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Research article

A higher order evolution inequality with a gradient term in the exterior of the half-ball

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Abstract: We study the existence and nonexistence of weak solutions to a semilinear higher order (in time) evolution inequality involving a convection term in the exterior of the half-ball, under Dirichlet-type boundary conditions. A weight function of the form $|x|^a$ is allowed in front of the power nonlinearity. When $a > -2$, we show that the dividing line with respect to existence or nonexistence is given by a critical exponent (Fujita critical exponent), which depends on the parameters of the problem, but independent of the order of the time-derivative. Our study yields naturally optimal nonexistence results for the corresponding stationary problem.

Keywords: evolution inequality; convection term; half-ball; critical exponent
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1. Introduction

Let $\Omega = \{x \in \overline{\mathbb{R}^N_+} : |x| \geq 1\}$, $\mathbb{R}^N_+ = \{x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N : x_N > 0\}$ and $N \geq 2$. The boundary of Ω is denoted by

$$\partial\Omega = \bigcup_{i=0}^{1} \Gamma_i,$$

where $\Gamma_0 = \{x \in \Omega : x_N = 0\}$ and $\Gamma_1 = \{x \in \Omega : x_N > 0, |x| = 1\}$. We are concerned with the existence an nonexistence of weak solutions to the evolution inequality

$$\frac{\partial^k u}{\partial t^k} - \Delta(|u|^{m-1}u) + \frac{\mu}{|x|^2} x \cdot \nabla(|u|^{m-1}u) \geq |x|^a |u|^p \quad \text{in } (0, \infty) \times \Omega, \tag{1.1}$$

where $u = u(t, x)$, $k \geq 1$ is an integer, $p > m \geq 1$, $\mu, a \in \mathbb{R}$ and $\cdot$ is the inner product in $\mathbb{R}^N$. Problem (1.1) is considered under the Dirichlet-type boundary conditions

$$\begin{cases} u \geq 0 \quad \text{on } (0,\infty) \times \Gamma_0, \\ |u|^{m-1}u \geq f \quad \text{on } (0,\infty) \times \Gamma_1, \end{cases} \tag{1.2}$$

where $f = f(x)$.

The issue of existence and nonexistence of solutions to higher order (in time) evolution inequalities has been studied in several papers. For instance, Hamidi and Laptev [1] investigated the nonexistence of weak solutions to higher-order evolution inequalities of the form

$$\begin{cases} \dfrac{\partial^k u}{\partial t^k} - \Delta u + \dfrac{\lambda}{|x|^2} u \geq |u|^p \quad \text{in } (0,\infty) \times \mathbb{R}^N, \\ \\ \dfrac{\partial^{k-1} u}{\partial t^{k-1}}(0, x) \geq 0 \quad \text{in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $N \geq 3$, $\lambda \geq -\left(\frac{N-2}{2}\right)^2$ and $p > 1$. Namely, it was shown that, if one of the following assumptions is satisfied:

$$\lambda \geq 0, \ 1 < p \leq 1 + \frac{2}{\frac{2}{k} + s^*};$$

or

$$-\left(\frac{N-2}{2}\right)^2 \leq \lambda < 0, \ 1 < p \leq 1 + \frac{2}{\frac{2}{k} - s_*},$$

where

$$s^* = \frac{N-2}{2} + \sqrt{\lambda + \left(\frac{N-2}{2}\right)^2}, \ s_* = s^* + 2 - N,$$

then (1.3) admits no nontrivial weak solution. In [2], Caristi considered evolution inequalities of the form

$$\frac{\partial^k u}{\partial t^k} - |x|^\sigma \Delta^m u \geq |u|^q \quad \text{in } (0,\infty) \times \mathbb{R}^N, \tag{1.4}$$

where m is a positive integer, $q > 1$ and $\sigma \leq 2m$. In the case $\sigma = 2m$ (critical degeneracy case) it was proven that, if $k \geq 2$, $\frac{\partial^{k-1} u}{\partial t^{k-1}}(0, \cdot)|x|^{-N} \in L^1(\mathbb{R}^N)$ with a positive average, and one of the following conditions holds:

(i) $N \neq 2(j+1)$ for $j = 0, m-1$ and $1 < q \leq k+1$;
(ii) $N = 2(j+1)$ with $j = 0, \cdots, m-1$ and $q > 1$,

then (1.4) has no weak solution. In the case $\sigma < 2m$ (the subcritical degeneracy case), it was shown that, if $k \geq 2$, $\frac{\partial^j u}{\partial t^j}(0, \cdot)|x|^{-\sigma} \in L^1_{\text{loc}}(\mathbb{R}^N)$ for $j = 0, k-2$, $\frac{\partial^{k-1} u}{\partial t^{k-1}}(0, \cdot)|x|^{-\sigma} \in L^1(\mathbb{R}^N)$ with a positive average, and

$$q\,(k(N-2m) + 2m - \sigma) \leq Nk + 2m - \sigma(k+1),$$

then (1.4) has no weak solution. Very recently, Filippucci and Ghergu [3] investigated evolution inequalities of the form

$$\frac{\partial^k u}{\partial t^k} + (-\Delta)^m u \geq (K * |u|^p)|u|^q, \quad \text{in } (0,\infty) \times \mathbb{R}^N, \tag{1.5}$$

where $N, k, m \geq 1$ are integers, $p, q > 0$ and $K \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies: $K(|x|) \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\inf_{0<r<R} K(r) = K(R)$ for sufficiently large R. Namely, the authors proved the following results:

(i) If k is an even integer and $q \geq 1$, then (1.5) admits some positive solutions $u \in C^\infty((0,\infty) \times \mathbb{R}^N)$ which verify $\frac{\partial^{k-1} u}{\partial t^{k-1}}(0,\cdot) < 0$ in $\mathbb{R}^N$;

(ii) If $p + q > 2$ and

$$\limsup_{R\to\infty} K(R) R^{\frac{2N+\frac{2m}{k}}{p+q} - N + 2m\left(1-\frac{1}{k}\right)} > 0,$$

then (1.5) has no nontrivial solutions such that

$$\frac{\partial^{k-1} u}{\partial t^{k-1}} \geq 0; \quad \text{or} \quad \frac{\partial^{k-1} u}{\partial t^{k-1}}(0,\cdot) \in L^1(\mathbb{R}^N), \ \int_{\mathbb{R}^N} \frac{\partial^{k-1} u}{\partial t^{k-1}}(0,x)\, dx > 0.$$

Other contributions related to higher order (in time) evolution equations and inequalities can be found in [4–7].

In [8], Zheng and Wang studied the large time behavior of nonnegative solutions to parabolic equations of the form

$$\frac{\partial u}{\partial t} - \Delta u^m - \lambda \frac{x}{|x|^2} \cdot \nabla u^m = |x|^\sigma u^p \ (u \geq 0) \text{ in } (0,\infty) \times \mathbb{R}^N \backslash \overline{\omega}, \tag{1.6}$$

where $k \in \mathbb{R}$, $\sigma > -2$, $p > m \geq 1$ and ω is a bounded domain in $\mathbb{R}^N$ containing the origin with a smooth boundary $\partial\omega$. Problem (1.6) was investigated under the homogeneous Neumann boundary condition

$$\frac{\partial u^m}{\partial \nu}(t,x) = 0 \quad \text{on } (0,\infty) \times \partial\omega \tag{1.7}$$

and the homogeneous Dirichlet boundary condition

$$u(t,x) = 0 \quad \text{on } (0,\infty) \times \partial\omega. \tag{1.8}$$

For problem (1.6) under the boundary condition (1.7), it was shown that (under a certain regularity on the geometry of ω)

$$p^* = \begin{cases} m + \frac{\sigma+2}{N+\lambda}, & \text{if } \lambda > -N, \\ \infty, & \text{if } \lambda \leq -N \end{cases} \tag{1.9}$$

is critical in the sense of Fujita. When $\lambda > 2 - N$, it was proven that problem (1.6) under the boundary condition (1.8), admits the same Fujita critical exponent p^*. Other contributions related to parabolic equations involving terms of the form $b(x) \cdot \nabla u^m$ can be found in [9–13] (see also the references therein). Notice that in all the above mentioned references, only the parabolic case has been treated. Moreover, the considered solutions have been assumed to be positive. Very recently, in [14], the authors considered evolution inequalities of the form

$$\frac{\partial^k u}{\partial t^k} - \Delta(|u|^{m-1}u) - \lambda \frac{x}{|x|^2} \cdot \nabla(|u|^{m-1}u) \geq |x|^\sigma |u|^p \text{ in } (0,\infty) \times B_1^c, \tag{1.10}$$

under different types of boundary conditions, where $p > m \geq 1$, B_1 denotes the open ball of radius 1 centered at the origin point in $\mathbb{R}^N$ with $N \geq 2$ and B_1^c denotes the complement of B_1. For instance, under the Dirichlet-type boundary condition

$$|u(t,x)|^{m-1} u(t,x) \geq f(x) \quad \text{on } (0,\infty) \times \partial B_1,$$

where $f \in L^1(\partial B_1)$ has a positive average, the authors proved that when $\sigma > -2$, (1.10) admits as Fujita critical exponent

$$p_{cr} = \begin{cases} m + \dfrac{m(\sigma+2)}{\lambda+N-2} & \text{if} \quad \lambda > 2-N, \\ \infty & \text{if} \quad \lambda \le 2-N. \end{cases}$$

More precisely, the authors proved the following results:

(i) If $\lambda \le 2-N$ and $f \in L^1(\partial B_1)$ has a positive average, then for all $p > m$, (1.10) admits no weak solution;
(ii) If $\lambda > 2-N$ and $f \in L^1(\partial B_1)$ has a positive average, then for all $m < p \le p_{cr}$, (1.10) admits no weak solution;
(iii) If $\lambda > 2-N$ and $p > p_{cr}$, then (1.10) admits (stationary) solutions for some $f > 0$.

For more contributions related to the issue of existence and nonexistence of solutions to evolution equations and inequalities in exterior domains, see e.g., [15–19].

Our aim in this paper is to study the influence of the obstacle domain on the critical behavior of (1.10) by considering the half-unit ball instead of the unit ball. Before presenting our main results, we need to define weak solutions to (1.1) and (1.2).

Let

$$D = (0,\infty)\times\Omega \quad \text{and} \quad \partial D_i = (0,\infty)\times\Gamma_i,\ i = 0,1.$$

Notice that $\partial D_i \subset D$ for all $i = 0,1$. We introduce the functional space $\mathbb{V}$ defined as follows.

Definition 1.1. A function $\varphi = \varphi(t,x)$ belongs to $\mathbb{V}$, if the following conditions are satisfied:

(i) $\varphi \in C_{t,x}^{k,2}(D)$, $\varphi \ge 0$;
(ii) $\operatorname{supp}(\varphi) \subset\subset D$;
(iii) $\varphi = 0$ on ∂D_i, $i = 0,1$;
(iv) $\dfrac{\partial \varphi}{\partial \nu_i} \le 0$ on ∂D_i, $i = 0,1$, where ν_i denotes the outward unit normal vector on Γ_i, relative to D.

Using standard integrations by parts, we define weak solutions to (1.1) and (1.2) as follows.

Definition 1.2. We say that $u \in L^p_{\text{loc}}(D)$ is a weak solutions to (1.1) and (1.2), if

$$\begin{aligned} &\int_D |x|^a |u|^p \varphi\, dx\, dt - \int_{\partial D_1} \frac{\partial \varphi}{\partial \nu_1} f(x)\, d\sigma\, dt \\ &\le (-1)^k \int_D u \frac{\partial^k \varphi}{\partial t^k}\, dx\, dt - \int_D |u|^{m-1} u \left(\Delta\varphi + \mu \operatorname{div}\left(\frac{\varphi x}{|x|^2}\right)\right) dx\, dt \end{aligned} \tag{1.11}$$

for every $\varphi \in \mathbb{V}$.

For $\mu \in \mathbb{R}$, let us introduce the parameter

$$\alpha_\mu = -\frac{N+\mu}{2} + \sqrt{\mu + \left(\frac{N-\mu}{2}\right)^2}. \tag{1.12}$$

Our main results are stated in the following theorem.

Theorem 1.3. *Let $N \geq 2$, $k \geq 1$ (an integer) and $\mu, a \in \mathbb{R}$.*

(I) *Let $f \in L^1(\Gamma_1)$ be such that*

$$\int_{\Gamma_1} f(x) x_N \, d\sigma > 0. \tag{1.13}$$

Assume that

$$p > m, \ (\alpha_\mu + N - 1)p - m(\alpha_\mu + 1 + a + N) \leq 0. \tag{1.14}$$

Then (1.1) *and* (1.2) *admits no weak solution.*

(II) *If*

$$p > m, \ (\alpha_\mu + N - 1)p - m(\alpha_\mu + 1 + a + N) > 0, \tag{1.15}$$

then (1.1) *and* (1.2) *admits (stationary) solutions in the sense of Definition 1.2, for some $f > 0$.*

The proof of part (I) of Theorem 1.3 is based on nonlinear capacity estimates specifically adapted to the domain, the operator $-\Delta + \frac{\mu}{|x|^2} x \cdot \nabla$ and the boundary conditions (1.2). Part (II) is established by the construction of expilicit solutions.

Remark 1.4. (i) Let us point out that the used method in [8] for proving the blow-up of solutions to (1.6) requires the positivity of u. Namely, the authors used that functions of the form

$$\ell \mapsto w_\ell(t) := \int_{\mathbb{R}^N \setminus \omega} u(t, x) \psi_l(|x|) \, dx$$

are nondecreasing for sufficiently large ℓ, where $\psi_\ell \geq 0$ is a certain cut-off function. In this paper, no restriction on the sign of solutions is imposed. Moreover, even in the case of positive solutions, it is difficult to use the method in [8] for proving the blow-up of solutions in the hyperbolic case. Namely, in order to show the blow-up of solutions to (1.6), the authors proved that the function w_ℓ defined above, satisfies the differential inequality

$$\frac{dw_\ell}{dt} \geq \gamma w_\ell^p,$$

for a certain constant $\gamma > 0$. A such inequality is related essentially to the parabolic nature of the problem.

(ii) The emphasis of this paper is on blow up results. The existence result provided by part (II) of Theorem 1.3 is a consequence of elliptic results. We refer to [20, 21], where some regularization methods to deal with the degeneracy were used to obtain the strong solution with latent singularity. We refer also to [22, 23], where global solutions have been obtained following the standard gradient flow method. It will be interested to see if such methods can be adapted to the case of problem (1.1).

(iii) It is not difficult to show that for all $\mu \in \mathbb{R}$, one has

$$\alpha_\mu + N - 1 > 0.$$

Hence, (1.14) is equivalent to

$$m < p \leq m + \frac{m(a+2)}{\alpha_\mu + N - 1}, \ a > -2.$$

(iv) From the above remark, we observe that (1.15) is equivalent to

$$a \leq -2; \quad \text{or} \quad p > m + \frac{m(a+2)}{\alpha_\mu + N - 1}, \ a > -2.$$

Remark 1.5. (i) From Remark 1.4, we deduce that, if $a \leq -2$, then (1.1) and (1.2) admits no critical behavior. However, if $a > -2$, then (1.1) and (1.2) admits as Fujita critical exponent the real number

$$p^* = p^*(m, a, \mu, N) = m + \frac{m(a+2)}{\alpha_\mu + N - 1}.$$

(ii) It is interesting to observe that p^* is independent on k. This implies that Theorem 1.3 holds true in the parabolic ($k = 1$) as well as hyperbolic ($k = 2$) case.

Clearly, Theorem 1.3 yields existence and nonexistence results for the corresponding stationary problem

$$-\Delta(|u|^{m-1}u) + \frac{\mu}{|x|^2} x \cdot \nabla(|u|^{m-1}u) \geq |x|^a |u|^p \quad \text{in } \Omega \tag{1.16}$$

under the Dirichlet-type boundary conditions

$$\begin{cases} u \geq 0 & \text{on } \Gamma_0, \\ |u|^{m-1}u \geq f & \text{on } \Gamma_1. \end{cases} \tag{1.17}$$

Corollary 1.6. *Let $N \geq 2$ and $\mu, a \in \mathbb{R}$.*

(I) *Let $f \in L^1(\Gamma_1)$ be such that* (1.13) *holds. If* (1.14) *is satisfied, then* (1.16) *and* (1.17) *admits no weak solution.*

(II) *If* (1.15) *holds, then* (1.16) *and* (1.17) *admits solutions for some $f > 0$.*

The rest of the paper is organized as follows. In Section 2, we establish some preliminary lemmas that will be useful in the proof of our main results. Namely, we first prove an a priori estimate for problems (1.1) and (1.2). Next, we construct two families of functions belonging to $\mathbb{V}$. The first family will be used in the proof of part (I) of Theorem 1.3 in the case $(\alpha_\mu + N - 1)p - m(\alpha_\mu + 1 + a + N) < 0$, and the second family will be used in the proof of the critical case $(\alpha_\mu + N - 1)p - m(\alpha_\mu + 1 + a + N) = 0$. Finally, Section 3 is devoted to the proof of Theorem 1.3.

Throughout this paper, the letters C, C_i denote always generic positive constants whose values are unimportant and may vary at different occurrences.

2. Preliminaries

Let $N \geq 2$, $k \geq 1$ (an integer), $p > m \geq 1$, $\mu, a \in \mathbb{R}$ and $f \in L^1(\Gamma_1)$. We denote by L_μ the differential operator given by

$$L_\mu \phi = \Delta \phi + \mu \operatorname{div}\left(\frac{\phi x}{|x|^2}\right).$$

2.1. A priori estimate

For $\varphi \in \mathbb{V}$, we introduce the integral terms

$$\omega_1(\varphi) = \int_{\operatorname{supp}(\varphi)} |x|^{\frac{-a}{p-1}} \varphi^{\frac{-1}{p-1}} \left|\frac{\partial^k \varphi}{\partial t^k}\right|^{\frac{p}{p-1}} dx\, dt \tag{2.1}$$

and

$$\omega_2(\varphi) = \int_{\operatorname{supp}(\varphi)} |x|^{\frac{-am}{p-m}} \varphi^{\frac{-m}{p-m}} |L_\mu \varphi|^{\frac{p}{p-m}} dx\, dt. \tag{2.2}$$

We have the following a priori estimate.

Lemma 2.1. *Let $u \in L^p_{\text{loc}}(D)$ be a weak solution to* (1.1) *and* (1.2)*. Then*

$$-\int_{\partial D_1} \frac{\partial \varphi}{\partial \nu_1} f(x)\, d\sigma\, dt \le C \sum_{i=1}^{2} \omega_i(\varphi), \tag{2.3}$$

for every $\varphi \in \mathbb{V}$, provided that $\omega_i(\varphi) < \infty$, $i = 1, 2$.

Proof. Let $u \in L^p_{\text{loc}}(D)$ be a weak solution to (1.1) and (1.2) and $\varphi \in \mathbb{V}$ be such that $\omega_i(\varphi) < \infty$, $i = 1, 2$. Then, by (1.11), there holds

$$\int_D |x|^a |u|^p \varphi\, dx\, dt - \int_{\partial D_1} \frac{\partial \varphi}{\partial \nu_1} f(x)\, d\sigma\, dt \le \int_D |u| \left| \frac{\partial^k \varphi}{\partial t^k} \right| dx\, dt + \int_D |u|^m |L_\mu \varphi|\, dx\, dt. \tag{2.4}$$

Making use of Young's inequality, we obtain

$$\begin{aligned} \int_D |u| \left| \frac{\partial^k \varphi}{\partial t^k} \right| dx\, dt &= \int_D \left(|x|^{\frac{a}{p}} |u| \varphi^{\frac{1}{p}} \right) \left(|x|^{\frac{-a}{p}} \varphi^{\frac{-1}{p}} \left| \frac{\partial^k \varphi}{\partial t^k} \right| \right) dx\, dt \\ &\le \frac{1}{2} \int_D |x|^a |u|^p \varphi\, dx\, dt + C\omega_1(\varphi). \end{aligned} \tag{2.5}$$

Similarly, we obtain

$$\int_D |u|^m |L_\mu \varphi|\, dx\, dt \le \frac{1}{2} \int_D |x|^a |u|^p \varphi\, dx\, dt + C\omega_2(\varphi). \tag{2.6}$$

Therefore, combining (2.4)–(2.6), we obtain (2.3). □

2.2. Construction of families of functions belonging to $\mathbb{V}$

Let us introduce the function

$$F(x) = x_N |x|^{\alpha_\mu} \left(1 - |x|^{-(N+\mu)-2\alpha_\mu} \right), \quad x \in \Omega, \tag{2.7}$$

where the parameter α_μ is given by (1.12). Elementary calculations show that

$$F \ge 0, \quad L_\mu F = 0 \text{ in } \Omega, \quad F|_{\Gamma_0 \cup \Gamma_1} = 0 \tag{2.8}$$

and

$$\frac{\partial F}{\partial \nu_1}|_{\Gamma_1} = -(N + \mu + 2\alpha_\mu) x_N < 0, \ \frac{\partial F}{\partial \nu_0}|_{\Gamma_0} = -|x|^{\alpha_\mu} \left(1 - |x|^{-(N+\mu)-2\alpha_\mu} \right) < 0. \tag{2.9}$$

Let $\xi, \vartheta, \iota \in C^\infty(\mathbb{R})$ be three cut-off functions satisfying respectively

$$0 \le \xi \le 1, \quad \xi(s) = 1 \text{ if } |s| \le 1, \quad \xi(s) = 0 \text{ if } |s| \ge 2, \tag{2.10}$$

$$0 \le \vartheta \le 1, \quad \vartheta(s) = 1 \text{ if } s \le 0, \quad \vartheta(s) = 0 \text{ if } s \ge 1 \tag{2.11}$$

and

$$\iota \ge 0, \quad \text{supp}(\iota) \subset\subset (0, 1). \tag{2.12}$$

For sufficiently large T, R and ℓ, let

$$\begin{aligned}
\iota_T(t) &= \iota^\ell\left(\frac{t}{T}\right), \quad t > 0, && (2.13)\\
\xi_R(x) &= F(x)\xi^\ell\left(\frac{|x|^2}{R^2}\right), \quad x \in \Omega, && (2.14)\\
\vartheta_R(x) &= F(x)\vartheta^\ell\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right), \quad x \in \Omega. && (2.15)
\end{aligned}$$

Next, we consider functions of the form

$$\varphi(t,x) = \iota_T(t)\xi_R(x), \quad (t,x) \in D \tag{2.16}$$

and

$$\psi(t,x) = \iota_T(t)\vartheta_R(x), \quad (t,x) \in D. \tag{2.17}$$

Lemma 2.2. *For sufficiently large T, R and ℓ, the function φ defined by* (2.16) *belongs to* $\mathbb{V}$.

Proof. By (2.8), (2.10), (2.12)–(2.14) and (2.16), it can be easily seen that properties (i)–(iii) of Definition 1.1 are satisfied. Moreover, for $(t,x) \in \partial D_i$, $i = 0, 1$, one has

$$\begin{aligned}
\frac{\partial \varphi}{\partial \nu_i}(t,x) &= \iota_T(t)\frac{\partial \xi_R}{\partial \nu_i}(x)\\
&= \iota_T(t)\frac{\partial F}{\partial \nu_i}(x), && (2.18)
\end{aligned}$$

which implies by (2.9) that

$$\frac{\partial \varphi}{\partial \nu_i}(t,x) \leq 0, \quad (t,x) \in \partial D_i.$$

This shows that property (iv) of Definition 1.1 is also satisfied. Therefore, $\varphi \in \mathbb{V}$. □

Similarly, using (2.8), (2.9), (2.11), (2.12), (2.15) and (2.17), we obtain the following result.

Lemma 2.3. *For sufficiently large T, R and ℓ, the function ψ defined by* (2.17) *belongs to* $\mathbb{V}$.

2.3. Estimates of $\omega_i(\varphi)$

For sufficiently large T, R and ℓ, let φ be the function defined by (2.16).

Lemma 2.4. *The following estimate holds:*

$$\omega_1(\varphi) \leq CT^{1-\frac{kp}{p-1}}\left(\ln R + R^{\alpha_\mu - \frac{a}{p-1}+N+1}\right). \tag{2.19}$$

Proof. By (2.1) and (2.16), we obtain

$$\omega_1(\varphi) = \left(\int_0^T \iota_T^{\frac{-1}{p-1}}\left|\frac{d^k\iota_T}{dt^k}\right|^{\frac{p}{p-1}}\,dt\right)\left(\int_{1<|x|<\sqrt{2}R,\, x_N>0} |x|^{\frac{-a}{p-1}}\xi_R(x)\,dx\right). \tag{2.20}$$

On the other hand, by (2.12) and (2.13), we obtain

$$\left|\frac{d^k \iota_T}{dt^k}\right| \le CT^{-k}\iota^{\ell-k}\left(\frac{t}{T}\right), \quad 0 < t < T,$$

which yields

$$\begin{aligned}\int_0^T \iota_T^{\frac{-1}{p-1}} \left|\frac{d^k \iota_T}{dt^k}\right|^{\frac{p}{p-1}} dt &\le CT^{\frac{-kp}{p-1}} \int_0^T \iota^{\ell-\frac{kp}{p-1}}\left(\frac{t}{T}\right) dt \\ &= CT^{1-\frac{kp}{p-1}} \int_0^1 \iota^{\ell-\frac{kp}{p-1}}(s)\, ds,\end{aligned}$$

that is,

$$\int_0^T \iota_T^{\frac{-1}{p-1}} \left|\frac{d^k \iota_T}{dt^k}\right|^{\frac{p}{p-1}} dt \le CT^{1-\frac{kp}{p-1}}. \tag{2.21}$$

Moreover, by (2.14), we have

$$\int_{1<|x|<\sqrt{2}R,\, x_N>0} |x|^{\frac{-a}{p-1}} \xi_R(x)\, dx = \int_{1<|x|<\sqrt{2}R,\, x_N>0} |x|^{\frac{-a}{p-1}} F(x)\xi^\ell\left(\frac{|x|^2}{R^2}\right) dx. \tag{2.22}$$

Using (2.7) and (2.10), we obtain

$$\begin{aligned}\int_{1<|x|<\sqrt{2}R,\, x_N>0} |x|^{\frac{-a}{p-1}} F(x)\xi^\ell\left(\frac{|x|^2}{R^2}\right) dx &\le \int_{1<|x|<\sqrt{2}R,\, x_N>0} |x|^{\frac{-a}{p-1}} F(x)\, dx \\ &\le \int_{1<|x|<\sqrt{2}R} |x|^{\alpha_\mu+1-\frac{a}{p-1}}\, dx \\ &= C\int_{r=1}^{\sqrt{2}R} r^{\alpha_\mu-\frac{a}{p-1}+N}\, dr \\ &\le C\left(\ln R + R^{\alpha_\mu-\frac{a}{p-1}+N+1}\right).\end{aligned} \tag{2.23}$$

Hence, in view of (2.20)–(2.23), we obtain (2.19). □

Lemma 2.5. *The following estimate holds:*

$$\omega_2(\varphi) \le CTR^{\frac{(\alpha_\mu+N-1)p-m(\alpha_\mu+1+a+N)}{p-m}}. \tag{2.24}$$

Proof. By (2.2) and (2.16), we have

$$\omega_2(\varphi) = \left(\int_0^T \iota_T\, dt\right)\left(\int_{1<|x|<\sqrt{2}R,\, x_N>0} |x|^{\frac{-am}{p-m}} \xi_R^{\frac{-m}{p-m}} |L_\mu \xi_R|^{\frac{p}{p-m}}\, dx\right). \tag{2.25}$$

By (2.13), we obtain

$$\begin{aligned}\int_0^T \iota_T\, dt &= \int_0^T \iota^\ell\left(\frac{t}{T}\right) dt \\ &= T\int_0^1 \iota^\ell(s)\, ds,\end{aligned}$$

that is,

$$\int_0^T \iota_T \, dt = CT. \tag{2.26}$$

Moreover, by (2.14), for $< |x| < \sqrt{2}R,\ x_N > 0$, we have

$$\begin{aligned}
& L_\mu \xi_R(x) \\
&= L_\mu\left(F(x)\xi^\ell\left(\frac{|x|^2}{R^2}\right)\right) \\
&= \Delta\left(F(x)\xi^\ell\left(\frac{|x|^2}{R^2}\right)\right) + \mu \operatorname{div}\left(\frac{(F(x)\xi^\ell\left(\frac{|x|^2}{R^2}\right)x}{|x|^2}\right) \\
&= \xi^\ell\left(\frac{|x|^2}{R^2}\right)\Delta F(x) + F(x)\Delta\left(\xi^\ell\left(\frac{|x|^2}{R^2}\right)\right) + 2\nabla F(x)\cdot\nabla\left(\xi^\ell\left(\frac{|x|^2}{R^2}\right)\right) \\
&+ \mu\xi^\ell\left(\frac{|x|^2}{R^2}\right)\operatorname{div}\left(\frac{F(x)x}{|x|^2}\right) + \frac{F(x)}{|x|^2}x\cdot\nabla\left(\xi^\ell\left(\frac{|x|^2}{R^2}\right)\right) \\
&= \xi^\ell\left(\frac{|x|^2}{R^2}\right)L_\mu F(x) + F(x)\Delta\left(\xi^\ell\left(\frac{|x|^2}{R^2}\right)\right) + \left(2\nabla F(x) + \frac{F(x)}{|x|^2}x\right)\cdot\nabla\left(\xi^\ell\left(\frac{|x|^2}{R^2}\right)\right) \\
&= \xi^\ell\left(\frac{|x|^2}{R^2}\right)L_\mu F(x) + F(x)\Delta\left(\xi^\ell\left(\frac{|x|^2}{R^2}\right)\right) \\
&+ 2\ell R^{-2}|x|\xi^{\ell-1}\left(\frac{|x|^2}{R^2}\right)\xi'\left(\frac{|x|^2}{R^2}\right)\left(2\nabla F(x)\cdot\frac{x}{|x|} + |x|^{-1}F(x)\right).
\end{aligned}$$

In view of (2.8) ($L_\mu F = 0$), we obtain

$$\begin{aligned}
L_\mu \xi_R(x) \;=\;& F(x)\Delta\left(\xi^\ell\left(\frac{|x|^2}{R^2}\right)\right) \\
&+2\ell R^{-2}|x|\xi^{\ell-1}\left(\frac{|x|^2}{R^2}\right)\xi'\left(\frac{|x|^2}{R^2}\right)\left(2\nabla F(x)\cdot\frac{x}{|x|} + |x|^{-1}F(x)\right),
\end{aligned} \tag{2.27}$$

which implies by (2.10) that

$$\int_{1<|x|<\sqrt{2}R,\ x_N>0} |x|^{\frac{-am}{p-m}}\xi_R^{\frac{-m}{p-m}}|L_\mu\xi_R|^{\frac{p}{p-m}}\,dx = \int_{R<|x|<\sqrt{2}R,\ x_N>0} |x|^{\frac{-am}{p-m}}\xi_R^{\frac{-m}{p-m}}|L_\mu\xi_R|^{\frac{p}{p-m}}\,dx. \tag{2.28}$$

On the other hand, by (2.7) and (2.10), for $R < |x| < \sqrt{2}R,\ x_N > 0$, we obtain

$$C_1 x_N R^{\alpha_\mu} \le F(x) \le C_2 x_N R^{\alpha_\mu},\ \left|2\nabla F(x)\cdot\frac{x}{|x|} + |x|^{-1}F(x)\right| \le C x_N R^{\alpha_\mu - 1} \tag{2.29}$$

and

$$\left|\Delta\left(\xi^\ell\left(\frac{|x|^2}{R^2}\right)\right)\right| \le CR^{-2}\xi^{\ell-2}\left(\frac{|x|^2}{R^2}\right). \tag{2.30}$$

Hence, in view of (2.27), (2.29), (2.30) and using that $0 \le \xi \le 1$, there holds

$$|L_\mu\xi_R(x)| \le C x_N R^{\alpha_\mu - 2}\xi^{\ell-2}\left(\frac{|x|^2}{R^2}\right), \quad R < |x| < \sqrt{2}R,\ x_N > 0. \tag{2.31}$$

Thus, using (2.14), (2.28), (2.29) and (2.31), we get

$$\begin{aligned}
&\int_{1<|x|<\sqrt{2}R,\, x_N>0} |x|^{\frac{-am}{p-m}} \xi_R^{\frac{-m}{p-m}} |L_\mu \xi_R|^{\frac{p}{p-m}}\, dx \\
&\leq CR^{\frac{(\alpha_\mu-2)p}{p-m}} \int_{R<|x|<\sqrt{2}R,\, x_N>0} |x|^{\frac{-am}{p-m}} F^{\frac{-m}{p-m}}(x) x_N^{\frac{p}{p-m}} \xi^{\ell-\frac{2p}{p-m}} \left(\frac{|x|^2}{R^2}\right) dx \\
&\leq CR^{\frac{(\alpha_\mu-2)p-\alpha_\mu m}{p-m}} \int_{R<|x|<\sqrt{2}R,\, x_N>0} x_N |x|^{\frac{-am}{p-m}}\, dx \\
&\leq CR^{\frac{(\alpha_\mu-2)p-\alpha_\mu m}{p-m}} \int_{R<|x|<\sqrt{2}R} |x|^{1-\frac{am}{p-m}}\, dx \\
&\leq CR^{\frac{(\alpha_\mu-2)p-\alpha_\mu m}{p-m}} R^{1-\frac{am}{p-m}} R^N,
\end{aligned}$$

that is,

$$\int_{1<|x|<\sqrt{2}R,\, x_N>0} |x|^{\frac{-am}{p-m}} \xi_R^{\frac{-m}{p-m}} |L_\mu \xi_R|^{\frac{p}{p-m}}\, dx \leq CR^{\frac{(\alpha_\mu+N-1)p-m(\alpha_\mu+1+a+N)}{p-m}}. \tag{2.32}$$

Finally, (2.24) follows from (2.25), (2.26) and (2.32). □

2.4. Estimates of $\omega_i(\psi)$

For sufficiently large T, R and ℓ, let ψ be the function defined by (2.17).

Lemma 2.6. *The following estimate holds:*

$$\omega_1(\psi) \leq CT^{1-\frac{kp}{p-1}} \left(\ln R + R^{\alpha_\mu-\frac{a}{p-1}+N+1}\right). \tag{2.33}$$

Proof. By (2.1) and (2.17), we obtain

$$\omega_1(\psi) = \left(\int_0^T \iota_T^{\frac{-1}{p-1}} \left|\frac{d^k \iota_T}{dt^k}\right|^{\frac{p}{p-1}} dt\right)\left(\int_{1<|x|<R,\, x_N>0} |x|^{\frac{-a}{p-1}} \vartheta_R(x)\, dx\right). \tag{2.34}$$

Moreover, by (2.15), we have

$$\int_{1<|x|<R,\, x_N>0} |x|^{\frac{-a}{p-1}} \vartheta_R(x)\, dx = \int_{1<|x|<R,\, x_N>0} |x|^{\frac{-a}{p-1}} F(x) \vartheta^\ell \left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right) dx. \tag{2.35}$$

Using (2.7) and (2.11), we obtain

$$\begin{aligned}
\int_{1<|x|<R,\, x_N>0} |x|^{\frac{-a}{p-1}} F(x) \vartheta^\ell \left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right) dx &\leq \int_{1<|x|<R,\, x_N>0} |x|^{\frac{-a}{p-1}} F(x)\, dx \\
&\leq C\left(\ln R + R^{\alpha_\mu-\frac{a}{p-1}+N+1}\right).
\end{aligned} \tag{2.36}$$

Hence, in view of (2.21), (2.34)–(2.36), we obtain (2.33). □

Lemma 2.7. *Let $(\alpha_\mu + N - 1)p = m(\alpha_\mu + 1 + a + N)$. Then, the following estimate holds:*

$$\omega_2(\psi) \leq CT(\ln R)^{\frac{-m}{p-m}}. \tag{2.37}$$

Proof. By (2.2) and (2.17), we have

$$\omega_2(\psi) = \left(\int_0^T \iota_T\, dt\right)\left(\int_{1<|x|<R,\, x_N>0} |x|^{\frac{-am}{p-m}} \vartheta_R^{\frac{-m}{p-m}} |L_\mu \vartheta_R|^{\frac{p}{p-m}}\, dx\right). \tag{2.38}$$

Similar calculations to those done in the proof of Lemma 2.5 give us

$$\begin{aligned} & L_\mu \vartheta_R(x) \\ &= F(x)\Delta\left(\vartheta^\ell\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right)\right) \\ &+ \frac{\ell}{\ln(\sqrt{R})|x|}\vartheta^{\ell-1}\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right)\vartheta'\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right)\left(2\nabla F(x)\cdot\frac{x}{|x|} + |x|^{-1}F(x)\right), \end{aligned} \tag{2.39}$$

which implies by (2.11) that

$$\int_{1<|x|<R,\, x_N>0} |x|^{\frac{-am}{p-m}} \vartheta_R^{\frac{-m}{p-m}} |L_\mu \vartheta_R|^{\frac{p}{p-m}}\, dx = \int_{\sqrt{R}<|x|<R,\, x_N>0} |x|^{\frac{-am}{p-m}} \vartheta_R^{\frac{-m}{p-m}} |L_\mu \vartheta_R|^{\frac{p}{p-m}}\, dx. \tag{2.40}$$

Moreover, by (2.7) and (2.11), we obtain, as $|x| \to \infty$,

$$C_1 x_N |x|^{\alpha_\mu} \le F(x) \le C_2 x_N |x|^{\alpha_\mu}, \quad \left|2\nabla F(x)\cdot\frac{x}{|x|} + |x|^{-1}F(x)\right| \le C x_N |x|^{\alpha_\mu - 1} \tag{2.41}$$

and

$$\left|\Delta\left(\vartheta^\ell\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right)\right)\right| \le C(\ln R)^{-1}|x|^{-2}\vartheta^{\ell-2}\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right), \quad \sqrt{R} < |x| < R,\ x_N > 0. \tag{2.42}$$

In view of (2.39), (2.41), (2.42) and using that $0 \le \vartheta \le 1$, we get

$$|L_\mu \vartheta_R(x)| \le C x_N |x|^{\alpha_\mu - 2}(\ln R)^{-1}\vartheta^{\ell-2}\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right), \quad \sqrt{R} < |x| < R,\ x_N > 0. \tag{2.43}$$

Next, it follows from (2.40), (2.41) and (2.43) that

$$\begin{aligned} & \int_{1<|x|<R,\, x_N>0} |x|^{\frac{-am}{p-m}} \vartheta_R^{\frac{-m}{p-m}} |L_\mu \vartheta_R|^{\frac{p}{p-m}}\, dx \\ &\le C(\ln R)^{\frac{-p}{p-m}} \int_{\sqrt{R}<|x|<R,\, x_N>0} |x|^{\frac{(\alpha_\mu-2)p-m(a+\alpha_\mu)}{p-m}} x_N \vartheta^{\ell-\frac{2p}{p-m}}\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right) dx \\ &\le C(\ln R)^{\frac{-p}{p-m}} \int_{\sqrt{R}<|x|<R} |x|^{\frac{(\alpha_\mu-1)p-m(a+\alpha_\mu+1)}{p-m}}\, dx. \end{aligned}$$

Using that $(\alpha_\mu + N - 1)p = m(\alpha + 1 + a + N)$, we get

$$\begin{aligned} \int_{1<|x|<R,\, x_N>0} |x|^{\frac{-am}{p-m}} \vartheta_R^{\frac{-m}{p-m}} |L_\mu \vartheta_R|^{\frac{p}{p-m}}\, dx &\le C(\ln R)^{\frac{-p}{p-m}} \int_{\sqrt{R}<|x|<R} |x|^{-N}\, dx \\ &= C(\ln R)^{\frac{-p}{p-m}} \int_{r=\sqrt{R}}^{R} r^{-1}\, dr \\ &\le C(\ln R)^{\frac{-m}{p-m}}. \end{aligned} \tag{2.44}$$

Finally, (2.37) follows from (2.26), (2.38) and (2.44). □

3. Proof of Theorem 1.3

3.1. Proof of part (I)

We use the contradiction argument. Namely, we suppose that $u \in L^p_{\mathrm{loc}}(D)$ is a weak solutions to (1.1) and (1.2). We first consider the case

$$p > m, \ (\alpha_\mu + N - 1)p - m(\alpha_\mu + 1 + a + N) < 0. \tag{3.1}$$

By Lemmas 2.1 and 2.2, for sufficiently large T, R and ℓ, there holds

$$-\int_{\partial D_1} \frac{\partial \varphi}{\partial \nu_1} f(x)\, d\sigma\, dt \le C \sum_{i=1}^{2} \omega_i(\varphi), \tag{3.2}$$

where φ is the function defined by (2.16). On the other hand, by (2.9), (2.18) and (2.26), we have

$$\begin{aligned} -\int_{\partial D_1} \frac{\partial \varphi}{\partial \nu_1} f(x)\, d\sigma\, dt &= -\int_{\partial D_1} \iota_T(t) f(x) \frac{\partial F}{\partial \nu_1}(x)\, d\sigma\, dt \\ &= (N + \mu + 2\alpha_\mu) \left(\int_0^T \iota_T(t)\, dt \right) \int_{\Gamma_1} f(x) x_N\, d\sigma \\ &= CT \int_{\Gamma_1} f(x) x_N\, d\sigma. \end{aligned} \tag{3.3}$$

Then, using Lemmas 2.4 and 2.5, (3.2) and (3.3), we obtain

$$T \int_{\Gamma_1} f(x) x_N\, d\sigma \le C \left(T^{1-\frac{kp}{p-1}} \left(\ln R + R^{\alpha_\mu - \frac{a}{p-1} + N + 1} \right) + T R^{\frac{(\alpha_\mu + N - 1)p - m(\alpha_\mu + 1 + a + N)}{p-m}} \right),$$

that is,

$$\int_{\Gamma_1} f(x) x_N\, d\sigma \le C \left(T^{\frac{-kp}{p-1}} \left(\ln R + R^{\alpha_\mu - \frac{a}{p-1} + N + 1} \right) + R^{\frac{(\alpha_\mu + N - 1)p - m(\alpha_\mu + 1 + a + N)}{p-m}} \right).$$

Next, taking $T = R^\theta$, where

$$\theta > \max \left\{ 0, \frac{p-1}{kp} \left(\alpha_\mu - \frac{a}{p-1} + N + 1 \right) \right\}, \tag{3.4}$$

the above estimate reduces to

$$\int_{\Gamma_1} f(x) x_N\, d\sigma \le C \left(R^{\frac{-\theta kp}{p-1}} \ln R + R^{\zeta_1} + R^{\zeta_2} \right), \tag{3.5}$$

where

$$\zeta_1 = \alpha_\mu - \frac{a}{p-1} + N + 1 - \frac{\theta kp}{p-1}, \ \zeta_2 = \frac{(\alpha_\mu + N - 1)p - m(\alpha_\mu + 1 + a + N)}{p - m}.$$

Notice that due to (3.4), one has $\zeta_1 < 0$. Moreover, by (3.1), we get $\zeta_2 < 0$. Therefore, passing to the limit as $R \to \infty$ in (3.5), we obtain $\int_{\Gamma_1} f(x) x_N\, d\sigma \le 0$, which contradicts (1.13).

Next, we consider the case

$$p > m, \ (\alpha_\mu + N - 1)p - m(\alpha_\mu + 1 + a + N) = 0. \tag{3.6}$$

By Lemmas 2.1 and 2.3, for sufficiently large T, R and ℓ, there holds

$$-\int_{\partial D_1} \frac{\partial \psi}{\partial \nu_1} f(x)\, d\sigma\, dt \le C \sum_{i=1}^{2} \omega_i(\psi), \tag{3.7}$$

where ψ is the function defined by (2.17). As in the previous case, using Lemmas 2.6 and 2.7, (2.9), (2.17) and (3.7), we obtain

$$T \int_{\Gamma_1} f(x) x_N\, d\sigma \le C\left(T^{1-\frac{kp}{p-1}}\left(\ln R + R^{\alpha_\mu - \frac{a}{p-1}+N+1}\right) + T(\ln R)^{\frac{-m}{p-m}}\right),$$

that is,

$$\int_{\Gamma_1} f(x) x_N\, d\sigma \le C\left(T^{-\frac{kp}{p-1}}\left(\ln R + R^{\alpha_\mu - \frac{a}{p-1}+N+1}\right) + (\ln R)^{\frac{-m}{p-m}}\right). \tag{3.8}$$

Hence, taking $T = R^\theta$, where the parameter θ satisfies (3.4), and passing to the limit as $R \to \infty$ in (3.8), we reach a contradiction with (1.13). This completes the proof of part (I) of Theorem 1.3. □

3.2. Proof of part (II)

Assume that (1.15) holds. Let us consider a parameter δ satisfying

$$\max\left\{-\mu - \alpha_\mu, 1 + \frac{m(a+2)}{p-m}, 1\right\} < \delta < N + \alpha_\mu. \tag{3.9}$$

Notice that $-\mu - \alpha_\mu < N + \alpha_\mu$ and $1 < N + \alpha_\mu$. Moreover, due to (1.15), one has $1 + \frac{m(a+2)}{p-m} < N + \alpha_\mu$. Hence, the set of δ satisfying (3.9) is nonempty. Let

$$0 < \varepsilon < \left[(N + \alpha_\mu - \delta)(\delta + \mu + \alpha_\mu)\right]^{\frac{1}{p-m}}. \tag{3.10}$$

We consider functions of the form

$$u_{\delta,\varepsilon}(x) = \varepsilon x_N^{\frac{1}{m}} |x|^{\frac{-\delta}{m}}, \quad x \in \Omega. \tag{3.11}$$

Elementary calculations show that

$$-\Delta u_{\delta,\varepsilon}^m + \frac{\mu}{|x|^2} x \cdot \nabla u_{\delta,\varepsilon}^m = \varepsilon^m (N + \alpha_\mu - \delta)(\delta + \mu + \alpha_\mu) x_N |x|^{-\delta-2}, \quad x \in \Omega.$$

Hence, using (3.9)–(3.11), for all $x \in \Omega$, we obtain

$$\begin{aligned}
&-\Delta u_{\delta,\varepsilon}^m + \frac{\mu}{|x|^2} x \cdot \nabla u_{\delta,\varepsilon}^m \\
&= \left(|x|^a \varepsilon^p x_N^{\frac{p}{m}} |x|^{\frac{-\delta p}{m}}\right) \varepsilon^{m-p} (N + \alpha_\mu - \delta)(\delta + \mu + \alpha_\mu) x_N^{1-\frac{p}{m}} |x|^{-\delta-2-a+\frac{\delta p}{m}} \\
&= |x|^a u_{\delta,\varepsilon}^p(x) \varepsilon^{m-p} (N + \alpha_\mu - \delta)(\delta + \mu + \alpha_\mu) x_N^{1-\frac{p}{m}} |x|^{-\delta-2-a+\frac{\delta p}{m}} \\
&\ge |x|^a u_{\delta,\varepsilon}^p(x) |x|^{(\delta-1)\left(\frac{p}{m}-1\right)-(a+2)} \\
&\ge |x|^a u_{\delta,\varepsilon}^p(x).
\end{aligned}$$

Therefore, $u_{\delta,\varepsilon}$ is a stationary solution to (1.1) and (1.2) with $f(x) = \varepsilon^m x_N$, $x \in \Gamma_1$. This completes the proof of part (II) of Theorem 1.3. □

4. Conclusions

We investigated The existence and nonexistence of weak solutions to the evolution inequality (1.1) under the Dirichlet-type boundary conditions (1.2). When $a \leq -2$, we proved that (1.1) and (1.2) admit no critical behavior, namely, for all $p > m \geq 1$, (1.1) and (1.2) admit stationary solutions for some $f > 0$. When $a > -2$, we proved that (1.1) and (1.2) admit a critical exponent

$$p^* = p^*(m, a, \mu, N) = m + \frac{m(a+2)}{\alpha_\mu + N - 1},$$

in the following sense:

(i) If

$$m < p \leq p^*,$$

then (1.1) and (1.2) admit no weak solution, provided that $f \in L^1(\Gamma_1)$ and

$$\int_{\Gamma_1} f(x) x_N \, d\sigma > 0.$$

(ii) If

$$p > p^*,$$

then (1.1) and (1.2) admit (stationary) solutions, for some $f > 0$.

It is interesting to observe that in the case $a > -2$, the critical exponent p^* depends only on m, a, μ and N, but it is independent of k, the order of the time-derivative. Therefore, our obtained results hold in both parabolic and hyperbolic cases. Finally, let us mention that comparing with previous existing results in the literature, in this study no restriction on the sign of solutions is imposed.

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Conflict of interest

The authors declare no conflict of interest.

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