Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 9352–9364.
DOI: 10.3934/math.2023470
Received: 03 December 2022
Revised: 31 January 2023
Accepted: 05 February 2023
Published: 15 February 2023

Research article

Synchronization and fluctuation of a stochastic coupled systems with additive noise

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Abstract: The synchronization and fluctuation of a stochastic coupled system with additive noise were investigated in this paper. According to the relationship between the stochastic coupled system and multi-scale system, an averaging principle in which the multi-scale system with singular coefficients was established, thereby the synchronization of stochastic coupled systems was obtained. Then the convergence rate of synchronization was also obtained. In addition, to prove fluctuation of multi-scale system, the martingale approach method was used. And then the fluctuation of the stochastic coupled systems was got. In the end, we give an example to illustrate the utility of our results.

Keywords: synchronization; fluctuation; multi-scale system; averaging principle; stochastic coupled system
Mathematics Subject Classification: 34F05, 60H10

1. Introduction

Caraballo and Kloeden [1] considered the following two stochastic differential equations (SDEs) in $\mathbb{R}^{2d}$

$$\begin{cases} dX_t = f(X_t)dt + \alpha dW_t^1, \\ dY_t = g(Y_t)dt + \beta dW_t^2, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}^d$ are constant vectors with no components equal to zero, W_t^1, W_t^2 are independent two-sided scalar Wiener processes and the continuously differentiable functions f, g satisfy the one-sided dissipative Lipschitz conditions. And then the corresponding coupled system is

$$\begin{cases} dX_t^\nu = f(X_t^\nu)dt + \nu(Y_t^\nu - X_t^\nu)dt + \alpha dW_t^1, \\ dY_t^\nu = g(Y_t^\nu)dt + \nu(X_t^\nu - Y_t^\nu)dt + \beta dW_t^2, \end{cases}$$

with a coupling coefficient $\nu > 0$. They proved that the coupled system has a unique stationary solution (X_t^ν, Y_t^ν), which is pathwise globally asymptotically stable. Moreover,

$$(X_t^\nu, Y_t^\nu) \to (Z_t, Z_t) \text{ as } \nu \to \infty,$$

where Z_t is the unique stationary solution of the averaged system

$$dZ_t = \frac{1}{2}[f(Z_t) + g(Z_t)]dt + \frac{1}{2}\alpha dW_t^1 + \frac{1}{2}\beta dW_t^2.$$

This phenomenon is that the unique asymptotically stationary solution of the coupled system converges to the unique asymptotically stationary solution of the averaged system, which also called synchronization.

Synchronization is motivated by a wide range of applications in physics, control and biology, see e.g., [2–4]. The synchronization of deterministic coupled dynamical systems has been presented in both autonomous systems [5, 6] and nonautonomous systems [7]. Caraballo and Kloeden [1] and Al-Azzawi et al. [8] investigated the effect of additive noise on the synchronization of coupled dissipative systems through the theory of stochastic dynamical systems. Besides, a almost everywhere convergence rate of convergence is established in [8]. Liu and Zhao did research on synchronization of coupled systems with additive fractional Brownian motion [9] and normal deviation of synchronization of stochastic coupled systems [10]. It is worth mentioning that all the above problems are studied from the perspective of dynamic systems.

To be more precise, in this paper we consider the following system.

$$\begin{cases} dX_t = f(X_t)dt + \alpha dW_t, X_0 = x_0, \\ dY_t = g(Y_t)dt + \beta dW_t, Y_0 = y_0, \end{cases}$$

where $\alpha,\beta \in \mathbb{R}^{d\times n}$ are constant matrices, W_t is a two-sided $\mathbb{R}^n$ valued Wiener process and the continuously differentiable functions f, g satisfy some assumption. And then the corresponding coupled system is

$$\begin{cases} dX_t^\nu = f(X_t^\nu)dt + \nu(Y_t^\nu - X_t^\nu)dt + \alpha dW_t, X_0 = x_0, \\ dY_t^\nu = g(Y_t^\nu)dt + \nu(X_t^\nu - Y_t^\nu)dt + \beta dW_t, Y_0 = y_0, \end{cases} \tag{1.1}$$

with a coupling coefficient $\nu > 0$. We will proved that

$$E|X_t^\nu - Z_t|^4 + E|Y_t^\nu - Z_t|^4 \leqslant \frac{C}{\nu},$$

where Z_t is the unique solution of the averaged system

$$dZ_t = \frac{1}{2}[f(Z_t) + g(Z_t)]dt + \frac{1}{2}(\alpha + \beta)dW_t, Z_0 = \frac{1}{2}(x_0 + y_0). \tag{1.2}$$

This result can be viewed as a version of the law of large numbers. The central limit theorem corresponds to the law of large numbers, so that the following problem is to prove the central limit theorem for the coupled system.

From the above, to the best knowledge of the authors, the existing literature about synchronization only shows the results of synchronization and the corresponding convergence rate, leaving the central limit theorem of synchronization unsolved. Therefore, this paper mainly introduces the central limit theorem of synchronized system. We show the normalized difference $\nu^{\frac{1}{4}}(X_t^\nu - Z_t)$ converges weakly to Z_t^∞ as ν tends to infinity, where $Z^\infty(t)$ is the unique solution of the SDE

$$dZ_t^\infty = \frac{1}{2}[D_x f(Z_t) + D_x g(Z_t)]Z_t^\infty dt, Z_0^\infty = 0.$$

Comparing with the synchronization conclusions in previous articles, these results provide a better approximation of the limit behavior of the synchronized system.

In order to solve these problems, we mainly transforms the coupled system (1.1) to a multi-scale system and then discusses the synchronization under the framework of the averaging principle of the multi-scale system. We can construct some equivalence relations and convert the synchronized system (1.1) into the multi-scale system, as shown below.

Substituting $\hat{X}_t^\nu = X_t^\nu$ and $\hat{Y}_t^\nu = X_t^\nu - Y_t^\nu$ into the SDEs (1.1), and then

$$\begin{cases} d\hat{X}_t^\nu = f(\hat{X}_t^\nu)dt - \nu\hat{Y}_t^\nu + \alpha dW_t, \hat{X}_0^\nu = x_0, \\ d\hat{Y}_t^\nu = \left(f(\hat{X}_t^\nu) - g\left(\hat{X}_t^\nu - \hat{Y}_t^\nu\right)\right)dt - 2\nu\hat{Y}_t^\nu dt + (\alpha - \beta)dW_t, \hat{Y}_0^\nu = x_0 - y_0. \end{cases}$$

Let $\frac{1}{\nu} = \epsilon$, $\tilde{X}_t^\epsilon = \hat{X}_t^\nu$ and $\tilde{Y}_t^\epsilon = \sqrt{\nu}\hat{Y}_t^\nu$,

$$\begin{cases} d\tilde{X}_t^\epsilon = f(\tilde{X}_t^\epsilon)dt - \frac{1}{\sqrt{\varepsilon}}\tilde{Y}_t^\epsilon dt + \alpha dW_t, \tilde{X}_0^\epsilon = x_0, \\ d\tilde{Y}_t^\epsilon = \frac{1}{\sqrt{\epsilon}}[f(\tilde{X}_t^\epsilon) - g(\tilde{X}_t^\epsilon - \sqrt{\epsilon}\tilde{Y}_t^\epsilon)]dt - \frac{2}{\epsilon}\tilde{Y}_t^\epsilon dt + \frac{1}{\sqrt{\epsilon}}(\alpha - \beta)dW_t, \tilde{Y}_0^\epsilon = \frac{1}{\sqrt{\epsilon}}(x_0 - y_0). \end{cases} \tag{1.3}$$

Thus, to achieve the synchronization and fluctuation of the coupled system (1.1), one needs to verify when ϵ tends to zero, X_t^ϵ converges in four square sense to Z_t, and to verify when ϵ tends to zero, $\frac{1}{\epsilon^{\frac{1}{4}}}(\tilde{X}_t^\epsilon - Z_t)$ converge weakly to a SDE

$$dZ_t^0 = \frac{1}{2}[D_x f(Z_t) + D_x g(Z_t)]Z_t^0 dt, Z_0^0 = 0. \tag{1.4}$$

Similarly, the synchronization and fluctuation result of Y_t^ν is obtained only by $\frac{1}{\nu} = \epsilon$, $\tilde{Y}_t^\epsilon = \hat{Y}_t^\nu$ and $\tilde{X}_t^\epsilon = \sqrt{\nu}\hat{X}_t^\nu$.

The theory of averaging principle which can be regarded as the law of large numbers has been intensively studied in both the deterministic $\alpha = \beta = 0$, see e.g., [11,12] and the references therein. For the fluctuation of multi-scale system with singular coefficients, refer to [9, 13, 14]. Note that we can not directly apply the arguments about the averaging principle that have been presented in the previous literature. The key reason is that the relation between singular parameters of fast slow system is not satisfied in the above literature. When $\alpha_\epsilon = \sqrt{\epsilon}$ and $\gamma_\epsilon = \sqrt{\epsilon}$, the $\lim_{\epsilon\to 0}\frac{\alpha_\epsilon}{\gamma_\epsilon} = 1 \neq 0$. So that we cannot solve such problems by constructing proper Poisson's equation.

We will make some assumptions.

Assumption 1.1. *(Lipschitz condition) For all x, y, there exists a constant $L > 0$ such that*

$$|f(x) - f(y)|^2 + |g(x) - g(y)|^2 \leqslant L|x - y|^2.$$

Assumption 1.2. *(Linear growth condition) For all x, there exists a constant $K > 0$ such that*

$$|f(x)|^2 + |g(x)|^2 \leqslant K(1 + |x|^2).$$

Throughout this paper, the capital letter C denotes a constant (independent of ϵ) whose value may change from line to line.

A brief outline of the paper is as follows. Section 2 contains proofs of results related to synchronization of coupled system (1.1) as the coupled coefficient ν tends to infinity, including supporting lemma. Section 3 introduces the central limit theorem of synchronized system. Moreover, we give an example to illustrate the utility of our results in Section 4 and a conclusion of this paper in Section 5.

2. Synchronization of stochastic coupled system

In this section, we will prove that the unique solution to coupled system (1.1) converges in an L^4 to the unique solution of averaged system (1.2). Moreover, the convergence rate of synchronization is obtained respectively.

Theorem 2.1. *Let $\tilde{X}_t^\epsilon$ and Z_t be the unique solutions of (1.3) and (1.2) respectively. If Assumptions 1.1 and 1.2 are satisfied, then*

$$E|\tilde{X}_t^\epsilon - Z_t|^4 \leqslant C\epsilon.$$

Lemma 2.2. *Let (X_t^ν, Y_t^ν) and Z_t be the unique solutions of (1.1) and (1.2) respectively. If Assumptions 1.1 and 1.2 are satisfied, then*

$$E|X_t^\nu - Z_t|^4 + E|Y_t^\nu - Z_t|^4 \leqslant \frac{C}{\nu}.$$

Before discussing the synchronization of the stochastic coupled system in detail, we give some conclusions which are used in the next proof.

Lemma 2.3. *Let $\tilde{Y}_t^\epsilon$ be the unique solutions of (1.3). Assume that Assumptions 1.1 and 1.2 hold, there exists a constant $C > 0$, such that for any $t \in [0, T]$,*

$$E|\tilde{Y}_t^\epsilon|^4 \leqslant \frac{C}{\epsilon}, E|\tilde{X}_t^\epsilon|^4 \leqslant C.$$

Proof. By (1.3), a simple computation shows that

$$\begin{aligned}
\tilde{Y}_t^\epsilon \quad = \quad & \frac{1}{\sqrt{\epsilon}} e^{-\frac{2}{\epsilon}t}(x_0 - y_0) + \frac{1}{\sqrt{\epsilon}} \int_0^t e^{-\frac{2}{\epsilon}(t-s)} \left(f(\tilde{X}_s^\epsilon) - g(\tilde{X}_s^\epsilon - \sqrt{\epsilon}\tilde{Y}_s^\epsilon) \right) ds \\
& + \frac{1}{\sqrt{\epsilon}} \int_0^t e^{-\frac{2}{\epsilon}(t-s)} (\alpha - \beta) dW_s.
\end{aligned}$$

One gets

$$\begin{aligned}
\tilde{X}_t^\epsilon &= x_0+\int_0^t f(\tilde{X}_s^\epsilon)ds-\frac{1}{\sqrt{\epsilon}}\int_0^t \tilde{Y}_s^\epsilon ds+\int_0^t \alpha dW_t \\
&= x_0+\int_0^t f(\tilde{X}_s^\epsilon)ds-\frac{1}{\epsilon}\int_0^t e^{-\frac{2}{\epsilon}s}(x_0-y_0)ds+\int_0^t \alpha dW_t-\frac{1}{\epsilon}\int_0^t ds\int_0^s e^{-\frac{2}{\epsilon}(s-r)}(\alpha-\beta)dW_r \\
&\quad -\frac{1}{\epsilon}\int_0^t ds\int_0^s e^{-\frac{2}{\epsilon}(s-r)}\left(f(\tilde{X}_r^\epsilon)-g(\tilde{X}_r^\epsilon-\sqrt{\epsilon}\tilde{Y}_r^\epsilon)\right)dr \\
&= x_0+\int_0^t f(\tilde{X}_s^\epsilon)ds-\frac{1}{\epsilon}(x_0-y_0)\left(\frac{\epsilon}{2}-\frac{\epsilon}{2}e^{-\frac{2}{\epsilon}t}\right)+\alpha W_t \\
&\quad -\frac{1}{\epsilon}\int_0^t dr\int_r^t e^{-\frac{2}{\epsilon}(s-r)}\left(f(\tilde{X}_r^\epsilon)-g(\tilde{X}_r^\epsilon-\sqrt{\epsilon}\tilde{Y}_r^\epsilon)\right)ds-\frac{1}{\epsilon}\int_0^t dW_r\int_r^t e^{-\frac{2}{\epsilon}(s-r)}(\alpha-\beta)ds \\
&= x_0-\frac{x_0-y_0}{2}+\frac{\sqrt{\epsilon}}{2}\tilde{Y}_t^\epsilon+\int_0^t f(\tilde{X}_s^\epsilon))ds-\frac{1}{2}\int_0^t\left(f(\tilde{X}_s^\epsilon)-g(\tilde{X}_s^\epsilon-\sqrt{\epsilon}\tilde{Y}_s^\epsilon)\right)ds \\
&\quad +\frac{1}{2}\int_0^t(\alpha+\beta)dW_s \\
&= \frac{x_0+y_0}{2}+\frac{\sqrt{\epsilon}}{2}\tilde{Y}_t^\epsilon+\frac{1}{2}\int_0^t\left(f(\tilde{X}_s^\epsilon)+g(\tilde{X}_s^\epsilon-\sqrt{\epsilon}\tilde{Y}_s^\epsilon)\right)ds+\frac{1}{2}\int_0^t(\alpha+\beta)dW_s. \qquad (2.1)
\end{aligned}$$

And then,

$$\begin{aligned}
E|\tilde{X}_t^\epsilon|^4+E|\tilde{Y}_t^\epsilon|^4 &\leqslant C\frac{(x_0+y_0)^4}{16}+C\epsilon^2E|\tilde{Y}_t^\epsilon|^4+CE\left|\int_0^t\left(f(\tilde{X}_s^\epsilon)+g(\tilde{X}_s^\epsilon-\sqrt{\epsilon}\tilde{Y}_s^\epsilon)\right)ds\right|^4 \\
&\quad +C\frac{1}{\epsilon^2}e^{-\frac{16}{\epsilon}t}(x_0-y_0)^4+C\frac{1}{\epsilon^2}E\left|\int_0^t e^{-\frac{2}{\epsilon}(t-s)}\left(f(\tilde{X}_s^\epsilon)-g(\tilde{X}_s^\epsilon-\sqrt{\epsilon}\tilde{Y}_s^\epsilon)\right)ds\right|^4 \\
&\quad +\frac{1}{\epsilon^2}\int_0^t e^{-\frac{16}{\epsilon}(t-s)}(\alpha-\beta)^4ds+Ct\int_0^t(\alpha+\beta)^2ds \\
&\leqslant C+\frac{C}{\epsilon}+Ct^2E\left|\int_0^t\left(f(\tilde{X}_s^\epsilon)+g(\tilde{X}_s^\epsilon-\sqrt{\epsilon}\tilde{Y}_s^\epsilon)\right)^2ds\right|^2 \\
&\quad +C\frac{1}{\epsilon^2}\left|\int_0^t e^{-\frac{4}{\epsilon}(t-s)}ds\right|^2E\left|\int_0^t\left(f(\tilde{X}_s^\epsilon)-g(\tilde{X}_s^\epsilon-\sqrt{\epsilon}\tilde{Y}_s^\epsilon)\right)^2ds\right|^2 \\
&\leqslant \frac{C}{\epsilon}+Ct\int_0^t\left(E|\tilde{X}_t^\epsilon|^4+\epsilon^2E|\tilde{Y}_t^\epsilon|^4\right)ds \\
&\leqslant \frac{C}{\epsilon}+Ct\int_0^t\left(E|\tilde{X}_t^\epsilon|^4+E|\tilde{Y}_t^\epsilon|^4\right)ds.
\end{aligned}$$

The Grownall lemma yields that

$$E|\tilde{X}_t^\epsilon|^4+E|\tilde{Y}_t^\epsilon|^4\leqslant\left(x_0+\frac{1}{\sqrt{\epsilon}}(x_0-y_0)\right)e^{ct}-\frac{C}{\epsilon}(1-e^{ct}).$$

It then follows that

$$E|\tilde{Y}_t^\epsilon|^4\leqslant E|\tilde{X}_t^\epsilon|^4+E|\tilde{Y}_t^\epsilon|^4\leqslant\frac{C}{\epsilon}.$$

Since the estimate of $E|X_t^\epsilon|^4$ is quite similar to that of $E|Y_t^\epsilon|^4$. Substitute the above inequality obtain

$$E|X_t^\epsilon|^4 \leqslant C.$$

The proof is completed. □

With the help of the preceding lemma, Theorem 2.1 is proved.

Proof of Theorem 2.1. Note from (2.1) that

$$\begin{aligned} E\left|\tilde{X}_t^\epsilon - Z_t\right|^4 &= E\left|\frac{\sqrt{\epsilon}}{2}\tilde{Y}_t^\epsilon + \frac{1}{2}\int_0^t \left(f(\tilde{X}_s^\epsilon) + g(\tilde{X}_s^\epsilon - \sqrt{\epsilon}\tilde{Y}_s^\epsilon)\right) ds - \frac{1}{2}\int_0^t (f(Z_s) + g(Z_s))\, ds\right|^4 \\ &\leqslant C\epsilon^2 E|\tilde{Y}_t^\epsilon|^4 + CE\left|\int_0^t \left(f(\tilde{X}_s^\epsilon) - f(Z_s)\right) ds\right|^4 + CE\left|\int_0^t \left(g(\tilde{X}_s^\epsilon - \sqrt{\epsilon}\tilde{Y}_s^\epsilon) - g(Z_s)\right) ds\right|^4 \\ &\leqslant C\epsilon + ct^{\frac{4}{3}}\int_0^t E\left|\tilde{X}_s^\epsilon - Z_s\right|^4 ds + C\int_0^t E\left|\tilde{X}_s^\epsilon - Z_s\right|^4 + \epsilon^2 E|\tilde{Y}_s^\epsilon|^4 ds \\ &\leqslant C\epsilon + C\int_0^t E\left|\tilde{X}_s^\epsilon - Z_s\right|^4 ds. \end{aligned}$$

Thus

$$E\left|\tilde{X}_t^\epsilon - Z_t\right|^4 \leqslant C\epsilon.$$

The proof is completed. □

Theorem 2.1 implies in particular that synchronization of the stochastic coupled system. In addition, through a simple example in Section 4 will explicitly illustrate that synchronization for SDE is valid.

3. Fluctuation of stochastic coupled system

In this section, we will establish a limit in distribution of the fluctuation of X_t^γ about its typical behavior Z_t. Before discussing the synchronization of stochastic coupled system in detail, we give some conclusions which are used in the next proof.

Lemma 3.1. *The family of process* $\{Z_t^\epsilon, 0 \leqslant t \leqslant T, 0 < \epsilon \leqslant 1\}$ *is weakly compact in* $C([0,T];\mathbb{R}^d)$.

Proof. There exists a convenient criterion for tightness: Kolmogorov's criterion of Remark A.5 in [15]. What we only need to verify is that there exist α, β, $C > 0$ such that $E|Z_{t+h}^\epsilon - Z_t^\epsilon|^\beta \leqslant Ch^{1+\alpha}$ for all $t \in [0,T]$.

By (1.3), a simple computation shows that

$$\begin{aligned} \tilde{Y}_{t+h}^\epsilon - \tilde{Y}_t^\epsilon &= \frac{1}{\sqrt{\epsilon}}e^{-\frac{2}{\epsilon}(t+h)}(x_0 - y_0) + \frac{1}{\sqrt{\epsilon}}\int_0^{t+h} e^{-\frac{2}{\epsilon}(t+h-s)}\left(f(\tilde{X}_s^\epsilon) - g(\tilde{X}_s^\epsilon - \sqrt{\epsilon}\tilde{Y}_s^\epsilon)\right) ds \\ &\quad + \frac{1}{\sqrt{\epsilon}}\int_0^{t+h} e^{-\frac{2}{\epsilon}(t+h-s)}(\alpha - \beta)dW_s - \frac{1}{\sqrt{\epsilon}}\int_0^t e^{-\frac{2}{\epsilon}(t-s)}\left(f(\tilde{X}_s^\epsilon) - g(\tilde{X}_s^\epsilon - \sqrt{\epsilon}\tilde{Y}_s^\epsilon)\right) ds \\ &\quad - \frac{1}{\sqrt{\epsilon}}\int_0^t e^{-\frac{2}{\epsilon}(t-s)}(\alpha - \beta)dW_s - \frac{1}{\sqrt{\epsilon}}e^{-\frac{2}{\epsilon}t}(x_0 - y_0). \end{aligned}$$

Using Hölder's inequality, Jensen's inequality, some elementary inequalities and the linear growth conditions of f and g, one gets

$$\begin{aligned}
\frac{1}{\epsilon}E\left|\sqrt{\epsilon}\tilde{Y}^{\epsilon}_{t+h}-\sqrt{\epsilon}\tilde{Y}^{\epsilon}_{t}\right|^4 \leqslant\ & \frac{C}{\epsilon}\left|e^{-\frac{2}{\epsilon}(t+h)}(x_0-y_0)-e^{-\frac{2}{\epsilon}t}(x_0-y_0)\right|^4 \\
&+\frac{C}{\epsilon}E\left|\int_0^{t+h}e^{-\frac{2}{\epsilon}(t+h-s)}(\alpha-\beta)dW_s-\int_0^t e^{-\frac{2}{\epsilon}(t-s)}(\alpha-\beta)dW_s\right|^4 \\
&+\frac{C}{\epsilon}E\Big|\int_0^{t+h}e^{-\frac{2}{\epsilon}(t+h-s)}\left(f(\tilde{X}^{\epsilon}_s)-g(\tilde{X}^{\epsilon}_s-\sqrt{\epsilon}\tilde{Y}^{\epsilon}_s)\right)ds \\
&-\int_0^t e^{-\frac{2}{\epsilon}(t-s)}\left(f(\tilde{X}^{\epsilon}_s)-g(\tilde{X}^{\epsilon}_s-\sqrt{\epsilon}\tilde{Y}^{\epsilon}_s)\right)ds\Big|^4 \\
\leqslant\ & Ch^4+\frac{C}{\epsilon}E\left|\int_0^{t+h}\left(e^{-\frac{2}{\epsilon}(t+h-s)}-e^{-\frac{2}{\epsilon}(t-s)}\right)(\alpha-\beta)dW_s\right|^4 \\
&+\frac{C}{\epsilon}E\left|\int_t^{t+h}e^{-\frac{2}{\epsilon}(t-s)}(\alpha-\beta)dW_s\right|^4 \\
&+\frac{C}{\epsilon}E\left|\int_0^{t+h}\left(e^{-\frac{2}{\epsilon}(t+h-s)}-e^{-\frac{2}{\epsilon}(t-s)}\right)\left(f(\tilde{X}^{\epsilon}_s)-g(\tilde{X}^{\epsilon}_s-\sqrt{\epsilon}\tilde{Y}^{\epsilon}_s)\right)ds\right|^4 \\
&+\frac{C}{\epsilon}E\left|\int_t^{t+h}e^{-\frac{2}{\epsilon}(t-s)}\left(f(\tilde{X}^{\epsilon}_s)-g(\tilde{X}^{\epsilon}_s-\sqrt{\epsilon}\tilde{Y}^{\epsilon}_s)\right)ds\right|^4 \\
\leqslant\ & Ch^4+\frac{C(t+h)}{\epsilon}E\int_0^{t+h}\left|\left(e^{-\frac{2}{\epsilon}(t+h-s)}-e^{-\frac{2}{\epsilon}(t-s)}\right)(\alpha-\beta)\right|^4 ds \\
&+\frac{Ch}{\epsilon}\int_t^{t+h}E\left|e^{-\frac{2}{\epsilon}(t-s)}(\alpha-\beta)\right|^4 ds \\
&+\frac{C(t+h)^3}{\epsilon}E\int_0^{t+h}\left|\left(e^{-\frac{2}{\epsilon}(t+h-s)}-e^{-\frac{2}{\epsilon}(t-s)}\right)\left(f(\tilde{X}^{\epsilon}_s)-g(\tilde{X}^{\epsilon}_s-\sqrt{\epsilon}\tilde{Y}^{\epsilon}_s)\right)\right|^4 ds \\
&+\frac{Ch^3}{\epsilon}E\int_t^{t+h}e^{-\frac{8}{\epsilon}(t-s)}\left|\left(f(\tilde{X}^{\epsilon}_s)-g(\tilde{X}^{\epsilon}_s-\sqrt{\epsilon}\tilde{Y}^{\epsilon}_s)\right)\right|^4 ds.
\end{aligned}$$

Taking Lemma 2.3 into consideration, then

$$\frac{1}{\epsilon}E\left|\sqrt{\epsilon}\tilde{Y}^{\epsilon}_{t+h}-\sqrt{\epsilon}\tilde{Y}^{\epsilon}_{t}\right|^4\leqslant Ch^2.$$

Using Hölder's inequality, Jensen's inequality, some elementary inequalities and the Lipschitz conditions of f and g, then

$$\begin{aligned}
& E|Z^{\epsilon}_{t+h}-Z^{\epsilon}_t|^4 \\
=\ & \frac{1}{\epsilon}E\left|\frac{\sqrt{\epsilon}}{2}\tilde{Y}^{\epsilon}_{t+h}-\frac{\sqrt{\epsilon}}{2}\tilde{Y}^{\epsilon}_t+\frac{1}{2}\int_t^{t+h}\left(f(\tilde{X}^{\epsilon}_s)+g(\tilde{X}^{\epsilon}_s-\sqrt{\epsilon}\tilde{Y}^{\epsilon}_s)\right)ds-\frac{1}{2}\int_t^{t+h}\left(f(Z_s)+g(Z_s)\right)ds\right|^4 \\
\leqslant\ & \frac{C}{\epsilon}E\left|\frac{\sqrt{\epsilon}}{2}\tilde{Y}^{\epsilon}_{t+h}-\frac{\sqrt{\epsilon}}{2}\tilde{Y}^{\epsilon}_t\right|^4+\frac{Ch}{\epsilon}E\left|\int_t^{t+h}\left(f(\tilde{X}^{\epsilon}_s)-f(Z_s)\right)^2ds\right|^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{Ch}{\epsilon}E\left|\int_t^{t+h}\left(g(\tilde{X}_s^\epsilon-\sqrt{\epsilon}\tilde{Y}_s^\epsilon)-g(Z_s)\right)^2ds\right|^2\\
\leqslant\ & Ch^2+\frac{Ch^2}{\epsilon}E\left|\int_t^{t+h}\left|\tilde{X}_s^\epsilon-Z_s\right|^2ds\right|^2+\frac{Ch^2}{\epsilon}E\left|\int_t^{t+h}\left|\tilde{X}_s^\epsilon-Z_s\right|^2+\epsilon\left|\tilde{Y}_s^\epsilon\right|^2ds\right|^2\\
\leqslant\ & Ch^2+\frac{Ch^3}{\epsilon}\int_t^{t+h}E\left|\tilde{X}_s^\epsilon-Z_s\right|^4ds+\frac{Ch^3}{\epsilon}\int_t^{t+h}\left(E\left|\tilde{X}_s^\epsilon-Z_s\right|^4+\epsilon^2E\left|\tilde{Y}_s^\epsilon\right|^4\right)ds.
\end{aligned}
$$

Taking Theorem 2.1 and Lemma 2.3 into consideration, then

$$E|Z_{t+h}^\epsilon-Z_t^\epsilon|^4\leqslant Ch^2.$$

This estimate guarantees the weak compactness of the family of the processes $\{Z_t^\epsilon, 0\leqslant t\leqslant T, 0<\epsilon\leqslant 1\}$. □

Denote $\lambda_t^\epsilon=\frac{1}{2}\left(f(\tilde{X}_t^\epsilon)+g(\tilde{X}_t^\epsilon)\right)$ and $\lambda_t=\frac{1}{2}\left(f(Z_t)+g(Z_t)\right)$. By Taylor's theorem for λ_t, one can then derive that

$$\lambda_t^\epsilon=\lambda_t+D\lambda_t\left(\tilde{X}_t^\epsilon-Z_t\right)+o(\tilde{X}_t^\epsilon-Z_t).$$

We then have the following decomposition

$$Z_t^\epsilon=\mathrm{I}_t^\epsilon+\mathrm{II}_t^\epsilon+\mathrm{III}_t^\epsilon,$$

where, for $0\leqslant t\leqslant T$,

$$
\begin{aligned}
\mathrm{I}_t^\epsilon=&\frac{1}{\epsilon^{\frac{1}{4}}}\int_0^t(\lambda_s^\epsilon-\lambda_s)\,ds=\int_0^tD_x\lambda_sZ_s^\epsilon ds+\frac{1}{\epsilon^{\frac{1}{4}}}\int_0^to(\tilde{X}_s^\epsilon-Z_s)ds,\\
\mathrm{I}_t^\epsilon=&\frac{1}{2\epsilon^{\frac{1}{4}}}\int_0^t\left(g(\tilde{X}_s^\epsilon-\sqrt{\epsilon}\tilde{Y}_s^\epsilon)-g(\tilde{X}_s^\epsilon)\right)ds,\\
\mathrm{III}_t^\epsilon=&\frac{\epsilon^{\frac{1}{4}}}{2}\tilde{Y}_t^\epsilon\\
=&\frac{1}{2\epsilon^{\frac{1}{4}}}e^{-\frac{2}{\epsilon}t}(x_0-y_0)+\frac{1}{2\epsilon^{\frac{1}{4}}}\int_0^te^{-\frac{2}{\epsilon}(t-s)}\left(f(\tilde{X}_s^\epsilon)-g(\tilde{X}_s^\epsilon-\sqrt{\epsilon}\tilde{Y}_s^\epsilon)\right)ds\\
&+\frac{1}{2\epsilon^{\frac{1}{4}}}\int_0^te^{-\frac{2}{\epsilon}(t-s)}(\alpha-\beta)dW_s.
\end{aligned}
$$

Theorem 3.2. *Let $\tilde{X}_t^\epsilon$ and Z_t be the unique solutions of (1.3) and (1.2) respectively. If Assumptions 1.1 and 1.2 are satisfied, then $Z^\epsilon(t):=\frac{X_t^\epsilon-Z_t}{\epsilon^{\frac{1}{4}}}$ converges weakly to Z_t^0 in the space $C([0,T];\mathbb{R}^d)$, where Z_t^0 is the solution of*

$$dZ_t^0=\frac{1}{2}(D_xf(Z_t)+D_xg(Z_t))Z_t^0dt,\ Z_0^0=0.$$

Proof. We split the proof of theorem into two subsection. To begin, the process $\{Z^\epsilon, 0\leqslant t\leqslant T\}$ is tight in $C([0,T])$ in Lemma 3.1. In view of Prohorov's theorem, we can extract every sequence of such process contains a subsequence converging to a process; Next, we identify the limit via martingale problem.

Let Q^ϵ be the probability measure of $Z^\epsilon(t)$ in $C([0,T];\mathbb{R}^d)$. By the Itô formula to a function $\phi \in C_b^2(\mathbb{R}^d)$ with $Z^\epsilon(t)$, one has

$$\begin{aligned}
\phi(Z^\epsilon(t)) &= \int_0^t D_x\lambda_s Z_s^\epsilon D\phi(Z^\epsilon(s))ds + \frac{1}{\epsilon^{\frac{1}{4}}}\int_0^t o(\tilde{X}_s^\epsilon - Z_s)D\phi(Z^\epsilon(s))ds \\
&+\frac{1}{2\epsilon^{\frac{1}{4}}}\int_0^t e^{-\frac{2}{\epsilon}s}(x_0-y_0)D\phi(Z^\epsilon(s))ds + \frac{1}{2\epsilon^{\frac{1}{4}}}\int_0^t \left(g(\tilde{X}_s^\epsilon - \sqrt{\epsilon}\tilde{Y}_s^\epsilon) - g(\tilde{X}_s^\epsilon)\right)D\phi(Z^\epsilon(s))ds \\
&+\frac{1}{2\epsilon^{\frac{1}{4}}}\int_0^t\int_0^s e^{-\frac{2}{\epsilon}(s-r)}\left(f(\tilde{X}_r^\epsilon) - g(\tilde{X}_r^\epsilon - \sqrt{\epsilon}\tilde{Y}_r^\epsilon)\right)D\phi(Z^\epsilon(s))drds \\
&+\frac{1}{2\epsilon^{\frac{1}{4}}}\int_0^t\int_0^s e^{-\frac{2}{\epsilon}(s-r)}(\alpha-\beta)D\phi(Z^\epsilon(s))dW_r ds \\
&+\frac{1}{8\sqrt{\epsilon}}\int_0^t\int_0^s e^{-\frac{4}{\epsilon}(s-r)}(\alpha-\beta)^2 D_{xx}\phi(Z^\epsilon(s))drds \\
&:= \int_0^t D_x\lambda_s Z_s^\epsilon D\phi(Z^\epsilon(s))ds + R_1(\epsilon,0,t) + R_2(\epsilon,0,t) + R_3(\epsilon,0,t),
\end{aligned}$$

where

$$\begin{aligned}
R_1(\epsilon,0,t) &= \frac{1}{\epsilon^{\frac{1}{4}}}\int_0^t o(\tilde{X}_s^\epsilon - Z_s)D\phi(Z^\epsilon(s))ds + \frac{1}{2\epsilon^{\frac{1}{4}}}\int_0^t e^{-\frac{2}{\epsilon}s}(x_0-y_0)D\phi(Z^\epsilon(s))ds \\
&+\frac{1}{2\epsilon^{\frac{1}{4}}}\int_0^t \left(g(\tilde{X}_s^\epsilon - \sqrt{\epsilon}\tilde{Y}_s^\epsilon) - g(\tilde{X}_s^\epsilon)\right)D\phi(Z^\epsilon(s))ds \\
&+\frac{1}{2\epsilon^{\frac{1}{4}}}\int_0^t\int_0^s e^{-\frac{2}{\epsilon}(s-r)}\left(f(\tilde{X}_r^\epsilon) - g(\tilde{X}_r^\epsilon - \sqrt{\epsilon}\tilde{Y}_r^\epsilon)\right)D\phi(Z^\epsilon(s))drds,
\end{aligned}$$

$$R_2(\epsilon,0,t) = \frac{1}{2\epsilon^{\frac{1}{4}}}\int_0^t\int_0^s e^{-\frac{2}{\epsilon}(s-r)}(\alpha-\beta)D\phi(Z^\epsilon(s))dW_r ds,$$

$$R_3(\epsilon,0,t) = \frac{1}{8\sqrt{\epsilon}}\int_0^t\int_0^s e^{-\frac{4}{\epsilon}(s-r)}(\alpha-\beta)^2 D_{xx}\phi(Z^\epsilon(s))drds.$$

We will consider the above cases separately. Firstly, by the linear growth of f and g, $E|X_s^\epsilon - Z_s|^4 \leqslant C\epsilon$, Lemma 2.3, one gets that

$$\lim_{\epsilon\to 0} E\left|R_1(\epsilon,0,t)\right| = 0. \tag{3.1}$$

Secondly, the stochastic integrals in $R_2(\epsilon,t,\omega)$ are square integrable. The expected value vanishes by Doob's inequality, that is

$$\lim_{\epsilon\to 0} E\left[\int_s^t R_2(\epsilon,0,r)\,dr\middle|\mathscr{F}_s\right] = 0. \tag{3.2}$$

Thirdly,

$$\lim_{\epsilon\to 0} E\left|R_3(\epsilon, 0, t)\right| = 0. \tag{3.3}$$

Above all, combine (3.1)–(3.3), then

$$\lim_{\epsilon\to 0} E\Big[\phi\left(Z_t^\epsilon\right) - \phi(Z_s^\epsilon) - \int_s^t \Big[D_x\lambda_r Z_r^\epsilon D\phi(Z^\epsilon(r))\Big]dr\Big|\mathscr{F}_s\Big] = 0.$$

This concludes the proof of Theorem 3.2. □

Theorem 3.3. *Let (X_t^ν, Y_t^ν) and Z_t be the unique solutions of (1.1) and (1.2) respectively. If Assumptions 1.1 and 1.2 are satisfied, then $Z^\nu(t) := \nu^{\frac{1}{4}}\left(X_t^\nu - Z_t\right)$ converges weakly to Z_t^∞ in the space $C([0,T];\mathbb{R}^d)$, where Z_t^∞ is the solution of*

$$dZ_t^\infty = \frac{1}{2}(D_x f(Z_t) + D_x g(Z_t))Z_t^\infty dt,\ Z_0^\infty = 0.$$

Proof. Because of $\nu = \frac{1}{\epsilon}$,

$$Z^\nu(t) := \nu^{\frac{1}{4}}\left(X_t^\nu - Z_t\right) = \frac{X_t^\epsilon - Z_t}{\epsilon^{\frac{1}{4}}}.$$

The result of this theorem follows from Theorem 3.2. □

Theorem 3.3 implies in particular that fluctuation of the stochastic coupled system.

4. Numerical simulations

Example 4.1. *Consider the following equation*

$$\begin{cases} dX_t^\epsilon &= 2X_t^\epsilon dt - \frac{1}{\sqrt{\epsilon}}Y_t^\epsilon dt + 2dW_t,\ X_0^\epsilon = x_0,\\ dY_t^\epsilon &= \frac{1}{\sqrt{\epsilon}}\left(X_t^\epsilon + \sqrt{\epsilon}Y_t^\epsilon\right)dt - \frac{2}{\epsilon}Y_t^\epsilon dt + \frac{1}{\sqrt{\epsilon}}dW_t,\ Y_0^\epsilon = y_0, \end{cases}$$

where W_t is a two-sided Wiener process.

The corresponding averaged SDE is

$$dZ_t = \frac{3}{2}Z_t dt + \frac{3}{2}dW_t,\ Z_0 = \frac{1}{2}(x_0 + y_0).$$

Let's first illustrate the averaging principle through images.

Figure 1 are slow variable X_t^ϵ and averaged variable Z_t between the same initial value $X_0^\epsilon = 1, Z_0 = 1$ and the different ϵ $((a)\epsilon = 0.1, (b)\epsilon = 0.01)$, respectively. This means that the smaller ϵ is, the closer Z_t is to X_t^ϵ.

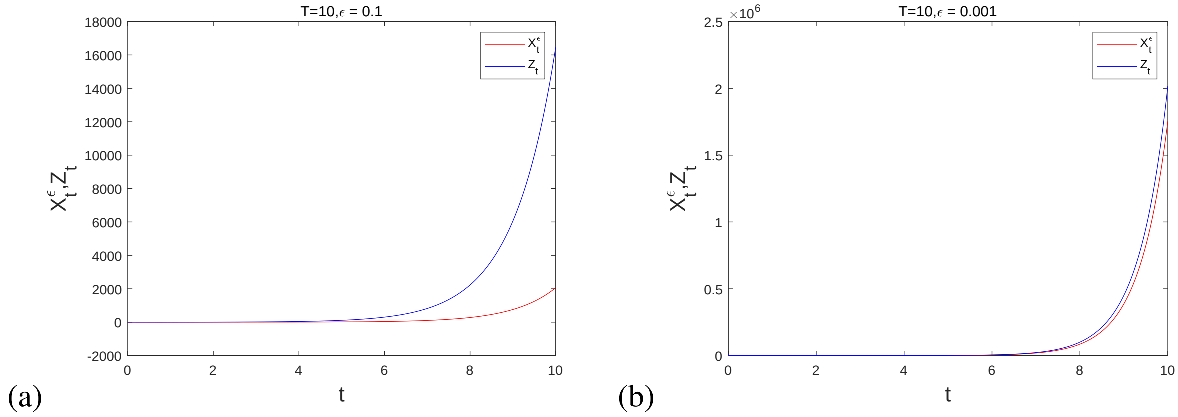

Figure 1. Phase diagram

5. Conclusions

This paper investigates the synchronization and fluctuation of stochastic coupled system with additive Gaussian noise. Through a transformation, such stochastic coupled system is converted into stochastic slow-fast system. The synchronization of the stochastic coupled system is then viewed as the convergence of the corresponding stochastic slow-fast system to its averaged system. The fluctuation considered in this paper is a central limit-type result of the fluctuation between the coupled system and its averaged system. Moreover, we derives the fluctuation of synchronization for the stochastic coupled system by verifying the conditions proposed for the stochastic fast-slow system. In the further research, we will try using the multiscale analysis to consider the synchronization, bifurcation (codimension-1 and codimension-2) which based on the the existence and uniqueness [16], stability [17, 18], bifurcation [19] of the stochastic reaction-diffusion system. In addition, the numerical solution method in [20] may provides theoretical support for our numerical simulation.

Acknowledgments

This work was supported by the Scientific Research Foundation of Anhui Provincial Education Department (No.KJ2020A0483) and the PhD Research Startup Fund for Anhui Jianzhu University (NO.2019QDZ25,2022QDZ19).

Conflict of interest

The authors declare that there is no conflict of interest.

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