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Research article

A Derivative Hilbert operator acting from Bergman spaces to Hardy spaces

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Abstract: Let μ be a positive Borel measure on the interval $[0, 1)$. The Hankel matrix $\mathcal{H}_\mu = (\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where $\mu_n = \int_{[0,1)} t^n d\mu(t)$, formally induces the operator as follows:

$$\mathcal{DH}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)(n+1)z^n,\ z\in\mathbb{D},$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function in $\mathbb{D}$. In this article, we characterize those positive Borel measures on $[0, 1)$ such that $\mathcal{DH}_\mu$ is bounded (resp., compact) from Bergman spaces $\mathcal{A}^p$ into Hardy spaces H^q, where $0 < p, q < \infty$.

Keywords: Derivative-Hilbert operator; Bergman space; Hardy space; Carleson measure
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1. Introduction

Suppose that μ is a positive Borel measure on [0,1). $\mathcal{H}_\mu$ is defined as the Hankel matrix $(\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where $\mu_n = \int_{[0,1)} t^n d\mu(t)$. The matrix $\mathcal{H}_\mu$ can be seen as an operator on $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{D})$ by its action on the Taylor coefficients: $\{a_n\}_{n\geq 0} \to \{\sum_{k=0}^{\infty}\mu_{n,k}a_k\}_{n\geq 0}$. Furthermore, we can formally define the Hankel operator $\mathcal{H}_\mu$ as follows:

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}(\sum_{k=0}^{\infty}\mu_{n,k}a_k)z^n, z\in\mathbb{D},$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$. If we take the measure to be the Lebesgue measure, $\mathcal{H}_\mu$ is the classical Hilbert operator. This is why $\mathcal{H}_\mu$ is called a generalized Hilbert operator.

In recent decades, the operator $\mathcal{H}_\mu$ has been studied extensively in [1–6]. Galanopoulos and Peláez [5] characterized those measures μ supported on $[0, 1)$ such that the generalized Hilbert operator

$\mathcal{H}_\mu$ is well defined and it is bounded on H^1. Chatzifountas, et al. [1] described those measures μ for which $\mathcal{H}_\mu$ is a bounded operator from H^p into H^q, where $0 < p < \infty$ and $0 < q < \infty$. Diamantopoulos [3] gave many results about the operator on Dirichlet space. Girela [2] introduced the operators $\mathcal{H}_\mu$ acting on certain conformally invariant spaces.

Ye and Zhou [7, 8] defined the derivative-Hilbert operator $\mathcal{DH}_\mu$ as follows:

$$\mathcal{DH}_\mu(f)(z) = \sum_{n=0}^{\infty}(\sum_{k=0}^{\infty}\mu_{n,k}a_k)(n+1)z^n, \quad z \in \mathbb{D}, \tag{1.1}$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{D})$. It is closed related to the generalized Hilbert operator, that is,

$$\mathcal{DH}_\mu(f)(z) = (z\mathcal{H}_\mu(f)(z))'.$$

Another generalized Hilbert-integral operator related to $\mathcal{DH}_\mu$ denoted by $\mathcal{I}_{\mu_\alpha}(\alpha \in \mathbb{N}^+)$ is defined by

$$\mathcal{I}_{\mu_\alpha}(f)(z) = \int_{[0,1)} \frac{f(t)}{(1-tz)^\alpha} d\mu(t). \tag{1.2}$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$. We can easily check that the case $\alpha = 1$ is the integral representation of the generalized Hilbert operator. Ye and Zhou characterized the measure μ for which $\mathcal{I}_{\mu_2}$ and $\mathcal{DH}_\mu$ is bounded (resp., compact) on Bloch spaces [7] and Bergman spaces [8].

In this article, we characterize the positive Borel measure μ such that $\mathcal{DH}_\mu$ is bounded (resp. compact) from the Bergman space $\mathcal{A}^p$ into the Hardy space H^q, where $0 < p < \infty, 0 < q < \infty$.

2. Preliminaries and notation

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $H(\mathbb{D})$ denote the class of all analytic functions in $\mathbb{D}$.

For $0 < p < \infty$, the Bergman space $\mathcal{A}^p$ consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{A}^p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} dxdy$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. We refer to [4] for the theory of Bergman spaces.

The Bloch space $\mathscr{B}$ consists of those functions $f \in H(\mathbb{D})$ with

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

We mention [9, 10] as general references for Bloch spaces.

For $0 < p < \infty$ and $f \in H(\mathbb{D})$, set

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right)^{\frac{1}{p}},$$
$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|, \ 0 < r < 1.$$

The Hardy space H^p consists of those functions $f \in H(\mathbb{D})$ with

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f) < \infty, \ 0 < p < \infty.$$

We refer to [11] for the the theory of Hardy spaces. In particular, if $0 < q < 1$, let B_q [12] denote the space consisting of those functions $f \in H(\mathbb{D})$ with

$$\|f\|_{B_q} = \int_0^1 (1-r)^{\frac{1}{q}-2} M_1(r, f) dr < \infty.$$

We refer to [12] as general references for the B_q spaces. The Banach space B_q is the "containing Banach space" of H^q, that is, H^q is a dense subspace of B_q, and the two spaces have the same continuous linear functionals.

Let $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ be a *Möbius* transformations. If $f \in H(\mathbb{D})$, then $f \in BMOA$ if and only if

$$\|f\|_{BMOA} = |f(0)| + \|f\|_\star < \infty,$$

where

$$\|f\|_\star = \sup_{a\in\mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^2}.$$

It is clear that the seminorm $\|\cdot\|_\star$ is conformally invariant. If those functions $f \in H(\mathbb{D})$ for which

$$\lim_{|a|\to 1} \|f \circ \varphi_a - f(a)\|_{H^2} = 0,$$

then we call that $f \in VMOA$. We refer to [13] for the the theory of $BMOA$ spaces.

The relation between these spaces we introduced above is well known, that is,

$$H^\infty \subsetneq BMOA \subsetneq \bigcap_{0<p<\infty} H^p \quad \text{and} \quad BMOA \subsetneq \mathscr{B}.$$

Let us recall the knowledge of Carleson measure, which is a very useful tool in the study of Banach spaces of analytic functions. For $0 < s < \infty$, a positive Borel measure μ on $\mathbb{D}$ will be called a s-Carleson measure, if there exists a positive constant C such that

$$\sup_I \frac{\mu(S(I))}{|I|^s} \le C. \tag{2.1}$$

The Carleson square $S(I)$ is defined as follows:

$$S(I) = \left\{ z = re^{i\theta} : e^{i\theta} \in I; 1 - \frac{|I|}{2\pi} \le r \le 1 \right\},$$

where I is an interval of $\partial\mathbb{D}$, $|I|$ denotes the length of I. If μ satisfies $\mu(S(I)) = o(|I|^s)$, as $|I| \to 0$, we say that μ is a vanishing s-Carleson measure [14, 15].

A positive Borel measure on $[0, 1)$ also can be seen as a Borel measure on $\mathbb{D}$ by identifying it with the measure μ defined by

$$\tilde{\mu}(E) = \mu(E \bigcap [0, 1)).$$

for any Borel subset E of $\mathbb{D}$. In this way, we say that a positive Borel measure μ on $[0,1)$ can be seen as a s-Carleson measure on $\mathbb{D}$ if and only if there exists a positive constant C such that

$$\mu([t,1)) \leq C(1-t)^s, \quad t \in [0,1).$$

Also, μ is a vanishing s-Carleson measure if μ satisfies

$$\lim_{t\to 1^-} \frac{\mu([t,1))}{(1-t)^s} = 0.$$

Other Carleson type measures on $[0,1)$ have the similar definitions.

Throughout this work, C denotes a positive constant that only depends on the displayed parameters but not necessarily the same from one occurrence to the next. For any given $p > 1$, p' will denote the conjugate index of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

3. Bounededness of $\mathcal{DH}_\mu$ from $\mathcal{A}^p$ into H^q

In this section, we characterize those measure μ such that $\mathcal{DH}_\mu$ is a bounded operator from $\mathcal{A}^p$ into H^q by applying the equivalence relation between $\mathcal{DH}_\mu(f)$ and $I_{\mu_2}(f)$, where $0 < p < \infty, 0 < q < \infty$.

Lemma 3.1. [11] If $f \in H^p(0 < p < \infty)$,

$$|g(z)| \leq C\frac{\|g\|_{H^p}}{(1-|z|)^{\frac{1}{p}}}, \tag{3.1}$$

and

$$|g'(z)| \leq C\frac{\|g\|_{H^p}}{(1-|z|)^{\frac{1}{p}+1}}, \quad z \in \mathbb{D}. \tag{3.2}$$

Lemma 3.2. [8] Suppose $0 < p < \infty$, and let μ be a positive Borel measure on $[0,1)$. Then the power series in (1.1) defines a well defined analytic function in $\mathbb{D}$ for every $f \in \mathcal{A}^p$ in any of the following cases:

(i) μ is a $\frac{2}{p}$-Carleson measure, if $0 < p \leq 1$.
(ii) μ is a $\frac{2-(p-1)^2}{p}$-Carleson measure, if $1 \leq p \leq 2$.
(iii) μ is a $\frac{1}{p}$-Carleson measure, if $2 \leq p < \infty$.

Furthermore, in such cases we obtain that

$$\mathcal{DH}_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{(1-tz)^2} d\mu(t) = I_{\mu_2}(f)(z), \quad z \in \mathbb{D}. \tag{3.3}$$

Theorem 3.1. *Suppose* $0 < p \leq 1$*, and let* μ *be a positive Borel measure on* $[0,1)$*, which satisfies the conditions in Lemma* 3.2.

(i) *If* $q \geq p$ *and* $q > 1$*,* $\mathcal{DH}_\mu$ *is a bounded operator from* $\mathcal{A}^p$ *into* H^q *if and only if* μ *is a* $(\frac{2}{p} + \frac{1}{q'} + 1)$*-Carleson measure.*

(*ii*) *If* $q \ge p$ *and* $q = 1$, $\mathcal{DH}_\mu$ *is a bounded operator from* $\mathcal{A}^p$ *into* H^q *if and only if* μ *is a* $(\frac{2}{p}+1)$-*Carleson measure.*

(*iii*) *If* $q \ge p$ *and* $0 < q < 1$, $\mathcal{DH}_\mu$ *is a bounded operator from* $\mathcal{A}^p$ *into* B_q *if and only if* μ *is a* $(\frac{2}{p}+1)$-*Carleson measure.*

Proof. We recall that the well known result of Hastings [16]: For $0 < p \le q < \infty$, μ is a $\frac{2q}{p}$-Carleson measure if and only if there exists a positive constant C such that

$$\left\{\int_{\mathbb{D}} |f(z)|^q d\mu(z)\right\}^{\frac{1}{q}} \le C\|f\|_{\mathcal{A}^p}, \qquad \text{for all } f \in \mathcal{A}^p. \tag{3.4}$$

Suppose $0 < p < \infty$. Since μ satisfies the conditions in Lemma 3.2, as in the proof of Lemma 3.2, we obtain that

$$\int_{[0,1)} |f(t)| d\mu(t) < \infty, \quad \text{for any } f \in \mathcal{A}^p.$$

Hence, it implies that

$$\begin{aligned}
&\int_0^{2\pi}\int_{[0,1)} \left|\frac{f(t)g(re^{i\theta})}{(1-tre^{i\theta})^2}\right| d\mu(t)d\theta \\
\le&\frac{1}{(1-r)^2}\int_{[0,1)} |f(t)|\, d\mu(t)\int_0^{2\pi} |g(re^{i\theta})|d\theta \\
\le&\frac{C}{(1-r)^2}\|g\|_{H^1}, \quad 0 \le r < 1, f \in \mathcal{A}^p, g \in H^1.
\end{aligned} \tag{3.5}$$

Using (3.5), Fubini's theorem and Cauchy's integral representation of H^1 [11], for any $f \in \mathcal{A}^p$ and $g \in H^1$, we obtain that

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \overline{\mathcal{D}H_\mu(f)(re^{i\theta})} g(re^{i\theta})d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\int_{[0,1)} \frac{\overline{f(t)}}{(1-tre^{-i\theta})^2} d\mu(t) g(re^{i\theta})d\theta \\
&= \frac{1}{2\pi}\int_{[0,1)} \overline{f(t)} \int_{|e^{i\theta}|=1} \frac{g(re^{i\theta})e^{i\theta}}{(e^{i\theta}-tr)^2} d(e^{i\theta})d\mu(t) \\
&= \int_{[0,1)} \overline{f(t)}\,(tg(rt))'\, d\mu(t) \\
&= \int_{[0,1)} \overline{f(t)}\,(g(rt)+tg'(rt))\, d\mu(t), \quad 0 \le r < 1.
\end{aligned} \tag{3.6}$$

(*i*) Consider the case $q > 1$. Using (3.6) and the duality theorem in [11], that is, $(H^q)^* \cong H^{q'}$ and $(H^{q'})^* \cong H^q$, where $q > 1$, under the pairing

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(e^{i\theta})} g(e^{i\theta})d\theta, \quad f \in H^q, g \in H^{q'}, \tag{3.7}$$

it implies that $\mathcal{DH}_\mu$ is a bounded operator from $\mathcal{A}^p$ into H^q if and only if

$$\left|\int_{[0,1)} \overline{f(t)}\,(g(rt)+tg'(rt))\, d\mu(t)\right| \le C\|f\|_{\mathcal{A}^p}\|g\|_{H^{q'}}, f \in \mathcal{A}^p, g \in H^{q'}.$$

Suppose that $\mathcal{DH}_\mu$ is a bounded operator from $\mathcal{A}^p$ into H^q. For $0 < a < 1$, take test functions

$$f_a(z) = \left(\frac{1-a^2}{(1-az)^2}\right)^{\frac{2}{p}}, g_a(z) = \left(\frac{1-a^2}{(1-az)^2}\right)^{\frac{1}{q'}}, \ z \in \mathbb{D}.$$

A simply calculation shows that $f_a(z) \in \mathcal{A}^p, g_a(z) \in H^{q'}$, and $\sup_{0<a<1} \|f_a\|_{\mathcal{A}^p} < \infty$, $\sup_{0<a<1} \|g_a\|_{H^{q'}} < \infty$. Hence, it follows that

$$\begin{aligned}
\infty &> C \sup_{0<a<1} \|f_a\|_{\mathcal{A}^p} \sup_{0<a<1} \|g_a\|_{H^{q'}} \\
&\geq C \left| \int_{[0,1)} \overline{f_a(t)} \left(g_a(rt) + t g_a'(rt)\right) d\mu(t) \right| \\
&\geq C \int_{[a,1)} \left(\frac{1-a^2}{(1-at)^2}\right)^{\frac{2}{p}} \left(\left(\frac{1-a^2}{(1-art)^2}\right)^{\frac{1}{q'}} + \frac{2at}{q'} \left(\frac{(1-a^2)}{(1-art)^{q'+2}}\right)^{\frac{1}{q'}} \right) d\mu(t) \\
&\geq C \frac{\mu([a,1))}{(1-a^2)^{\frac{2}{p}+\frac{1}{q'}+1}}.
\end{aligned}$$

This is equivalent to saying that μ is a $(\frac{2}{p} + \frac{1}{q'} + 1)$-Carleson measure.

On the contrary, suppose that μ is a $(\frac{2}{p} + \frac{1}{q'} + 1)$-Carleson measure, and let $d\nu(t) = \frac{1}{(1-t)^{\frac{1}{q'}+1}} d\mu(t)$. By [17, Theorem 3.2], we obtain that ν is a $\frac{2}{p}$-Carleson measure. Using (3.4), (3.6) and Lemma 3.1, it follows that

$$\begin{aligned}
\left| \int_{[0,1)} \overline{f(t)} \left(g(rt) + t g'(rt)\right) d\mu(t) \right| &\leq C \|g\|_{H^{q'}} \int_{[0,1)} \left(\frac{1}{(1-t)^{\frac{1}{q'}}} + \frac{t}{(1-t)^{\frac{1}{q'}+1}} \right) |f(t)| \, d\mu(t) \\
&\leq C \|g\|_{H^{q'}} \int_{[0,1)} |f(t)| \, d\nu(t) \\
&\leq C \|f\|_{\mathcal{A}^p} \|g\|_{H^{q'}}, \quad f \in \mathcal{A}^p, g \in H^{q'}.
\end{aligned}$$

This is equivalent to saying that $\mathcal{DH}_\mu$ is a bounded operator from $\mathcal{A}^p$ into H^q.

(ii) Consider the case $q = 1$. Using (3.6) and Fefferman's duality theorem, which says that $(H^1)^* \cong BMOA$ and $(VMOA)^* \cong H^1$, under the Cauchy pairing

$$\langle f, g \rangle = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \overline{f(re^{i\theta})} g(re^{i\theta}) d\theta, \quad f \in H^1, \ g \in BMOA \ (resp., VMOA), \tag{3.8}$$

it implies that $\mathcal{DH}_\mu$ is a bounded operator from $\mathcal{A}^p$ into H^1 if and only if

$$\left| \int_{[0,1)} \overline{f(t)} \left(g(rt) + t g'(rt)\right) d\mu(t) \right| \leq C \|f\|_{\mathcal{A}^p} \|g\|_{BMOA}, \quad f \in \mathcal{A}^p, \ g \in VMOA, \ 0 \leq r < 1.$$

Suppose that $\mathcal{DH}_\mu$ is a bounded operator from $\mathcal{A}^p$ into H^1. For $0 < a < 1$, take test functions

$$\begin{aligned}
f_a(z) &= \left(\frac{1-a^2}{(1-az)^2}\right)^{\frac{2}{p}}, \\
g_a(z) &= \log \frac{e}{1-az}, \quad z \in \mathbb{D}.
\end{aligned}$$

A simply calculation shows that $\sup_{0<a<1}\|f_a\|_{\mathcal{A}^p}<\infty$ and $\sup_{0<a<1}\|g_a\|_{BMOA}<\infty$. Hence, it follows that

$$\begin{aligned}\infty &> C\sup_{0<a<1}\|f_a\|_{\mathcal{A}^p}\sup_{0<a<1}\|g_a\|_{BMOA}\\ &\geq C\left|\int_{[0,1)}\overline{f_a(t)}\,(g_a(rt)+tg_a'(rt))\,d\mu(t)\right|\\ &\geq C\int_{[a,1)}\left(\frac{1-a^2}{(1-at)^2}\right)^{\frac{2}{p}}\left(\log\frac{e}{1-art}+\frac{art}{1-art}\right)d\mu(t)\\ &\geq C\frac{\mu([a,1))}{(1-a^2)^{\frac{2}{p}+1}}.\end{aligned}$$

This is equivalent to saying that μ is a $(\frac{2}{p}+1)$-Carleson measure.

On the contrary, suppose that μ is a $(\frac{2}{p}+1)$-Carleson measure, and let $d\nu(t)=\frac{1}{1-t}d\mu(t)$. By [17, Theorem 3.2], we obtain that ν is a $\frac{2}{p}$-Carleson measure. For any function $g\in\mathscr{B}$, it is well known that

$$|g(z)|\leq C\log\frac{e}{1-|z|}\|g\|_{\mathscr{B}},\quad |g'(z)|\leq\frac{C\|g\|_{\mathscr{B}}}{1-|z|},\quad z\in\mathbb{D}. \tag{3.9}$$

Using this, (3.4), (3.6) and $BMOA\subset\mathscr{B}$, we obtain that

$$\begin{aligned}\left|\int_{[0,1)}\overline{f(t)}\,(g(rt)+tg'(rt))\,d\mu(t)\right| &\leq C\|g\|_{\mathscr{B}}\int_{[0,1)}\left(\log\frac{1}{1-t}+\frac{t}{1-t}\right)|f(t)|\,d\mu(t)\\ &\leq C\|g\|_{BMOA}\int_{[0,1)}|f(t)|\,d\nu(t)\\ &\leq C\|g\|_{BMOA}\|f\|_{\mathcal{A}^p},\quad f\in\mathcal{A}^p, g\in BMOA.\end{aligned}$$

This is equivalent to saying that $\mathcal{DH}_\mu$ is a bounded operator from $\mathcal{A}^p$ into H^1.

(*iii*) Consider the case $0<q<1$. We recall that B_q can be identified with the dual of a certain subspace X of H^∞ under the pairing

$$\langle f,g\rangle=\lim_{r\to1^-}\frac{1}{2\pi}\int_0^{2\pi}\overline{f(re^{i\theta})}g(e^{i\theta})d\theta,\quad f\in B_q, g\in X. \tag{3.10}$$

Using this and (3.6), it implies that $\mathcal{DH}_\mu$ is a bounded operator from $\mathcal{A}^p$ into B_q if and only if

$$\left|\int_{[0,1)}\overline{f(t)}\,(g(t)+tg'(t))\,d\mu(t)\right|\leq C\|f\|_{\mathcal{A}^p}\|g\|_X,\quad f\in\mathcal{A}^p,\ g\in X,\ 0\leq r<1.$$

Suppose that $\mathcal{DH}_\mu$ is a bounded operator from $\mathcal{A}^p$ into B_q. For $0<a<1$, take test functions

$$\begin{aligned}f_a(z)&=\left(\frac{1-a^2}{(1-az)^2}\right)^{\frac{2}{p}},\\ g_a(z)&=\frac{1-a^2}{1-az},\quad z\in\mathbb{D}.\end{aligned}$$

A simply calculation shows that $\sup_{0<a<1}\|f_a\|_{\mathcal{A}^p}<\infty$ and $\sup_{0<a<1}\|g_a\|_X<\infty$. Hence, it follows that

$$\begin{aligned}\infty &> C\sup_{0<a<1}\|f_a\|_{\mathcal{A}^p}\sup_{0<a<1}\|g_a\|_X\\ &\geq C\left|\int_{[0,1)}\overline{f_a(t)}\,(g_a(t)+tg_a'(t))\,d\mu(t)\right|\\ &\geq C\int_{[a,1)}\left(\frac{1-a^2}{(1-az)^2}\right)^{\frac{2}{p}}\left(\frac{1-a^2}{1-at}+\frac{a(1-a^2)}{(1-at)^2}\right)d\mu(t)\\ &\geq C\frac{\mu([a,1))}{(1-a^2)^{\frac{2}{p}+1}}.\end{aligned}$$

This is equivalent to saying that μ is a $(\frac{2}{p}+1)$-Carleson measure.

On the contrary, suppose that μ is a $(\frac{2}{p}+1)$-Carleson measure, and let $d\nu(t)=\frac{1}{1-t}d\mu(t)$. By [17, Theorem 3.2], we obtain that ν is a $\frac{2}{p}$-Carleson measure. Hence, it follows that

$$\begin{aligned}\left|\int_{[0,1)}\overline{f(t)}\,(g(t)+tg'(t))\,d\mu(t)\right| &\leq C\|g\|_X\int_{[0,1)}\left(1+\frac{t}{1-t}\right)|f(t)|\,d\mu(t)\\ &\leq C\|g\|_X\int_{[0,1)}|f(t)|\,d\nu(t)\\ &\leq C\|g\|_X\|f\|_{\mathcal{A}^p},\quad f\in\mathcal{A}^p, g\in X.\end{aligned}$$

This is equivalent to saying that $\mathcal{DH}_\mu$ is a bounded operator from $\mathcal{A}^p$ into B_q. □

Theorem 3.2. *Suppose* $1<p\leq q<\infty$, *and let* μ *be a positive Borel measure on* $[0,1)$, *which satisfies the conditions in Lemma* 3.2.

(*i*) *If* $\mathcal{DH}_\mu$ *is a bounded operator from* $\mathcal{A}^p$ *into* H^q, *then* μ *is a* $(\frac{2}{p}+\frac{1}{q'}+1)$-*Carleson measure.*
(*ii*) *If* μ *is a* $(\frac{2}{p}+\frac{1}{q'}+1+\varepsilon)$-*Carleson measure for any* $\varepsilon>0$, *then* $\mathcal{DH}_\mu$ *is a bounded operator from* $\mathcal{A}^p$ *into* H^q.

Proof. (*i*) The proof is the same as that of Theorem 3.1(i) and we omit the details here.

(*ii*) Suppose that μ is a $(\frac{2}{p}+\frac{1}{q'}+1+\varepsilon)$-Carleson measure and let $d\nu(t)=\frac{1}{1-t}d\mu(t)$, then we obtain that ν is a $(\frac{2}{p}+\frac{1}{q'}+\varepsilon)$-Carleson measure. Take $s=1+\frac{p}{2q'}$, then $s'=1+\frac{2q'}{p}$ and $\frac{2s}{p}=\frac{s'}{q'}=\frac{2}{p}+\frac{1}{q'}$. By (3.4), we obtain that

$$\left(\int_{[0,1)}|f(t)|^s d\nu(t)\right)^{\frac{1}{s}}\leq C\|f\|_{\mathcal{A}^p},\qquad \text{for any } f\in\mathcal{A}^p.$$

and

$$\int_{[0,1)}\left(\frac{1}{(1-t)^{\frac{1}{q'}}}\right)^{s'}d\nu(t)<\infty.$$

Using Hölder's inequality, it follows that

$$\begin{aligned}\left|\int_{[0,1)} \overline{f(t)}\,(g(rt)+tg'(rt))\,d\mu(t)\right| &\lesssim \|g\|_{H^{q'}} \int_{[0,1)} |f(t)| \left(\frac{1}{(1-t)^{\frac{1}{q'}+1}}\right) d\mu(t) \\ &= \|g\|_{H^{q'}} \int_{[0,1)} |f(t)| \left(\frac{1}{(1-t)^{\frac{1}{q'}}}\right) d\nu(t) \\ &\le C\|g\|_{H^{q'}} \left(\int_{[0,1)} |f(t)|^{s}\, d\nu(t)\right)^{\frac{1}{s}} \left(\int_{[0,1)} \frac{1}{(1-t)^{\frac{s'}{q'}}} d\nu(t)\right)^{\frac{1}{s'}} \\ &\le C\|f\|_{\mathcal{A}^p}\|g\|_{H^{q'}}.\end{aligned}$$

This is equivalent to saying that $\mathcal{DH}_\mu$ is a bounded operator from $\mathcal{A}^p$ into H^q. □

4. Compactness of $\mathcal{DH}_\mu$ from $\mathcal{A}^p$ into H^q

In this section, we characterize those measure μ such that $\mathcal{DH}_\mu$ is a compact operator from $\mathcal{A}^p$ into H^q, where $0 < p < \infty, 0 < q < \infty$.

Lemma 4.1. *Let $0 < p < \infty, 0 < q < \infty$ and $\mathcal{DH}_\mu$ be a bounded operator from $\mathcal{A}^p$ into H^q. Then $\mathcal{DH}_\mu$ is a compact operator if and only if $\mathcal{DH}_\mu(f_n) \to 0$ in H^q, for any bounded sequence $\{f_n\}$ in $\mathcal{A}^p$, which converges to 0 uniformly on every compact subset of $\mathbb{D}$.*

Proof. The proof is similar to that of in [18, Proposition 3.11], and we omit the details. □

Theorem 4.1. *Suppose $0 < p \le 1$, and let μ be a positive Borel measure on $[0, 1)$, which satisfies the conditions in Lemma 3.2.*

(i) If $q \ge p$ and $q > 1$, $\mathcal{DH}_\mu$ is a compact operator from $\mathcal{A}^p$ into H^q if and only if μ is a vanishing $(\frac{2}{p} + \frac{1}{q'} + 1)$-Carleson measure.
(ii) If $q \ge p$ and $q = 1$, $\mathcal{DH}_\mu$ is a compact operator from $\mathcal{A}^p$ into H^q if and only if μ is a vanishing $(\frac{2}{p} + 1)$-Carleson measure.
(iii) If $q \ge p$ and $0 < q < 1$, $\mathcal{DH}_\mu$ is a compact operator from $\mathcal{A}^p$ into B_q if and only if μ is a vanishing $(\frac{2}{p} + 1)$-Carleson measure.

Before giving the proof of Theorem 4.1, we recall some facts about Carleson measures. If μ is a s-Carleson measure, then we define $\mathcal{N}_1(\mu_r) = \sup_{I\subset\partial\mathcal{D}} \frac{\mu(S(I))}{|I|^s}$ to be the Carleson norm of μ and $\mathcal{N}_2(\mu_r)$ denote the norm of identity mapping i from H^q into $\mathcal{L}^q(\mathbb{D}, \mu)$. It is well known that the norms $\mathcal{N}_1(\mu_r)$ and $\mathcal{N}_2(\mu_r)$ are equivalent. For $r \in (0, 1)$, set $d\mu_r(z) = \chi_{r<|z|<1}(t)d\mu(t)$. Then $d\mu(t)$ is a vanishing s-Carleson measure if and only if

$$\mathcal{N}_1(\mu_r) \to 0 \ \text{(equivalently, } \mathcal{N}_2(\mu_r) \to 0), \quad \text{as} \quad r \to 1^-.$$

Proof. (*i*) Consider the case $q > 1$. Assume that $\mathcal{DH}_\mu$ is a compact operator from $\mathcal{A}^p$ into H^q. Let $\{a_n\} \subset (0, 1)$ be any sequence with $a_n \to 1$. Set

$$f_{a_n}(z) = \left(\frac{1-a_n^2}{(1-a_nz)^2}\right)^{\frac{2}{p}}, \quad z \in \mathbb{D}.$$

We obtain that $f_{a_n}(z) \in \mathcal{A}^p$ and $\sup_{n\geq 1} \|f_{a_n}\|_{\mathcal{A}^p} < \infty$, so $\{f_{a_n}\}$ is a bounded sequence on $\mathcal{A}^p$, which converges to 0 on any compact subset of $\mathbb{D}$. Then we obtain that $\mathcal{DH}_\mu(f_{a_n}) \to 0$ in H^q by applying Theorem 4.1. This and (3.6) imply that

$$\begin{aligned} &\lim_{n\to\infty} \left| \int_{[0,1)} f_{a_n}(t)\,(g(rt) + tg'(rt))\, d\mu(t) \right| \\ &= \lim_{n\to\infty} \left| \int_0^{2\pi} \overline{\mathcal{DH}_\mu(f_{a_n})(re^{i\theta})} g(re^{i\theta}) d\theta \right| = 0, \quad g \in H^{q'}. \end{aligned} \tag{4.1}$$

Next, take the function

$$g_{a_n}(z) = \left(\frac{1 - a_n^2}{(1 - a_n z)^2} \right)^{\frac{1}{q'}}, \quad z \in \mathbb{D}.$$

It is easy to check that $g_{a_n}(z) \in H^{q'}$ and $\sup_{n\geq 1} \|g_{a_n}\|_{H^{q'}} < \infty$. Then it follows that

$$\begin{aligned} &\int_{[0,1)} f_{a_n}(t) \left(g_{a_n}(rt) + tg'_{a_n}(rt) \right) d\mu(t) \\ \geq &C \int_{a_n}^1 \left(\frac{1 - a_n^2}{(1 - a_n t)^2} \right)^{\frac{2}{p}} \left(\left(\frac{1 - a_n^2}{(1 - a_n t)^2} \right)^{\frac{1}{q'}} + \frac{2a_n t}{q'} \left(\frac{(1 - a_n^2)}{(1 - a_n t)^{q'+2}} \right)^{\frac{1}{q'}} \right) d\mu(t) \\ \geq &C \frac{\mu([a_n, 1))}{(1 - a_n^2)^{\frac{2}{p} + \frac{1}{q'} + 1}}. \end{aligned}$$

Hence, we obtain that

$$\lim_{a_n \to 1^-} \frac{\mu([a_n, 1))}{(1 - a_n^2)^{\frac{2}{p} + \frac{1}{q'} + 1}} = 0.$$

This is equivalent to saying that μ is a vanishing $(\frac{2}{p} + \frac{1}{q'} + 1)$-Carleson measure.

Suppose that μ is a vanishing $(\frac{2}{p} + \frac{1}{q'} + 1)$-Carleson measure, and let $d\nu(t) = \frac{1}{(1-t)^{\frac{1}{q'}+1}} d\mu(t)$. By applying [17, Lemma 3.2], we obtain that ν is a vanishing $\frac{2}{q}$-Carleson measure. For $r \in (0, 1)$, let $d\nu_r(z) = \chi_{r<|z|<1}(t) d\nu(t)$ and $\mathcal{N}$ be the norm of identity mapping i, then $\mathcal{N}(\nu_r) \to 0 (r \to 1^-)$. Take a bounded sequence $\{f_n\}_{n=1}^\infty$ in $\mathcal{A}^p$, and $\{f_n\}_{n=1}^\infty$ uniformly converges to 0 on each compact subset of $\mathbb{D}$. For $0 < r < 1$ and $g \in H^{q'}$, it follows that

$$\begin{aligned} &\int_{[0,1)} |f_n(t)| \left| (g(rt) + tg'(rt)) \right| d\mu(t) \\ \leq &\int_{[0,r)} |f_n(t)| \left| (g(rt) + tg'(rt)) \right| d\mu(t) + C\|g\|_{H^{q'}} \int_{[r,1)} |f_n(t)| \left(\frac{1}{(1 - t)^{\frac{1}{q'}}} + \frac{t}{(1 - t)^{\frac{1}{q'}+1}} \right) d\mu(t) \\ \leq &\int_{[0,r)} |f_n(t)| \left| (g(rt) + tg'(rt)) \right| d\mu(t) + C\|g\|_{H^{q'}} \int_{[0,1)} |f_n(t)|\, d\nu_r(t) \\ \leq &\int_{[0,r)} |f_n(t)| \left| (g(rt) + tg'(rt)) \right| d\mu(t) + C\|g\|_{H^{q'}} \|g\|_{\mathcal{A}^p} \mathcal{N}(\nu_r). \end{aligned}$$

Then $\mathcal{N}(\nu_r) \to 0 (r \to 1^-)$ and the condition $\{f_n\} \to 0$ uniformly on each compact subset of $\mathbb{D}$ imply that

$$\lim_{n\to\infty} \int_{[0,1)} |f_n(t)| \left| (g(rt) + tg'(rt)) \right| d\mu(t) = 0.$$

combining this, (3.6) and (4.1), we obtain that

$$\lim_{n\to\infty}\left|\int_0^{2\pi}\overline{\mathcal{DH}_\mu(f_{a_n})(re^{i\theta})}g(re^{i\theta})d\theta\right| = 0.$$

Hence, $\mathcal{DH}_\mu(f_n) \to 0$ in H^q. This is equivalent to saying that $\mathcal{DH}_\mu$ is a compact operator from $\mathcal{A}^p$ into H^q.

(ii) Consider the case $q = 1$. Suppose that μ is a vanishing $(\frac{2}{p}+1)$-Carleson measure and let $d\nu(t) = \frac{1}{1-t}d\mu(t)$. By applying [17, Lemma 3.2], we obtain that ν is a vanishing $\frac{2}{q}$-Carleson measure. For $r \in (0,1)$, let $d\nu_r(z) = \chi_{r<|z|<1}(t)d\nu(t)$ and $\mathcal{N}$ be the norm of identity mapping i, then $\mathcal{N}(\nu_r) \to 0(r \to 1^-)$. Take a bounded sequence $\{f_n\}_{n=1}^\infty$ in $\mathcal{A}^p$, and $\{f_n\}_{n=1}^\infty$ uniformly converges to 0 on each compact subset of $\mathbb{D}$. For $0 < r < 1$ and $g \in VMOA$, arguing as in the proof of Theorem 3.1(ii), it follows that

$$\begin{aligned}&\int_{[0,1)}|f_n(t)|\left|(g(rt)+tg'(rt))\right|d\mu(t)\\ \leq&\int_{[0,r)}|f_n(t)|\left|(g(rt)+tg'(rt))\right|d\mu(t)+C\|g\|_{BMOA}\int_{[0,1)}|f_n(t)|\,d\nu_r(t)\\ \leq&\int_{[0,r)}|f_n(t)|\left|(g(rt)+tg'(rt))\right|d\mu(t)+C\|g\|_{BMOA}\|f\|_{\mathcal{A}^p}\mathcal{N}(\nu_r).\end{aligned}$$

Then $\mathcal{N}(\nu_r) \to 0(r \to 1^-)$ and the condition $\{f_n\} \to 0$ uniformly on each compact subset of $\mathbb{D}$ impliy that

$$\begin{aligned}&\lim_{n\to\infty}\left|\int_0^{2\pi}\overline{\mathcal{DH}_\mu(f_{a_n})(re^{i\theta})}g(re^{i\theta})d\theta\right|\\ =&\lim_{n\to\infty}\int_{[0,1)}|f_n(t)|\left|(g(rt)+tg'(rt))\right|d\mu(t)=0.\end{aligned}$$

Hence, $\mathcal{DH}_\mu(f_n) \to 0$ in H^1. This is equivalent to saying that $\mathcal{DH}_\mu$ is a compact operator from $\mathcal{A}^p$ into H^1.

Suppose that $\mathcal{DH}_\mu$ is a compact operator from $\mathcal{A}^p$ into H^1. Let $\{a_n\} \subset (0,1)$ be any sequence with $a_n \to 1$ and f_{a_n} as defined in (i), then we obtain that $\mathcal{DH}_\mu(f_{a_n}) \to 0$ in H^1. Next, set

$$g_{a_n}(z) = \log\frac{e}{1-a_n z}.$$

It is easy to check that $g_{a_n}(z) \in VMOA$. Then it implies that

$$\begin{aligned}&\int_{[0,1)}f_{a_n}(t)\left(g_{a_n}(rt)+tg'_{a_n}(rt)\right)d\mu(t)\\ \geq&C\int_{a_n}^1\left(\frac{1-a_n^2}{(1-a_nt)^2}\right)^{\frac{2}{p}}\left(\log\frac{e}{1-a_nrt}+\frac{a_nrt}{1-a_nrt}\right)d\mu(t)\\ \geq&C\frac{\mu([a_n,1))}{(1-a_n^2)^{\frac{2}{p}+1}}.\end{aligned}$$

Hence, we obtain that

$$\lim_{a_n\to1^-}\frac{\mu([a_n,1))}{(1-a_n^2)^{\frac{2}{p}+1}} = 0.$$

This is equivalent to saying that μ is a vanishing $(\frac{2}{p}+1)$-Carleson measure.

(iii) From now on, it is similar to the proof of (ii), we omit the details. □

Theorem 4.2. *Suppose* $1 < p \leq q < \infty$, *and let* μ *be a positive Borel measure on* $[0, 1)$, *which satisfies the conditions in Lemma* 3.2.

(i) *If* $\mathcal{DH}_\mu$ *is a compact operator from* $\mathcal{A}^p$ *into* H^q, *then* μ *is a vanishing* $(\frac{2}{p}+\frac{1}{q'}+1)$-*Carleson measure.*

(ii) *If* μ *is a vanishing* $(\frac{2}{p}+\frac{1}{q'}+1+\varepsilon)$-*Carleson measure for any* $\varepsilon > 0$, *then* $\mathcal{DH}_\mu$ *is a compact operator from* $\mathcal{A}^p$ *into* H^q.

Proof. The proof is similar to that of Theorem 3.2 and Theorem 4.1(i), we omit the details here. □

5. Conclusions

In this article, we characterize the positive Borel measure μ such that $\mathcal{DH}_\mu$ is bounded (resp. compact) from the Bergman space $\mathcal{A}^p$ into the Hardy space H^q, where $0 < p < \infty, 0 < q < \infty$.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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