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Research article

New fixed point results in controlled metric type spaces based on new contractive conditions

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Abstract: In the present work, we will establish and prove some fixed point theorems for mappings that satisfy a set of conditions in controlled metric type spaces introduced by Mlaiki et al. [N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled metric type spaces and the related contraction principle. Mathematics 2018, 6, 194]. Our technique in constructing our new contraction conditions is to insert the control function $\theta(u, l)$ that appears on the right hand side of the triangular inequality of the definition of the controlled metric spaces in the right hand side of our proposed contraction conditions. Our results enrich the field of fixed point theory with novel findings that generalize many findings found in the literature. We provide an example to show the usefulness of our results. Also, we present an application to our results to show their significance.

Keywords: fixed point; controlled metric space of type (γ, β); extended b-metric space
Mathematics Subject Classification: 37C25, 47H10, 54H25

1. Introduction

The fixed point (FP) theory technique is widely used by scientists to prove the existence of solutions to problems in science involving integral equations or differential equations. So, the appeal of fixed point theory to a large number of scientists is understandable. After Banach [1] launched and proved Banach's contraction theorem, many mathematicians extended this well-known theorem into more general forms either by enhancing Banach's contraction into more general forms or by extending metric space (MS) into new ones, such as cone MS, G-MS, partial MS and so on.

One of the important generalizations of MS is the idea of b-MS introduced by Baktain [2] and Czerwik [3]. Some authors have obtained many FP theorems in b-MS; for some results see [4–8].

Abdeljawad et al. [9] used the idea of partial b-MS to enhance some known FP results. Shatanawi et al. [10] made use of ordered relations to present a new type of Banach's contraction theorem.

Rasham et al. [11] established a generalization of Banach's contraction theorem on fuzzy metric spaces. Also, Gupta at el. [12–14] initiated several fixed point results in the setting of fuzzy metric spaces. Gamal et al. [15] took the advantage of weakly compatible maps to present new fixed point findings via various contractions in multiplicative metric spaces and to examine some applications. Meanwhile, other authors introduced different types of contraction conditions, and to examine some applications in their obtained results, see for example [16, 17].

In the last few years, Kamran et al. [18] presented a good idea to extend the concept of b-MS in a clever way based on a control function with domain $[1, +\infty)$ and named their concept "extended b-metric spaces (EbMS)". Recently, Mlaiki et al. [19] extended the idea of b-MS to a new idea, which they named "controlled metric type space (CMTS)" by inserting a control function θ in the triangular inequality of the definition of the metric space in a luminous way. Also, Mlaiki et al. [19] provided an example showing that the concept of a CMTS is not an EbMS. For more results in extended b-metric spaces and controlled metric spaces, see [20–23].

From now on, F stands for a non-empty set.

Definition 1.1. *[2, 3] For $b \geq 1$, the function $\nu : F \times F \to [0, \infty)$ is called a b-metric if $\forall\ v, l, s \in F$, we have*

(1) $\nu(v, l) = 0 \iff l = v$,

(2) $\nu(v, l) = \nu(l, v)$,

(3) $\nu(v, l) \leq b[\nu(v, s) + \nu(s, l)]$.

The pair (F, ν) is called a b-MS.

The above concept has been generalized by two different ways. The first way was given by Kamran et al. [18] as follows:

Definition 1.2. *[18] Consider the function $\theta : F \times F \to [1, \infty)$, and the function $\nu : F \times F \to [0, \infty)$ is called an extended b-metric if $\forall\ v, l, s \in F$, we have*

(1) $\nu(v, l) = 0 \iff l = v$,

(2) $\nu(v, l) = \nu(l, v)$,

(3) $\nu(v, l) \leq \theta(v, l)[\nu(v, s) + \nu(s, l)]$.

The pair (F, ν) is referred to as an EbMS.

For some examples on EbMS, see [6, 18].

The second way for generalizing the b-MS was given by Mlaiki et al. [19] as follows:

Definition 1.3. *[19] Consider the function $\theta : F \times F \to [1, \infty)$, and the function $\nu : F \times F \to [0, \infty)$ is called a controlled metric type if $\forall\ v, l, s \in F$, we have*

(1) $\nu(v, l) = 0 \iff l = v$,

(2) $\nu(v, l) = \nu(l, v)$,

(3) $\nu(v, l) \leq \theta(v, s)\nu(v, s) + \theta(s, l)\nu(s, l)$.

The pair (F, ν) *is called a CMTS.*

Mlaiki et al. [19] introduced the following notable example to show the big difference between the EbMS and the CMTS.

Example 1.1. *Let* $F = \{1, 2, 3, \ldots\}$. *Define* $\theta : F \times F \to [1, \infty)$ *by*

$$\theta(v, l) = \begin{cases} v, & \text{if } v \text{ is even and } l \text{ is odd} \\ l, & \text{if } v \text{ is odd and } l \text{ is even} \\ 1, & \text{otherwise.} \end{cases}$$

Also, define $\nu : F \times F \to [0, +\infty)$ *via*

$$\nu(v, l) = \begin{cases} v, & \text{if } v \text{ is even and } l \text{ is odd} \\ l, & \text{if } v \text{ is odd and the } l \text{ is even} \\ 1, & \text{otherwise.} \end{cases}$$

Then (F, ν) *is a CMTS which is not an EbMS.*

The aim of the present work is to take advantage of the notion of CMTS to present new contractive conditions and making use of our new contractions to formulate new results related to FP of a mapping that satisfies a set of conditions.

2. Main results

From now on, CCMTS is a complete controlled metric type space, and CbMS is a complete b-metric space with constant b.

Theorem 2.1. *On CCMTS* (F, ν), *assume there exists* $r \in (0, 1]$ *such that* $Q : F \to F$ *satisfies*

$$\nu(Ql, Qv) \leq r\theta(l, v)\nu(l, v), \tag{2.1}$$

for all $l, v \in F$. *Assume*

$$\limsup_{i\to\infty} \theta(l_{i+1}, l_m)\theta(l_{i+1}, l_{i+2}) \text{ exists and less than } \frac{1}{r}, \tag{2.2}$$

where $l_i = Q^i l_0$ *for* $l_0 \in F$. *Also, suppose that for any* $v, l \in F$, *we have*

$$\limsup_{i\to+\infty} \theta(v, Q^i l) \text{ and } \limsup_{i\to+\infty} \theta(Q^i l, v) \text{ exist and are finite.}$$

Then, Q has a FP in F.

Proof. Let $l_0 \in Q$. Then, we construct a sequence (l_t) in Q by putting $l_t = Q^t l_0$. For $t \in \mathbf{N}$, Condition (2.1) gives

$$\nu(l_t, l_{t+1}) = \nu(Ql_{t-1}, Ql_t)$$

$$
\begin{aligned}
&\leq r\theta(l_{t-1}, l_t)\nu(l_{t-1}, l_t)\\
&\leq r^2\theta(l_{t-1}, l_t)\theta(l_{t-2}, l_{t-1})\nu(l_{t-2}, l_{t-1})\\
&\vdots\\
&\leq r^t\theta(l_{t-1}, l_t)\theta(l_{t-2}, l_{t-1})\ldots\theta(l_0, l_1)\nu(l_0, l_1)\\
&= r^t\prod_{j=1}^{t}\theta(l_{j-1}, l_j)\nu(l_0, l_1). \qquad (2.3)
\end{aligned}
$$

For $t, m \in \mathbf{N}$ with $m > t$, we choose $k \in \mathbf{N}$ with $m = t + k$. The triangular inequality of the definition ν produces

$$
\begin{aligned}
\nu(l_t, l_{t+k}) &\leq \theta(l_t, l_{t+1})\nu(l_t, l_{t+1}) + \theta(l_{t+1}, l_{t+k})\nu(l_{t+1}, l_{t+k})\\
&\leq \theta(l_t, l_{t+1})\nu(l_t, l_{t+1}) + \theta(l_{t+1}, l_{t+k})\theta(l_{t+1}, l_{t+2})\nu(l_{t+1}, l_{t+2})\\
&+ \theta(l_{t+1}, l_{t+k})\theta(l_{t+2}, l_{t+k})\nu(l_{t+2}, l_{t+k})\\
&\leq \theta(l_t, l_{t+1})\nu(l_t, l_{t+1}) + \theta(l_{t+1}, l_{t+k})\theta(l_{t+1}, l_{t+2})\nu(l_{t+1}, l_{t+2})\\
&+ \theta(l_{t+1}, l_{t+k})\theta(l_{t+2}, l_{t+3})\theta(l_{t+2}, l_{t+3})\nu(l_{t+2}, l_{t+3})\\
&+ \theta(l_{t+1}, l_{t+k})\theta(l_{t+2}, l_{t+k})\theta(l_{t+3}, l_{t+k})\nu(l_{t+3}, l_{t+k})\\
&\leq\\
&\vdots\\
&\leq \theta(l_t, l_{t+1})\nu(l_t, l_{t+1}) + \theta(l_{t+1}, l_{t+k})\theta(l_{t+1}, l_{t+2})\nu(l_{t+1}, l_{t+2})\\
&+ \theta(l_{t+1}, l_{t+k})\theta(l_{t+2}, l_{t+k})\theta(l_{t+2}, l_{t+3})\nu(l_{t+2}, l_{t+3})\\
&+ \theta(l_{t+1}, l_{t+k})\theta(l_{t+2}, l_{t+k})\theta(l_{t+3}, l_{t+k})\theta(l_{t+3}, l_{t+4})\nu(l_{t+3}, l_{t+4})\\
&+\\
&\vdots\\
&+ \theta(l_{t+1}, l_{t+k})\theta(l_{t+2}, l_{t+k})\ldots\theta(l_{t+k-2}, l_{t+k})\theta(l_{t+k-2}, l_{t+k-1})\nu(l_{t+k-2}, l_{t+k-1})\\
&+ \theta(l_{t+1}, l_{t+k})\theta(l_{t+2}, l_{t+k})\ldots\theta(l_{t+k-2}, l_{t+k})\theta(l_{t+k-1}, l_{t+k})\nu(l_{t+k-1}, l_{t+k}).
\end{aligned}
$$

In light of the values of $\theta(l_t, l_{t+k}) \geq 1$ and $\theta(l_{t+k-1}, l_{t+k}) \geq 1$, the above inequalities imply

$$
\begin{aligned}
&\nu(l_t, l_{t+k}) \leq \theta(l_t, l_{t+k})\theta(l_t, l_{t+1})\nu(l_t, l_{t+1})\\
&+\theta(l_t, l_{t+k})\theta(l_{t+1}, l_{t+k})\theta(l_{t+1}, l_{t+2})\nu(l_{t+1}, l_{t+2})\\
&+\theta(l_t, l_{t+k})\theta(l_{t+1}, l_{t+k})\theta(l_{t+2}, l_{t+k})\theta(l_{t+2}, l_{t+3})\nu(l_{t+2}, l_{t+3})\\
&+\theta(l_t, l_{t+k})\theta(l_{t+1}, l_{t+k})\theta(l_{t+2}, l_{t+k})\theta(l_{t+3}, l_{t+k})\theta(l_{t+3}, l_{t+4})\nu(l_{t+3}, l_{t+4})\\
&+\\
&\vdots\\
&+\theta(l_t, l_{t+k})\theta(l_{t+1}, l_{t+k})\theta(l_{t+2}, l_{t+k})\ldots\theta(l_{t+k-2}, l_{t+k})\theta(l_{t+k-2}, l_{t+k-1})\nu(l_{t+k-2}, l_{t+k-1})\\
&+\theta(l_t, l_{t+k})\theta(l_{t+1}, l_{t+k})\theta(l_{t+2}, l_{t+k})\ldots\theta(l_{t+k-2}, l_{t+k})\theta(l_{t+k-1}, l_{t+k})\theta(l_{t+k-1}, l_{t+k})\nu(l_{t+k-1}, l_{t+k})\\
&= \sum_{j=t}^{t+k-1}\prod_{i=t}^{j}\theta(l_i, l_{t+k})\theta(l_j, l_{j+1})\nu(l_j, l_{j+1}). \qquad (2.4)
\end{aligned}
$$

Taking advantage of inequalities (2.3) and (2.4) yields

$$\nu(l_t, l_m) \leq \sum_{j=t}^{t+k-1} \prod_{i=t}^{j} \theta(l_i, l_{t+k})\theta(l_j, l_{j+1}) r^j \prod_{y=1}^{j} \theta(l_{y-1}, l_y)\nu(l_0, l_1). \tag{2.5}$$

Define

$$\prod_{i=t}^{j} \theta(l_i, l_{t+k})\theta(l_j, l_{j+1}) r^j \prod_{y=1}^{j} \theta(l_{y-1}, l_y)\nu(l_0, l_1) := A_j. \tag{2.6}$$

Then,

$$\lim_{j\to+\infty} \frac{A_{j+1}}{A_j} = \lim_{j\to+\infty} \theta(l_{j+1}, l_{t+k})\theta(l_j, l_{j+1}) r < 1.$$

As $t \to +\infty$, the ratio test implies that

$$S_t = \sum_{j=t}^{+\infty} \prod_{i=t}^{j} \theta(l_i, l_{t+k})\theta(l_j, l_{j+1}) r^j \prod_{y=1}^{j+1} \theta(l_{y-1}, l_y)\nu(l_0, l_1) \to S = \sum_{j=1}^{+\infty} \prod_{i=t}^{j} \theta(l_i, l_{t+k})\theta(l_j, l_{j+1}) r^j \prod_{y=1}^{j+1} \theta(l_{y-1}, l_y)\nu(l_0, l_1).$$

Inequality (2.5) implies that

$$\lim_{t,m\to+\infty} \nu(l_t, l_m) = 0,$$

which means that the sequence (l_t) is Cauchy in (F, ν). As a result of the completeness of (F, ν), $\exists\ l' \in F$ such that

$$\lim_{t\to\infty} \nu(l_t, l') = 0. \tag{2.7}$$

Now, the triangular inequality and (2.1) yield

$$\begin{aligned} \nu(l', Ql') &\leq \theta(l', l_{t+1})\nu(l', l_{t+1}) + \theta(l_{t+1}, Ql')\nu(l_{t+1}, Ql') \\ &\leq \theta(l', l_{t+1})\nu(l', l_{t+1}) + r\theta(l_{t+1}, Ql')\theta(l_t, l')\nu(l_t, l'). \end{aligned} \tag{2.8}$$

Permitting $t \to +\infty$ and keeping in our mind that

$$\limsup_{t\to+\infty} \theta(l', l_{t+1}) \text{ and } \limsup_{t\to+\infty} \theta(t_{t+1}, Ql') \text{ exist and are finite,}$$

(2.8) implies $\nu(l', Ql') = 0$, and hence $Ql' = l'$. □

In Theorem 2.1, we can remove the conditions

$$\lim_{t\to+\infty} \theta(v, Q^t l) \text{ and } \lim_{t\to+\infty} \theta(Q^t l, v) \text{ both exist and are finite}$$

from the context if θ is assumed to be continuous in its variables. So, we have the following theorem:

Theorem 2.2. *On CCMTS (F, ν), assume there exists $r \in (0, 1]$ such that $Q : F \to F$ satisfies*

$$\nu(Ql, Qv) \leq r\theta(l, v)\nu(l, v), \tag{2.9}$$

for all $l, v \in Q$. Suppose that for any $m \in \mathbf{N}$,

$$\limsup_{i\to\infty} \theta(l_i, l_m)\theta(l_i, l_{i+1}) \text{ exists and is less than } \frac{1}{r},$$

where $l_i = Q^i l_0$ for $l_0 \in F$. If θ is continuous in its variables, then Q has a FP in F.

Proof. Create a sequence $(l_t = Q^t l_0)$ in F in similar way to Theorem 2.1 such that $l_t \to l' \in F$ and

$$\lim_{t\to+\infty} \nu(l_t, l_{t+1}) = \lim_{t\to+\infty} \nu(l_t, l') = \lim_{t\to+\infty} \nu(l, l_t) = 0.$$

Take advantage of the continuity of θ in its variables to obtain:

$$\lim_{t\to+\infty} \theta(l_t, Ql') = \theta(l', Ql'),$$

and

$$\lim_{t\to+\infty} \theta(l', l_t) = \lim_{t\to+\infty} \theta(l_t, l') = \theta(l', l').$$

Claim: $Ql' = l'$. To achieve that, we benefit from the triangular inequality of ν and (2.9) to get

$$\begin{aligned} \nu(l', Ql') &\leq \theta(l', l_{t+1})\nu(l', l_{t+1}) + \theta(l_{t+1}, Ql')\nu(l_{t+1}, Ql') \\ &\leq \theta(l', l_{t+1})\nu(l', l_{t+1}) + r\theta(l_{t+1}, Ql')\theta(l_t, l')\nu(l_t, l'). \end{aligned} \tag{2.10}$$

Allow $t \to +\infty$ in (2.10) to obtain

$$\begin{aligned} \nu(l', Ql') &\leq \theta(l', l') \lim_{t\to+\infty} \nu(l', l_{t+1}) + r\theta(l', Ql')\theta(l', l') \lim_{t\to+\infty} \nu(l_t, l'). \\ &= 0. \end{aligned}$$

This means that $Ql' = l'$. Thus, the desired result is obtained. □

The uniqueness of the FP in Theorem 2.1 or 2.2 can be obtained if an appropriate condition is added.

Theorem 2.3. *On CCMTS (F, ν), assume there exists $r \in (0, 1]$ such that $Q : F \to F$ satisfies*

$$\nu(Ql, Qv) \leq r\theta(l, v)\nu(l, v),$$

for all $l, v \in F$. Assume that

$$\limsup_{i\to\infty} \theta(l_i, l_m)\theta(l_i, l_{i+1}) \text{ exists and is less than } \frac{1}{r},$$

where $l_i = Q^i l_0$ for $l_0 \in Q$. Moreover, assume that for any $l, s_0 \in M$,

$$\limsup_{i\to+\infty} \theta(l, Q^i l_0) \text{ exists and is finite, or } \theta \text{ is continuous.}$$

If $\forall l, s \in F$, we have

$$\limsup_{i\to+\infty} \theta(Q^i v, Q^i l) \text{ exists and is less than } \frac{1}{r},$$

then T has only one FP in Q.

Proof. The existence of the FP of Q in F follows from Theorem 2.1 (Theorem 2.2), say, $s' \in Q$. So, $Qs' = s'$.
To verify that Q has only one FP, let $l' \in F$ such that $Ql' = l'$ with $s' \neq l'$. Now,

$$\begin{aligned} \nu(l', s') = \nu(Ql', Qs') &\leq r\theta(l', s')\nu(l', s') \\ &= r\theta(Q^t l', Q^t s')\nu(l', s'). \end{aligned}$$

Once allowing $t \to +\infty$ in the above inequality, we get the following contradiction:

$$\nu(l', s') < \nu(l', s').$$

Thus $l' = s'$, and we deduce that T has only one FP. □

The following known result can be obtained immediately from our Theorem 2.3 by simply defining θ to be the constant function b.

Corollary 2.1. *On CbMS (F, ν), assume there exists $r \in (0, 1]$ with $b^2 r < 1$ such that $Q : F \to F$ satisfies*

$$\nu(Ql, Qv) \leq rb\nu(l, v), \tag{2.11}$$

for all $l, v \in F$. Then, Q has only one FP in F.

Proof. Define $\theta : F \times F \to [0, +\infty)$ via $\theta(s, p) = b \ \forall\ l, v \in F$. Now, for $l_0 \in F$, we have

$$\limsup_{i\to\infty} \sup_{m\geq 1} \theta(l_i, l_m)\theta(l_i, l_{i+1}) = b^2 < \frac{1}{r}.$$

Moreover, for $v \in F$, we notice

$$\limsup_{i\to+\infty} \theta(v, Q^i l_0) = b < \frac{1}{r}.$$

So, all conditions of Theorem 2.3 are met. So, the result also follows. □

Theorem 2.4. *On CCMTS (F, ν), assume there exist $r, a \in [0, 1]$ (both are not 0) and $h \in [0, 1)$ such that $Q : F \to F$ satisfies*

$$\nu(Ql, Qv) \leq r\theta(l, v)\nu(l, v) + a\theta(l, Ql)\nu(l, Ql) + h\nu(v, Qv), \tag{2.12}$$

for all $l, v \in F$. Also, suppose that for any $m \in \mathbf{N}$,

$$\limsup_{j\to+\infty} \theta(l_{j+1}, l_m)\theta(l_{j+1}, l_{j+2}) < \frac{1-h}{r+a}, \tag{2.13}$$

where $l_i = Q^i l_0$ for $l_0 \in F$. Moreover, assume that for any $v \in F$, we have $\limsup_{i\to+\infty} \theta(v, l_i)$ exists and is finite, and $\limsup_{i\to+\infty} \theta(l_i, v)$ exists, is less than $\frac{1}{h}$ and is finite. Then, T has a FP in F.

Proof. Construct a sequence (l_t) in F by choosing $l_o \in F$ and putting $l_t = Q^t l_0$.
For $t \in \mathbf{N}$, condition (2.12) gives

$$\nu(l_t, l_{t+1}) \ = \ \nu(Ql_{t-1}, Ql_t)$$

$$\begin{aligned} &\leq r\theta(l_{t-1}, l_t)\nu(l_{t-1}, l_t) + a\theta(l_{t-1}, Ql_{t-1})\nu(l_{t-1}, Ql_{t-1}) + h\nu(l_t, Ql_t) \\ &= r\theta(l_{t-1}, l_t)\nu(l_{t-1}, l_t) + a\theta(l_{t-1}, l_t)\nu(l_{t-1}, l_t) + h\nu(l_t, l_{t+1}). \end{aligned} \tag{2.14}$$

Inequality (2.14) yields

$$\nu(l_t, l_{t+1}) \leq \left(\frac{r+a}{1-h}\right)\theta(l_{t-1}, l_t)\nu(l_{t-1}, l_t). \tag{2.15}$$

The induction leads to

$$\nu(s_t, s_{t+1}) \leq \prod_{y=1}^{t}\left(\frac{r+a}{1-h}\right)^t \theta(l_{y-1}, l_y)\nu(l_0, l_1). \tag{2.16}$$

Choose $t, m \in \mathbf{N}$ in such a way that $m > t$. Select $k \in \mathbf{N}$ such that $m = t + k$. Similar to those arguments given in the proof of Theorem 2.1, at the end of the day, we get to:

$$\nu(l_t, l_m) \leq \sum_{j=t}^{t+k-1}\prod_{i=t}^{j}\theta(l_i, l_{t+k})\theta(l_j, l_{j+1})\left(\frac{r+a}{1-h}\right)^j \prod_{y=1}^{j}\theta(l_{y-1}, l_y)\nu(l_0, l_1).$$

Define

$$\prod_{i=t}^{j}\theta(l_i, l_{t+k})\theta(l_j, l_{j+1})\left(\frac{r+a}{1-h}\right)^j \prod_{y=1}^{j}\theta(l_{y-1}, l_y)\nu(l_0, l_1) := I_j. \tag{2.17}$$

Then,

$$\lim_{j\to+\infty}\frac{I_{j+1}}{I_j} = \lim_{j\to+\infty}\theta(l_{j+1}, l_{t+k})\theta(l_{j+1}, l_{j+2})\left(\frac{r+a}{1-h}\right) < 1.$$

Ratio test implies that

$$\lim_{t,m\to+\infty}\nu(l_t, l_m) = 0,$$

and hence (l_t) is Cauchy in (F, ν). As a result of the completeness of (F, ν), we find $l' \in F$ such that $l_t \to l'$; that is,

$$\lim_{t\to\infty}\nu(l_t, l') = \lim_{t\to\infty}\nu(l', l_t) = 0. \tag{2.18}$$

Our task is to verify $Ql' = l'$. Now, triangular inequality implies that

$$\nu(l_t, l_{t+1}) \leq \theta(l_t, l')\nu(l_t, l') + \theta(l', l_{t+1})\nu(l', l_{t+1}).$$

By allowing $n \to +\infty$ in the above inequality, we get

$$\lim_{n\to+\infty}\nu(l_t, l_{t+1}) = 0.$$

Also, employ the triangular inequality to get

$$\begin{aligned} \nu(l_{t+1}, Ql') &= \nu(Ql_t, Ql') \\ &\leq r\theta(l_t, l')\nu(l_t, l') + a\theta(l_t, Ql_t)\nu(l_t, Ql_t) + h\nu(l', Ql') \\ &= r\theta(l_t, l')\nu(l_t, l') + a\theta(l_t, l_{t+1})\nu(l_t, l_{t+1}) + h\nu(l', Ql') \end{aligned}$$

$$\leq \quad r\theta(l_t, l')v(l_t, l') + a\theta(l_t, l_{t+1})v(l_t, l_{t+1}) + h\theta(l', l_{t+1})v(l', l_{t+1}) + h\theta(l_{t+1}, Ql')v(l_{t+1}, Ql').$$

By allowing $n \to +\infty$ in the above inequalities and taking into account that $\limsup_{t\to+\infty} \theta(l_t, l')$, $\limsup_{t\to+\infty} \theta(l', l_t)$ and $\limsup_{n\to+\infty} \theta(l_t, l_{t+1})$ exist and are bounded, we get

$$\lim_{t\to+\infty} v(l_{t+1}, Ql') \leq h \lim_{t\to+\infty} \theta(l_{t+1}, Ql') \lim_{t\to+\infty} v(l_{t+1}, Ql').$$

Since

$$h \lim_{t\to+\infty} \theta(l_{t+1}, Ql') < 1,$$

we get

$$\lim_{t\to+\infty} v(l_{t+1}, Ql') = 0.$$

On the other hand,

$$v(l', Ql') \leq \theta(l', l_{n+1})v(l', l_{n+1}) + \theta(l_{n+1}, Ql')v(l_{n+1}, Ql').$$

Again, by allowing $t \to +\infty$ in above inequality, we get $v(l', Ql') = 0$. Accordingly, $Ql' = l'$. □

In our next result, we assume that θ is continuous in its variables.

Theorem 2.5. *On CCMTS* (F, v)*, assume there exist* $r, a \in [0, 1]$ (*both are not 0*) *and* $h \in [0, 1)$ *such that* $Q : F \to F$ *satisfies*

$$v(Ql, Qv) \leq r\theta(l, v)v(l, v) + a\theta(l, Ql)v(l, Ql) + hv(v, Qv), \tag{2.19}$$

for all $l, v \in F$*. Also, suppose that for any* $m \in \mathbf{N}$,

$$\limsup_{j\to+\infty} \theta(l_{j+1}, l_m)\theta(l_{j+1}, l_{j+2}) < \frac{1-h}{r+a}, \tag{2.20}$$

where $l_i = Q^i l_0$ *for* $l_0 \in F$*. Also, suppose for* $v \in Q$*, we have* $\theta(v, Qv) < \frac{1}{h}$*. If* θ *is continuous in its variables, then* Q *has a FP in* F*.*

Proof. Begin with $l_0 \in F$ to construct a sequence (l_n) as in the proof of Theorem 2.4 such that there exists $l' \in F$ with

$$\lim_{t\to+\infty} v(l_t, l') = \lim_{t\to+\infty} v(l', l_t) = \lim_{t\to+\infty} v(l_t, l_{t+1}) = 0.$$

Now, we show that $Ql' = l'$. Benefiting from the triangular inequality, we get

$$\begin{aligned} v(l', Ql') &\leq \theta(l', l_{t+1})v(l', l_{t+1}) + \theta(l_{t+1}, Ql')v(l_{t+1}, Ql') \\ &= \theta(l', l_{t+1})v(l', l_{t+1}) + \theta(l_{t+1}, Ql')v(Ql_t, Ql') \\ &\leq \theta(l', l_{t+1})v(l', l_{t+1}) + r\theta(l_{t+1}, Ql')\theta(l_t, l')v(l_t, l') + a\theta(l_{t+1}, Ql')\theta(l_t, l_{t+1})v(l_t, l_{t+1}) + h\theta(l_{t+1}, Ql')v(l', Ql'). \end{aligned}$$

Permitting $t \to +\infty$ in above the inequalities yields

$$v(l', Ql') \leq h\theta(l', Ql')v(l', Ql').$$

Since $h\theta(l', Ql') < 1$, we deduce $v(l', Ql') = 0$, and hence $l' = Ql'$. □

The uniqueness of FP can be achieved in Theorem 2.4 or 2.5 if a suitable condition is added.

Theorem 2.6. *On CCMTS (F, ν), assume there exist $r \in (0, 1]$, $a \in [0, 1]$ and $h \in [0, 1)$ such that $Q : F \to F$ satisfies*

$$\nu(Ql, Qv) \leq r\theta(l, v)\nu(l, v) + a\theta(l, Ql)\nu(l, Ql) + h\nu(v, Qv), \tag{2.21}$$

for all $l, v \in F$. To the addition of all conditions in Theorem 2.4 or 2.5, suppose Q satisfies

$$\limsup_{t\to+\infty} \theta(Q^t l, Q^t v) < \frac{1}{r},$$

for all $l, v \in F$. Then, Q has only one FP in F.

Proof. Theorem 2.4 [Theorem 2.5] ensures that there exists $l' \in F$ with $Ql' = l'$. To verify that Q achieves only one fixed point, we suppose there exists $v' \in F$ with $l' \neq v'$ such that $Qv' = v'$. Now,

$$\begin{aligned} \nu(l', v') &= \nu(Ql', Qv') \\ &\leq r\theta(l', v')\nu(l', v') + a\theta(l', l')\nu(l', l') + h\nu(v', v') \\ &= r\theta(l', v')\nu(l', v') \\ &= r \limsup_{t\to+\infty} \theta(Q^t l', Q^t v')\nu(l', v'). \end{aligned}$$

Since

$$\limsup_{t\to+\infty} \theta(Q^t l', Q^t v') < \frac{1}{r},$$

we get a contradiction. Thus, $\nu(l', v') = 0$, and hence $l' = v'$. Thus, T has only one FP in Q. □

Corollary 2.2. *On CCMTS (F, ν), assume there exist $a \in (0, 1]$ and $h \in [0, 1)$ such that $Q : F \to F$ satisfies*

$$\nu(Ql, Qv) \leq a\theta(l, Ql)\nu(l, Ql) + h\nu(v, Qv),$$

for all $l, v \in F$. Also, suppose that for any $m \in \mathbf{N}$,

$$\lim_{j\to+\infty} \theta(l_{j+1}, l_m)\theta(l_{j+1}, l_{j+2}) < \frac{1-h}{a},$$

where $l_i = Q^i l_0$ for $l_0 \in F$. Moreover, assume that for any $v \in F$, we have $\limsup_{i\to+\infty} \theta(v, l_i)$ exists, and is finite and $\limsup_{i\to+\infty} \theta(l_i, v)$ exists, is less than $\frac{1}{h}$ and is finite. Then, T has a FP in F.

Proof. The result follows from Theorem 2.4 by taking $a = 0$. □

The following known result can be obtained immediately from our Theorem 2.6 by simply defining θ to be the constant function b.

Corollary 2.3. *On CbMS (F, ν), assume there exist $r, a \in [0, 1]$ (both are not zero) and $h \in [0, 1)$ with $b^2 r + b^2 a + h < 1$ such that $Q : F \to F$ satisfies*

$$\nu(Ql, Qv) \leq rb\nu(s, v) + ab\nu(l, Ql) + h\nu(v, Qv), \tag{2.22}$$

for all $l, v \in F$. Then, Q has only one FP in F.

Proof. Define $\theta : F \times F \to [0, +\infty)$ by $\theta(l, v) = b$. For $m \in \mathbf{N}$ and $l_0 \in F$, we have

$$\lim_{j\to+\infty} \theta(l_{j+1}, l_m)\theta(l_{j+1}, l_{j+2}) = b^2 < \frac{1-h}{r+a}.$$

Also, from $br \leq b^2 r + b^2 a + h < 1$, we figure out

$$\theta(v, Qv) = b < \frac{1}{r},$$

and

$$\lim_{t\to+\infty} \theta(Q^t l, Q^t v) < \frac{1}{r}.$$

Also, note that θ is continuous in its variables. So, all conditions of Theorem 2.6 are met. So, the result also follows. □

Now, we present the following example to show the significance of our results.

Example 2.1. *Let $F = \{0, 1, 2, 3, \ldots\}$. Define $Q : F \to F$ via*

$$Q(v) = \begin{cases} \sqrt{v}, & \text{if } v \text{ is a perfect square and } v \neq 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $\theta : F \times F \to [1, \infty)$ by

$$\theta(v, l) = \begin{cases} v + l, & \text{if } (v, l) \neq (0, 0), \\ 1, & \text{if } (v, l) = (0, 0). \end{cases}$$

Also, define $\nu : F \times F \to [0, +\infty)$ via

$$\nu(v, l) = \begin{cases} 0, & \text{if } v = l, \\ 1, & \text{if one is even and the other is odd,} \\ \max\{v, l\}, & \text{if both are even or both are odd.} \end{cases}$$

Then:

(1) (F, ν) is CCMTS.

(2) Let $l_0 \in F$, and take $(l_t) = (Q^t l_0)$. Then, for $m \in \mathbf{N}$, we have

$$\limsup_{i\to+\infty} \theta(l_i, l_{i+m})\theta(l_i, l_{i+1}) = 1 < 2 = \frac{1}{r}.$$

(3) For any $v, l_0 \in F$, we have

$$\limsup_{t\to+\infty} \theta(v, Q^t l_0) = v \text{ exists and is finite.}$$

(4) For any $v, l \in F$, we have

$$\limsup_{t\to+\infty} \theta(Q^t v, Q^t l) = 1 < 2 = \frac{1}{r}.$$

(5) For $v, l \in F$, we have

$$\nu(Qv, Ql) \leq \frac{1}{2}\theta(v, l)\nu(v, l).$$

We note that the hypotheses of Theorem 2.3 have been fulfilled for $r = \frac{1}{2}$.

Proof. The proof that ν is a controlled metric type has been left to the reader. To prove (F, ν) is complete, let (l_t) be Cauchy in F. For $\epsilon = \frac{1}{2}$, there exists $t_0 \in F$ such that

$$\nu(l_t, l_m) \leq \frac{1}{2} \forall m \geq t \geq t_0.$$

Thus, (l_t) has a constant tail, say, l'. So, (l_t) converges to l'. Thus, (F, ν) is complete.
To prove (2), let $l_0 \in F$. Then,
Case 1: If $l_0 = 1$ or $l_0 = 0$, then $l_t = Q^t l_0 = 0$ for all $t \in \mathbf{N}$. Thus,

$$\limsup_{i \to +\infty} \theta(l_i, l_{i+m})\theta(l_i, l_{i+1}) = 1 < 2 = \frac{1}{r}.$$

Case 2: If $l_0 \in \{2, 3, 4, \ldots\}$, then we find $j \in \mathbf{N}$ such that l_j is not a perfect square. So $l_t = 0$ for all $t > j$, so

$$\limsup_{i \to +\infty} \theta(l_i, l_{i+m})\theta(l_i, l_{i+1}) = 1 < 2 = \frac{1}{r}.$$

To prove (3), given $v, l_0 \in F$, there exists a large integer number j such that $Q^j l_0 = l_j$ is not a perfect square. So, $Q^t l_0 = 0$ for all $t > j$. Thus,

$$\limsup_{t \to +\infty} \theta(v, Q^t l_0) = \theta(v, 0) \text{ exists and is finite.}$$

To prove (4), given $v, l \in F$, then,
Case 1: If $v = l = 1$ or $v = l = 0$, then $Q^t v = Q^t l = 0$. Thus,

$$\limsup_{t \to +\infty} \theta(Q^t v, Q^t l) = \theta(0, 0) = 1 < 2 = \frac{1}{r}.$$

Case 2: If $v, l \in \{2.3.4. \ldots\}$, then we can find two integers j_0, j_1 such that $Q_0^j v$ and $Q_1^j l$ are not perfect squares. Then, $Q^t v = Q^t l = 0$ for all $v, t \geq \max\{j_0, j_1\}$. Thus,

$$\limsup_{t \to +\infty} \theta(Q^t v, Q^t l) = \theta(0, 0) = 1 < 2 = \frac{1}{r}.$$

To prove (5), given $v, l \in F$, then,
Case 1: If $v = l$, then

$$\nu(Qv, Ql) = 0 \leq \frac{1}{2}\theta(v, l)\nu(v, l) = 0.$$

Case 2: If v and l are not perfect squares, then $Qv = Ql = 0$. So,

$$\nu(Qv, Ql) = 0 \leq \frac{1}{2}\theta(v, l)\nu(v, l).$$

Case 3: If v is a perfect square, $v \neq 1$, and l is not a perfect square, then $Qv = \sqrt{v}$, and $Ql = 0$.
Sub-case I: If $\sqrt{v}$ is even, then v is even, and hence $v \geq 4$. Thus, $\sqrt{v} \leq \frac{1}{2}v$. Therefore,

$$\nu(Qv, Ql) = \nu(\sqrt{v}, 0) = \sqrt{v} \leq \frac{1}{2}v \leq \frac{1}{2}(v + l)\nu(v, l) = \frac{1}{2}\theta(v, l)\nu(v, l).$$

Sub-case II: If $\sqrt{v}$ is odd, then v is odd. Thus, $v \geq 9$, and hence $\sqrt{v} \leq \frac{1}{3}v$. Therefore,

$$\nu(Qv, Ql) = \nu(\sqrt{v}, 0) = 1 \leq \frac{1}{2}v \leq \frac{1}{2}(v + l)\nu(v, l).$$

Case 4: If $v = 1$, and l is not a perfect square, then $Qv = 0$, and $Ql = 0$. Thus,

$$\nu(Qv, Ql) = 0 \leq \frac{1}{2}\theta(v, l)\nu(v, l).$$

Case 5: If $v = 1$, and l is a perfect square, then $Qv = 0$, and $Ql = \sqrt{l}$. So,
Sub-case 1: If $\sqrt{l}$ is even, then l is even. Thus, $l \geq 4$, and hence $\sqrt{l} \leq \frac{1}{2}l$. Therefore,

$$\nu(Qv, Ql) = \sqrt{l} \leq \frac{1}{2}l \leq \frac{1}{2}\theta(v, l)\nu(v, l).$$

Sub-case 2: If $\sqrt{l}$ is odd, then l is odd. Thus, $l \geq 9$, and hence $\sqrt{l} \leq \frac{1}{3}l$. Therefore,

$$\nu(Qv, Ql) = \nu(0, \sqrt{l}) = 1 \leq \frac{1}{2}l \leq \frac{1}{2}\theta(v, l)\nu(v, l).$$

Case 6: If v and l are perfect squares with $v > l$, then $Qv = \sqrt{v}$, and $Ql = \sqrt{l}$. So,
Sub-case I: If v and l are both even, then $\sqrt{v}$ and $\sqrt{l}$ are both even, $v \geq 16$, and $l \geq 4$. So,

$$\nu(Qv, Ql) = \nu(\sqrt{v}, \sqrt{l}) = \sqrt{v} \leq \frac{1}{4}v \leq \frac{1}{2}(v + l)v = \frac{1}{2}\theta(v, l)\nu(v, l).$$

Sub-case II: If v and l are both odd, and $l \neq 1$, then $\sqrt{v}$ and $\sqrt{l}$ are both odd, $v \geq 25$, and $l \geq 9$. So,

$$\nu(Qv, Ql) = \nu(\sqrt{v}, \sqrt{l}) = \sqrt{v} \leq \frac{1}{5}v \leq \frac{1}{2}(v + l)v = \frac{1}{2}\theta(v, l)\nu(v, l).$$

Sub-case III: If v is odd, and $l = 1$, then $\sqrt{v}$, is odd and $v \geq 9$. So,

$$\nu(Qv, Q1) = \nu(\sqrt{v}, 0) = 1 \leq \frac{1}{2}(v + 1)v = \frac{1}{2}\theta(v, 1)\nu(v, 1).$$

Sub-case IV: If v, is even and $l = 1$, then $\sqrt{v}$ is even, and $v \geq 4$. So $\sqrt{v} \leq \frac{1}{2}v$. So,

$$\nu(Qv, Q1) = \nu(\sqrt{v}, 0) = \sqrt{v} \leq \frac{1}{2}v \leq \frac{1}{2}(v + 1)v = \frac{1}{2}\theta(v, 1)\nu(v, 1). \quad \square$$

3. Application

Now, we will support our results with the following application:

Theorem 3.1. *For an integer m with $m \geq 2$, the equation*

$$(v+1)^m + 1 = (4^m+1)v(v+1)^m + 4^m v$$

has a unique real solution v' in $[0, +\infty)$.

Proof. Let $F = [0, +\infty)$. Define $Q : F \to F$ by

$$Qv = \frac{(v+1)^m + 1}{(4^m+1)(v+1)^m + 4^m}.$$

Also, define $\theta : Q \times Q \to [1, +\infty)$ by

$$\theta(v, l) = (v+1)^{m-1} + (v+1)^{m-2}(l+1) + (v+1)^{m-3}(l+1)^2 + \ldots + (v+1)(l+1)^{m-2} + (l+1)^{m-1}.$$

Now, consider the CCMT (F, ν), where $\nu : Q \times Q \to [0, +\infty)$ is defined by

$$\nu(v, l) = |v - l|.$$

Then,

(1) For $l, v \in Q$, we have

$$\nu(Qv, Ql) \leq \frac{1}{16^m}\theta(v, l)\nu(v, l).$$

Indeed,

$$\begin{aligned}
\nu(Qv, Ql) &= |Qv - Ql| \\
&= \left|\frac{(v+1)^m + 1}{(4^m+1)(v+1)^m + 4^m} - \frac{(l+1)^m + 1}{(4^m+1)(l+1)^m + 4^m}\right| \\
&= \left|\frac{(v+1)^m - (l+1)^m}{((4^m+1)(v+1)^m + 4^m)((4^m+1)(l+1)^m + 4^m)}\right| \\
&= \left|\frac{((v+1)^{m-1} + (v+1)^{m-2}(l+1) + \ldots + (v+1)(l+1)^{m-2} + (l+1)^{m-1})(v-l)}{((4^m+1)(v+1)^m + 4^m)((4^m+1)(l+1)^m + 4^m)}\right| \\
&\leq \frac{1}{16^m}((v+1)^{m-1} + (v+1)^{m-2}(l+1) + \ldots + (v+1)(l+1)^{m-2} + (l+1)^{m-1})|v-l| \\
&= \frac{1}{16^m}\theta(v, l)\nu(v, l).
\end{aligned}$$

(2) For $l_0 \in Q$, put $l_{i+1} = Q^i l_0$. Then, for $j \in \mathbb{N}$, we have

$$\limsup_{i\to\infty} \theta(l_{i+1}, l_j)\theta(l_{i+1}, l_{i+2}) \text{ exists and less than } \frac{1}{r} = 16^m.$$

Indeed, note that $Q(v) < 1$ for each $v \in M$. Thus, $l_i = Q^i l_0 < 1$ for all $i \in \mathbb{N}$. Thus,

$$\theta(l_{i+1}, l_j) = (l_{i+1} + 1)^{m-1} + (l_{i+1} + 1)^{m-2}(l_j + 1) + \ldots + (l_{i+1} + 1)(l_j + 1)^{m-2} + (l_j + 1)^{m-1} \le m(2)^{m-1}$$

and

$$\theta(l_{i+1}, l_{i+2}) = (l_{i+1}+1)^{m-1} + (l_{i+1}+1)^{m-2}(l_{i+2}+1) + \ldots + (l_{i+1}+1)(l_{i+1}+1)^{m-2} + (l_{i+2}+1)^{m-1} \le m(2)^{m-1}.$$

Thus,

$$\limsup_{i\to\infty} \theta(l_{i+1}, l_j)\theta(l_{i+1}, l_{i+2}) \le m^2(4)^{m-1} < (4^m)(4^m) = 16^m = \frac{1}{r}.$$

(3) It is clear that θ is continuous in its variables.

(4) For $v, l \in Q$, we have

$$\limsup_{i\to+\infty} \theta(Q^i v, Q^i l) \text{ exists and less than } \frac{1}{r} = 16^m.$$

Indeed, for $v, l \in M$, we have $Qi^v < 1$ and $Q^i l < 1$. So,

$$\limsup_{i\to+\infty} \theta(Q^i v, Q^i l) \le m(2)^{m-1} < 16.$$

Thus, all conditions of Theorem 2.3 are met. Hence, Q has a unique fixed point. □

Example 3.1. *The equation*

$$(257v - 1)(v + 1)^4 + 256v - 1 = 0$$

has a unique real solution v' *in* $[0, +\infty)$.

Proof. The equation

$$(257v - 1)(v + 1)^4 + 256v - 1 = 0$$

is equivalent to

$$(v + 1)^4 + 1 = (4^4 + 1)v(v + 1)^4 + 4^4 v.$$

The result follows from Theorem 3.1 by taking $m = 4$. □

4. Conclusions

In our work, we established and proved some fixed point theorems for mappings that satisfy a set of conditions in controlled metric type spaces. We relied on the function θ that appears in the triangular inequality of the definition of the controlled metric type function to construct our contraction conditions. Our results enriched the field of fixed point theory with novel findings that generalize many findings found in the literature. We provided an example to show the usefulness of our results. Also, we presented an application to our results to show their significance.

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Conflict of interest

The authors declare no conflicts of interest.

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