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Research article

A counterexample to the new iterative scheme of Rezapour et al.: Some discussions and corrections

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Abstract: In this paper, we show a counterexample to the new iterative scheme introduced by Rezapour et al. in "A new modified iterative scheme for finding common fixed points in Banach spaces: application in variational inequality problems" [2]. We propose a modified iteration to conclude the convergence result. Moreover, some of our results are established under a weaker assumption.

Keywords: common fixed point; generalized α-nonexpansive mapping; weak convergence; strong convergence; uniformly convex Banach space; strictly convex Banach space
Mathematics Subject Classification: 47H09, 47H10

1. Introduction

Let C be a closed and convex subset of a Banach space $X := (X, \|\cdot\|)$. Suppose that $\alpha \in [0, 1)$. We say that a mapping $T : C \to C$ is generalized α-nonexpansive [1], if the following inequality holds

$$\|Tx - Ty\| \le \alpha\|Tx - y\| + \alpha\|x - Ty\| + (1 - 2\alpha)\|x - y\|,$$

for all $x, y \in C$ with $\frac{1}{2}\|x - Tx\| \le \|x - y\|$.

Recently, Rezapour et al. [2] proposed the following result for finding a common fixed point of three generalized α-nonexpansive mappings. Recall that a point $p \in C$ is a fixed point of a mapping $T : C \to C$ if $p = Tp$ and the set of all fixed points of T is denoted by $\mathrm{Fix}(T)$. The interested reader is referred to [3, 4] for some further discussion on the fixed point theory.

Theorem 1.1. *Suppose that C is a closed and convex subset of a uniformly convex Banach space X and $T_1, T_2, T_3 : C \to C$ are three generalized α-nonexpansive mappings such that*

$$\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \cap \mathrm{Fix}(T_3) \ne \emptyset.$$

Suppose that $\{x_n\}_{n=1}^{\infty}$ *is generated by the following iterative scheme:*

$$\begin{aligned} x_1 &\in C; \\ z_n &:= T_1((1-\alpha_n)x_n + \alpha_n T_1 x_n); \\ y_n &:= T_2 z_n; \\ x_{n+1} &:= T_3 y_n \end{aligned}$$

where $\alpha_n \in (0,1)$ *for all* $n \geq 1$. *Then the following statements are true:*

(a) *The sequence* $\{x_n\}_{n=1}^{\infty}$ *is bounded and* $\lim_{n\to\infty} \|x_n - T_1 x_n\| = \lim_{n\to\infty} \|x_n - T_2 x_n\| = \lim_{n\to\infty} \|x_n - T_3 x_n\| = 0$. *(See Theorem 3.2 of [2].)*

(b) *If X fulfills the Opial's condition or C is compact, then* $\{x_n\}_{n=1}^{\infty}$ *converges weakly to a common fixed point of* T_1, T_2, *and* T_3. *(See Theorems 3.3 and 3.4 of [2].)*

Recall that

- X is *uniformly convex* [5] if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ whenever $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two sequences in X such that
$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = \lim_{n\to\infty} \frac{1}{2}\|x_n + y_n\| = 1.$$
- X fulfills the *Opial's condition* [6] if $\liminf_{n\to\infty} \|x_n\| < \liminf_{n\to\infty} \|x_n - x\|$ whenever $\{x_n\}_{n=1}^{\infty}$ is a weakly null sequence in X and $x \neq 0$.

2. A counterexample, some discussions and corrections

2.1. A counterexample

Let $X := \mathbb{R}$ be equipped with the usual norm $|\cdot|$ and $C := [-1,1]$. It follows that $X := (\mathbb{R}, |\cdot|)$ is uniformly convex with the Opial's criterion and C is compact. Define $T_1, T_2, T_3 : C \to C$ by

$$T_1 x = T_2 x := x \quad \text{and} \quad T_3 x := -x,$$

for all $x \in C$. It follows that all T_i's are nonexpansive, that is, they are generalized 0-nonexpansive. Moreover, $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \cap \mathrm{Fix}(T_3) = \{0\}$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is generated by the iterative scheme given in Theorem 1.1 where $x_1 := 1$. It follows that $y_1 = z_1 = 1$ and $x_2 = -1$. Processing the scheme again gives $y_2 = z_2 = -1$ and $x_3 = 1$. It follows that $\{x_n\}_{n=1}^{\infty} = \{1, -1, 1, -1, \ldots\}$ and it is not convergent.

2.2. Some discussions

Let us discuss why the conclusion of Theorem 1.1 is not correct. In fact, it is the invalidation of Theorem 3.2 which is stated as Theorem 1.1(a) in this paper. The proof of Theorem 3.2 given there is not correct. To apply Lemma 2.5 which is a consequence of the uniform convexity of the space, we must assume that the sequence of parameters $\{\alpha_n\}_{n=1}^{\infty}$ is bounded away from zero and one. In fact, if we assume that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, then we can follow the original proof to conclude that $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$. Unfortunately, the authors of [2] claimed that the conclusion $\lim_{n\to\infty} \|x_n - T_2 x_n\| = \lim_{n\to\infty} \|x_n - T_3 x_n\| = 0$ follows similarly. As shown by the counterexample above, this is also not true.

2.3. Some corrections

It is clear from the definition that every generalized α-nonexpansive mapping is quasi-nonexpansive. Recall that $T : C \to C$ is quasi-nonexpansive [7], if $\mathrm{Fix}(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $(x, p) \in C \times \mathrm{Fix}(T)$. We propose the following correction of Theorem 1.1. It is inspired by the result of Kim [8]. Since Kim proved the result in a Hilbert space, we give a sketch proof in a more general space.

The following lemma (see [9]) is a correction of Lemma 2.5 of [2]. Note that the condition $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ cannot be discarded from the statement.

Lemma 2.1. *Let $X := (X, \|\cdot\|)$ be a uniformly convex Banach space. Suppose that $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1]$ such that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ and suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two sequences in X. If $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\limsup_{n\to\infty} \|(1 - \alpha_n)x_n + \alpha_n y_n\| = r$ where $r \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Theorem 2.1. *Suppose that C is a closed and convex subset of a uniformly convex Banach space X and $T_1, T_2, T_3 : C \to C$ are three quasi-nonexpansive mappings such that*

$$\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \cap \mathrm{Fix}(T_3) \neq \emptyset.$$

Suppose that $\{x_n\}_{n=1}^{\infty}$ is generated by the following iterative scheme:

$$\begin{aligned} &x_1 \in C; \\ &x_{n+1} := (1 - \alpha_n)x_n + \alpha_n \left(\frac{1}{3} T_1 x_n + \frac{1}{3} T_2 x_n + \frac{1}{3} T_3 x_n \right), \end{aligned}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1]$ such that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Then the following statements are true:

(a) *The sequence $\{x_n\}_{n=1}^{\infty}$ is bounded and $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, 3$.*

(b) *Suppose, in addition, that $I - T_i$ is demiclosed at zero for all $i = 1, 2, 3$. If X fulfills the Opial's condition, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to a common fixed point of T_1, T_2, and T_3.*

Recall that $I - T$ is demiclosed at zero if $p \in \mathrm{Fix}(T)$ whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in C which is weakly convergent to $p \in C$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. It is easy to see that if $T : C \to C$ is generalized α-nonexpansive and X fulfills the Opial's condition, then $I - T$ is demiclosed at zero (see [1]).

Proof. (a) Let $F := \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \cap \mathrm{Fix}(T_3) \neq \emptyset$ and $p \in F$. For each $n \geq 1$ and $i = 1, 2, 3$, we have $\|T_i x_n - p\| \leq \|x_n - p\|$ and hence

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n \left\| \frac{1}{3}(T_1 x_n - p) + \frac{1}{3}(T_2 x_n - p) + \frac{1}{3}(T_3 x_n - p) \right\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \frac{\alpha_n}{3}\|T_1 x_n - p\| + \frac{\alpha_n}{3}\|T_2 x_n - p\| + \frac{\alpha_n}{3}\|T_3 x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

In particular, the limit $\lim_{n\to\infty} \|x_n - p\| := r$ exists and the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Moreover, $\limsup_{n\to\infty} \|T_i x_n - p\| \le r$ for all $i = 1, 2, 3$. For convenience, let $y_n := \frac{1}{3}(T_1 x_n + T_2 x_n + T_3 x_n)$. It follows that $\limsup_{n\to\infty} \|y_n - p\| \le r$. Moreover, we have

$$\lim_{n\to\infty} \|(1 - \alpha_n)(x_n - p) + \alpha_n(y_n - p)\| = \lim_{n\to\infty} \|x_{n+1} - p\| = r.$$

It follows from Lemma 2.1 that $\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|(x_n - p) - (y_n - p)\| = 0$. Hence $\lim_{n\to\infty} \|y_n - p\| = r$. This implies that

$$\begin{aligned} r &= \lim_{n\to\infty} \|y_n - p\| \\ &= \lim_{n\to\infty} \left\| \frac{2}{3}\left(\frac{1}{2}(T_1 x_n - p) + \frac{1}{2}(T_2 x_n - p)\right) + \frac{1}{3}(T_3 x_n - p) \right\| \\ &\le \frac{2}{3} \liminf_{n\to\infty} \left\| \frac{1}{2}(T_1 x_n - p) + \frac{1}{2}(T_2 x_n - p) \right\| + \frac{1}{3} \limsup_{n\to\infty} \|T_2 x_n - p\| \\ &\le \frac{2}{3} \liminf_{n\to\infty} \left\| \frac{1}{2}(T_1 x_n - p) + \frac{1}{2}(T_2 x_n - p) \right\| + \frac{r}{3}. \end{aligned}$$

In particular, $\liminf_{n\to\infty} \left\| \frac{1}{2}(T_1 x_n - p) + \frac{1}{2}(T_2 x_n - p) \right\| \ge r$. It is clear that

$$\limsup_{n\to\infty} \left\| \frac{1}{2}(T_1 x_n - p) + \frac{1}{2}(T_2 x_n - p) \right\| \le r.$$

This implies that $\lim_{n\to\infty} \left\| \frac{1}{2}(T_1 x_n - p) + \frac{1}{2}(T_2 x_n - p) \right\| = r$. It follows from Lemma 2.1 again that

$$\lim_{n\to\infty} \|T_1 x_n - T_2 x_n\| = \lim_{n\to\infty} \|(T_1 x_n - p) - (T_2 x_n - p)\| = 0.$$

Similarly, we can show that $\lim_{n\to\infty} \|T_2 x_n - T_3 x_n\| = 0$. Since $\lim_{n\to\infty} \|x_n - y_n\| = 0$, we have $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, 3$. This completes the proof of (a).

(b) Suppose, in addition, that $I - T_i$ is demiclosed at zero for all $i = 1, 2, 3$ and X fulfills the Opial's criterion. Since every uniformly convex Banach space is reflexive and the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded, we assume that there are two weakly convergent subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ and $\{x_{m_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ with the weak limits z_1 and z_2, respectively. Since each $I - T_i$ is demiclosed at zero, we have $z_1, z_2 \in F$. To conclude that the whole sequence is weakly convergent, we suppose that $z_1 \ne z_2$ to reach a contradiction. Since $z_1 \ne z_2$, we have

$$\lim_{n\to\infty} \|x_n - z_1\| = \lim_{k\to\infty} \|x_{n_k} - z_1\| < \lim_{k\to\infty} \|x_{n_k} - z_2\| = \lim_{n\to\infty} \|x_n - z_2\|.$$

Similarly, we also have

$$\lim_{n\to\infty} \|x_n - z_2\| = \lim_{k\to\infty} \|x_{m_k} - z_2\| < \lim_{k\to\infty} \|x_{m_k} - z_1\| = \lim_{n\to\infty} \|x_n - z_1\|.$$

This completes the proof of (b). □

The following result is established in the presence of a weaker assumption than the uniform convexity. Recall that a Banach space $X := (X, \|\cdot\|)$ is strictly convex [5], if $x = y$ whenever $x, y \in x$ are such that $\|x\| = \|y\| = \frac{1}{2}\|x + y\| = 1$. Note that every uniformly convex Banach space is strictly convex but the converse is not true.

Lemma 2.2. *Let $X := (X, \|\cdot\|)$ be a strictly convex Banach space. Suppose that $0 < \alpha < 1$ and suppose that $x, y \in X$. If $\|x\| \le r$, $\|y\| \le r$ and $\|(1-\alpha)x + \alpha y\| = r$ where $r \ge 0$, then $x = y$.*

Theorem 2.2. *Suppose that C is a compact and convex subset of a strictly convex Banach space X and $T_1, T_2, T_3 : C \to C$ are three quasi-nonexpansive mappings such that*

$$\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \cap \mathrm{Fix}(T_3) \neq \varnothing.$$

Suppose that $\{x_n\}_{n=1}^{\infty}$ is generated by the following iterative scheme:

$$\begin{aligned} &x_1 \in C;\\ &x_{n+1} := (1-\alpha_n)x_n + \alpha_n\left(\frac{1}{3}T_1x_n + \frac{1}{3}T_2x_n + \frac{1}{3}T_3x_n\right) \end{aligned}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ such that $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$. Suppose, in addition, that $I - T_i$ is strongly closed at zero for all $i = 1,2,3$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of T_1, T_2, and T_3.

Recall that $I - T$ is *strongly closed at zero* if $p \in \mathrm{Fix}(T)$ whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in C which is strongly convergent to $p \in C$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. Let $p \in F := \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \cap \mathrm{Fix}(T_3) \neq \varnothing$. We follow the first part of the proof of Theorem 2.1(a) and obtain that $\lim_{n\to\infty} \|x_n - p\| := r$ exists and $\limsup_{n\to\infty} \|T_ix_n - p\| \le r$ for all $i = 1,2,3$ where $r \ge 0$. Since all the sequences $\{x_n\}_{n=1}^{\infty}$, $\{T_1x_n\}_{n=1}^{\infty}$, $\{T_2x_n\}_{n=1}^{\infty}$, and $\{T_3x_n\}_{n=1}^{\infty}$ are in the compact set C, we may assume that there exists a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers such that $\lim_{k\to\infty} |\alpha_{n_k} - \alpha| = 0$ and

$$\lim_{k\to\infty} \|x_{n_k} - z_0\| = \lim_{k\to\infty} \|T_1x_{n_k} - z_1\| = \lim_{k\to\infty} \|T_2x_{n_k} - z_2\| = \lim_{k\to\infty} \|T_3x_{n_k} - z_3\| = 0$$

for some $\alpha \in (0,1)$ and $z_0, z_1, z_2, z_3 \in C$. For convenience, let $y_n := \frac{1}{3}(T_1x_n + T_2x_n + T_3x_n)$ and $y := \frac{1}{3}(z_1 + z_2 + z_3)$. It follows that $\limsup_{n\to\infty} \|y_n - p\| \le r$. In particular, we have $\|z_0 - p\| = r = \|(1-\alpha)(z_0 - p) + \alpha(y - p)\|$ and $\|y - p\| \le r$. It follows from Lemma 2.2 that $z_0 - p = y - p$ and hence $z_0 = y$. This implies that $\|y - p\| = r$ and hence

$$\begin{aligned} r = \|y - p\| &= \left\|\frac{2}{3}\left(\frac{1}{2}(z_1 - p) + \frac{1}{2}(z_2 - p)\right) + \frac{1}{3}(z_3 - p)\right\|\\ &\le \frac{2}{3}\left\|\frac{1}{2}(z_1 - p) + \frac{1}{2}(z_2 - p)\right\| + \frac{1}{3}\|z_3 - p\| \le \frac{2}{3}\left\|\frac{1}{2}(z_1 - p) + \frac{1}{2}(z_2 - p)\right\| + \frac{r}{3}. \end{aligned}$$

Now, we have $\left\|\frac{1}{2}(z_1 - p) + \frac{1}{2}(z_2 - p)\right\| \ge r$. It is easy to see that $\left\|\frac{1}{2}(z_1 - p) + \frac{1}{2}(z_2 - p)\right\| \le r$. This implies that $\left\|\frac{1}{2}(z_1 - p) + \frac{1}{2}(z_2 - p)\right\| = r$. We apply Lemma 2.2 again and obtain that $z_1 - p = z_2 - p$

and hence $z_1 = z_2$. Similarly, we can prove that $z_2 = z_3$. This implies that $z_0 = y = \frac{1}{3}(z_1 + z_2 + z_3) = z_1$ and hence $z_0 = z_1 = z_2 = z_3$. In particular,

$$\lim_{k\to\infty} \|x_{n_k} - z_0\| = \lim_{k\to\infty} \|T_1 x_{n_k} - z_0\| = \lim_{k\to\infty} \|T_2 x_{n_k} - z_0\| = \lim_{k\to\infty} \|T_3 x_{n_k} - z_0\| = 0.$$

This implies that $\lim_{k\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0$ for all $i = 1, 2, 3$. Since each $I - T_i$ is strongly closed at zero, we have $z_0 \in F$. Note that $\lim\limits_{k\to\infty} \|x_{n_k} - z_0\| = 0$ and $\lim\limits_{n\to\infty} \|x_n - z_0\|$ exists. We can conclude that $\lim\limits_{n\to\infty} \|x_n - z_0\| = 0$. Hence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of T_1, T_2, and T_3. □

Proposition 2.1. *Suppose that C is a subset of a Banach space $X := (X, \|\cdot\|)$. If $T : C \to C$ is a generalized α-nonexpansive mapping where $0 \le \alpha < 1$, then $I - T$ is strongly closed at zero.*

Proof. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in C which is strongly convergent to $p \in C$ and $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$. It follows from [1] that

$$\|x_n - Tp\| \le \frac{3+\alpha}{1-\alpha}\|x_n - Tx_n\| + \|x_n - p\|.$$

Taking $n \to \infty$ gives $\lim\limits_{n\to\infty} \|x_n - Tp\| = 0$ and hence $p = Tp$. □

We now summarize our result for generalized α-nonexpansive mappings.

Theorem 2.3. *Suppose that C is a closed and convex subset of a Banach space X and $T_1, T_2, T_3 : C \to C$ are three generalized α-nonexpansive mappings such that $0 \le \alpha < 1$ and*

$$\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \cap \mathrm{Fix}(T_3) \ne \emptyset.$$

Suppose that $\{x_n\}_{n=1}^{\infty}$ is generated by the following iterative scheme:

$$\begin{aligned} &x_1 \in C; \\ &x_{n+1} := (1-\alpha_n)x_n + \alpha_n\left(\frac{1}{3}T_1x_n + \frac{1}{3}T_2x_n + \frac{1}{3}T_3x_n\right) \end{aligned}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ such that $0 < \liminf\limits_{n\to\infty} \alpha_n \le \limsup\limits_{n\to\infty} \alpha_n < 1$. Then the following statements are true:

(a) *If X is uniformly convex and fulfills the Opial's condition, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to a common fixed point of T_1, T_2, and T_3.*

(b) *If X is strictly convex and C is compact, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of T_1, T_2, and T_3.*

Remark 2.1. *Compare our Theorem 2.3(b) and Theorem 3.4 of [2], we find that our assumption is more general and our conclusion is better than theirs.*

3. Conclusions

After a careful reading the paper of Rezapour et al. [2], we find a gap in their result. We show by a counterexample that their results are not correct. We also discuss a reason why their conclusion does not hold. Moreover, we propose a correction and reestablish the corresponding result with a more general assumption.

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Conflict of interest

The author declares no conflicts of interest.

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