Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 9385–9399.
DOI: 10.3934/math.2023472
Received: 22 January 2022
Revised: 27 April 2022
Accepted: 24 May 2022
Published: 15 February 2023

Research article

Generalized q-convex functions characterized by q-calculus

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Abstract: The objective of the current examination is to present new sub-classes of q-convex and q-starlike functions inside $\mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}$, by q-difference operator. We determined connections of these classes and acquired a few fundamental properties, for instance, inclusion relation, subordination properties and q-limits on real part.

Keywords: analytic functions; second-order q-differential; convolution; subordination; q-number
Mathematics Subject Classification: 30C45, 30C10

1. Introduction

The mathematical study of q-calculus has been a subject of top importance for researchers due to its huge applications in unique fields. Few recognized work at the application of q-calculus firstly added through Jackson [4]. Later, q-analysis with geometrical interpretation become diagnosed. Currently, q-calculus has attained the attention researchers due to its massive applications in mathematics and physics. The in-intensity evaluation of q-calculus changed into first of all noted with the aid of Jackson [4, 5], wherein he defined q-derivative and q-integral in a totally systematic way. Recently, authors are utilizing the q-integral and q-derivative to study some new sub-families of univalent functions and obtain certain new results, see for example Nadeem et al. [8], Obad et al. [11] and reference therein.

Assume that $f \in \mathbb{C}$. Furthermore, f is normalized analytic, if f along $f(0) = 0$, $f'(0) = 1$ and characterized as

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j. \tag{1.1}$$

We denote by $\mathcal{A}$, the family of all such functions. Let $f \in \mathcal{A}$ be presented as (1.1). Furthermore,

$$f \textit{ is univalent} \Longleftrightarrow \xi_1 \neq \xi_2 \Longrightarrow f(\xi_1) \neq f(\xi_2), \quad \forall\ \xi_1, \xi_2 \in \mathcal{E}.$$

We present by $\mathcal{S}$ the family of all univalent functions. Let $\tilde{p} \in \mathbb{C}$ be analytic. Furthermore, $\tilde{p} \in \mathcal{P}$, iff $\Re(\tilde{p}(z)) > 0$, along $\tilde{p}(0) = 1$ and presented as follows:

$$\tilde{p}(z) = 1 + \sum_{j=1}^{\infty} c_j z^j. \tag{1.2}$$

Broadening the idea of $\mathcal{P}$, the family $\mathcal{P}(\alpha_0)$, $0 \leq \alpha_0 < 1$ defined by

$$(1 - \alpha_0)p_1 + \alpha_0 = p(z) \Longleftrightarrow p \in \mathcal{P}(\alpha_0), \quad p_1 \in \mathcal{P},$$

for further details one can see [2].

Assume that C, $\mathcal{K}$ and $\mathcal{S}^*$ signify the common sub-classes of $\mathcal{A}$, which contains convex, close-to-convex and star-like functions in $\mathcal{E}$. Furthermore, by $\mathcal{S}^*(\alpha_0)$, we meant the class of starlike functions of order α_0, $0 \leq \alpha_0 < 1$, for details, see [1, 2] and references therein. Main motivation behind this research work is to extend the concept of Kurki and Owa [6] into q-calculus.

The structure of this paper is organized as follows. For convenience, Section 2 give some material which will be used in upcoming sections along side some recent developments in q-calculus. In Section 3, we will introduce our main classes $C_q(\alpha_0, \beta_0)$ and $\mathcal{S}_q^*(\alpha_0, \beta_0)$. In Section 4, we will discuss our main result which include, inclusion relations, q-limits on real parts and integral invariant properties. At the end, we conclude our work.

2. Materials and methods

The concept of Hadamard product (convolution) is critical in GFT and it emerged from

$$\Phi\left(r^2 e^{i\theta}\right) = (g * f)\left(r^2 e^{i\theta}\right) = \frac{1}{2\pi} \int_0^{2\pi} g\left(re^{i(\theta - t)}\right) f\left(re^{it}\right) dt, \quad r < 1,$$

and

$$H(z) = \int_0^z \xi^{-1} h(\xi)\, d\xi, \quad |\xi| < 1$$

is integral convolution. Let f be presented as in (1.1), the convolution $(f * g)$ is characterize as

$$(f * g)(\zeta) = \zeta + \sum_{j=2}^{\infty} a_j b_j \zeta^j, \quad \zeta \in \mathcal{E},$$

where

$$g(\zeta) = \zeta + b_2 \zeta^2 + \cdots = \zeta + \sum_{j=2}^{\infty} b_j \zeta^j, \tag{2.1}$$

for details, see [2].

Let h_1 and h_2 be two functions. Then, $h_1 \prec h_2 \Longleftrightarrow \exists$, ϖ analytic such that $\varpi(0) = 0$, $|\varpi(z)| < 1$, with $h_1(z) = (h_2 \circ \varpi)(z)$. It can be found in [1] that, if $h_2 \in \mathcal{S}$, then

$$h_1(0) = h_2(0) \textit{ and } h_1(E) \subset h_2(E) \Longleftrightarrow h_1 \prec h_2,$$

for more information, see [7].

Assume that $q \in (0,1)$. Furthermore, q-number is characterized as follows:

$$[\upsilon]_q = \begin{cases} \frac{1-q^{\upsilon}}{1-q}, & \textit{if } \upsilon \in \mathbb{C}, \\ \sum\limits_{k=0}^{j-1} q^k = 1 + q + q^2 + \ldots + q^{j-1}, & \textit{if } \upsilon = j \in \mathbb{N}. \end{cases} \tag{2.2}$$

Utilizing the q-number defined by (2.2), we define the shifted q-factorial as the following:

Assume that $q \in (0,1)$. Furthermore, the shifted q-factorial is denoted and given by

$$[j]_q! = \begin{cases} 1 & \textit{if } j = 0, \\ \prod\limits_{k=1}^{j} [k]_q & \textit{if } j \in \mathbb{N}. \end{cases}$$

Let $f \in \mathbb{C}$. Then, utilizing (2.2), the q-derivative of the function f is denoted and defined in [4] as

$$\left(D_q f\right)(\zeta) = \begin{cases} \frac{f(\zeta) - f(q\zeta)}{(1-q)\zeta}, & \textit{if } \zeta \neq 0, \\ f'(0), & \textit{if } \zeta = 0, \end{cases} \tag{2.3}$$

provided that $f'(0)$ exists.

That is
$$\lim_{q \longrightarrow 1^-} \frac{f(\zeta) - f(q\zeta)}{(1-q)\zeta} = \lim_{q \longrightarrow 1^-} \left(D_q f\right)(\zeta) = f'(\zeta).$$
If $f \in \mathcal{A}$ defined by (1.1), then,

$$\left(D_q f\right)(\zeta) = 1 + \sum_{j=2}^{\infty} [j]_q a_j \zeta^j, \quad \zeta \in \mathcal{E}. \tag{2.4}$$

Also, the q-integral of $f \in \mathbb{C}$ is defined by

$$\int_0^{\zeta} f(t) d_q t = \zeta(1-q) \sum_{i=0}^{\infty} q^i f\left(q^i \zeta\right), \tag{2.5}$$

provided that the series converges, see [5].

The q-gamma function is defined by the following recurrence relation:

$$\Gamma_q(\zeta + 1) = [\zeta]_q \Gamma_q(\zeta) \text{ and } \Gamma_q(1) = 1.$$

In recent years, researcher are utilizing the q-derivative defined by (2.3), in various branches of mathematics very effectively, especially in Geometric Function Theory (GFT). For further developments and discussion about q-derivative defined by (2.3), we can obtain excellent articles produced by famous mathematician like [3, 8–10, 12–14] and many more.

Ismail et al. [3] investigated and study the class C_q as

$$C_q = \left\{ f \in \mathcal{A} : \Re\left[\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\right] > 0, \quad 0 < q < 1, \quad z \in \mathcal{E} \right\}.$$

If $q \longrightarrow 1^-$, then $C_q = C$.

Later, Ramachandran et al. [12] discussed the class $C_q(\alpha_0)$, $0 \le \alpha_0 < 1$, given by

$$C_q(\alpha_0) = \left\{ f \in \mathcal{A} : \Re\left(\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\right) > \alpha_0, \quad 0 < q < 1, \quad z \in \mathcal{E} \right\}.$$

For $\alpha_0 = 0$, $C_q(\alpha_0) = C_q$.

2.1. On the classes $C_q(\alpha_0, \beta_0)$ and $\mathcal{S}_q^*(\alpha_0, \beta_0)$

Now, extending the idea of [13] and by utilizing the q-derivative defined by (2.3), we define the families $C_q(\alpha_0, \beta_0)$ and $\mathcal{S}_q^*(\alpha_0, \beta_0)$ as follows:

Definition 2.1. *Let $f \in \mathcal{A}$ and $\alpha_0, \beta_0 \in \mathbb{R}$ such that $0 \le \alpha_0 < 1 < \beta_0$. Then,*

$$f \in C_q(\alpha_0, \beta_0) \Longleftrightarrow \alpha_0 < \Re\left(\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\right) < \beta_0, \quad z \in \mathcal{E}. \tag{2.6}$$

It is obvious that if $q \longrightarrow 1^-$, then $C_q(\alpha_0, \beta_0) \longrightarrow C(\alpha_0, \beta_0)$, see [13]. This means that

$$C_q(\alpha_0, \beta_0) \subset C(\alpha_0, \beta_0) \subset C.$$

Definition 2.2. *Let $\alpha_0, \beta_0 \in \mathbb{R}$ such that $0 \le \alpha_0 < 1 < \beta_0$ and $f \in \mathcal{A}$ defined by (1.1). Then,*

$$f \in \mathcal{S}_q^*(\alpha_0, \beta_0) \Longleftrightarrow \alpha_0 < \Re\left(\frac{zD_qf(z)}{f(z)}\right) < \beta_0, \quad z \in \mathcal{E}. \tag{2.7}$$

Or equivalently, we can write

$$f \in C_q(\alpha_0, \beta_0) \Longleftrightarrow zD_qf \in \mathcal{S}_q^*(\alpha_0, \beta_0), \quad z \in \mathcal{E}. \tag{2.8}$$

Remark 2.1. *From Definitions 2.1 and 2.2, it follows that $f \in C_q(\alpha_0, \beta_0)$ or $f \in \mathcal{S}_q^*(\alpha_0, \beta_0)$ iff f fulfills*

$$\begin{aligned} 1 + \frac{zD_q^2f(z)}{D_qf(z)} &\prec \frac{1-(2\alpha_0-1)z}{1-qz}, \\ 1 + \frac{zD_q^2f(z)}{D_qf(z)} &\prec \frac{1-(2\beta_0-1)z}{1-qz}, \end{aligned}$$

or

$$\begin{aligned} \frac{zD_q f(z)}{f(z)} &\prec \frac{1-(2\alpha_0-1)z}{1-qz}, \\ \frac{zD_q f(z)}{f(z)} &\prec \frac{1-(2\beta_0-1)z}{1-qz}, \end{aligned}$$

for all $z \in \mathcal{E}$.

We now consider q-analogue of the function p defined by [13] as

$$p_q(z) = 1 + \frac{\beta_0-\alpha_0}{\pi} i \log\left(\frac{1-qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z}{1-qz}\right). \tag{2.9}$$

Firstly, we fined the series form of (2.9).

Consider

$$\begin{aligned} p_q(z) &= 1 + \frac{\beta_0-\alpha_0}{\pi} i \log\left(\frac{1-qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z}{1-qz}\right) && (2.10) \\ &= 1 + \frac{\beta_0-\alpha_0}{\pi} i \left[\log\left(1-qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z\right) - \log(1-qz)\right]. && (2.11) \end{aligned}$$

If we let $w = qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z$, then,

$$\log\left(1-qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z\right) = \log(1-w) = -w - \sum_{j=2}^{\infty} \frac{w^j}{j}.$$

This implies that

$$\log\left(1-qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z\right) = -\left(qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z\right) - \sum_{j=2}^{\infty} \frac{\left(qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z\right)^j}{j},$$

and

$$-\log(1-qz) = qz + \sum_{j=2}^{\infty} \frac{(qz)^j}{j}.$$

Utilizing these, Eq (2.11) can be written as

$$p_q(z) = 1 + \sum_{j=1}^{\infty} \frac{\beta_0-\alpha_0}{j\pi} iq^j \left(1-e^{2n\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}\right) z^j. \tag{2.12}$$

This shows that the $p_q \in \mathcal{P}$.

Motivated by this work and other aforementioned articles, the aim in this paper is to keep with the research of a few interesting properties of $C_q(\alpha_0,\beta_0)$ and $\mathcal{S}_q^*(\alpha_0,\beta_0)$.

3. Results and discussion

Utilizing the meaning of subordination, we can acquire the accompanying Lemma, which sum up the known results in [6].

Lemma 3.1. *Let* $f \in \mathcal{A}$ *be defined by* (1.1)*,* $0 \le \alpha_0 < 1 < \beta_0$ *and* $0 < q < 1$*. Then,*

$$f \in \mathcal{S}_q^*(\alpha_0,\beta_0) \Longleftrightarrow \left(\frac{zD_q f(z)}{f(z)}\right) \prec 1 + \frac{\beta_0 - \alpha_0}{\pi} i \log\left(\frac{1 - qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z}{1 - qz}\right), \quad z \in \mathcal{E}. \tag{3.1}$$

Proof. Assume that F be characterized as

$$F(z) = 1 + \frac{\beta_0 - \alpha_0}{\pi} i \log\left(\frac{1 - qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z}{1 - qz}\right), \quad 0 \le \alpha_0 < 1 < \beta_0.$$

At that point it can without much of a stretch seen that function F ia simple and analytic along $F(0) = 1$ in $\mathcal{E}$. Furthermore, note

$$\begin{aligned} F(z) &= 1 + \frac{\beta_0 - \alpha_0}{\pi} i \log\left(\frac{1 - qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z}{1 - qz}\right) \\ &= 1 + \frac{\beta_0 - \alpha_0}{\pi} i \log\left[\frac{e^{\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}}{i}\left(\frac{ie^{-\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)} - qie^{\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z}{1 - qz}\right)\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} F(z) &= 1 + \frac{\beta_0 - \alpha_0}{\pi} i \left[\log\left(e^{\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}\right) - \log i\right] + \frac{\beta_0 - \alpha_0}{\pi} i \log\left[\frac{ie^{-\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)} - qie^{\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z}{1 - qz}\right] \\ &= 1 + \frac{\beta_0 - \alpha_0}{\pi} i \left[\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right) - \left(\frac{\pi i}{2}\right)\right] + \frac{\beta_0 - \alpha_0}{\pi} i \log\left[\frac{ie^{-\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)} - qie^{\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z}{1 - qz}\right] \\ &= \frac{\alpha_0 + \beta_0}{2} + \frac{\beta_0 - \alpha_0}{\pi} i \log\left[\frac{ie^{-\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)} - qie^{\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z}{1 - qz}\right]. \end{aligned}$$

A simple calculation leads us to conclude that F maps $\mathcal{E}$ onto the domain Ω defined by

$$\Omega = \{w : \alpha_0 < \Re(w) < \beta_0\}. \tag{3.2}$$

Therefore, it follows from the definition of subordination that the inequalities (2.7) and (3.1) are equivalent. This proves the assertion of Lemma 3.1. □

Lemma 3.2. *Let $f \in \mathcal{A}$ and $0 \leq \alpha_0 < 1 < \beta_0$. Then,*

$$f \in C_q(\alpha_0,\beta_0) \Longleftrightarrow \left(\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\right) \prec 1 + \frac{\beta_0 - \alpha_0}{\pi} i \log\left(\frac{1 - qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z}{1 - qz}\right),$$

and if p presented as in (2.9) *has the structure*

$$p_q(z) = 1 + \sum_{j=1}^{\infty} B_j(q) z^j, \tag{3.3}$$

then,

$$B_j(q) = \frac{\beta_0 - \alpha_0}{j\pi} i q^j \left(1 - e^{2j\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}\right), \quad j \in \mathbb{N}. \tag{3.4}$$

Proof. Proof directly follows by utilizing (2.8), (2.12) and Lemma 3.1. □

Example 3.1. Let f be defined as

$$f(z) = z\exp\left\{\frac{\beta_0 - \alpha_0}{\pi} i \int_0^z \frac{1}{t} \log\left(\frac{1 - qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}t}{1 - qt}\right) d_qt\right\}. \tag{3.5}$$

This implies that

$$\frac{zD_qf(z)}{f(z)} = 1 + \frac{\beta_0 - \alpha_0}{\pi} i \log\left(\frac{1 - qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z}{1 - qz}\right), \quad z \in \mathcal{E}.$$

According to the proof of Lemma 3.1, it can be observed that f given by (3.5) satisfies (2.7), which means that $f \in S_q^*(\alpha_0,\beta_0)$. Similarly, it can be seen by utilizing Lemma 3.2 that

$$f(z) = \int_0^z z\exp\left\{\frac{\beta_0 - \alpha_0}{\pi} i \int_0^u \frac{1}{t} \log\left(\frac{1 - qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}t}{1 - qt}\right) d_qt\right\} d_qu, \tag{3.6}$$

belongs to the class $C_q(\alpha_0,\beta_0)$.

Inclusion relations:

In this segment, we study some inclusion relations and furthermore acquire some proved results as special cases. For this, we need below mentioned lemma which is the q-analogue of known result in [7].

Lemma 3.3. *Let $\mathbf{u}, \mathbf{v} \in \mathbb{C}$, such that $\mathbf{u} \neq 0$ and what's more, $\hbar \in \mathcal{H}$ such that $\Re[\mathbf{u}\hbar(z) + \mathbf{v}] > 0$. Assume that $\wp \in \mathcal{P}$, fulfill*

$$\wp(z) + \frac{zD_qp(z)}{\mathbf{u}\wp(z) + \mathbf{v}} \prec \hbar(z) \Longrightarrow \wp(z) \prec \hbar(z), \quad z \in \mathcal{E}.$$

Theorem 3.1. *For $0 \leq \alpha_0 < 1 < \beta_0$ and $0 < q < 1$,*

$$C_q(\alpha_0,\beta_0) \subset \mathcal{S}_q^*(\alpha_0,\beta_0), \quad z \in \mathcal{E}.$$

Proof. Let $f \in C_q(\alpha_0,\beta_0)$. Consider

$$p(z) = \frac{zD_q f(z)}{f(z)}, \quad p \in \mathcal{P}.$$

Differentiating q-logarithamically furthermore, after some simplifications, we get

$$p(z) + \frac{zD_q p(z)}{p(z)} = \frac{D_q\left(zD_q f(z)\right)}{D_q f(z)} \prec 1 + \frac{\beta_0 - \alpha_0}{\pi} i \log\left(\frac{1 - qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z}{1 - qz}\right), \quad z \in \mathcal{E}.$$

Note that by utilizing Lemma 3.3 with $u = 1$ and $v = 0$, we have

$$p(z) \prec 1 + \frac{\beta_0 - \alpha_0}{\pi} i \log\left(\frac{1 - qe^{2\pi i\left(\frac{1-\alpha_0}{\beta_0-\alpha_0}\right)}z}{1 - qz}\right), \quad z \in \mathcal{E}.$$

Consequently,

$$f \in S_q^*(\alpha_0,\beta_0), \quad z \in \mathcal{E}.$$

This completes the proof. □

Note for distinct values of parameters in Theorem 3.1, we obtain some notable results, see [2,6,13].

Corollary 3.1. *For $q \longrightarrow 1^-$, $0 \le \alpha_0 < 1 < \beta_0$, we have*

$$C(\alpha_0,\beta_0) \subset \mathcal{S}^*(\alpha_0,\beta_0), \quad z \in \mathcal{E}.$$

Corollary 3.2. *For $q \longrightarrow 1^-$, $\alpha_0 = 0$ and $\beta_0 > 1$, we have*

$$C(\beta_0) \subset \mathcal{S}^*(\beta_1), \quad z \in \mathcal{E},$$

where

$$\beta_1 = \frac{1}{4}\left[(2\beta_0 - 1) + \sqrt{4\beta_0^2 - 4\beta_0 + 9}\right].$$

q-limits on real parts:

In this section, we discuss some q-bounds on real parts for the function f in $C_q(\alpha_0,\beta_0)$ and following lemma will be utilize which is the q-analogue of known result of [7].

Lemma 3.4. *Let $U \subset \mathbb{C} \times \mathbb{C}$ and let $c \in \mathbb{C}$ along $\Re(b) > 0$. Assume that $\mho : \mathbb{C}^2 \times E \longrightarrow \mathbb{C}$ fulfills*

$$\mho(i\rho,\sigma;z) \notin U, \ \forall \rho,\sigma \in \mathbb{R}, \quad \sigma \le -\frac{|b - i\rho|^2}{(2\Re(b))}.$$

If $p(z) = c + c_1 z + c_2 z^2 + \ldots$ is in $\mathcal{P}$ along

$$\mho\left(p(z), zD_q p(z); z\right) \in U \Longrightarrow \Re(p(z)) > 0, \quad z \in \mathcal{E}.$$

Lemma 3.5. *Let* $p(z) = \sum_{j=1}^{\infty} C_j z^j$ *and assume that* $p(E)$ *is a convex domain. Furthermore, let* $q(z) = \sum_{j=1}^{\infty} A_j z^j$ *is analytic and if* $q \prec p$ *in* E*. Then,*

$$\left|A_j\right| \leq |C_1|, \quad j = 1, 2, \cdots .$$

Theorem 3.2. *Suppose* $f \in \mathcal{A}$, $0 \leq \alpha_0 < 1$ *and*

$$\Re\left(\frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\right) > \alpha_0, \quad z \in \mathcal{E}. \tag{3.7}$$

Then,

$$\Re\left(\sqrt{D_q f(z)}\right) > \frac{1}{2-\alpha_0}, \quad z \in \mathcal{E}. \tag{3.8}$$

Proof. Let $\gamma = \frac{1}{2-\alpha_0}$ and for $0 \leq \alpha_0 < 1$ implies $\frac{1}{2} \leq \gamma < 1$. Let

$$\sqrt{D_q f(z)} = (1-\gamma) p(z) + \gamma, \quad p \in \mathcal{P}.$$

Differentiating q-logrithmically, we obtain

$$D_q \frac{\left(zD_q f(z)\right)}{D_q f(z)} = 1 + \frac{2(1-\gamma) zD_q p(z)}{(1-\gamma) p(z) + \gamma}.$$

Let us construct the functional $\mho$ such that

$$\mho(r, s; z) = 1 + \frac{2(1-\gamma) s}{(1-\gamma) r - \gamma}, \quad r = p(z), \quad s = zD_q p(z).$$

Utilizing (3.7), we can write

$$\left\{\mho\left(p(z), zD_q p(z); z \in \mathcal{E}\right)\right\} \subset \{w \in \mathbb{C} : \Re(w) > \alpha_0\} = \Omega_{\alpha_0}.$$

Now, $\rho, \delta \in \mathbb{R}$ with $\delta \leq -\frac{(1+\rho^2)}{2}$, we have

$$\Re(\mho(i\rho, \delta)) = 1 + \frac{2(1-\gamma)\delta}{(1-\gamma)(i\rho) + r}.$$

This implies that

$$\Re(\mho(i\rho, \delta; z)) = \Re\left(1 + \frac{2(1-\gamma)\delta}{(1-\gamma)^2 \rho^2 + \gamma^2}\right).$$

Utilizing $\delta \leq -\frac{(1+\rho^2)}{2}$, we can write

$$\Re(\mho(i\rho, \delta; z)) \leq 1 - \frac{\gamma(1-\gamma)\left(1+\rho^2\right)}{(1-\gamma)^2 \rho^2 + \gamma^2}. \tag{3.9}$$

Let

$$g(\rho)=\frac{1+\rho^2}{(1-\gamma)^2\rho^2+\gamma^2}.$$

Then, $g(-\rho)=g(\rho)$, which shows that g is even continuous function. Thus,

$$D_q(g(\rho))=\frac{[2]_q(2\gamma-1)\rho}{\left[(1-\gamma)^2\rho^2+\gamma^2\right]\left[(1-\gamma)^2q^2\rho^2+\gamma^2\right]},$$

and $D_q(g(0))=0$. Also, it can be seen that g is increasing function on $(0,\infty)$. Since $\frac{1}{2}\le\gamma<1$, therefore,

$$\frac{1}{\gamma^2}\le g(\rho)<\frac{1}{(1-\gamma)^2},\quad \rho\in\mathbb{R}. \tag{3.10}$$

Now by utilizing (3.9) and (3.10), we have

$$\Re(\mho(i\rho,\delta;z))\le 1-\gamma(1-\gamma)g(\rho)\le 2-\frac{1}{\gamma}=\alpha_0.$$

This means that $\Re(\mho(i\rho,\delta;z))\notin\Omega_{\alpha_0}$ for all $\rho,\delta\in\mathbb{R}$ with $\delta\le-\frac{(1+\rho^2)}{2}$. Thus, by utilizing Lemma 3.4, we conclude that $\Re p(z)>0$, $\forall z\in\mathcal{E}$. □

Theorem 3.3. *Suppose $f\in\mathcal{A}$ be defined by* (1.1) *and* $1<\beta_0<2$*,*

$$\Re\left(\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\right)<\beta_0,\quad z\in\mathcal{E}.$$

Then,

$$\Re\left(\sqrt{D_qf(z)}\right)>\frac{1}{2-\beta_0},\quad z\in\mathcal{E}.$$

Proof. Continuing as in Theorem 3.2, we have the result. □

Combining Theorems 3.2 and 3.3, we obtain the following result.

Theorem 3.4. *Suppose $f\in\mathcal{A}$, $0\le\alpha_0<1<\beta_0<2$ and*

$$\alpha_0<\Re\left(\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\right)<\beta_0,\quad z\in\mathcal{E}.$$

$$\frac{1}{2-\alpha_0}<\Re\left\{\sqrt{D_qf(z)}\right\}<\frac{1}{2-\beta_0},\quad z\in\mathcal{E}.$$

Theorem 3.5. *Let $f\in\mathcal{A}$ be defined by* (1.1) *and $\alpha_0,\beta_0\in\mathbb{R}$ such that $0\le\alpha_0<1<\beta_0$. If $f\in C_q(\alpha_0,\beta_0)$, then,*

$$\left|a_j\right|\le\begin{cases}\frac{|B_1|}{[2]_q}, & \text{if } j=2,\\ \frac{|B_1|}{[j]_q[j-1]_q}\prod\limits_{k=1}^{j-2}\left(1+\frac{|B_1|}{[k]_q}\right), & \text{if } j=3,4,5,\cdots,\end{cases}$$

where $|B_1|$ is given by

$$|B_1(q)|=\frac{2q(\beta_0-\alpha_0)}{\pi}\sin\frac{\pi(1-\alpha_0)}{\beta_0-\alpha_0}. \tag{3.11}$$

Proof. Assume that

$$q(z)=\left(\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\right),\quad q\in\mathcal{P},\quad z\in\mathcal{E}. \tag{3.12}$$

Then, by definition of $C_q(\alpha_0,\beta_0)$, we obtain

$$q(z)\prec p_q(z),\quad z\in\mathcal{E}. \tag{3.13}$$

Let p_q be defined by (3.3) and $B_n(q)$ is given as in (3.4). If

$$q(z)=1+\sum_{j=2}^{\infty}A_j(q)z^j, \tag{3.14}$$

by (3.12), we have

$$D_q\left(zD_qf(z)\right)=q(z)D_q(f(z)).$$

Note that by utilizing (1.1), (2.4) and (3.14), one can obtain

$$1+\sum_{j=2}^{\infty}[j]_q[j-1]_q a_jz^{j-1}=\left(1+\sum_{j=1}^{\infty}A_j(q)z^j\right)\left(1+\sum_{j=2}^{\infty}[j]_q a_jz^{j-1}\right).$$

Comparing the coefficient of of z^{j-1} on both sides, we have

$$\begin{aligned}
&[j]_q[j-1]_q a_j\\
=\ & A_{j-1}(q)+[j]_q a_j+\sum_{k=2}^{j-1}[k]_q a_kA_{j-k}(q)\\
=\ & A_{j-1}(q)+[j]_q a_q+[2]_q a_2A_{j-2}(q)+[3]_q a_3A_{j-3}(q)+\cdots+[j-1]_q a_{j-1}A_1(q).
\end{aligned} \tag{3.15}$$

This implies that by utilizing Lemma 3.5 with (3.13), we can write

$$\left|A_j(q)\right|\le|B_1(q)|,\ \text{for } j=1,2,3,\cdots \tag{3.16}$$

Now by utilizing (3.16) in (3.15) and after some simplifications, we have

$$\begin{aligned}
\left|a_j\right| &\le \frac{|B_1(q)|}{[j]_q[j-1]_q}\sum_{k=2}^{j-1}[k-1]_q|a_{k-1}|,\\
&\le \frac{|B_1|}{[j]_q[j-1]_q}\prod_{k=1}^{j-2}\left(1+\frac{|B_1|}{[k]_q}\right).
\end{aligned}$$

Furthermore, for $j=2,3,4,$

$$\begin{aligned}
|a_2| &\le \frac{|B_1(q)|}{[2]_q},\\
|a_3| &\le \frac{|B_1(q)|}{[3]_q[2]_q}[1+|B_1|],
\end{aligned}$$

$$|a_4| \leq \frac{|B_1(q)|}{[4]_q [3]_q} \left[(1 + |B_1(q)|) \left(1 + \frac{|B_1(q)|}{[2]_q} \right) \right].$$

By utilizing mathematical induction for q-calculus, it can be observed that

$$\left| a_j \right| \leq \frac{|B_1(q)|}{[j]_q [j-1]_q} \prod_{k=1}^{j-2} \left(1 + \frac{|B_1(q)|}{[k]_q} \right),$$

which is required. □

Remark 3.1. *Note that by taking $q \longrightarrow 1^-$ in Theorems 3.2–3.4, we attain remarkable results in ordinary calculus discussed in [6].*

Integral invariant properties:

In this portion, we show that the family $C_q(\alpha_0, \beta_0)$ is invariant under the q-Bernardi integral operator defined and discussed in [9] is given by

$$B_q(f(z)) = F_{c,q}(z) = \frac{[1+c]_q}{z^c} \int_0^z t^{c-1} f(t) d_q t, \quad 0 < q < 1, \quad c \in \mathbb{N}. \tag{3.17}$$

Making use of (1.1) and (2.5), we can write

$$\begin{aligned} F_{c,q}(z) = B_q(f(z)) &= \frac{[1+c]_q}{z^c} z(1-q) \sum_{i=0}^{\infty} q^i \left(zq^i \right)^{c-1} f\left(zq^i \right) \\ &= [1+c]_q (1-q) \sum_{i=0}^{\infty} q^{ic} \sum_{j=1}^{\infty} q^{ij} a_j z^j \\ &= \sum_{j=1}^{\infty} [1+c]_q \left[\sum_{j=0}^{\infty} (1-q) q^{i(j+c)} \right] a_j z^j \\ &= \sum_{j=1}^{\infty} [1+c]_q \left(\frac{1-q}{1-q^{j+c}} \right) a_j z^j. \end{aligned}$$

Finally, we obtain

$$F_{c,q}(z) = B_q(f(z)) = z + \sum_{j=2}^{\infty} \left(\frac{[1+c]_q}{[j+c]_q} \right) a_j z^j. \tag{3.18}$$

For $c = 1$, we obtain

$$\begin{aligned} F_{1,q}(z) &= \frac{[2]_q}{z} \int_0^z f(t) d_q t, \quad 0 < q < 1, \\ &= z + \sum_{j=2}^{\infty} \left(\frac{[2]_q}{[j+1]_q} \right) a_j z^j. \end{aligned}$$

It is well known [9] that the radius of convergence R of

$$\sum_{j=1}^{\infty}\left(\frac{[1+c]_q}{[j+c]_q}\right)a_j z^j \quad and \quad \sum_{j=1}^{\infty}\left(\frac{[2]_q}{[j+1]_q}\right)a_j z^j$$

is q and the function given by

$$\phi_q(z)=\sum_{j=1}^{\infty}\left(\frac{[1+c]_q}{[j+c]_q}\right)z^j, \tag{3.19}$$

belong to the class C_q of q-convex function introduced by [3].

Theorem 3.6. *Let $f\in\mathcal{A}$. If $f\in C_q(\alpha_0,\beta_0)$, then $F_{c,q}\in C_q(\alpha_0,\beta_0)$, where $F_{c,q}$ is defined by* (3.17).

Proof. Let $f\in C_q(\alpha_0,\beta_0)$ and set

$$p(z)=\frac{D_q\left(zD_qF_{c,q}(z)\right)}{D_qF_{c,q}(z)},\quad p\in\mathcal{P}. \tag{3.20}$$

q-differentiation of (3.17) yields

$$zD_qF_{c,q}(z)+cF_{c,q}(z)=[1+c]_q\,f(z).$$

Again q-differentiating and utilizing (3.20), we obtain

$$[1+c]_q\,D_qf(z)=D_qF_{c,q}(z)\,(c+p(z)).$$

Now, logarithmic q-differentiation of this yields

$$p(z)+\frac{zD_qp(z)}{c+p(z)}=\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)},\quad z\in\mathcal{E}.$$

By utilizing the definition of the class $C_q(\alpha_0,\beta_0)$, we have

$$p(z)+\frac{zD_qp(z)}{c+p(z)}=\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\prec p_q(z).$$

Therefore,

$$p(z)+\frac{zD_qp(z)}{c+p(z)}\prec p_q(z),\quad z\in\mathcal{E}.$$

Consequently, utilizing Lemma 3.3, we have

$$p(z)\prec p_q(z),\quad z\in\mathcal{E}.$$

The proof is complete. □

Remark 3.2. *Letting $q\longrightarrow 1^-$, in Theorem* 3.6*, we obtain a known result from [13].*

Corollary 3.3. *Let $f \in \mathcal{A}$. If $f \in C(\alpha_0, \beta_0)$, then $F_c \in C(\alpha_0, \beta_0)$, where F_c is Bernardi integral operator defined in [1].*

Also, for $q \longrightarrow 1^{-}$, $\alpha_0 = 0$ and $\beta_0 = 0$, we obtain the well known result proved by [1]. It is well known [9] that for $0 \leq \alpha_0 < 1 < \beta_0$, $0 < q < 1$ and $c \in \mathbb{N}$, the function (3.19) belong to the class C_q. Utilizing this, we can prove

$$\begin{aligned} f &\in C_q(\alpha_0, \beta_0), \ \phi_q \in C_q \Longrightarrow \left(f * \phi_q\right) \in C_q(\alpha_0, \beta_0), \\ f &\in \mathcal{S}_q^*(\alpha_0, \beta_0), \ \phi_q \in C_q \Longrightarrow \left(f * \phi_q\right) \in \mathcal{S}_q^*(\alpha_0, \beta_0). \end{aligned}$$

Remark 3.3. *As an example consider the function $f \in C_q(\alpha_0, \beta_0)$ defined by* (3.6) *and $\phi_q \in C_q$ given by* (3.19)*, implies $\left(f * \phi_q\right) \in C_q(\alpha_0, \beta_0)$.*

4. Conclusions

In this article, we mainly focused on q-calculus and utilized this is to study new generalized sub-classes $C_q(\alpha_0, \beta_0)$ and $\mathcal{S}_q^*(\alpha_0, \beta_0)$ of q-convex and q-star-like functions. We discussed and study some fundamental properties, for example, inclusion relation, q-coefficient limits on real part, integral preserving properties. We have utilized traditional strategies alongside convolution and differential subordination to demonstrate main results. This work can be extended in post quantum calculus. The path is open for researchers to investigate more on this discipline and associated regions.

Acknowledgments

The work was supported by Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R52), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare that they have no conflicts of interest.

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