\upsilon, \Im\hbar) \leq \rho\varsigma(\upsilon, \hbar),$$

for all $\upsilon, \hbar \in \mathcal{F}$, then $\Im$ has a unique fixed point.

In [3], Kannan proved a result for the mapping, which is not necessarily continuous in this way.

Theorem 2. *[3] Let $(\mathcal{F}, \varsigma)$ be a complete metric space and let $\Im : \mathcal{F} \to \mathcal{F}$. If there exists a nonnegative constant $\rho \in [0, \frac{1}{2})$ such that*

$$\varsigma(\Im\upsilon, \Im\hbar) \leq \rho(\varsigma(\upsilon, \Im\upsilon) + \varsigma(\hbar, \Im\hbar)),$$

for all $\upsilon, \hbar \in \mathcal{F}$, then $\Im$ has a unique fixed point.

In 1972, Chaterjea [4] commuted the terms and established the following result.

Theorem 3. *[4] Let $(\mathcal{F}, \varsigma)$ be a complete metric space and let $\Im : \mathcal{F} \to \mathcal{F}$. If there exists a nonnegative constant $\rho \in [0, \frac{1}{2})$ such that*

$$\varsigma(\Im\upsilon, \Im\hbar) \leq \rho(\varsigma(\upsilon, \Im\hbar) + \varsigma(\hbar, \Im\upsilon)),$$

for all $\upsilon, \hbar \in \mathcal{F}$, then $\Im$ has a unique fixed point.

In [5], Fisher proved the following result in this way.

Theorem 4. *[5] Let $(\mathcal{F},\varsigma)$ be a complete metric space and let $\Im:\mathcal{F}\to\mathcal{F}$. If there exist nonnegative constants $\rho,\kappa\in[0,1)$ such that $\rho+\kappa<1$ and*

$$\varsigma(\Im\upsilon,\Im\hbar)\leq\rho\varsigma(\upsilon,\hbar)+\kappa\frac{\varsigma(\upsilon,\Im\upsilon)\varsigma(\hbar,\Im\hbar)}{1+\varsigma(\upsilon,\hbar)},$$

for all $\upsilon,\hbar\in\mathcal{F}$, then $\Im$ has a unique fixed point.

Czerwik [7] gave the notion of b-metric space as follows:

Definition 1. *[7] Let $\mathcal{F}\neq\emptyset$ and $s\geq1$ be a constant. A function $\varsigma:\mathcal{F}\times\mathcal{F}\to[0,\infty)$ is called a b-metric if the following assertions hold:*

$(b1)$ $\varsigma(\upsilon,\hbar)\geq0$ and $\varsigma(\upsilon,\hbar)=0\Leftrightarrow\upsilon=\hbar$;
$(b2)$ $\varsigma(\upsilon,\hbar)=\varsigma(\hbar,\upsilon)$;
$(b3)$ $\varsigma(\upsilon,\varphi)\leq s[\varsigma(\upsilon,\hbar)+\varsigma(\hbar,\varphi)]$;
for all $\upsilon,\hbar,\varphi\in\mathcal{F}$.

The pair $(\mathcal{F},\varsigma)$ is then said to be a b-metric space.

It is clear from the notion of b-metric that every metric space is b-metric for $s=1$, but the converse is not true.

Example 1. *[14] Let $0<p<1$ and define $\mathcal{F}=L_p[a,b]$ by*

$$L_p[a,b]=\left\{\upsilon:\int_a^b|\upsilon(t)|^p\,dt<\infty\right\},$$

and $\varsigma:\mathcal{F}\times\mathcal{F}\to\mathbb{R}^+$ is mapping defined by

$$\varsigma(\upsilon,\hbar)=\left(\int_a^b|\upsilon(t)-\hbar(t)|^p\,dt\right)^{\frac{1}{p}},$$

for all $\upsilon=\upsilon(t)$ and $\hbar=\hbar(t)\in\mathcal{F}$. Then $(\mathcal{F},\varsigma)$ is a b-metric space with $s=2^{\frac{1}{p}-1}$.

Definition 2. *[7] Let $(\mathcal{F},\varsigma)$ be a b-metric space*
(i) a sequence $\{\upsilon_\jmath\}$ in $\mathcal{F}$ is said to converges to $\upsilon\in\mathcal{F}$, if

$$\lim_{\jmath\to\infty}\varsigma(\upsilon_\jmath,\upsilon)=0,$$

(ii) a sequence $\{\upsilon_\jmath\}$ is said to be Cauchy sequence, if

$$\lim_{\jmath,m\to\infty}\varsigma(\upsilon_\jmath,\upsilon_m)=0,$$

(iii) if every Cauchy sequence in $\mathcal{F}$ is convergent to a point of $\mathcal{F}$, then $(\mathcal{F},\varsigma)$ is said to be complete.

Throughout this article, we consider ς as a continuous functional.

Czerwik [7] proved the following result in a b-metric space.

Theorem 5. *[7] Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im : \mathcal{F} \to \mathcal{F}$. If there exists nonnegative constant $\rho \in [0, 1)$ such that*

$$\varsigma(\Im\upsilon, \Im\hbar) \leq \rho\varsigma(\upsilon, \hbar),$$

for all $\upsilon, \hbar \in \mathcal{F}$, then $\Im$ has a unique fixed point.

Theorem 6. *[15] Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$. If $\Im : \mathcal{F} \to \mathcal{F}$ satisfies the inequality*

$$\varsigma(\Im\upsilon, \Im\hbar) \leq \rho_1\varsigma(\upsilon, \hbar) + \rho_2\varsigma(\upsilon, \Im\upsilon) + \rho_3\varsigma(\hbar, \Im\hbar) + \rho_4(\varsigma(\upsilon, \Im\hbar) + (\hbar, \Im\upsilon)),$$

where $\rho_i \geq 0$ for all $i = 1, 2, 3, 4$ and $\rho_1 + \rho_2 + \rho_3 + 2\rho_4 < 1$ for $s \in [1, 2]$ and $\frac{s}{2} < \rho_1 + \rho_2 + \rho_3 + 2\rho_4 < 1$ for $s \in (2, +\infty)$, then $\Im$ has a unique fixed point.

Recently, Seddik et al. [13] established a common fixed point result for rational contraction.

Theorem 7. *[13] Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$. If there exist nonnegative constant $\rho, \kappa \in [0, 1)$ such that $\rho + \kappa < 1$ and*

$$\varsigma(\Im_1\upsilon, \Im_2\hbar) \leq \rho\varsigma(\upsilon, \hbar) + \kappa\frac{\varsigma(\upsilon, \Im_1\upsilon)\varsigma(\upsilon, \Im_2\hbar) + \varsigma(\hbar, \Im_1\upsilon)\varsigma(\hbar, \Im_2\hbar)}{\varsigma(\upsilon, \Im_2\hbar) + \varsigma(\hbar, \Im_1\upsilon)},$$

for all $\upsilon, \hbar \in \mathcal{F}$, with $\varsigma(\upsilon, \Im_2\hbar) + \varsigma(\hbar, \Im_1\upsilon) \neq 0$, then $\Im_1$ and $\Im_2$ have a unique common fixed point. For more details in this direction, we refer the readers to (see [14–30]).

3. Main results

We state and prove the following proposition, which is required in the proof of our main result:

Proposition 1. *Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$. Let $\upsilon_0 \in \mathcal{F}$. Define the sequence $\{\upsilon_\iota\}$ by*

$$\upsilon_{2\iota+1} = \Im_1\upsilon_{2\iota} \text{ and } \upsilon_{2\iota+2} = \Im_2\upsilon_{2\iota+1}$$

for all $\iota = 0, 1, 2, ...$

Assume that there exists a control function $\rho : \mathcal{F}\times\mathcal{F}\to[0, 1)$ satisfying

$$\rho(\Im_2\Im_1\upsilon, \hbar) \leq \rho(\upsilon, \hbar) \text{ and } \rho(\upsilon, \Im_1\Im_2\hbar) \leq \rho(\upsilon, \hbar)$$

for all $\upsilon, \hbar \in \mathcal{F}$. Then

$$\rho(\upsilon_{2\iota}, \hbar) \leq \rho(\upsilon_0, \hbar) \text{ and } \rho(\upsilon, \upsilon_{2\iota+1}) \leq \rho(\upsilon, \upsilon_1)$$

for all $\upsilon, \hbar \in \mathcal{F}$ and $\iota = 0, 1, 2, ...$

Proof. Let $\upsilon, \hbar \in \mathcal{F}$ and $\iota = 0, 1, 2, ...$ Then we have

$$\begin{aligned} \rho(\upsilon_{2\iota}, \hbar) &= \rho(\Im_2\Im_1\upsilon_{2\iota-2}, \hbar) \leq \rho(\upsilon_{2\iota-2}, \hbar) \\ &= \rho(\Im_2\Im_1\upsilon_{2\iota-4}, \hbar) \leq \rho(\upsilon_{2\iota-4}, \hbar) \end{aligned}$$

$$\leq \quad \cdots \leq \rho(\upsilon_0, \hbar).$$

Similarly, we have

$$\begin{aligned}
\rho(\upsilon, \upsilon_{2\iota+1}) &= \rho(\upsilon, \Im_1\Im_2\upsilon_{2\iota-1}) \leq \rho(\upsilon, \upsilon_{2\iota-1}) \\
&= \rho(\upsilon, \Im_1\Im_2\upsilon_{2\iota-3}) \leq \rho(\upsilon, \upsilon_{2\iota-3}) \\
&\leq \cdots \leq \rho(\upsilon, \upsilon_1).
\end{aligned}$$

□

Definition 3. *Let $(\mathcal{F}, \varsigma)$ be a b-metric space with coefficient $s \geq 1$. The mappings $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$ are said to be generalized rational contractive mappings if there exist the control functions $\rho, \varrho, \kappa : \mathcal{F} \times \mathcal{F} \to [0, 1)$ such that*

$$\begin{aligned}
\varsigma(\Im_1\upsilon, \Im_2\hbar) \leq\ & \rho(\upsilon, \hbar)\varsigma(\upsilon, \hbar) + \varrho(\upsilon, \hbar)[\varsigma(\upsilon, \Im_2\hbar) + \varsigma(\hbar, \Im_1\upsilon)] \\
& + \kappa(\upsilon, \hbar)\frac{\varsigma(\upsilon, \Im_1\upsilon)\varsigma(\hbar, \Im_2\hbar)}{1 + \varsigma(\upsilon, \Im_2\hbar) + \varsigma(\hbar, \Im_1\upsilon) + \varsigma(\upsilon, \hbar)}, \qquad (3.1)
\end{aligned}$$

for all $\upsilon, \hbar \in \mathcal{F}$.

Theorem 8. *Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$ be generalized rational contractive mappings satisfying the following conditions: (a) $\rho(\Im_2\Im_1\upsilon, \hbar) \leq \rho(\upsilon, \hbar)$ and $\rho(\upsilon, \Im_1\Im_2\hbar) \leq \rho(\upsilon, \hbar)$*

$\varrho(\Im_2\Im_1\upsilon, \hbar) \leq \varrho(\upsilon, \hbar)$ and $\varrho(\upsilon, \Im_1\Im_2\hbar) \leq \varrho(\upsilon, \hbar)$

$\kappa(\Im_2\Im_1\upsilon, \hbar) \leq \kappa(\upsilon, \hbar)$ and $\kappa(\upsilon, \Im_1\Im_2\hbar) \leq \kappa(\upsilon, \hbar)$;

(b) $\rho(\upsilon, \hbar) + 2s\varrho(\upsilon, \hbar) + s\kappa(\upsilon, \hbar) < 1$,

then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Let υ_0 be an arbitrary point in $\mathcal{F}$ and the sequence $\{\upsilon_\iota\}$ be defined by

$$\upsilon_{2\iota+1} = \Im_1\upsilon_{2\iota} \text{ and } \upsilon_{2\iota+2} = \Im_2\upsilon_{2\iota+1}$$

for all $\iota = 0, 1, 2, \ldots$ Now by (3.1), we have

$$\begin{aligned}
\varsigma(\upsilon_{2\iota+1}, \upsilon_{2\iota+2}) &= \varsigma(\Im_1\upsilon_{2\iota}, \Im_2\upsilon_{2\iota+1}) \\
&\leq \rho(\upsilon_{2\iota}, \upsilon_{2\iota+1})\varsigma(\upsilon_{2\iota}, \upsilon_{2\iota+1}) \\
&\quad + \varrho(\upsilon_{2\iota}, \upsilon_{2\iota+1})[\varsigma(\upsilon_{2\iota}, \Im_2\upsilon_{2\iota+1}) + \varsigma(\upsilon_{2\iota+1}, \Im_1\upsilon_{2\iota})] \\
&\quad + \kappa(\upsilon_{2\iota}, \upsilon_{2\iota+1})\frac{\varsigma(\upsilon_{2\iota}, \Im_1\upsilon_{2\iota})\varsigma(\upsilon_{2\iota+1}, \Im_2\upsilon_{2\iota+1})}{1 + \varsigma(\upsilon_{2\iota}, \Im_2\upsilon_{2\iota+1}) + \varsigma(\upsilon_{2\iota+1}, \Im_1\upsilon_{2\iota}) + \varsigma(\upsilon_{2\iota}, \upsilon_{2\iota+1})} \\
&\leq \rho(\upsilon_{2\iota}, \upsilon_{2\iota+1})\varsigma(\upsilon_{2\iota}, \upsilon_{2\iota+1}) \\
&\quad + \varrho(\upsilon_{2\iota}, \upsilon_{2\iota+1})[\varsigma(\upsilon_{2\iota}\upsilon, \upsilon_{2\iota+2}) + \varsigma(\upsilon_{2\iota+1}, \upsilon_{2\iota+1})] \\
&\quad + \kappa(\upsilon_{2\iota}, \upsilon_{2\iota+1})\frac{\varsigma(\upsilon_{2\iota}, \upsilon_{2\iota+1})\varsigma(\upsilon_{2\iota+1}, \upsilon_{2\iota+2})}{1 + \varsigma(\upsilon_{2\iota}, \upsilon_{2\iota+2}) + \varsigma(\upsilon_{2\iota}, \upsilon_{2\iota+1})} \\
&\leq \rho(\upsilon_{2\iota}, \upsilon_{2\iota+1})\varsigma(\upsilon_{2\iota}, \upsilon_{2\iota+1}) \\
&\quad + \varrho(\upsilon_{2\iota}, \upsilon_{2\iota+1})[\varsigma(\upsilon_{2\iota}, \upsilon_{2\iota+2})]
\end{aligned}$$

$$
\begin{aligned}
&+s\kappa\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\frac{\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)}{s+\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)}\\
\leq\ &\rho\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)+s\varrho\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\\
&+s\varrho\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)\\
&+s\kappa\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right).
\end{aligned}
$$

Now by Proposition 1, we have

$$
\begin{aligned}
\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)\leq\ &\rho\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\\
&+s\varrho\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\\
&+s\varrho\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)\\
&+s\kappa\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\\
\leq\ &\rho\left(\upsilon_0,\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\\
&+s\varrho\left(\upsilon_0,\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\\
&+s\varrho\left(\upsilon_0,\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)\\
&+s\kappa\left(\upsilon_0,\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\\
\leq\ &\rho\left(\upsilon_0,\upsilon_1\right)\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)\\
&+s\varrho\left(\upsilon_0,\upsilon_1\right)\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)+s\varrho\left(\upsilon_0,\upsilon_1\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)\\
&+s\kappa\left(\upsilon_0,\upsilon_1\right)\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right)
\end{aligned}
$$

which implies that

$$
\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)\leq\frac{\rho\left(\upsilon_0,\upsilon_1\right)+s\varrho\left(\upsilon_0,\upsilon_1\right)+s\kappa\left(\upsilon_0,\upsilon_1\right)}{1-s\varrho\left(\upsilon_0,\upsilon_1\right)}\varsigma\left(\upsilon_{2\imath},\upsilon_{2\imath+1}\right). \tag{3.2}
$$

Similarly, we have

$$
\begin{aligned}
\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+3}\right)=\ &\varsigma\left(\Im_2\upsilon_{2\imath+1},\Im_1\upsilon_{2\imath+2}\right)\\
=\ &\varsigma\left(\Im_1\upsilon_{2\imath+2},\Im_2\upsilon_{2\imath+1}\right)\\
\leq\ &\rho\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\\
&+\varrho\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\left[\varsigma\left(\upsilon_{2\imath+2},\Im_2\upsilon_{2\imath+1}\right)+\varsigma\left(\upsilon_{2\imath+1},\Im_1\upsilon_{2\imath+2}\right)\right]\\
&+\kappa\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\frac{\varsigma\left(\upsilon_{2\imath+2},\Im_1\upsilon_{2\imath+2}\right)\varsigma\left(\upsilon_{2\imath+1},\Im_2\upsilon_{2\imath+1}\right)}{1+\varsigma\left(\upsilon_{2\imath+2},\Im_2\upsilon_{2\imath+1}\right)+\varsigma\left(\upsilon_{2\imath+1},\Im_1\upsilon_{2\imath+2}\right)+\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)}\\
\leq\ &\rho\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\\
&+\varrho\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\left[\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+2}\right)+\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+3}\right)\right]\\
&+\kappa\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\frac{\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+3}\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)}{1+\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+3}\right)+\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)}\\
\leq\ &\rho\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\\
&+s\varrho\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)+s\varrho\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+3}\right)\\
&+s\kappa\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\frac{\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+3}\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)}{s+\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+3}\right)}\\
\leq\ &\rho\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)
\end{aligned}
$$

$$+s\varrho\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)+s\varrho\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+3}\right)$$
$$+s\kappa\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right).$$

Now by Proposition 1, we have

$$\begin{aligned}
\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+3}\right) &\leq \rho\left(\upsilon_0,\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)\\
&\quad +s\varrho\left(\upsilon_0,\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)+s\varrho\left(\upsilon_0,\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+3}\right)\\
&\quad +s\kappa\left(\upsilon_0,\upsilon_{2\imath+1}\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)\\
&\leq \rho\left(\upsilon_0,\upsilon_1\right)\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+1}\right)\\
&\quad +s\varrho\left(\upsilon_0,\upsilon_1\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)+s\varrho\left(\upsilon_0,\upsilon_1\right)\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+3}\right)\\
&\quad +s\kappa\left(\upsilon_0,\upsilon_1\right)\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right)
\end{aligned}$$

which implies that

$$\varsigma\left(\upsilon_{2\imath+2},\upsilon_{2\imath+3}\right)\leq\frac{\rho\left(\upsilon_0,\upsilon_1\right)+s\varrho\left(\upsilon_0,\upsilon_1\right)+s\kappa\left(\upsilon_0,\upsilon_1\right)}{1-s\varrho\left(\upsilon_0,\upsilon_1\right)}\varsigma\left(\upsilon_{2\imath+1},\upsilon_{2\imath+2}\right). \tag{3.3}$$

Let $\lambda=\frac{\rho(\upsilon_0,\upsilon_1)+s\varrho(\upsilon_0,\upsilon_1)+s\kappa(\upsilon_0,\upsilon_1)}{1-s\varrho(\upsilon_0,\upsilon_1)}<1$. Then from (3.2) and (3.3), we have

$$\varsigma\left(\upsilon_\imath,\upsilon_{\imath+1}\right)\leq\lambda\varsigma\left(\upsilon_{\imath-1},\upsilon_\imath\right)$$

for all $\imath\in\mathbb{N}$. By induction, we build a sequence $\{\upsilon_\imath\}$ in $\mathcal{F}$ so that

$$\varsigma\left(\upsilon_\imath,\upsilon_{\imath+1}\right)\leq\lambda\varsigma\left(\upsilon_{\imath-1},\upsilon_\imath\right)\leq\lambda^2\varsigma\left(\upsilon_{\imath-2},\upsilon_{\imath-1}\right)\leq...\leq\lambda^\imath\varsigma\left(\upsilon_0,\upsilon_1\right) \tag{3.4}$$

$\forall\ \imath\in\mathbb{N}$. Now, for $m>\imath$, we have

$$\begin{aligned}
\varsigma\left(\upsilon_\imath,\upsilon_m\right) &\leq s\left[\varsigma\left(\upsilon_\imath,\upsilon_{\imath+1}\right)+\varsigma\left(\upsilon_{\imath+1},\upsilon_m\right)\right]\\
&= s\varsigma\left(\upsilon_\imath,\upsilon_{\imath+1}\right)+s\varsigma\left(\upsilon_{\imath+1},\upsilon_m\right)\\
&\leq s\varsigma\left(\upsilon_\imath,\upsilon_{\imath+1}\right)+s^2\left[\varsigma\left(\upsilon_{\imath+1},\upsilon_{\imath+2}\right)+\varsigma\left(\upsilon_{\imath+2},\upsilon_m\right)\right]\\
&= s\varsigma\left(\upsilon_\imath,\upsilon_{\imath+1}\right)+s^2\varsigma\left(\upsilon_{\imath+1},\upsilon_{\imath+2}\right)+s^2\varsigma\left(\upsilon_{\imath+2},\upsilon_m\right)\\
&\leq ...\leq s\varsigma\left(\upsilon_\imath,\upsilon_{\imath+1}\right)+s^2\varsigma\left(\upsilon_{\imath+1},\upsilon_{\imath+2}\right)+...+s^{m-\imath}\varsigma\left(\upsilon_{m-1},\upsilon_m\right).
\end{aligned}$$

By (3.4), we have

$$\begin{aligned}
\varsigma\left(\upsilon_\imath,\upsilon_m\right) &\leq s\lambda^\imath\varsigma\left(\upsilon_0,\upsilon_1\right)+s^2\lambda^{\imath+1}\varsigma\left(\upsilon_0,\upsilon_1\right)+s^3\lambda^{\imath+2}\varsigma\left(\upsilon_0,\upsilon_1\right)\\
&\quad +...+s^{m-\imath}\lambda^{m-1}\varsigma\left(\upsilon_0,\upsilon_1\right)\\
&\leq \left[s\lambda^\imath+s^2\lambda^{\imath+1}+...+s^{m-\imath}\lambda^{m-1}\right]\varsigma\left(\upsilon_0,\upsilon_1\right)\\
&\leq s\lambda^\imath\left[1+(s\lambda)^1+(s\lambda)^2+...+(s\lambda)^{m-\imath-1}\right]\varsigma\left(\upsilon_0,\upsilon_1\right)\\
&\leq \frac{s\lambda^\imath}{1-s\lambda}\varsigma\left(\upsilon_0,\upsilon_1\right).
\end{aligned}$$

Letting $\imath\to\infty$, we have

$$\varsigma\left(\upsilon_\imath,\upsilon_m\right)\to 0.$$

Hence the sequence $\{\upsilon_\iota\}$ is Cauchy. Since $\mathcal{F}$ is complete, there is υ^* so that $\upsilon_\iota \to \upsilon^* \in \mathcal{F}$ as $\iota \to \infty$, that is,

$$\lim_{\iota\to\infty} \upsilon_\iota = \upsilon^*.$$

Then

$$\lim_{\iota\to\infty} \upsilon_{2\iota+1} = \upsilon^* \text{ and } \lim_{\iota\to\infty} \upsilon_{2\iota+2} = \upsilon^*.$$

Now, we show that υ^* is a fixed point of $\Im_1$. From (3.1), we have

$$\begin{array}{rcl}
\varsigma(\upsilon^*, \Im_1\upsilon^*) & \leq & s\left(\varsigma(\upsilon^*, \Im_2\upsilon_{2\iota+1}) + \varsigma(\Im_2\upsilon_{2\iota+1}, \Im_1\upsilon^*)\right) \\
& = & s\left(\varsigma(\upsilon^*, \Im_2\upsilon_{2\iota+1}) + \varsigma(\Im_1\upsilon^*, \Im_2\upsilon_{2\iota+1})\right) \\
& \leq & s\begin{pmatrix} \varsigma(\upsilon^*, \upsilon_{2\iota+2}) + \rho(\upsilon^*, \upsilon_{2\iota+1})\varsigma(\upsilon^*, \upsilon_{2\iota+1}) \\ +\varrho(\upsilon^*, \upsilon_{2\iota+1})\left[\varsigma(\upsilon^*, \Im_2\upsilon_{2\iota+1}) + \varsigma(\upsilon_{2\iota+1}, \Im_1\upsilon^*)\right] \\ +\kappa(\upsilon^*, \upsilon_{2\iota+1})\frac{\varsigma(\upsilon^*, \Im_1\upsilon^*)\varsigma(\upsilon_{2\iota+1}, \Im_2\upsilon_{2\iota+1})}{1+\varsigma(\upsilon^*, \Im_2\upsilon_{2\iota+1})+\varsigma(\upsilon_{2\iota+1}, \Im_1\upsilon^*)+\varsigma(\upsilon^*, \upsilon_{2\iota+1})} \end{pmatrix} \\
& \leq & s\begin{pmatrix} \varsigma(\upsilon^*, \upsilon_{2\iota+2}) + \rho(\upsilon^*, \upsilon_1)\varsigma(\upsilon^*, \upsilon_{2\iota+2}) \\ +\varrho(\upsilon^*, \upsilon_1)\left[\varsigma(\upsilon^*, \upsilon_{2\iota+2}) + \varsigma(\upsilon_{2\iota+1}, \Im_1\upsilon^*)\right] \\ +\kappa(\upsilon^*, \upsilon_1)\frac{\varsigma(\upsilon^*, \Im_1\upsilon^*)\varsigma(\upsilon_{2\iota+1}, \upsilon_{2\iota+2})}{1+\varsigma(\upsilon^*, \upsilon_{2\iota+2})+\varsigma(\upsilon_{2\iota+1}, \Im_1\upsilon^*)+\varsigma(\upsilon^*, \upsilon_{2\iota+1})} \end{pmatrix}.
\end{array}$$

Letting $\iota \to \infty$ in the above inequality, we get

$$\begin{array}{rcl}
\varsigma(\upsilon^*, \Im_1\upsilon^*) & \leq & s\varrho(\upsilon^*, \upsilon_1)\varsigma(\upsilon^*, \Im_1\upsilon^*) \\
& \leq & (\rho(\upsilon^*, \upsilon_1) + 2s\varrho(\upsilon^*, \upsilon_1) + s\kappa(\upsilon^*, \upsilon_1))\varsigma(\upsilon^*, \Im_1\upsilon^*) \\
& < & \varsigma(\upsilon^*, \Im_1\upsilon^*),
\end{array}$$

which is a contradiction. Thus, $\upsilon^* = \Im_1\upsilon^*$. Now, we show that υ^* is a fixed point of $\Im_2$. From (3.1), we have

$$\begin{array}{rcl}
\varsigma(\upsilon^*, \Im_2\upsilon^*) & \leq & s\left(\varsigma(\upsilon^*, \upsilon_{2\iota+1}) + \varsigma(\upsilon_{2\iota+1}, \Im_2\upsilon^*)\right) \\
& = & s\left(\varsigma(\upsilon^*, \upsilon_{2\iota+1}) + \varsigma(\Im_1\upsilon_{2\iota}, \Im_2\upsilon^*)\right) \\
& \leq & s\begin{pmatrix} \varsigma(\upsilon^*, \upsilon_{2\iota+1}) + \rho(\upsilon_{2\iota}, \upsilon^*)\varsigma(\upsilon_{2\iota}, \upsilon^*) \\ +\varrho(\upsilon_{2\iota}, \upsilon^*)\left[\varsigma(\upsilon_{2\iota}, \Im_2\upsilon^*) + \varsigma(\upsilon^*, \Im_1\upsilon_{2\iota})\right] \\ +\kappa(\upsilon_{2\iota}, \upsilon^*)\frac{\varsigma(\upsilon_{2\iota}, \Im_1\upsilon_{2\iota})\varsigma(\upsilon^*, \Im_2\upsilon^*)}{1+\varsigma(\upsilon_{2\iota}, \Im_2\upsilon^*)+\varsigma(\upsilon^*, \Im_1\upsilon_{2\iota})+\varsigma(\upsilon_{2\iota}, \upsilon^*)} \end{pmatrix} \\
& \leq & s\begin{pmatrix} \varsigma(\upsilon^*, \upsilon_{2\iota+1}) + \rho(\upsilon_0, \upsilon^*)\varsigma(\upsilon_{2\iota}, \upsilon^*) \\ +\varrho(\upsilon_0, \upsilon^*)\left[\varsigma(\upsilon_{2\iota}, \Im_2\upsilon^*) + \varsigma(\upsilon^*, \upsilon_{2\iota+1})\right] \\ +\kappa(\upsilon_0, \upsilon^*)\frac{\varsigma(\upsilon_{2\iota}, \upsilon_{2\iota+1})\varsigma(\upsilon^*, \Im_2\upsilon^*)}{1+\varsigma(\upsilon_{2\iota}, \Im_2\upsilon^*)+\varsigma(\upsilon^*, \upsilon_{2\iota+1})+\varsigma(\upsilon_{2\iota}, \upsilon^*)} \end{pmatrix}.
\end{array}$$

Letting $\iota \to \infty$ in the above inequality, we get

$$\begin{array}{rcl}
\varsigma(\upsilon^*, \Im_2\upsilon^*) & \leq & s\varrho(\upsilon_0, \upsilon^*)\varsigma(\upsilon^*, \Im_2\upsilon^*) \\
& \leq & (\rho(\upsilon_0, \upsilon^*) + 2s\varrho(\upsilon_0, \upsilon^*) + s\kappa(\upsilon_0, \upsilon^*))\varsigma(\upsilon^*, \Im_2\upsilon^*) \\
& < & \varsigma(\upsilon^*, \Im_2\upsilon^*),
\end{array}$$

which is a contradiction. Thus $\upsilon^* = \Im_2\upsilon^*$. □

Now, we prove that υ^* is a unique. We assume that there exists another common fixed of $\upsilon^/$ of $\Im_1$ and $\Im_2$, i.e.,

$$\upsilon^/ = \Im_1\upsilon^/ = \Im_2\upsilon^/$$

but $\upsilon^* \neq \upsilon^/$. Now, from (3.1), we have

$$\begin{aligned}
\varsigma\left(\upsilon^*, \upsilon^/\right) &= \varsigma\left(\Im_1\upsilon^*, \Im_2\upsilon^/\right) \\
&\leq \rho\left(\upsilon^*, \upsilon^/\right)\varsigma\left(\upsilon^*, \upsilon^/\right) \\
&\quad +\varrho\left(\upsilon^*, \upsilon^/\right)\left[\varsigma\left(\upsilon^*, \Im_2\upsilon^/\right) + \varsigma\left(\upsilon^/, \Im_1\upsilon^*\right)\right] \\
&\quad +\kappa\left(\upsilon^*, \upsilon^/\right)\frac{\varsigma\left(\upsilon^*, \Im\upsilon^*\right)\varsigma\left(\upsilon^/, \Im_2\upsilon^/\right)}{1 + \varsigma\left(\upsilon^*, \Im_2\upsilon^/\right) + \varsigma\left(\upsilon^/, \Im_1\upsilon^*\right) + \varsigma\left(\upsilon^*, \upsilon^/\right)} \\
&= \rho\left(\upsilon^*, \upsilon^/\right)\varsigma\left(\upsilon^*, \upsilon^/\right) + 2\varrho\left(\upsilon^*, \hbar\right)\varsigma\left(\upsilon^*, \upsilon^/\right) \\
&\leq \rho\left(\upsilon^*, \upsilon^/\right)\varsigma\left(\upsilon^*, \upsilon^/\right) + 2s\varrho\left(\upsilon^*, \hbar\right)\varsigma\left(\upsilon^*, \upsilon^/\right) \\
&= \left(\rho\left(\upsilon^*, \upsilon^/\right) + 2s\varrho\left(\upsilon^*, \hbar\right)\right)\varsigma\left(\upsilon^*, \upsilon^/\right).
\end{aligned}$$

As $\rho\left(\upsilon^*, \upsilon^/\right) + 2s\varrho\left(\upsilon^*, \hbar\right) < 1$, we have

$$\varsigma\left(\upsilon^*, \upsilon^/\right) = 0.$$

Thus, $\upsilon^* = \upsilon^/$.

Corollary 1. *Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$. If there exist control functions $\rho, \varrho : \mathcal{F} \times \mathcal{F} \to [0, 1)$ such that*

(a) $\rho\left(\Im_2\Im_1\upsilon, \hbar\right) \leq \rho\left(\upsilon, \hbar\right)$ and $\rho\left(\upsilon, \Im_1\Im_2\hbar\right) \leq \rho\left(\upsilon, \hbar\right)$
$\varrho\left(\Im_2\Im_1\upsilon, \hbar\right) \leq \varrho\left(\upsilon, \hbar\right)$ and $\varrho\left(\upsilon, \Im_1\Im_2\hbar\right) \leq \varrho\left(\upsilon, \hbar\right)$,

(b) $\rho\left(\upsilon, \hbar\right) + 2s\varrho\left(\upsilon, \hbar\right) < 1$,

(c) $\varsigma\left(\Im_1\upsilon, \Im_2\hbar\right) \leq \rho\left(\upsilon, \hbar\right)\varsigma\left(\upsilon, \hbar\right) + \varrho\left(\upsilon, \hbar\right)\left[\varsigma\left(\upsilon, \Im_2\hbar\right) + \varsigma\left(\hbar, \Im_1\upsilon\right)\right]$,

then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Take $\kappa\left(\upsilon, \hbar\right) = 0$ in Theorem 8. □

Theorem 9. *Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$. If there exist control functions $\rho, \kappa : \mathcal{F} \times \mathcal{F} \to [0, 1)$ such that*

(a) $\rho\left(\Im_2\Im_1\upsilon, \hbar\right) \leq \rho\left(\upsilon, \hbar\right)$ and $\rho\left(\upsilon, \Im_1\Im_2\hbar\right) \leq \rho\left(\upsilon, \hbar\right)$
$\kappa\left(\Im_2\Im_1\upsilon, \hbar\right) \leq \kappa\left(\upsilon, \hbar\right)$ and $\kappa\left(\upsilon, \Im_1\Im_2\hbar\right) \leq \kappa\left(\upsilon, \hbar\right)$,

(b) $\rho\left(\upsilon, \hbar\right) + s\kappa\left(\upsilon, \hbar\right) < 1$,

(c) $\varsigma\left(\Im_1\upsilon, \Im_2\hbar\right) \leq \rho\left(\upsilon, \hbar\right)\varsigma\left(\upsilon, \hbar\right) + \kappa\left(\upsilon, \hbar\right)\frac{\varsigma\left(\upsilon, \Im_1\upsilon\right)\varsigma\left(\hbar, \Im_2\hbar\right)}{1+\varsigma\left(\upsilon, \Im_2\hbar\right)+\varsigma\left(\hbar, \Im_1\upsilon\right)+\varsigma\left(\upsilon, \hbar\right)}$,

then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Take $\varrho\left(\upsilon, \hbar\right) = 0$ in Theorem 8. □

Corollary 2. *Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefcient $s \geq 1$ and let $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$. If there exist control function $\rho : \mathcal{F} \times \mathcal{F} \to [0, 1)$ such that*

(a) $\rho\left(\Im_2\Im_1\upsilon, \hbar\right) \leq \rho\left(\upsilon, \hbar\right)$ and $\rho\left(\upsilon, \Im_1\Im_2\hbar\right) \leq \rho\left(\upsilon, \hbar\right)$

(b) $\rho\left(\upsilon, \hbar\right) < 1$,

(c) $\varsigma(\Im_1\upsilon, \Im_2\hbar) \leq \rho(\upsilon, \hbar)\varsigma(\upsilon, \hbar)$,
then $\Im_1$ *and* $\Im_2$ *have a unique common fixed point.*

Proof. Take $\varrho(\upsilon, \hbar) = \kappa(\upsilon, \hbar) = 0$ in Theorem 8. □

Corollary 3. *Let* $(\mathcal{F}, \varsigma)$ *be a complete b-metric space with coefcient* $s \geq 1$ *and let* $\Im : \mathcal{F} \to \mathcal{F}$. *If there exist control function* $\rho : \mathcal{F} \times \mathcal{F} \to [0, 1)$ *such that*
(a) $\rho(\Im\upsilon, \hbar) \leq \rho(\upsilon, \hbar)$ *and* $\rho(\upsilon, \Im\hbar) \leq \rho(\upsilon, \hbar)$
(b) $\rho(\upsilon, \hbar) < 1$,
(c) $\varsigma(\Im\upsilon, \Im\hbar) \leq \rho(\upsilon, \hbar)\varsigma(\upsilon, \hbar)$,
then $\Im$ *has a unique fixed point.*

Proof. Set $\Im_1 = \Im$ and $\Im_2 = I$ (Identity mapping) and $\varrho(\upsilon, \hbar) = \kappa(\upsilon, \hbar) = 0$ in Theorem 8. □

Corollary 4. *Let* $(\mathcal{F}, \varsigma)$ *be a complete b-metric space with coefficient* $s \geq 1$ *and let* $\Im : \mathcal{F} \to \mathcal{F}$. *If there exist control functions* $\rho, \varrho, \kappa : \mathcal{F} \times \mathcal{F} \to [0, 1)$ *such that*
(a) $\rho(\Im\upsilon, \hbar) \leq \rho(\upsilon, \hbar)$ *and* $\rho(\upsilon, \Im\hbar) \leq \rho(\upsilon, \hbar)$,
$\varrho(\Im\upsilon, \hbar) \leq \varrho(\upsilon, \hbar)$ *and* $\varrho(\upsilon, \Im\hbar) \leq \varrho(\upsilon, \hbar)$
$\kappa(\Im\upsilon, \hbar) \leq \kappa(\upsilon, \hbar)$ *and* $\kappa(\upsilon, \Im\hbar) \leq \kappa(\upsilon, \hbar)$,
(b) $\rho(\upsilon, \hbar) + 2s\varrho(\upsilon, \hbar) + s\kappa(\upsilon, \hbar) < 1$,
(c) $\varsigma(\Im\upsilon, \Im\hbar) \leq \rho(\upsilon, \hbar)\varsigma(\upsilon, \hbar) + \varrho(\upsilon, \hbar)[\varsigma(\upsilon, \Im\hbar) + \varsigma(\hbar, \Im\upsilon)] + \kappa(\upsilon, \hbar)\frac{\varsigma(\upsilon, \Im\upsilon)\varsigma(\hbar, \Im\hbar)}{1+\varsigma(\upsilon, \Im\hbar)+\varsigma(\hbar, \Im\upsilon)+\varsigma(\upsilon, \hbar)}$
for all $\upsilon, \hbar \in \mathcal{F}$, *then* $\Im$ *has a unique fixed point.*

Proof. Set $\Im_1 = \Im$ and $\Im_2 = I$ (Identity mapping) in Theorem 8. □

Example 2. *Let* $\mathcal{F} = [0, 1)$ *and* $\varsigma : \mathcal{F} \times \mathcal{F} \to \mathbb{C}$ *by*

$$\varsigma(\upsilon, \hbar) = |\upsilon - \hbar|^2$$

for all $\upsilon, \hbar \in \mathcal{F}$ *with* $s = 2$. *Then* $(\mathcal{F}, \varsigma)$ *is a complete b-metric space. Define the self mappings* $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$ *by*

$$\Im_1\upsilon = \frac{\upsilon}{3}$$

and

$$\Im_2\upsilon = \frac{\upsilon}{4}$$

Consider

$$\rho, \varrho, \kappa : \mathcal{F} \times \mathcal{F} \to [0, 1)$$

by

$$\rho(\upsilon, \hbar) = \frac{\upsilon}{17} + \frac{\hbar}{20}$$

and

$$\varrho(\upsilon, \hbar) = \frac{\upsilon}{16} + \frac{\hbar}{21}$$

and

$$\kappa(\upsilon, \hbar) = \frac{\upsilon\hbar}{39}.$$

Now, we satisfy the condition (a) as follows.

$\rho(\Im_2\Im_1\upsilon,\hbar) = \frac{\upsilon}{204} + \frac{\hbar}{20} \le \frac{\upsilon}{17} + \frac{\hbar}{20} = \rho(\upsilon,\hbar)$ and $\rho(\upsilon,\Im_1\Im_2\hbar) = \frac{\upsilon}{17} + \frac{\hbar}{240} \le \frac{\upsilon}{17} + \frac{\hbar}{20} = \rho(\upsilon,\hbar)$

$\varrho(\Im_2\Im_1\upsilon,\hbar) = \frac{\upsilon}{192} + \frac{\hbar}{21} \le \frac{\upsilon}{16} + \frac{\hbar}{21} = \varrho(\upsilon,\hbar)$ and $\varrho(\upsilon,\Im_1\Im_2\hbar) = \frac{\upsilon}{16} + \frac{\hbar}{252} \le \frac{\upsilon}{16} + \frac{\hbar}{21} = \varrho(\upsilon,\hbar)$

$\kappa(\Im_2\Im_1\upsilon,\hbar) = \frac{\upsilon\hbar}{468} \le \frac{\upsilon\hbar}{39} = \kappa(\upsilon,\hbar)$ and $\kappa(\upsilon,\Im_1\Im_2\hbar) = \frac{\upsilon\hbar}{468} \le \frac{\upsilon\hbar}{39} = \kappa(\upsilon,\hbar)$.

Also condition (b) holds, that is, $\rho(\upsilon,\hbar) + 2s\varrho(\upsilon,\hbar) + s\kappa(\upsilon,\hbar) < 1$.

Consider

$$\begin{aligned}
\varsigma(\Im_1\upsilon,\Im_2\hbar) &= \left|\frac{\upsilon}{3} - \frac{\hbar}{4}\right|^2 \\
&\le \left(\frac{\upsilon}{17} + \frac{\hbar}{20}\right)|\upsilon-\hbar|^2 + \left(\frac{\upsilon}{16} + \frac{\hbar}{21}\right)\left(\left|\upsilon - \frac{\hbar}{4}\right|^2 + \left|\hbar - \frac{\upsilon}{3}\right|^2\right) \\
&\quad + \left(\frac{\upsilon\hbar}{39}\right)\frac{\left|\upsilon - \frac{\upsilon}{3}\right|^2\left|\hbar - \frac{\hbar}{4}\right|^2}{1 + \left|\upsilon - \frac{\hbar}{4}\right|^2 + \left|\hbar - \frac{\upsilon}{3}\right|^2 + |\upsilon-\hbar|^2} \\
&= \rho(\upsilon,\hbar)\varsigma(\upsilon,\hbar) + \varrho(\upsilon,\hbar)\left[\varsigma(\upsilon,\Im_2\hbar) + \varsigma(\hbar,\Im_1\upsilon)\right] \\
&\quad + \kappa(\upsilon,\hbar)\frac{\varsigma(\upsilon,\Im_1\upsilon)\varsigma(\hbar,\Im_2\hbar)}{1 + \varsigma(\upsilon,\Im_2\hbar) + \varsigma(\hbar,\Im_1\upsilon) + \varsigma(\upsilon,\hbar)}.
\end{aligned}$$

Thus, all the assumptions of Theorem 8 are satisfied and 0 is a unique common fixed point of the mappings $\Im_1$ and $\Im_2$.

Corollary 5. *Let $(\mathcal{F},\varsigma)$ be a complete b-metric space with coefficient $s \ge 1$ and let $\Im : \mathcal{F} \to \mathcal{F}$. If there exist mappings $\rho,\kappa,\mu : \mathcal{F}\times\mathcal{F} \to [0,1)$ such that for all $\upsilon,\hbar \in \mathcal{F}$,*

(a) $\rho(\Im\upsilon,\hbar) \le \rho(\upsilon,\hbar)$ and $\rho(\upsilon,\Im\hbar) \le \rho(\upsilon,\hbar)$

$\kappa(\Im\upsilon,\hbar) \le \kappa(\upsilon,\hbar)$ and $\kappa(\upsilon,\Im\hbar) \le \kappa(\upsilon,\hbar)$

$\mu(\Im\upsilon,\hbar) \le \mu(\upsilon,\hbar)$ and $\mu(\upsilon,\Im\hbar) \le \mu(\upsilon,\hbar)$,

(b) $\rho(\upsilon,\hbar) + 2s\kappa(\upsilon,\hbar) + s\mu(\upsilon,\hbar) < 1$,

(c)

$$\begin{aligned}
\varsigma(\Im^n\upsilon,\Im^n\hbar) &\le \rho(\upsilon,\hbar)\varsigma(\upsilon,\hbar) + \varrho(\upsilon,\hbar)\left[\varsigma(\upsilon,\Im^n\hbar) + \varsigma(\hbar,\Im^n\upsilon)\right] \\
&\quad + \kappa(\upsilon,\hbar)\frac{\varsigma(\upsilon,\Im^n\upsilon)\varsigma(\hbar,\Im^n\hbar)}{1 + \varsigma(\upsilon,\Im^n\hbar) + \varsigma(\hbar,\Im^n\upsilon) + \varsigma(\upsilon,\hbar)}
\end{aligned} \tag{3.5}$$

for all $\upsilon,\hbar \in \mathcal{F}$, then there exists a unique point $\upsilon^* \in \mathcal{F}$ such that $\Im\upsilon^* = \upsilon^*$.

Proof. From Corollary 4, we have $\upsilon \in \mathcal{F}$ such that $\Im^n\upsilon = \upsilon$. Now, from

$$\begin{aligned}
\varsigma(\Im\upsilon,\upsilon) &= \varsigma(\Im\Im^n\upsilon,\Im^n\upsilon) = \varsigma(\Im^n\Im\upsilon,\Im^n\upsilon) \\
&\le \rho(\Im\upsilon,\upsilon)\varsigma(\Im\upsilon,\upsilon) + \varrho(\Im\upsilon,\upsilon)\left[\varsigma(\Im\upsilon,\Im^n\upsilon) + \varsigma(\upsilon,\Im^n\Im\upsilon)\right] \\
&\quad + \kappa(\Im\upsilon,\upsilon)\frac{\varsigma(\Im\upsilon,\Im^n\Im\upsilon)\varsigma(\upsilon,\Im^n\upsilon)}{1 + \varsigma(\Im\upsilon,\Im^n\upsilon) + \varsigma(\upsilon,\Im^n\Im\upsilon) + \varsigma(\Im\upsilon,\upsilon)} \\
&= \rho(\Im\upsilon,\upsilon)\varsigma(\Im\upsilon,\upsilon) + \varrho(\Im\upsilon,\upsilon)\left[\varsigma(\Im\upsilon,\upsilon) + \varsigma(\upsilon,\Im\upsilon)\right] \\
&\quad + \kappa(\Im\upsilon,\upsilon)\frac{\varsigma(\Im\upsilon,\Im\upsilon)\varsigma(\upsilon,\upsilon)}{1 + \varsigma(\Im\upsilon,\upsilon) + \varsigma(\upsilon,\Im\upsilon) + \varsigma(\Im\upsilon,\upsilon)} \\
&= \left(\rho(\Im\upsilon,\upsilon) + 2\varrho(\Im\upsilon,\upsilon)\right)\varsigma(\Im\upsilon,\upsilon)
\end{aligned}$$

which is possible only whenever $\varsigma(\Im\upsilon,\upsilon) = 0$. Thus, $\Im\upsilon = \upsilon$. □

4. Deduced results

Corollary 6. *Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$. If there exist control functions $\rho, \varrho, \kappa : \mathcal{F} \to [0, 1)$ such that*

(a) $\rho(\Im_1 \upsilon) \leq \rho(\upsilon)$ and $\rho(\Im_2 \upsilon) \leq \rho(\upsilon)$
$\varrho(\Im_1 \upsilon) \leq \varrho(\upsilon)$ and $\varrho(\Im_2 \upsilon) \leq \varrho(\upsilon)$
$\kappa(\Im_1 \upsilon) \leq \kappa(\upsilon)$ and $\kappa(\Im_2 \upsilon) \leq \kappa(\upsilon)$,
(b) $\rho(\upsilon) + 2s\varrho(\upsilon) + s\kappa(\upsilon) < 1$,
(c) $\varsigma(\Im_1 \upsilon, \Im_2 \hbar) \leq \rho(\upsilon) \varsigma(\upsilon, \hbar) + \varrho(\upsilon)[\varsigma(\upsilon, \Im_2 \hbar) + \varsigma(\hbar, \Im_1 \upsilon)] + \kappa(\upsilon) \frac{\varsigma(\upsilon, \Im_1 \upsilon)\varsigma(\hbar, \Im_2 \hbar)}{1+\varsigma(\upsilon, \Im_2 \hbar)+\varsigma(\hbar, \Im_1 \upsilon)+\varsigma(\upsilon, \hbar)}$,

then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Define $\rho, \varrho, \kappa : \mathcal{F} \times \mathcal{F} \to [0, 1)$ by

$$\rho(\upsilon, \hbar) = \rho(\upsilon), \quad \varrho(\upsilon, \hbar) = \varrho(\upsilon) \quad \text{and} \quad \kappa(\upsilon, \hbar) = \kappa(\upsilon)$$

for all $\upsilon, \hbar \in \mathcal{F}$. Then for all $\upsilon, \hbar \in \mathcal{F}$, we have

(a) $\rho(\Im_2 \Im_1 \upsilon, \hbar) = \rho(\Im_2 \Im_1 \upsilon) \leq \rho(\Im_1 \upsilon) \leq \rho(\upsilon) = \rho(\upsilon, \hbar)$ and $\rho(\upsilon, \Im_1 \Im_2 \hbar) = \rho(\upsilon) = \rho(\upsilon, \hbar)$
$\varrho(\Im_2 \Im_1 \upsilon, \hbar) = \varrho(\Im_2 \Im_1 \upsilon) \leq \varrho(\Im_1 \upsilon) \leq \varrho(\upsilon) = \varrho(\upsilon, \hbar)$ and $\varrho(\upsilon, \Im_1 \Im_2 \hbar) = \varrho(\upsilon) = \varrho(\upsilon, \hbar)$
$\kappa(\Im_2 \Im_1 \upsilon, \hbar) = \kappa(\Im_2 \Im_1 \upsilon) \leq \kappa(\Im_1 \upsilon) \leq \kappa(\upsilon) = \kappa(\upsilon, \hbar)$ and $\kappa(\upsilon, \Im_1 \Im_2 \hbar) = \kappa(\upsilon) = \kappa(\upsilon, \hbar)$
(b) $\rho(\upsilon, \hbar) + 2s\varrho(\upsilon, \hbar) + s\kappa(\upsilon, \hbar) = \rho(\upsilon) + 2s\varrho(\upsilon) + s\kappa(\upsilon) < 1$,
(c) $\varsigma(\Im_1 \upsilon, \Im_2 \hbar) \leq \rho(\upsilon) \varsigma(\upsilon, \hbar) + \varrho(\upsilon)[\varsigma(\upsilon, \Im_2 \hbar) + \varsigma(\hbar, \Im_1 \upsilon)] + \kappa(\upsilon) \frac{\varsigma(\upsilon, \Im_1 \upsilon)\varsigma(\hbar, \Im_2 \hbar)}{1+\varsigma(\upsilon, \Im_2 \hbar)+\varsigma(\hbar, \Im_1 \upsilon)+\varsigma(\upsilon, \hbar)}$
$\varsigma(\Im_1 \upsilon, \Im_2 \hbar) \leq \rho(\upsilon, \hbar) \varsigma(\upsilon, \hbar) + \varrho(\upsilon, \hbar)[\varsigma(\upsilon, \Im_2 \hbar) + \varsigma(\hbar, \Im_1 \upsilon)] + \kappa(\upsilon, \hbar) \frac{\varsigma(\upsilon, \Im_1 \upsilon)\varsigma(\hbar, \Im_2 \hbar)}{1+\varsigma(\upsilon, \Im_2 \hbar)+\varsigma(\hbar, \Im_1 \upsilon)+\varsigma(\upsilon, \hbar)}$.

By Theorem 8, $\Im_1$ and $\Im_2$ have a unique common fixed point. □

Remark 1. *It is notable that (a) and (b) of Theorem 8 above can be weakened by the condition $\rho(\Im_2 \Im_1 \upsilon) \leq \rho(\upsilon)$, $\varrho(\Im_2 \Im_1 \upsilon) \leq \varrho(\upsilon)$ and $\kappa(\Im_2 \Im_1 \upsilon) \leq \kappa(\upsilon)$ for all $\upsilon \in \mathcal{F}$.*

Corollary 7. *Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$. If there exist control functions $\rho, \varrho, \kappa : \mathcal{F} \to [0, 1)$ such that*

(a) $\rho(\Im_2 \Im_1 \upsilon) \leq \rho(\upsilon)$
$\varrho(\Im_2 \Im_1 \upsilon) \leq \varrho(\upsilon)$
$\kappa(\Im_2 \Im_1 \upsilon) \leq \kappa(\upsilon)$,
(b) $\rho(\upsilon) + 2s\varrho(\upsilon) + s\kappa(\upsilon) < 1$,
(c) $\varsigma(\Im_1 \upsilon, \Im_2 \hbar) \leq \rho(\upsilon) \varsigma(\upsilon, \hbar) + \varrho(\upsilon)[\varsigma(\upsilon, \Im_2 \hbar) + \varsigma(\hbar, \Im_1 \upsilon)] + \kappa(\upsilon) \frac{\varsigma(\upsilon, \Im_1 \upsilon)\varsigma(\hbar, \Im_2 \hbar)}{1+\varsigma(\upsilon, \Im_2 \hbar)+\varsigma(\hbar, \Im_1 \upsilon)+\varsigma(\upsilon, \hbar)}$,

then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Define $\rho, \varrho, \kappa : \mathcal{F} \times \mathcal{F} \to [0, 1)$ by

$$\rho(\upsilon, \hbar) = \rho(\upsilon), \quad \varrho(\upsilon, \hbar) = \varrho(\upsilon) \quad \text{and} \quad \kappa(\upsilon, \hbar) = \kappa(\upsilon)$$

for all $\upsilon, \hbar \in \mathcal{F}$. Then for all $\upsilon, \hbar \in \mathcal{F}$, we have

(a) $\rho(\Im_2 \Im_1 \upsilon, \hbar) = \rho(\Im_2 \Im_1 \upsilon) \leq \rho(\upsilon) = \rho(\upsilon, \hbar)$ and $\rho(\upsilon, \Im_1 \Im_2 \hbar) = \rho(\upsilon) = \rho(\upsilon, \hbar)$
$\varrho(\Im_2 \Im_1 \upsilon, \hbar) = \varrho(\Im_2 \Im_1 \upsilon) \leq \varrho(\upsilon) = \varrho(\upsilon, \hbar)$ and $\varrho(\upsilon, \Im_1 \Im_2 \hbar) = \varrho(\upsilon) = \varrho(\upsilon, \hbar)$
$\kappa(\Im_2 \Im_1 \upsilon, \hbar) = \kappa(\Im_2 \Im_1 \upsilon) \leq \kappa(\upsilon) = \kappa(\upsilon, \hbar)$ and $\kappa(\upsilon, \Im_1 \Im_2 \hbar) = \kappa(\upsilon) = \kappa(\upsilon, \hbar)$
(b) $\rho(\upsilon, \hbar) + 2s\varrho(\upsilon, \hbar) + s\kappa(\upsilon, \hbar) = \rho(\upsilon) + 2s\varrho(\upsilon) + s\kappa(\upsilon) < 1$,

(c) $\varsigma(\Im_1 \upsilon, \Im_2 \hbar) \leq \rho(\upsilon) \varsigma(\upsilon, \hbar) + \varrho(\upsilon) [\varsigma(\upsilon, \Im_2 \hbar) + \varsigma(\hbar, \Im_1 \upsilon)] + \kappa(\upsilon) \frac{\varsigma(\upsilon, \Im_1 \upsilon) \varsigma(\hbar, \Im_2 \hbar)}{1+\varsigma(\upsilon, \Im_2 \hbar)+\varsigma(\hbar, \Im_1 \upsilon)+\varsigma(\upsilon, \hbar)}$,
$\varsigma(\Im_1 \upsilon, \Im_2 \hbar) \leq \rho(\upsilon, \hbar) \varsigma(\upsilon, \hbar) + \varrho(\upsilon, \hbar) [\varsigma(\upsilon, \Im_2 \hbar) + \varsigma(\hbar, \Im_1 \upsilon)] + \kappa(\upsilon, \hbar) \frac{\varsigma(\upsilon, \Im_1 \upsilon) \varsigma(\hbar, \Im_2 \hbar)}{1+\varsigma(\upsilon, \Im_2 \hbar)+\varsigma(\hbar, \Im_1 \upsilon)+\varsigma(\upsilon, \hbar)}$.
By Theorem 8, $\Im_1$ and $\Im_2$ have a unique common fixed point. □

Corollary 8. *Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$. If there exist nonnegative real numbers ρ, ϱ and κ with $\rho + 2s\varrho + s\kappa < 1$ such that*

$$\begin{aligned} \varsigma(\Im_1 \upsilon, \Im_2 \hbar) \leq\ & \rho \varsigma(\upsilon, \hbar) + \varrho [\varsigma(\upsilon, \Im_2 \hbar) + \varsigma(\hbar, \Im_1 \upsilon)] \\ & + \kappa \frac{\varsigma(\upsilon, \Im_1 \upsilon) \varsigma(\hbar, \Im_2 \hbar)}{1 + \varsigma(\upsilon, \Im_2 \hbar) + \varsigma(\hbar, \Im_1 \upsilon) + \varsigma(\upsilon, \hbar)}, \end{aligned}$$

for all $\upsilon, \hbar \in \mathcal{F}$, then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Take $\rho(\cdot) = \rho$, $\varrho(\cdot) = \varrho$ and $\kappa(\cdot) = \kappa$ in Corollary 7. □

Corollary 9. *Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im : \mathcal{F} \to \mathcal{F}$. If there exist nonnegative real numbers ρ, ϱ and κ with $\rho + 2s\varrho + s\kappa < 1$ such that*

$$\begin{aligned} \varsigma(\Im \upsilon, \Im \hbar) \leq\ & \rho \varsigma(\upsilon, \hbar) + \varrho [\varsigma(\upsilon, \Im \hbar) + \varsigma(\hbar, \Im \upsilon)] \\ & + \kappa \frac{\varsigma(\upsilon, \Im \upsilon) \varsigma(\hbar, \Im \hbar)}{1 + \varsigma(\upsilon, \Im \hbar) + \varsigma(\hbar, \Im \upsilon) + \varsigma(\upsilon, \hbar)}, \end{aligned}$$

for all $\upsilon, \hbar \in \mathcal{F}$, then $\Im$ has a unique fixed point.

Proof. Take $\Im_1 = \Im_2 = \Im$ in above Corollary. □

Corollary 10. *Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$. If there exist nonnegative real numbers ρ and ϱ with $\rho + 2s\varrho < 1$ such that*

$$\varsigma(\Im_1 \upsilon, \Im_2 \hbar) \leq \rho \varsigma(\upsilon, \hbar) + \varrho [\varsigma(\upsilon, \Im_2 \hbar) + \varsigma(\hbar, \Im_1 \upsilon)],$$

for all $\upsilon, \hbar \in \mathcal{F}$, then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Take $\kappa = 0$ in Corollary 8. □

Corollary 11. *Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$. If there exist nonnegative real numbers ρ and κ with $\rho + s\kappa < 1$ such that*

$$\varsigma(\Im_1 \upsilon, \Im_2 \hbar) \leq \rho \varsigma(\upsilon, \hbar) + \kappa \frac{\varsigma(\upsilon, \Im_1 \upsilon) \varsigma(\hbar, \Im_2 \hbar)}{1 + \varsigma(\upsilon, \Im_2 \hbar) + \varsigma(\hbar, \Im_1 \upsilon) + \varsigma(\upsilon, \hbar)},$$

for all $\upsilon, \hbar \in \mathcal{F}$, then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Take $\varrho = 0$ in Corollary 8. □

Corollary 12. *Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$. If there exists nonnegative real number $\rho \in [0, 1)$ such that*

$$\varsigma(\Im_1 \upsilon, \Im_2 \hbar) \leq \rho \varsigma(\upsilon, \hbar)$$

for all $\upsilon, \hbar \in \mathcal{F}$, then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Take $\varrho = \kappa = 0$ in Corollary 8. $\square$

Corollary 13. *[7] Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im : \mathcal{F} \to \mathcal{F}$. If there exists nonnegative real number $\rho \in [0, 1)$ such that*

$$\varsigma(\Im\upsilon, \Im\hbar) \leq \rho\varsigma(\upsilon, \hbar)$$

for all $\upsilon, \hbar \in \mathcal{F}$, then $\Im$ has a unique fixed point.

Proof. Take $\Im_1 = \Im_2 = \Im$ in above Corollary. $\square$

Corollary 14. *Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im_1, \Im_2 : \mathcal{F} \to \mathcal{F}$. If there exists nonnegative real number ϱ with $2s\varrho < 1$ such that*

$$\varsigma(\Im_1\upsilon, \Im_2\hbar) \leq \varrho\left[\varsigma(\upsilon, \Im_2\hbar) + \varsigma(\hbar, \Im_1\upsilon)\right],$$

for all $\upsilon, \hbar \in \mathcal{F}$, then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Take $\rho = \kappa = 0$ in Corollary 8. $\square$

Corollary 15. *[8] Let $(\mathcal{F}, \varsigma)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Im : \mathcal{F} \to \mathcal{F}$. If there exists nonnegative real number ϱ with $2s\varrho < 1$ such that*

$$\varsigma(\Im\upsilon, \Im\hbar) \leq \varrho\left[\varsigma(\upsilon, \Im\hbar) + \varsigma(\hbar, \Im\upsilon)\right],$$

for all $\upsilon, \hbar \in \mathcal{F}$, then $\Im$ has a unique fixed point.

Proof. Take $\Im_1 = \Im_2 = \Im$ in above Corollary. $\square$

Remark 2. *If we take $s = 1$, then b-metric space reduced to metric space and Banach contraction principle [2] and Chaterjea fixed point theorem [4] are direct consequences of Corollaries 13 and 15 respectively.*

5. Application

In the present section, we discuss the existence of solution for the Fredholm integral equation

$$\upsilon(t) = \int_0^1 K(t, s, \upsilon(s))ds \tag{5.1}$$

where $K : [0, 1]\times[0, 1]\times\mathbb{R} \to \mathbb{R}^+$ is continuous function. Let $\mathcal{F} = C[0, 1]$ be the set of real continuous functions defined on $[0, 1]$ and

$$d(\upsilon(t), \hbar(t)) = \max_{t\in[0,1]} (\|\upsilon(t) - \hbar(t)\|)^m$$

for all $\upsilon, \hbar \in \mathcal{F}$, where $m \geq 1$. It is evident that $(\mathcal{F}, d)$ is a complete b-metric space with a parameter $s = 2^{m-1}$.

Theorem 10. *Consider Eq (5.1) and suppose that*

(i) $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}^{+}$ *is continuous function,*

(ii) there exists a continuous function $\lambda : [0,1] \times [0,1] \to \mathbb{R}^{+}$ *such that* $\int_0^1 \lambda(t,s)\, ds \leq 1$,

(iii) there exists a control function $\rho : \mathcal{F} \times \mathcal{F} \to [0,1)$ *such that* $\rho(\Im\upsilon, \hbar) \leq \rho(\upsilon, \hbar)$ *and* $\rho(\upsilon, \Im\hbar) \leq \rho(\upsilon, \hbar)$,

(iv) for all $(t,s) \in [0,1]^2$ *and* $\upsilon, \hbar \in \mathcal{F}$,

$$|K(t,s,\upsilon(s)) - K(t,s,\hbar(s))| \leq \rho(\upsilon,\hbar)^{\frac{1}{m}} \lambda(t,s) |\upsilon(s) - \hbar(s)|.$$

Then the integral Eq (5.1) has a unique solution $\upsilon \in \mathcal{F}$.

Proof. Define the mapping $\Im : \mathcal{F} \to \mathcal{F}$ by

$$\Im\upsilon(t) = \int_0^1 K(t,s,\upsilon(s))ds.$$

Now for $\upsilon, \hbar \in \mathcal{F}$ and $t \in [0,1]$, we have

$$\begin{aligned}
d(\Im\upsilon(t), \Im\hbar(t)) &= \left(\left|\Im\upsilon(t) - \Im\hbar(t)\right|\right)^m \\
&= \left(\left|\int_0^1 K(t,s,\upsilon(s))ds - \int_0^1 K(t,s,\hbar(s))ds\right|\right)^m \\
&= \left(\left|\int_0^1 \left(K(t,s,\upsilon(s)) - K(t,s,\hbar(s))\right) ds\right|\right)^m \\
&\leq \left(\int_0^1 |K(t,s,\upsilon(s)) - K(t,s,\hbar(s))|\, ds\right)^m \\
&\leq \left(\int_0^1 \rho(\upsilon,\hbar)^{\frac{1}{m}} \lambda(t,s) \left(|\upsilon(s) - \hbar(s)|^m\right)^{\frac{1}{m}} ds\right)^m \\
&= \left(\int_0^1 \rho(\upsilon,\hbar)^{\frac{1}{m}} \lambda(t,s)\, d(\upsilon(t),\hbar(t))^{\frac{1}{m}} ds\right)^m \\
&= \rho(\upsilon,\hbar)\, d(\upsilon(t),\hbar(t)) \left(\int_0^1 \lambda(t,s)\, ds\right)^m \\
&\leq \rho(\upsilon,\hbar)\, d(\upsilon(t),\hbar(t)).
\end{aligned}$$

Thus,

$$\varsigma(\Im\upsilon, \Im\hbar) \leq \rho(\upsilon,\hbar)\, d(\upsilon,\hbar).$$

Hence, all the assumptions of Corollary 3 are satisfied and $\Im$ has a unique fixed point in $\mathcal{F}$. Which is a solution of the integral equation in (5.1). □

6. Conclusions

In this article, we have obtained common fixed point results for rational contractions involving control functions of two variables in the the background of b-metric space. As outcomes of our main results, we have derived certain fixed points and common fixed points of self mappings for rational contractions presuming control functions of one variable and constants. As an application, we have investigated the solution of integral equation.

For future study, the obtained results in this article can be extended to multivalued mappings and fuzzy set valued mappings. As applications of these results for multivalued mappings in the setting of b-metric space, some differential and integral inclusions can be investigated. Moreover, these results can be proved in the background of graphical extended b-metric space.

Acknowledgments

This work funded by the University of Jeddah, Jeddah, Saudi Arabia (No. UJ-22-DR-19). The authors, therefore, acknowledge with thanks the University of Jeddah for its technical and financial support.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. M. Frechet, Sur quelques points du calcul fonctionnel, Rendiconti del circolo matematico di Palermo, *Rend. Circ. Matem. Palermo*, **22** (1906), 1–72. http://dx.doi.org/10.1007/BF03018603
2. S. Banach, Sur les operations dans les ensembles abstracts ET leur applications aux equations integrals, *Fund. Math.*, **3** (1922), 133–181.
3. R. Kannan, Some results on fixed points-II, *The American Mathematical Monthly*, **76** (1969), 405–408.
4. S. Chatterjea, Fixed point theorems, *C. R. Acad. Bulgara Sci.*, **25** (1972), 727–730.
5. B. Fisher, Mappings satisfying a rational inequality, *Bull. Math. de la Soc. Sci. Math. de la R. S. de Roumanie*, **24** (1980), 247–251.
6. I. Bakhtin, The contraction mapping principle in almost metric spaces, *Functional Analysis*, **30** (1989), 26–37.
7. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis*, **1** (1993), 5–11.
8. R. Koleva, B. Zlatanov, On fixed points for Chatterjea's maps in b-metric spaces, *Turkish Journal of Analysis and Number Theory*, **4** (2016), 31–34. http://dx.doi.org/10.12691/tjant-4-2-1
9. M. Abbas, I. Chema, A. Razani, Existence of common fixed point for b-metric rational type contraction, *Filomat*, **30** (2016), 1413–1429. http://dx.doi.org/10.2298/FIL1606413A
10. H. Hammad, M. la Sen, A solution of Fredholm integral equation by using the cyclic η_s^q-rational contractive mappings technique in b-metric-like spaces, *Symmetry*, **11** (2019), 1184. http://dx.doi.org/10.3390/sym11091184
11. H. Hammad, M. la Sen, Generalized contractive mappings and related results in b-metric like spaces with an application, *Symmetry*, **11** (2019), 667. http://dx.doi.org/10.3390/sym11050667

12. E. Ameer, H. Aydi, M. Arshad, M. la Sen, Hybrid Ćirić type graphic (Y-Λ)-contraction mappings with applications to electric circuit and fractional differential equations, *Symmetry*, **12** (2020), 467. http://dx.doi.org/10.3390/sym12030467
13. M. Seddik, H. Taieb, Some fixed point theorems of rational type contraction in b-metric spaces, *Moroccan Journal of Pure and Applied Analysis*, **7** (2021), 350–363. http://dx.doi.org/10.2478/mjpaa-2021-0023
14. H. Huang, G. Deng, S. Radenović, Fixed point theorems for C-class functions in b-metric spaces and applications, *J. Nonlinear Sci. Appl.*, **10** (2017), 5853–5868. http://dx.doi.org/10.22436/jnsa.010.11.23
15. H. Huang, Y. Singh, M. Khan, S. Radenović, Rational type contractions in extended b-metric spaces, *Symmetry*, **13** (2021), 614. http://dx.doi.org/10.3390/sym13040614
16. A. Shoaib, A. Asif, M. Arshad, E. Ameer, Generalized dynamic process for generalized multivalued F-contraction of Hardy Rogers type in b-metric spaces, *Turkish Journal of Analysis and Number Theory*, **6** (2018), 43–48. http://dx.doi.org/10.12691/tjant-6-2-2
17. H. Aydi, M. Bota, E. Karapınar, S. Mitrović, A fixed point theorem for set-valued quasi-contractions in b-metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 88. http://dx.doi.org/10.1186/1687-1812-2012-88
18. H. Aydi, M. Bota, E. Karapinar, S. Moradi, A common fixed point for weak φ-contractions on b-metric spaces, *Fixed Point Theory*, **13** (2012), 337–346.
19. H. Aydi, A. Felhi, S. Sahmim, Common fixed points via implicit contractions on b-metric-like spaces, *J. Nonlinear Sci. Appl.*, **10** (2017), 1524–1537.
20. H. Afshari, H. Aydi, E. Karapınar, On generalized α-ψ-Geraghty contractions on b-metric spaces, *Georgian Math. J.*, **27** (2020), 9–21. http://dx.doi.org/10.1515/gmj-2017-0063
21. Z. Ma, J. Ahmad, A. Al-Mazrooei, D. Lateef, Fixed point results for rational orbitally (Θ, δ_b)-contractions with an application, *J. Funct. Space.*, **2021** (2021), 9946125. http://dx.doi.org/10.1155/2021/9946125
22. K. Haji, K. Tola, M. Mamud, Fixed point results for generalized rational type α-admissible contractive mappings in the setting of partially ordered b-metric spaces, *BMC Res. Notes*, **15** (2022), 242. http://dx.doi.org/10.1186/s13104-022-06122-z
23. T. Stephen, Y. Rohen, M. Singh, K. Devi, Some rational F-contractions in b-metric spaces, *Nonlinear Functional Analysis and Applications*, **27** (2022), 309–322. http://dx.doi.org/10.22771/nfaa.2022.27.02.07
24. H. Huang, G. Deng, S. Radenović, Fixed point theorems in b-metric spaces with applications to differential equations, *J. Fixed Point Theory Appl.*, **20** (2018), 52. http://dx.doi.org/10.1007/s11784-018-0491-z
25. S. Aleksic, Z. Mitrovic, S. Radenovic, On some recent fixed point results for single and multivalued mappings in b-metric spaces, *Fasc. Math.*, **61** (2018), 5–16.
26. H. Hammad, M. la Sen, A technique of tripled coincidence points for solving a system of nonlinear integral equations in POCML spaces, *J. Inequal. Appl.*, **2020** (2020), 211. http://dx.doi.org/10.1186/s13660-020-02477-8

27. H. Hammad, M. la Sen, H. Aydi, Generalized dynamic process for an extended multi-valued F-contraction in metric-like spaces with applications, *Alex. Eng. J.*, **59** (2020), 3817–3825. http://dx.doi.org/10.1016/j.aej.2020.06.037

28. H. Hammad, M. la Sen, H. Aydi, Analytical solution for differential and nonlinear integral equations via F_{ω_ϱ}-suzuki contractions in modified ω_ϱ-metric-like spaces, *J. Funct. Space.*, **2021** (2021), 6128586. http://dx.doi.org/10.1155/2021/6128586

29. P. Debnath, M. la Sen, Set-valued interpolative Hardy-Rogers and set-valued Reich-Rus-Ćirić-type contractions in b-metric spaces, *Mathematics*, **7** (2019), 849. http://dx.doi.org/10.3390/math7090849

30. N. Alamgir, Q. Kiran, H. Aydi, Y. Gaba, Fuzzy fixed point results of generalized almost F-contractions in controlled metric spaces, *Adv. Differ. Equ.*, **2021** (2021), 476. http://dx.doi.org/10.1186/s13662-021-03598-0