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Research article

On minimal asymptotically nonexpansive mappings

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Abstract: In this paper we present the following two results: 1.- A characterization of the renorming invariant family of asymptotically nonexpansive mappings defined on a convex, closed and bounded set of a Banach space; 2.- A comparison of the renorming invariant family of asymptotically nonexpansive mappings with the renorming invariant family of nonexpansive mappings. Additionally, a series of examples are shown for general and particular cases.

Keywords: renorming; asymptotically nonexpansive mapping; lipschitzian mapping; nonexpansive mapping
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1. Introduction

The metric renorming theory for Banach spaces is dedicated to construct equivalent norms that do or do not satisfy a given metric property. As references to deepen the subject of study, [1–4] can be consulted. The properties that are preserved under isomorphisms necessarily are renorming invariant, whereas properties that depend heavily on the norm are called geometric properties, thus the latter ones are at an intermediate point between being invariant under isomorphisms and isometries. Some examples are the rotundity and smoothness of ball [5, 6], the packing of the ball [7–10], and the fixed point property [11, 12], which we will describe in detail. Given a Banach space $(X, \|\cdot\|)$ and C a convex, closed and bounded subset of X, we say that $T : C \to C$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \tag{1.1}$$

for each $x, y \in C$. We say that C has the fixed point property (FPP) if every nonexpansive operator defined from C to itself, has at least one fixed point, and we say that $(X, \|\cdot\|)$ has the FPP if every convex, closed and bounded subset of X has the FPP. A number of geometric properties have been linked to the FPP [11, 12], and it was for reflexivity that it took about 50 years to shed some light on their relationship with the FPP. In the year 2008 P.K. Lin [13] construct through a renorming, the first example of a nonreflexive Banach space with the FPP, and in the year 2009 Domínguez-Benavides [14] proved that every reflexive Banach space can be renormed in such a way it has the FPP. The FPP under renormings have been studied for other types of operators, see for example [15, 16]. But, why are FPP-type properties lost or gained when renorming? In essence, when we renorm a space the Lipschitz constants of operators change, hence the families of nonexpansive-like operators change. It is under this approach that there are a number of works which study the behavior of families of nonexpansive-type operators when renorming. On the one hand, there are works studying the genericity of FPP [17–20]. On the other hand. there are works studying the topological structure of the space of Lipschitzian mappings [21, 22]. Finally, there are works that compare and classify the invariant families of operators [23, 24]. This article is in the latter direction.

We will conclude this introduction by giving a brief summary of the contents of this article. In Section 2, the notation is introduced and known results are referred to. In Section 3, we characterize the family of asymptotically nonexpansive operators which are asymptotically nonexpansive with respect to a family of norms. In Section 4, we characterize the family of renorming invariant asymptotically nonexpansive mappings when the domain of definition is a one-dimensional convex set. Finally, in Section 5, we construct a series of examples and compare the families and minimal families of asymptotically nonexpansive operators with the respective families of nonexpansive operators.

2. Preliminaries

We will start by giving some definitions and notation that will be used throughout this article. Let $(X, \|\cdot\|_0)$ be a Banach space over the scalar field $\mathbb{F} = \mathbb{R} \vee \mathbb{C}$ and C a nonempty subset of X with at least two elements. We denote by $\mathcal{N}(X)$ the family of equivalent norms of X. We say that an operator $T : C \to X$ is $\|\cdot\|_0$-Lipschitz if it has finite Lipschitz constant:

$$K(T, \|\cdot\|_0) = \sup\left\{ \frac{\|Tx - Ty\|_0}{\|x - y\|_0} \;\middle|\; x, y \in C, x \neq y \right\}. \tag{2.1}$$

Note that if T is $\|\cdot\|$-Lipschitz for some $\|\cdot\| \in \mathcal{N}(X)$ then it is $\|\cdot\|'$-Lipschitz for all $\|\cdot\|' \in \mathcal{N}(X)$. Therefore we will simply say that an operator is Lipschitz without referring to the norm w.r.t which it is Lipschitz. If D is a nonempty subset of X we denote by $Lip(C, D)$ the family of Lipschitzian operators $T : C \to D$. In particular when $D = C$ we will write $Lip(C)$ instead of $Lip(C, C)$. It is well known that for every $\|\cdot\| \in \mathcal{N}(X)$ the functional $K(\,\cdot\,, \|\cdot\|)$ is a seminorm in $Lip(C, X)$, which also induces a pseudometric in $Lip(C, X)$ defined by

$$d_K(T, S, \|\cdot\|) = K(T - S, \|\cdot\|), \tag{2.2}$$

for each $S, T \in Lip(C, X)$. Since we are working with equivalent norms, then all seminorms $K(\,\cdot\,, \|\cdot\|)$ with $\|\cdot\| \in \mathcal{N}(X)$ are equivalent in the sense that they imply the same convergence and induces the same topology τ_K. Thus for τ_K-convergence purposes we use the notations

$$d_K(S_n, S) \to 0 \text{ as } n \to \infty, \tag{2.3}$$

or

$$S_n \to_K S, \tag{2.4}$$

when a sequence (S_n) in $Lip(C, X)$ converges to $S \in Lip(C, X)$ with respect to the topology τ_K. In other words, for each $\|\cdot\| \in \mathcal{N}(X)$ and $\varepsilon > 0$ exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have that

$$\begin{aligned} \|S_n x - S_n y - (Sx - Sy)\| &= \|(S_n - S)x - (S_n - S)y\| \\ &\leq \varepsilon\|x - y\|, \end{aligned} \tag{2.5}$$

for each $x, y \in C$. Note that (2.5) is the same as $d_K(S_n, S, \|\cdot\|) \leq \varepsilon$ for each $n \geq N$. Similarly if $\mathfrak{F}$, (S_n) and S are respectively a nonempty subset, a sequence and an element in $Lip(C, X)$ and $\|\cdot\| \in \mathcal{N}(X)$, then we define

$$\begin{aligned} d_K(S, \mathfrak{F}, \|\cdot\|) &= \inf\{d_K(S, F, \|\cdot\|) \,|\, F \in \mathfrak{F}\} \\ &= \inf\{K(S - F, \|\cdot\|) \,|\, F \in \mathfrak{F}\}, \end{aligned} \tag{2.6}$$

and the two introduced notations

$$d_K(S_n, \mathfrak{F}) \to 0 \text{ as } n \to \infty, \tag{2.7}$$

or

$$S_n \to \mathfrak{F}, \tag{2.8}$$

means that for each $\|\cdot\| \in \mathcal{N}(X)$ we have that $d_K(S_n, \mathfrak{F}, \|\cdot\|) \to 0$. Which in turn is equivalent to that for each $\|\cdot\| \in \mathcal{N}(X)$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $n \geq N$ exists $F_n \in \mathfrak{F}$ such that $d_K(S_n, F_n, \|\cdot\|) < \varepsilon$. Additionally, it is well known that due to the convexity of the seminorms we have that the convergence in seminorm implies punctual convergence. Thus for all $\|\cdot\| \in \mathcal{N}(X)$ we have that $S_n \to_K S$ implies

$$K(S_n, \|\cdot\|) \to K(S, \|\cdot\|). \tag{2.9}$$

Then if $\sup\{K(T, \|\cdot\|) \,|\, T \in \mathfrak{F}\} = M < \infty$ and $S_n \to_K \mathfrak{F}$, we have that

$$\limsup_n K(S_n, \|\cdot\|) \leq M. \tag{2.10}$$

It is important to note that the seminorms $K(\,\cdot\,, \|\cdot\|)$ do not distinguish between operators that differ by a constant, in other words, if $S, T \in Lip(C, X)$ satisfy $S - T = f_x$ where $x \in X$ and $f_x(c) = x$ for each $c \in C$, then $K(S - T, \|\cdot\|) = 0$, and it is not hard to check that the only operators that have Lipschitz constant equals to 0 are the constant functions.

An operator $T \in Lip(C, X)$ is said to be $\|\cdot\|$-nonexpansive if $K(T, \|\cdot\|) \leq 1$. For each $\|\cdot\| \in \mathcal{N}(X)$ we denote by $NE(C, \|\cdot\|)$ the family of all $\|\cdot\|$-nonexpansive mappings from C to itself. The next definition is due to Goebel and Kirk [25] in 1972 and is a natural extension of the concept of nonexpansive mapping.

Definition 2.1. Let C be a nonempty subset of a Banach space $(X, \|\cdot\|)$. A mapping $T : C \to C$ is said to be $\|\cdot\|$-asymptotically nonexpansive if

$$\limsup_n K(T^n, \|\cdot\|) \leq 1. \tag{2.11}$$

For each $\|\cdot\| \in \mathcal{N}(X)$ we denote by $ANE(C, \|\cdot\|)$ the family of all $\|\cdot\|$-asymptotically nonexpansive mappings from C to C.

With the notation introduced we notice that for each $\|\cdot\| \in \mathcal{N}(X)$ it is true that the set of $\|\cdot\|$-nonexpansive mappings from C to itself is equal to the $K(\,\cdot\,, \|\cdot\|)$-ball with center 0 and radii 1 intersected by $Lip(C)$, that is

$$NE(C, \|\cdot\|) = B(0, K(\,\cdot\,, \|\cdot\|), 1) \cap Lip(C), \tag{2.12}$$

where $B(0, K(\,\cdot\,, \|\cdot\|), 1) = \{S \in Lip(C, X) \,|\, K(S, \|\cdot\|) \leq 1\}$ is the closed unit ball in $(Lip(C, X), K(\,\cdot\,, \|\cdot\|))$. It is therefore equivalent to deal with families of nonexpansive mappings than with certain balls associated with seminorms. Even more from this approach can be treated families of operators whose definition involves a certain type of seminorm-like function, as in the case of asymptotically nonexpansive operators in which the space of study is

$$ULip(C) = \{T \in Lip(C) \,|\, \sup K(T^n, \|\cdot\|) < \infty\}, \tag{2.13}$$

and the family of seminorm-like functions of interest are

$$UK(T, \|\cdot\|) = \limsup_n K(T^n, \|\cdot\|), \tag{2.14}$$

for every $\|\cdot\| \in \mathcal{N}(X)$. Hence, the asymptotically nonexpansive sets $ANE(C, \|\cdot\|)$ are equals to the intersection of the unit UK-ball of $(Lip(C, X), UK(\,\cdot\,, \|\cdot\|))$ with $ULip(C)$

$$ANE(C, \|\cdot\|) = B(0, UK(\,\cdot\,, \|\cdot\|), 1) \cap ULip(C). \tag{2.15}$$

3. Asymptotically nonexpansive non one-dimensional case

In this section we study the family of renorming invariant asymptotically nonexpansive mappings and relate them with some special class of operators that asymptotically tends to behave as non rotating-like functions.

Theorem 3.1. *Let X be a Banach space, C a convex, closed and bounded subset of X with at least two elements, $\mathcal{I}$ a nonempty subset of $\mathcal{N}(X)$, for each $\|\cdot\| \in \mathcal{N}(X)$ a nonnegative $r_{\|\cdot\|} \geq 0$, and (T_n) a sequence in $Lip(C)$. Then the following statements are equivalent:*

(1) $T_n \to_K \{S \in Lip(C) \,|\, K(S, \|\cdot\|) \leq r_{\|\cdot\|}\}$ for each $\|\cdot\| \in \mathcal{I}$.
(2) $\limsup_n K(T_n, \|\cdot\|) \leq r_{\|\cdot\|}$ for each $\|\cdot\| \in \mathcal{I}$.

Moreover, they follow from

(3) $T_n \to_K \bigcap_{\|\cdot\|\in\mathcal{I}} \{S \in Lip(C) \,|\, K(S, \|\cdot\| \leq r_{\|\cdot\|}\}$.

Proof. First, the sets

$$\{S \in Lip(C) \,|\, K(S, \|\cdot\|) \leq r_{\|\cdot\|}\} \tag{3.1}$$

are nonempty and have nonempty intersection with respect to $\mathcal{I}$. Since the constant functions f_x with $x \in C$ are always elements of them.The one that (3) implies (1) follows directly from the definition of intersection. We will prove that (1) implies (2). By (2.5), (2.7) and (2.9), for each $\varepsilon > 0$ and $\|\cdot\| \in \mathcal{I}$ there exists $N \in \mathbb{N}$ such that for each $n \geq N$ exists $S_n \in \{S \in Lip(C) \,|\, K(S, \|\cdot\|) \leq r_{\|\cdot\|}\}$ with $K(T_n - S_n, \|\cdot\|) < \varepsilon$, Hence

$$|K(T_n, \|\cdot\|) - K(S_n, \|\cdot\|)| \leq K(T_n - S_n, \|\cdot\|) < \varepsilon. \tag{3.2}$$

Thus $K(T_n, \|\cdot\|) < r_{\|\cdot\|} + \varepsilon$ for each $n \geq N$, which implies (2). Now we prove that (2) implies (1). Let $\varepsilon > 0$, $\|\cdot\| \in \mathcal{I}$ and $x_0 \in C$. Then by the definition of upper limit there exists $N \in \mathbb{N}$ such that for each $n \geq N$ it is fulfilled that $K(T_n, \|\cdot\|) < r_{\|\cdot\|} + \varepsilon$. We define

$$\lambda = 1 - r_{\|\cdot\|}\left(r_{\|\cdot\|} + \varepsilon\right)^{-1} < 1, \tag{3.3}$$

and for each $n \geq N$

$$S_n = \lambda f_{x_0} + (1 - \lambda) T_n, \tag{3.4}$$

where f_{x_0} is the constant function x_0 defined on C. By the convexity of C we have that $S_n \in Lip(C)$ for each $n \geq N$. Then

$$\begin{aligned} K(S_n, \|\cdot\|) &= K(\lambda f_{x_0} + (1-\lambda) T_n, \|\cdot\|) \\ &\leq \lambda K(f_{x_0}, \|\cdot\|) + (1-\lambda) K(T_n, \|\cdot\|) \\ &\leq (1-\lambda)\left(r_{\|\cdot\|} + \varepsilon\right) \\ &= r_{\|\cdot\|}. \end{aligned} \tag{3.5}$$

Hence $S_n \in \{S \in Lip(C) \,|\, K(S, \|\cdot\|) \leq r_{\|\cdot\|}\}$. Moreover

$$\begin{aligned} K(S_n - T_n, \|\cdot\|) &= K(\lambda f_{x_0} + (1-\lambda) T_n - T_n, \|\cdot\|) \\ &\leq \lambda K(f_{x_0}, \|\cdot\|) + \lambda K(T_n, \|\cdot\|) \\ &= \lambda K(T_n, \|\cdot\|) \\ &\leq \lambda\left(r_{\|\cdot\|} + \varepsilon\right). \end{aligned} \tag{3.6}$$

By (3.3) it is clear that $\lambda \to 0$ when $\varepsilon \to 0$. Then

$$T_n \to_K \{S \in Lip(C) \,|\, K(S, \|\cdot\|) \leq r_{\|\cdot\|}\}, \tag{3.7}$$

for each $\|\cdot\| \in \mathcal{I}$. □

Corollary 3.2. *Let X be a Banach space, C a convex, closed and bounded subset of X with at least two elements, I a nonempty subset of $\mathcal{N}(X)$ and $T \in Lip(C)$. Then the following statements are equivalent:*

(1) $T \in ANE(C, \|\cdot\|)$ *for each* $\|\cdot\| \in I$.
(2) $T \in \bigcap_{\|\cdot\| \in I} ANE(C, \|\cdot\|)$.
(3) $T^n \to_K NE(C, \|\cdot\|)$ *for each* $\|\cdot\| \in I$.
(4) $\limsup_n K(T^n, \|\cdot\|) \leq 1$ *for each* $\|\cdot\| \in I$.

Moreover, they follow from

(5) $T^n \to_K \bigcap_{\|\cdot\| \in I} NE(C, \|\cdot\|)$.

Proof. By definition (1) and (2) are equivalent. While proposition (3) and (4) are equivalent by Theorem 3.1. Taking

$$NE(C, \|\cdot\|) = \{S \in Lip(C) \,|\, K(S, \|\cdot\|) \leq 1\}, \tag{3.8}$$

for each $\|\cdot\| \in \mathcal{I}$, and sequence (T_n) as the iterated sequence (T^n) of T. (4) is equivalent to (1) by Definition 2.1. Finally, (5) implies (1) by Theorem 3.1. □

Definition 3.3. We say that a convex C is one-dimensional if exist $x, y \in C$ with $x \neq y$ such that for each $z \in C$ there is a scalar $\alpha_z \in \mathbb{F} = \mathbb{R} \vee \mathbb{C}$ such that $z = \alpha_z x + (1 - \alpha_z) y$.

In [23] Acosta-Portilla, Hernández-Linâres and Pérez-García proved that the family of renorming-invariant Lipschitzian mappings

$$S'(C) = \bigcap_{\|\cdot\| \in \mathcal{N}(X)} NE(C, \|\cdot\|) \tag{3.9}$$

is made up of elements of the form $T = f_x + \alpha I$ for some $x \in X$ and $|\alpha| \leq 1$ when C is a non one-dimensional convex. Whereas $S'(C)$ is isometric isomorphic to the family $NE(A, |\cdot|)$ with A some convex in $\mathbb{F}$ when C is one-dimensional.

Definition 3.4. A sequence $(T_n) \in Lip(C)$ is r-asymptotically uniformly collinear if for each $\varepsilon > 0$ exists $N \in \mathbb{N}$ such that for each $n \geq N$ exists $|\alpha_n| \leq r$ with

$$\|T_n x - T_n y - \alpha_n (x - y)\| \leq \varepsilon \|x - y\|, \tag{3.10}$$

for each $x, y \in C$.

Definition 3.5. A sequence $(T_n) \in Lip(C)$ is r-asymptotically collinear if for each $\varepsilon > 0$ exists $N \in \mathbb{N}$ such that for each $n \geq N$ and $x, y \in C$ there exists $\alpha = \alpha(n, x, y)$ with $|\alpha| \leq r$, such that

$$\|T_n x - T_n y - \alpha (x - y)\| \leq \varepsilon \|x - y\|. \tag{3.11}$$

When the sequence coincides with the iterate sequence (T^n) of an operator T, We will simply say that T is r-asymptotically (uniformly) collinear.

Note that if a sequence is asymptotically uniformly collinear, then it is asymptotically collinear and if it is asymptotically (uniformly) collinear with respect to some norm, then it is asymptotically (uniformly) collinear with respect to all equivalent norm. Therefore the asymptomatically (uniformly) collinear is a pure algebraic and topological property and does not depend on the choice of the norm. Which was to be expected since as we show below, it characterizes some operators that are always asymptotically nonexpansive. The following lemma relates the r-asymptotically collinear property to the asymptotic behaviour of the Lipschitz constants of the sequence.

Lemma 3.6. *Let X be a normed space, C a nonempty subset of X with at least two elements, and (T_n) a sequence in $Lip(C)$ that is r-asymptotically collinear with respect to some norm $\|\cdot\|_0 \in \mathcal{N}(X)$. Then for each $\|\cdot\| \in \mathcal{N}(X)$ the sequence is r-asymptotically collinear and*

$$\limsup_n K(T_n, \|\cdot\|) \leq r. \tag{3.12}$$

Proof. Let $\varepsilon > 0$, $\|\cdot\| \in \mathcal{N}(X)$ and $l, u > 0$ be two constants such that for every $x \in X$

$$l\|x\| \leq \|x\|_0 \leq u\|x\|. \tag{3.13}$$

Then exists $N \in \mathbb{N}$ such that if $n \geq N$ and $x, y \in C$, there exists $\alpha = \alpha(n, x, y)$ with $|\alpha| \leq r$, such that

$$\|T_n x - T_n y - \alpha(x - y)\|_0 \leq \varepsilon\|x - y\|_0. \tag{3.14}$$

Thus

$$\begin{aligned} l\|T_n x - T_n y - \alpha(x - y)\| &\leq \|T_n x - T_n y - \alpha(x - y)\|_0 \\ &\leq \varepsilon\|x - y\|_0 \\ &\leq \varepsilon u\|x - y\|. \end{aligned} \tag{3.15}$$

Hence $\|T_n x - T_n y - \alpha(x - y)\| \leq \varepsilon\frac{u}{l}\|x - y\|$ for each $\|\cdot\| \in \mathcal{N}(X)$, that is, (T_n) is r-asymptotically collinear for each $\|\cdot\| \in \mathcal{N}(X)$. Moreover for each $n \geq N$ and $x, y \in C$ we have that

$$\|T_n x - T_n y\| - |\alpha|\|x - y\| \leq \varepsilon\frac{u}{l}\|x - y\|. \tag{3.16}$$

Therefore for $x \neq y$

$$\frac{\|T_n x - T_n y\|}{\|x - y\|} \leq |\alpha| + \varepsilon\frac{u}{l}. \tag{3.17}$$

Then

$$\begin{aligned} K(T_n, \|\cdot\|) &\leq \sup\left\{|\alpha| + \varepsilon\frac{u}{l} \,\middle|\, x, y \in C,\ x \neq y\right\} \\ &= \sup\{|\alpha| \mid x, y \in C,\ x \neq y\} + \varepsilon\frac{u}{l} \\ &\leq r + \varepsilon', \end{aligned} \tag{3.18}$$

where $\varepsilon' = \varepsilon\frac{u}{l}$. Thus $\limsup_n K(T_n, \|\cdot\|) \leq r$ for each $\|\cdot\| \in \mathcal{N}(X)$. □

Theorem 3.7. *Let X be a Banach space, C a nonempty non one-dimensional convex, closed and bounded subset of X and $T \in Lip(C)$ and the following statements:*

(1) $T^n \to_K S'(C)$.
(2) T is 1-asymptotically uniformly collinear.
(3) T is 1-asymptotically collinear.
(4) $T \in ANE(C, \|\cdot\|)$ for each $\|\cdot\| \in \mathcal{N}(X)$.

Then (1) ⇒ (2) ⇒ (3) ⇒ (4).

Proof. It is clear that (2) implies (3). First we will prove that (1) implies (2). Remember that

$$S'(C) = \bigcap_{\|\cdot\| \in \mathcal{N}(X)} NE(C, \|\cdot\|). \tag{3.19}$$

Let $\varepsilon > 0$ and $\|\cdot\| \in \mathcal{N}(X)$. Then exists $N \in \mathbb{N}$ such that for each $n \geq N$ exists $S_n \in S'(C)$ with $d_K(T^n, S, \|\cdot\|) < \varepsilon$. That is,

$$\|T^n x - T^n y - (S_n x - S_n y)\| \leq \varepsilon \|x - y\|, \tag{3.20}$$

for each $x, y \in C$. By Theorems 2 and 3 and Corollary 8 in [23] for each $n \geq N$ exist $x_n \in X$ and $|\alpha_n| \leq 1$ such that

$$S_n = f_x + \alpha_n I. \tag{3.21}$$

Then for each $x, y \in C$

$$\begin{aligned} \varepsilon\|x - y\| &\geq \|T^n x - T^n y - (S_n x - S_n y)\| \\ &= \|T^n x - T^n y - (f_{x_n} + \alpha_n I x - f_{x_n} - \alpha_n I y)\| \\ &= \|T^n x - T^n y - \alpha_n (x - y)\|. \end{aligned} \tag{3.22}$$

Hence T is 1-asymptotically uniformly collinear. Finally we will prove that (3) implies (4). By Lemma 3.6 we have that for each $\|\cdot\| \in \mathcal{N}(X)$ it is fulfilled

$$\limsup_n K(T^n, \|\cdot\|) \leq 1. \tag{3.23}$$

Hence $T \in ANE(C, \|\cdot\|)$ for every $\|\cdot\| \in \mathcal{N}(X)$. □

The intuition indicates that the difference between an asymptotically uniformly collinear operator and an asymptotically collinear operator, from the point of view of real functions, is similar to that between a function with a constant derivative and a differentiable function. However, for the non one dimensional case, we conjecture that AUC and AC operators are the same, since Lemma 4 in [23] proves that given three non collinear points x, y and z, and scalars $\alpha_{x,y}, \alpha_{x,z}$ and $\alpha_{y,z}$ with

$$
\begin{aligned}
Tx - Ty &= \alpha_{x,y}(x - y),\\
Ty - Tz &= \alpha_{x,z}(x - z),\\
Ty - Tz &= \alpha_{y,z}(y - z).
\end{aligned}
\tag{3.24}
$$

It necessarily holds that $\alpha_{x,y} = \alpha_{x,z} = \alpha_{y,z}$. Thus in the case of an asymptotically collinear operator T, most likely it must be fulfilled that

$$
\begin{aligned}
Tx - Ty &\approx \alpha_{x,y}(x - y),\\
Ty - Tz &\approx \alpha_{x,z}(x - z),\\
Ty - Tz &\approx \alpha_{y,z}(y - z),
\end{aligned}
\tag{3.25}
$$

implies $\alpha_{x,y} \approx \alpha_{x,z} \approx \alpha_{y,z}$. Therefore the AUC and AC properties would match. On the other hand, we also conjecture that in Theorems 3.1 and 3.7, and Corollary 3.2 the respective statements are equivalent. Result that is equivalent to proving the existence of a renorming that makes a countable family of operators T^n not non expansive. However, so far we do not know of a technique that allows us to construct that renorming.

We might think that the set of norm-invariant asymptotically nonexpansive operators

$$
\begin{aligned}
AS'(C) &= \bigcap_{\|\cdot\| \in \mathcal{N}(X)} ANE(C, \|\cdot\|)\\
&\supset \bigcap_{\|\cdot\| \in \mathcal{N}(X)} NE(C, \|\cdot\|)\\
&= S'(C),
\end{aligned}
\tag{3.26}
$$

coincides with that of norm-invariant nonexpansive operators $S'(C)$. Even so, the containment $S'(C) \subsetneq AS'(C)$ is always strict, as we shown in the next section.

4. Asymptotically nonexpansive one-dimensional case

As can be seen, in the results of the previous section it was considered that the domain of definition of the operators was a non one-dimensional convex C. So the natural question is: what happens in the one-dimensional case?. The proof of the following statements is found in Remark 6 and Theorem 7 of article [23]. Firstly, in the one-dimensional case there is only one renorming for the space, since all the others are a scalar multiple of it, thus the Lipschitz constant only depends of the operator. In addition, in the one-dimensional case in essence we are working with a convex subset of the field of scalars, so the study is equivalent to that of functions defined between convexes in $\mathbb{R}$ or $\mathbb{C}$, which added to the uniqueness of the norm and Lipschitz constant implies that the families studied are an invariant associated with the convex C. In order to formalize the above. Let C be a one-dimensional convex of $(X, \|\cdot\|)$ and $x, y \in C$ such that for each $z \in C$ there is a scalar α_z with $z = \alpha_z x + (1 - \alpha_z)y$. We define $\phi : C \to \mathbb{F}$ by

$$\phi z = \alpha_z \|x - y\|. \tag{4.1}$$

It can be proved that ϕ is an affine isometry between $(C, \|\cdot\|)$ and $(\phi(C), |\cdot|) \subset (\mathbb{F}, |\cdot|)$. Thus we have the following Lipschitz-preserving identification of $(Lip(C), K(\,\cdot\,, \|\cdot\|))$ with $(Lip(\phi(C)), K(\,\cdot\,, |\cdot|))$ defined by $T^{\phi} = \phi T \phi^{-1}$ for each $T \in Lip(C)$.

$$\begin{array}{ccc} C & \xrightarrow{\quad T \quad} & C \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \phi(C) & \xrightarrow{\quad T^{\phi} \quad} & \phi(C) \end{array} \tag{4.2}$$

We will say that a function ϕ between metric spaces (C_1, d_1) and (C_2, d_2) is an M-isometry for some $M > 0$ if for each $x, y \in C_1$ it is fulfilled

$$d_2(\phi x, \phi y) = M d_1(x, y), \tag{4.3}$$

the above Construction (4.2) is also valid for ϕ an M-isometry.

Lemma 4.1. *Let (C_1, d_1) and (C_2, d_2) be two metric spaces, $\phi : C_1 \to C_2$ a bijective M-isometry and a Lipschitzian function $T : C_1 \to C_1$ with Lipschitz constant $K(T, d_1)$. Then the function T^{ϕ} defined by the Diagram 4.2 has Lipschitz constant $K(T^{\phi}, d_2) = K(T, d_1)$. That is, the function ϕ induces a Lipschitz preserving identifications between $(Lip(C_1), K(\,\cdot\,, d_1))$ and $(Lip(C_2), K(\,\cdot\,, d_2))$. Moreover the identification is compatible with compositions. Thus if $S \in Lip(C_1)$, then*

$$(TS)^{\phi} = T^{\phi} S^{\phi} \tag{4.4}$$

and

$$K(T^{\phi} S^{\phi}, d_2) = K((TS)^{\phi}, d_2) = K(TS, d_1). \tag{4.5}$$

In particular for each $n \in \mathbb{N}$ we have that

$$K((T^{\phi})^n, d_2) = K((T^n)^{\phi}, d_2) = K(T^n, d_1). \tag{4.6}$$

If T is the constant function f_a for some $a \in C_1$, then T^{ϕ} is the constant function ϕa in C_2

$$T^{\phi} = (f_a)^{\phi} = f_{\phi a}, \tag{4.7}$$

and the identification induced by ϕ is a bijection between $Lip(C_1)$ and $Lip(C_2)$ with inverse the identification induced by ϕ^{-1}.

Proof. Since $\phi T \phi^{-1} \phi S \phi^{-1} = \phi T S \phi^{-1}$, then

$$(TS)^{\phi} = T^{\phi} S^{\phi}. \tag{4.8}$$

Moreover ϕ is a bijective M-isometry. Thus we have that

$$
\begin{aligned}
K(T, d_1) &= \sup\left\{ \frac{d_1(Tx, Ty)}{d_1(x, y)} \,\middle|\, x, y \in C_1, x \neq y \right\} \\
&= \sup\left\{ \frac{M^{-1} d_2(\phi T x, \phi T y)}{M^{-1} d_2(\phi x, \phi y)} \,\middle|\, x, y \in C_1, x \neq y \right\} \\
&= \sup\left\{ \frac{d_2(\phi T \phi^{-1}(\phi x), \phi T \phi^{-1}(\phi y))}{d_2(\phi x, \phi y)} \,\middle|\, \phi x, \phi y \in C_2, \phi x \neq \phi y \right\} \\
&= \sup\left\{ \frac{d_2(T^{\phi} a, T^{\phi} b)}{d_2(a, b)} \,\middle|\, a, b \in C_2, a \neq b \right\} \\
&= K(T^{\phi}, d_2).
\end{aligned}
\tag{4.9}
$$

Hence by (4.8) and (4.9)

$$K(T^{\phi} S^{\phi}, d_2) = K((TS)^{\phi}, d_2) = K(TS, d_1). \tag{4.10}$$

In particular, for every $n \in \mathbb{N}$ if is fulfilled that

$$(T^n)^{\phi} = (T^{\phi})^n, \tag{4.11}$$

and

$$K((T^{\phi})^n, d_2) = K((T^n)^{\phi}, d_2) = K(T^n, d_1). \tag{4.12}$$

If $T = f_a$, then

$$
\begin{aligned}
T^{\phi} x =& \phi(f_a(\phi^{-1} x)) \\
=& \phi(a).
\end{aligned}
\tag{4.13}
$$

Hence $(f_a)^{\phi} = f_{\phi a}$. Finally, we have the following diagram for each $T \in Lip(C_1)$

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\quad T \quad} & C_1 \\
\big\downarrow \phi & & \big\downarrow \phi \\
C_2 & \xrightarrow{\quad T^{\phi} \quad} & C_2 \\
\big\downarrow \phi^{-1} & & \big\downarrow \phi^{-1} \\
C_1 & \xrightarrow{\quad T \quad} & C_1
\end{array}
\tag{4.14}
$$

Thus $(T^{\phi})^{\phi^{-1}} = T$, that is, the identification induced by ϕ has inverse the identification induced by ϕ^{-1}. In a similar way it can be proved that for each $S \in Lip(C_2)$ it is fulfilled $(S^{\phi^{-1}})^{\phi} = S$. Then ϕ induces a bijection between $Lip(C_1)$ and $Lip(C_2)$. □

The following lemmas are special cases of Lemma 4.1 when C is a one dimensional convex subset of a normed space and will be used to prove Theorems 4.4 and 4.6 in which it is characterized the family of asymptotically nonexpansive operators defined on a one dimensional convex set.

Lemma 4.2. *Let C be a one dimensional convex subset of a normed space $(X, \|\cdot\|)$ with distinguished points $x, y \in C$. Then if $\alpha \in \mathbb{F}$ is such that $z = \alpha x + (1-\alpha)y$ for some $z \in C$, it necessarily holds $\alpha = \alpha_z$.*

Proof. Let $z \in C$ and α with $z = \alpha x + (1-\alpha)y$. Then

$$\alpha x + (1-\alpha)y = \alpha_z x + (1-\alpha_z)y. \tag{4.15}$$

Hence

$$\begin{aligned} 0 &= \|(\alpha_z - \alpha)x - (\alpha_z - \alpha)y\| \\ &= |\alpha_z - \alpha|\|x - y\|. \end{aligned} \tag{4.16}$$

Since $x \neq y$ it follows that $\alpha_z = \alpha$. □

Lemma 4.3. *Let C be a one dimensional convex subset of a normed space $(X, \|\cdot\|_1)$ with distinguished points $x, y \in C$. Then the function $\phi : C \to \phi(C) \subset \mathbb{F}$ defined by*

$$\phi z = \alpha_z \|x - y\| \tag{4.17}$$

is a bijective affine 1-isometry and the induced identification $(\cdot)^\phi$ of $Lip(C_1)$ with $Lip(C_2)$ is affine.

Proof. By Lemma 4.2 the function ϕ is well defined and is a bijection. Let $a, b \in C$. Then we have that

$$\begin{aligned} \|a - b\| &= \|\alpha_a x + (1-\alpha_a)y - [\alpha_b x + (1-\alpha_b)y]\| \\ &= |\alpha_a - \alpha_b|\|x - y\| \\ &= |\alpha_a\|x - y\| - \alpha_b\|x - y\|| \\ &= |\phi a - \phi b|. \end{aligned} \tag{4.18}$$

Hence ϕ is a 1-isometry between $(C, \|\cdot\|)$ and $(\phi(C), |\cdot|)$ with $\phi(C) \subset \mathbb{F}$. Moreover, for any $0 \leq \lambda \leq 1$

$$\begin{aligned} \alpha a + (1-\lambda)b &= \lambda(\alpha_a x + (1-\alpha_a)y) + (1-\lambda)(\alpha_b + (1-\alpha_b)y) \\ &= \lambda\alpha_a x + \lambda y - \lambda\alpha_a y + \alpha_b x - \lambda\alpha_b x + y - \alpha_b y - \lambda y + \lambda\alpha_b y \\ &= \lambda\alpha_a x + \alpha_b x - \lambda\alpha_b x + y - \lambda\alpha_a y - \alpha_b y + \lambda\alpha_b y \\ &= (\lambda\alpha_a + (1-\lambda)\alpha_b)x + [1 - (\lambda\alpha_a + (1-\lambda)\alpha_b)]y. \end{aligned} \tag{4.19}$$

Then by Lemma 4.2 and (4.19) we have that

$$\begin{aligned} \phi(\alpha a + (1-\lambda)b) &= \phi(\lambda(\alpha_a x + (1-\alpha_a)y) + (1-\lambda)(\alpha_b + (1-\alpha_b)y)) \\ &= \phi[(\lambda\alpha_a + (1-\lambda)\alpha_b)x + [1 - (\lambda\alpha_a + (1-\lambda)\alpha_b)]y] \\ &= [\lambda\alpha_a + (1-\lambda)\alpha_b]\|x - y\| \\ &= \lambda\alpha_a\|x - y\| + (1-\lambda)\alpha_b\|x - y\| \\ &= \lambda\phi a + (1-\lambda)\phi b. \end{aligned} \tag{4.20}$$

Thus ϕ is an affine operator. Finally, let $T, S \in Lip(C)$ and $0 \leq \lambda \leq 1$. Therefore we have that

$$\begin{aligned}(\lambda T + (1-\lambda)S)^{\phi} &= \phi(\lambda T + (1-\lambda)S)\phi^{-1} \\ &= \phi(\lambda T\phi^{-1} + (1-\lambda)S\phi^{-1}) \\ &= \lambda\phi T\phi^{-1} + (1-\lambda)\phi S\phi^{-1} \\ &= \lambda T^{\phi} + (1-\lambda)S^{\phi}.\end{aligned} \tag{4.21}$$

That is, the identification $(\cdot)^{\phi}$ is affine. □

Now we have the elements to characterize the families of asymptotically nonexpansive operators defined over a one dimensional convex set.

Theorem 4.4. *Let X be a Banach space and C a nonempty one dimensional convex, closed and bounded subset of $(X, \|\cdot\|)$. Then exist $D \subset \mathbb{F}$ and an affine isometry $\phi : C \to D$ such that the correspondence $T \mapsto T^{\phi}$ from $ANE(C, \|\cdot\|)$ to $ANE(D, |\cdot|)$ is an affine Lipschitz constant and composition preserving bijective mapping.*

Proof. Let $\phi : C \to \phi(C) \subset \mathbb{F}$ defined by $\phi z = \alpha_z \|x - y\|$. By Lemmas 4.1 and 4.3 only left to prove that $(\cdot)^{\phi}$ maps $ANE(C, \|\cdot\|)$ over $ANE(\phi(C), |\cdot|)$, and this is true since the identifications $(\cdot)^{\phi}$ and $(\cdot)^{\phi^{-1}}$ are inverse to each other, preserve Lipschitz constants, and are compatible with compositions. □

Lemma 4.5. *Let C be a one dimensional convex subset of a normed space X with distinguished points $x, y \in C$. Then for each $\|\cdot\|_1, \|\cdot\|_2 \in \mathcal{N}(X)$ there exists $r > 0$ such that $\|a - b\|_2 = r\|a - b\|_1$ for each $a, b \in C$.*

Proof. Let $\|\cdot\|_1, \|\cdot\|_2 \in \mathcal{N}(X)$ and

$$r = \frac{\|x - y\|_2}{\|x - y\|_1}. \tag{4.22}$$

Then for each $a, b \in C$ we have that

$$\begin{aligned}a - b &= \alpha_a x + (1-\alpha_a)y - [\alpha_b x + (1-\alpha_b)y] \\ &= \alpha_a x - \alpha_b x - \alpha_a y + \alpha_b y + y - y \\ &= (\alpha_a - \alpha_b)(x - y).\end{aligned} \tag{4.23}$$

For this reason

$$\begin{aligned}\|a - b\|_2 &= |\alpha_a - \alpha_b|\|x - y\|_2 \\ &= r|\alpha_a - \alpha_b|\|x - y\|_1 \\ &= r\|a - b\|_1.\end{aligned} \tag{4.24}$$

□

Theorem 4.6. *Let X be a Banach space and C a nonempty one dimensional convex, closed and bounded subset of X. Then for each* $\|\cdot\|_1, \|\cdot\|_2 \in \mathcal{N}(X)$ *it is fulfilled that*

$$ANE(C, \|\cdot\|_1) = ANE(C, \|\cdot\|_2). \tag{4.25}$$

Proof. Let $\|\cdot\|_1, \|\cdot\|_2 \in \mathcal{N}(X)$ and $T \in Lip(C)$. Then by Lemma 4.5 there exists $r > 0$ such that $\|a-b\|_2 = r\|a-b\|_1$. Thus for each $x, y \in C$ with $x \neq y$ we have that

$$\begin{aligned}\frac{\|Tx - Ty\|_2}{\|x-y\|_2} &= \frac{r\|Tx-Ty\|_1}{r\|x-y\|_1} \\ &= \frac{\|Tx-Ty\|_1}{\|x-\|_1}.\end{aligned} \tag{4.26}$$

In consequence for each $T \in Lip(C)$ it is fulfilled that $K(T, \|\cdot\|_2) = K(T, \|\cdot\|_1)$. In particular, for each $T \in ANE(C, \|\cdot\|_2)$ and $n \in \mathbb{N}$ we have that $K(T^n, \|\cdot\|_2) = K(T^n, \|\cdot\|_1)$. Hence $ANE(C, \|\cdot\|_2) \subset ANE(C, \|\cdot\|_1)$. Similarly the other containment can be proved. Then

$$ANE(C, \|\cdot\|_1) = ANE(C, \|\cdot\|_2), \tag{4.27}$$

for each $\|\cdot\|_1, \|\cdot\| \in \mathcal{N}(X)$. □

5. Comparing ANE sets

In this section, we will study the minimal family of asymptotically nonexpansive mappings compared to the minimal family of nonexpansive mappings, and how the collections of asymptotically nonexpansive operators relate to those of nonexpansive operators. However, before making such comparisons we will present some examples that we will make use of later.

Example 5.1. For every nontrivial convex $C \subset \mathbb{R}$ there exists an asymptotically nonexpansive function $g : C \to C$ that is not nonexpansive. Let $0 < \varepsilon < 1$ and $f : [0,1] \to [0,1]$ defined by

$$f(x) = \min\{2x + \varepsilon, 1\}. \tag{5.1}$$

We note that f is non decreasing and $K(f, |\cdot|) = 2$, thus f^n is non decreasing for each $n \in N$. We define recursively $\varepsilon_{n+1} = 2\varepsilon_n + \varepsilon$ with $\varepsilon_0 = 0$. The sequence (ε_n) is non decreasing and unbounded. We call $N_f = \min\{n \mid \varepsilon_n \geq 1\}$. Then for each $n < N_f$, $\varepsilon_n = f^n(0)$ and $f^{N_f}(0) = \min\{\varepsilon_{N_f}, 1\} = 1$. That is, $f^m(x) = 1$ for each $m \geq N_f$ and $x \in [0,1]$. Thus $K(f^m, |\cdot|) = 0$ for each $m \geq N_f$. Hence f is an asymptotically nonexpansive function that is not nonexpansive.

Now we have constructed an asymptotically nonexpansive operator from C to C. Since C is nontrivial, there exist $a, b \in C$ such that $a < y$ and $[a, b] \subset C$. We define $\rho : C \to [a, b]$ by

$$\rho(x) = \begin{cases} a, & \text{if } x \leq a \\ x, & \text{if } x \in [a,b] \\ b, & \text{if } x \geq b \end{cases} \tag{5.2}$$

and $\phi : [0,1] \to [a,b]$ by $\phi(\lambda) = \lambda b + (1-\lambda)a$ for each $0 \le \lambda \le 1$. It is clear that $\rho^2 = \rho$. We affirm that $g : C \to C$ defined by $g = \phi f \phi^{-1}\rho = f^{\phi}\rho$ is an asymptotically nonexpansive mapping that is not nonexpansive. In order to prove that assertion, we construct the following commutative diagram which summarize the functions:

$$\begin{array}{ccc} [0,1] & \xrightarrow{\ f\ } & [0,1] \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \phi} \\ [a,b] & \xrightarrow{\ f^{\phi}\ } & [a,b] \\ {\scriptstyle \rho}\big\uparrow & & \big\downarrow \\ C & \xrightarrow{\ g\ } & C \end{array} \tag{5.3}$$

Without lost of generality we may assume that $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 < \lambda_2$. Then

$$\begin{aligned} |\phi\lambda_2 - \phi\lambda_1| &= |\lambda_2 b + (1-\lambda_2)a - (\lambda_1 b + (1-\lambda_1)a)| \\ &= |\lambda_2 - \lambda_1||b-a|. \end{aligned} \tag{5.4}$$

Thus ϕ is an M-isometry with $M = |b-a|$. In consequence by Lemma 4.1 we have that

$$K(f^{\phi}, |\cdot|) = K(f, |\cdot|) = 2, \tag{5.5}$$

and for each $m \ge N_f$ we have that

$$\begin{aligned} (f^{\phi})^m &= (f^m)^{\phi} \\ &= (f_1)^{\phi} \\ &= f_{\phi(1)} \\ &= f_b. \end{aligned} \tag{5.6}$$

Where f_1 is the constant function 1 defined on $[0,1]$ and f_b the constant function b defined on $[a,b]$. Thus

$$\begin{aligned} g^m &= (f^{\phi}\rho)^m \\ &= (f^m)^{\phi}\rho \\ &= (f_b)\rho \\ &= f_b. \end{aligned} \tag{5.7}$$

Hence $K(g^m, |\cdot|) = 0$ for each $m \ge N_f$. That is, g is an asymptotically nonexpansive operator that is not nonexpansive.

Example 5.2. Each operator $T : C \to C$ that is a contraction in any of its iterations is asymptotically nonexpansive for each equivalent norm. In symbols, let X be a normed space, C a nonempty subset of X with at least two elements and $T : C \to C$ such that for some $N \in \mathbb{N}$ and $\|\cdot\|_0 \in \mathcal{N}(X)$ we have that $K(T^N, \|\cdot\|_0) = l < 1$. Then $T \in ANE(C, \|\cdot\|)$ for each $\|\cdot\| \in \mathcal{N}(X)$. By Banach contraction Theorem the operator T has a fixed point $a \in C$. Thus we affirm that $T^n \to_K f_a$. In fact, for each $x, y \in C$ we have that

$$\begin{aligned} \|T^{N+r}x - f_a x - (T^{N+r}y - f_a y)\|_0 &= \|T^{N+r}x - T^{N+r}y\|_0 \\ &\leq l^r \|x - y\|_0. \end{aligned} \tag{5.8}$$

Then $d_K(T^n, f_a, \|\cdot\|_0) \to 0$ as $n \to \infty$. That is, $T \to_K f_a$. Moreover $T^n \to_K f_b$ for each $b \in X$. In particular for each $b \in C$ and $\|\cdot\| \in \mathcal{N}(X)$ we have that

$$T^n \to_K f_b \in NE(C, \|\cdot\|). \tag{5.9}$$

Thus by Theorem 3.7, T is asymptotically nonexpansive for every equivalent norm.

In the following theorem we show that the invariant family of asymptotically nonexpansive mappings defined in (3.26) is a proper subset of the family of invariant nonexpansive operators (3.9).

Theorem 5.3. *Let X be a Banach space and C a nonempty convex, closed and bounded subset of X. Then $S'(C) \subsetneq AS'(C)$. Moreover there exist a non affine operator $T \in AS'(C) \setminus S'(C)$.*

Proof. Let $\|\cdot\|_0 \in \mathcal{N}(X)$. By Corollary 19 in [23] there exists a nonaffine mapping $T \in NE(C, \|\cdot\|_0)$. For a fixed $a \in C$ and $0 < \lambda < 1$ we define

$$T_\lambda = \lambda f_a + (1 - \lambda)T. \tag{5.10}$$

It is not hard to check that T_λ is a nonaffine mapping such that

$$\begin{aligned} K(T_\lambda, \|\cdot\|_0) &= \lambda K(T, \|\cdot\|_0) \\ &\leq \lambda < 1. \end{aligned} \tag{5.11}$$

Thus by the convexity of C it is clear that $T_\lambda : C \to C$. Then by Example 5.2 it is fulfilled that $T_\lambda \to_K NE(C, \|\cdot\|)$ for each $\|\cdot\| \in \mathcal{N}(X)$. Hence by Theorem 3.7 T_λ is a nonaffine asymptotically nonexpansive mapping with respect to every equivalent norm. □

Example 5.4. An asymptotically nonexpansive mapping which is only asymptotically nonexpansive with respect to one norm. Let $(\mathbb{R}^2, \|\cdot\|_2)$ be the two dimensional real space with the euclidean norm. It is well known that rotations around the origin with $2\pi\theta$ angle in counterclockwise direction are $\|\cdot\|_2$-isometries and have the form

$$A_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}. \tag{5.12}$$

Another well-known result is that the orbit of A_θ at any $x \in S^1$ is dense in S^1 whenever θ is an irrational number, see Theorem 3.2.3 of Kronecker in [26]. We consider

$$T = A_\theta|_B : B \to B, \tag{5.13}$$

for some irrational θ and B the euclidean ball. If $\|\cdot\| \in \mathcal{N}(\mathbb{R}^2)$ is not collinear with the euclidean norm on B, then by Lemma 14 in [23] there exist $x, y \in B \setminus \{0\}$ such that $\|x\|_2 = \|y\|_2$ and $\|y\| > \|x\|$.

Without loss of generality we may assume that $\|x\|_2 = \|y\|_2 = 1$. Given a small enough $\varepsilon > 0$ and a neighborhood U_y of y such that $\|z\| > \|x\| + \varepsilon\|x\|$ for each $z \in U \cap B$. We have by the density of the orbits that there exist a subsequence (n_k) such that $T^{n_k}x \in U \cap B$. That is, $\|T^{n_k}x\| > \|x\| + \varepsilon\|x\|$ for each $k \in \mathbb{N}$. Hence

$$\begin{aligned}\|T^{n_k}x - T^{n_k}0\| &= \|T^{n_k}x\| \\ &> \|x\| + \varepsilon\|x\| \\ &= (1+\varepsilon)\|x - 0\|.\end{aligned} \tag{5.14}$$

Thus $K(T^{n_k}, \|\cdot\|) > 1 + \varepsilon > 1$ for each $k \in \mathbb{N}$. Equivalently

$$\limsup_k K(T^n, \|\cdot\|) \geq 1 + \varepsilon. \tag{5.15}$$

Then for each $\|\cdot\|' \in \mathcal{N}(X)$ not collinear with the euclidean norm we have that

$$\limsup_k K(T^n, \|\cdot\|) > 1. \tag{5.16}$$

Hence T only is asymptotically nonexpansive with respect to the euclidean norm. Now we will consider the complex case, in such a situation that we have $\mathbb{R}^2$ is a one dimensional Banach space. Then the only one norm is the euclidean norm. Hence the operator T defined above is asymptotically nonexpansive with respect to all norms. This situation shows us the importance of differentiating complex and real cases when limited to one dimensional aspects.

Example 5.5. An operator T such that is asymptotically nonexpansive with respect to each equivalent norm, but the sequence (T^n) does not converge in the strong sense. Let C be a symmetric set of a normed space X and $T = -I|_C$. It is clear that $T : C \to C$ is an isometry with respect to all norms over X. Then T is asymptotically nonexpansive with respect to all norms, but $T^n = (-I)^n = (-1)^n I$ does not converge with the infinity norm or with the Lipschitz seminorm.

We will finish this paper separating families of asymptotically nonexpansive operators from nonexpansive ones through nonlinear functions.

Theorem 5.6. *Let X be a Banach space and C a nonempty convex, closed and bounded subset of X. Then for each $\|\cdot\| \in \mathcal{N}(X)$ exists a nonaffine mapping*

$$T \in ANE(C, \|\cdot\|) \setminus NE(C, \|\cdot\|). \tag{5.17}$$

Proof. Let $f : [0,1] \to [0,1]$ as in Example 5.1 and $a, b \in C$ with $x \neq y$. We define $\phi : [0,1] \to [a,b] \subset C$ by $\phi(\lambda) = \lambda b + (1-\lambda)a$ for each $0 \leq \lambda \leq 1$. We consider $\mathbb{R}$ endowed with the norm $|r|_0 = \|b - a\||r|$. Since $|\cdot|_0$ and $|\cdot|$ are collinear, then $K(f, |\cdot|_0) = K(f, |\cdot|)$. Hence

$$\begin{aligned}\|\phi\lambda_2 - \phi\lambda_1\| &= \|\lambda_2 b + (1-\lambda_2)a - (\lambda_1 b + (1-\lambda_1)a)\| \\ &= |\lambda_2 - \lambda_1|\|b - a\| \\ &= |\lambda_2 - \lambda_1|_0.\end{aligned} \tag{5.18}$$

Thus the operator ϕ is a bijective 1-isometry from $([0,1], |\cdot|_0)$ to $([a,b], \|\cdot\|)$. We define $f^{\phi} = \phi f \phi^{-1}$. Then By Theorem 18 in [23] there exists $T : C \to [a,b] \subset C$ such that $T|_{[a,b]} = f^{\phi}$ with

$$\begin{aligned} K(T, \|\cdot\|) &= K(f^{\phi}, \|\cdot\|) \\ &= K(f, |\cdot|) \\ &= 2. \end{aligned} \tag{5.19}$$

We summarize the construction made in the following diagram.

$$\begin{array}{ccc} [0,1] & \xrightarrow{\;f\;} & [0,1] \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ [a,b] & \xrightarrow{\;f^{\phi}\;} & [a,b] \\ \big\downarrow & & \big\downarrow \\ C & \xrightarrow{\;T\;} & [a,b] \end{array} \tag{5.20}$$

Since $T(C) \subset [a,b]$, then for each $n \in \mathbb{N}$ we have that

$$\begin{aligned} T^{n+1} &= T^n T \\ &= (f^{\phi})^n T \\ &= (f^n)^{\phi} T, \end{aligned} \tag{5.21}$$

in which the last inequality follows from Lemma 4.1. Hence using N_f as in Example 5.1 we have that for each $n \geq N_f$ it is fulfilled that $f^n = f_1$. In consequence for each $n \geq N_f$

$$\begin{aligned} K(T^n, \|\cdot\|) &= K((f^n)^{\phi} T, \|\cdot\|) \\ &= K((f_1)^{\phi} T, \|\cdot\|) \\ &= K(f_{\phi(1)} T, \|\cdot\|) \\ &= K(f_b T, \|\cdot\|) \\ &= K(f_b, \|\cdot\|) \\ &= 0. \end{aligned} \tag{5.22}$$

Then

$$\limsup_n K(T^n, \|\cdot\|) = 0. \tag{5.23}$$

That is, T is a $\|\cdot\|$-asymptotically nonexpansive mapping which is not $\|\cdot\|$-nonexpansive.

□

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Conflict of interest

All the authors affirmed that they have no conflicts of interest.

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