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Research article

On certain Ostrowski type integral inequalities for convex function via AB-fractional integral operator

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Abstract: We investigate and prove a new lemma for twice differentiable functions with the fractional integral operator *AB*. Based on this newly developed lemma, we derive some new results about this identity. These new findings provide some generalizations of previous findings. This research builds on a novel new auxiliary result that allows us to create new variants of Ostrowski type inequalities for twice differentiable convex mappings. Some of the newly presented results' special cases are also discussed. As applications, several estimates involving special means of real numbers and Bessel functions are depicted.

Keywords: convexity; Hermite Hadamard inequality; Hölder inequality; power mean inequality; Young's inequality; Hölder-İşcan; improved power means inequality; Atangana-Baleanu fractional integral operator
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1. Introduction

In 1938, Ostrowski inequality established the following useful and interesting integral inequality, (see [1], p.468).

Let $g:\mathcal{J}\subseteq\mathfrak{R}\rightarrow\mathfrak{R}$ be a differentiable mapping on $\mathcal{J}^{o}$, the interior of the interval $\mathcal{J}$, such that $g\in\mathcal{L}[\mathfrak{c},\mathfrak{d}]$, where $\mathfrak{c},\mathfrak{d}\in\mathcal{J}$ with $\mathfrak{d}>\mathfrak{c}$. If $|g'(z)|\leq\mathcal{K}\subseteq\mathfrak{R}$, for all $z\in[\mathfrak{c},\mathfrak{d}]$, then the following inequality holds:

$$\left|g(z)-\frac{1}{\mathfrak{d}-\mathfrak{c}}\int_{\mathfrak{c}}^{\mathfrak{d}}g(u)\,du\right|\leq\mathcal{K}(\mathfrak{d}-\mathfrak{c})\left[\frac{1}{4}+\frac{\left(z-\frac{\mathfrak{c}+\mathfrak{d}}{2}\right)^{2}}{(\mathfrak{d}-\mathfrak{c})^{2}}\right]\tag{1.1}$$

holds. This result in the literature as the Ostrowski inequality. For recent result and their related some generalizations, variants and extensions concerning Ostrowski inequality (see [19–27]). This inequality yields an upper bound for the approximation of the integral average $\frac{1}{\mathfrak{d}-\mathfrak{c}}\int_{\mathfrak{c}}^{\mathfrak{d}}g(u)\,du$ by the value of $g(u)$ at the point $u\in[\mathfrak{c},\mathfrak{d}]$.

Convexity has also played an important role in the advancement of inequalities theory. Many well known results in inequalities theory can be obtained by exploiting the functions of convexity. Hermite Hadamard's double inequality is one of the most extensively studied convex function results. This result provides us necessary and sufficient condition for a function to be convex. Hermite-Hadamard (H-H) inequality has been considered the most useful inequality in mathematical analysis in 1883. It is also known as classical equation of (H-H) inequality.

Definition 1. *A function* $g:\mathcal{J}\subseteq\mathfrak{R}\rightarrow\mathfrak{R}$ *is said to be convex function if*

$$g(\zeta\mathfrak{c}+(1-\zeta)\mathfrak{d})\leq\zeta g(\mathfrak{c})+(1-\zeta)g(\mathfrak{d})\tag{1.2}$$

holds for all $[\mathfrak{c},\mathfrak{d}]\in\mathcal{J}$ *and* $\zeta\in[0,1]$. *We say that g is concave if* $(-g)$ *is convex.*

The Hermite-Hadamard inequality assert that, if a mapping $g:\mathcal{J}\subset\mathfrak{R}\rightarrow\mathfrak{R}$ is convex in $\mathcal{J}$ for $\mathfrak{c},\mathfrak{d}\in\mathcal{J}$ and $\mathfrak{d}>\mathfrak{c}$, then

$$g\left(\frac{\mathfrak{c}+\mathfrak{d}}{2}\right)\leq\frac{1}{\mathfrak{d}-\mathfrak{c}}\int_{\mathfrak{c}}^{\mathfrak{d}}g(\zeta)d\zeta\leq\frac{g(\mathfrak{c})+g(\mathfrak{d})}{2}.\tag{1.3}$$

Fractional calculus may be defined as an extension of the derivative operator idea from integer order n to arbitrary order. Fractional integrals are strong tools in applied mathematics for solving a wide range of issues in science and engineering. Many mathematicians have merged and put effort and new ideas into fractional analysis in the present decade to create a new dimension with various qualities in the field of mathematical analysis and applied mathematics. Several studies have shown that fractional operators can accurately explain complex long-memory and multiscale phenomena in materials that are difficult to capture using standard mathematical methods including classical differential calculus [6,7]. The significance of fractional calculus can be more understandable to analyze real world problems and several works involving fractional calculus have been done.

In recent years, some researchers have been interested in the concept of fractional derivative. Nonlocal fractional derivatives are classified into two types: Those with singular kernels, such as the Riemann-Liouville and Caputo derivatives, and those with nonsingular kernels, such as the Caputo-Fabrizio and Atangana-Baleanu derivatives. However, fractional derivative operators with non-singular kernels are very effective in resolving non-locality in real-world problems. Later, we'll go through the Caputo-Fabrizio integral operator.

Definition 2. *[14] Let $g \in H^1(\mathfrak{c}, \mathfrak{d})$, $\mathfrak{d} > \mathfrak{c}$, $\delta \in [0,1]$, then the definition of the new Caputo fractional derivative is:*

$$ {}^{CF}D^{\delta} g(t) = \frac{M(\delta)}{1-\delta} \int_{\mathfrak{c}}^{t} g'(s) exp\left[-\frac{\delta}{(1-\delta)}(t-s) \right] ds, $$

where $M(\delta)$ is normalization function.

Moreover, the corresponding Caputo-Fabrizio fractional integral operator is given as:

Definition 3. *[16] Let $g \in H^1(\mathfrak{c}, \mathfrak{d})$(where H^1 is class of first order differentiable function), $\mathfrak{d} > \mathfrak{c}$, $\delta \in [0,1]$.*

$$ \left({}^{CF}_{\mathfrak{c}}I^{\delta} \right)(t) = \frac{1-\delta}{M(\delta)} g(t) + \frac{\delta}{M(\delta)} \int_{\mathfrak{c}}^{t} g(y) dy, $$

and

$$ \left({}^{CF}I^{\delta}_{\mathfrak{d}} \right)(t) = \frac{1-\delta}{M(\delta)} g(t) + \frac{\delta}{M(\delta)} \int_{t}^{\mathfrak{d}} g(y) dy. $$

Where $M(\delta)$ is normalization function.

Recently, Atangana Baleanu introduced a new fractional operator containing the Mittag Leffler function in the kernel, that solve the problem of retrieving original function (a clear advantage on Caputo Fabrizio operator). This made this operator more effective and helpful. As a result many researchers have shown keen interest in utilizing this operator. Atangana Baleanu introduced the derivative operator both in Caputo and Riemann-Liouville sense:

Definition 4. *[16] Let $\nu > \mu$, $\delta \in [0,1]$ and $g \in H^1(\mu, \nu)$. The new fractional derivative is given:*

$$ {}^{ABC}_{\mu}D^{\delta}_{\mathfrak{c}} [g(t)] = \frac{M(\delta)}{1-\delta} \int_{\mu}^{\nu} g'(x) E_{\delta} \left[-\delta \frac{(t-x)^{\delta}}{(1-\delta)} \right] dx. $$

Definition 5. *[16] Let $g \in H^1(\mathfrak{c}, \mathfrak{d})$, $\mu > \nu$, $\delta \in [0,1]$. The new fractional derivative is given:*

$$ {}^{ABR}_{\mathfrak{c}}D^{\delta}_{t} [g(t)] = \frac{M(\delta)}{1-\delta} \frac{d}{dt} \int_{\mathfrak{c}}^{\mathfrak{d}} g(x) E_{\delta} \left[-\delta \frac{(t-x)^{\delta}}{(1-\delta)} \right] dx. $$

However in the same paper they give corresponding Atangana-Baleanu (AB-) fractional integral operator as:

Definition 6. *[16] The fractional integral operator with non-local kernel of a function $g \in H^1(\mathfrak{c}, \mathfrak{d})$ is defined as:*

$$ {}^{AB}_{\mathfrak{c}}I^{\delta}_{t} \{g(t)\} = \frac{1-\delta}{M(\delta)} g(t) + \frac{\delta}{M(\delta)\Gamma(\delta)} \int_{\mathfrak{c}}^{t} g(y)(t-y)^{\delta-1} dy $$

where $\mathfrak{d} > \mathfrak{c}, \delta \in [0,1]$.

In [16], the right hand side of AB-fractional integral operator as following;

$$ {}^{AB}_{\mathfrak{d}}I^{\delta}_{t} \{g(t)\} = \frac{1-\delta}{M(\delta)} g(t) + \frac{\delta}{M(\delta)\Gamma(\delta)} \int_{t}^{\mathfrak{d}} g(y)(y-t)^{\delta-1} dy. $$

Here, $\Gamma(\delta)$ is the Gamma function. The positivity of the normalization function $M(\delta)$ implies that the fractional AB-integral of a positive function is positive. It is worth noticing that the case when the order $\delta \to 1$ yields the classical integral and the case when $\delta \to 0$ provides the initial function.

2. Results

In this section, we give Ostrowski inequalities for AB-fractional integrals operator are obtained for twice differentiable functions on $(\mathfrak{c}, \mathfrak{d})$. For this purpose, we give a new identity that involve AB-fractional integrals operator whose second derivatives are convex functions.

Lemma 1. *Suppose a mapping* $g : \mathcal{J} = [\mathfrak{c}, \mathfrak{d}] \to \mathfrak{R}$ *is twice differentiable on* $(\mathfrak{c}, \mathfrak{d})$ *with* $\mathfrak{c} < \mathfrak{d}$. *If* $g'' \in \mathcal{L}_1[\mathfrak{c}, \mathfrak{d}]$, *then for all* $\hat{w} \in [\mathfrak{c}, \mathfrak{d}]$ *and* $\delta \in (0, 1]$, *the following equality for AB-fractional integrals*

$$\begin{aligned}
&\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} g'(\mathfrak{L}) - \frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})} g(\mathfrak{L}) \qquad (2.1)\\
&-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{ {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\, g(\mathfrak{c}) + {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\, g(\mathfrak{d}) \right\} - \frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}} g(\mathfrak{L})\\
&= \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} \int_0^1 v^{\delta+1} g''(v\mathfrak{L}+(1-v)\mathfrak{c})\, dv\\
&+ \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} \int_0^1 v^{\delta+1} g''(v\mathfrak{L}+(1-v)\mathfrak{d})\, dv,
\end{aligned}$$

holds for $v \in [0, 1]$.

Proof. Let us suppose that

$$\begin{aligned}
&\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} \int_0^1 v^{\delta+1} g''(v\mathfrak{L}+(1-v)\mathfrak{c})\, dv\\
&+ \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} \int_0^1 v^{\delta+1} g''(v\mathfrak{L}+(1-v)\mathfrak{d})\, dv\\
I &= \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} I_1 + \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} I_2, \qquad (2.2)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_0^1 v^{\delta+1} g''(v\mathfrak{L}+(1-v)\mathfrak{c})\, dv\\
&= \frac{v^{\delta+1} g'(v\mathfrak{L}+(1-v)\mathfrak{c})}{\mathfrak{L}-\mathfrak{c}}\Big|_0^1 - \int_0^1 \frac{(\delta+1)v^{\delta}\, g'(v\mathfrak{L}+(1-v)\mathfrak{c})}{\mathfrak{L}-\mathfrak{c}} dv\\
&= \frac{g'(\mathfrak{L})}{\mathfrak{L}-\mathfrak{c}} - \frac{\delta+1}{\mathfrak{L}-\mathfrak{c}} \int_0^1 v^{\delta}\, g'(v\mathfrak{L}+(1-v)\mathfrak{c})\, dv\\
&= \frac{g'(\mathfrak{L})}{\mathfrak{L}-\mathfrak{c}} - \frac{\delta+1}{(\mathfrak{L}-\mathfrak{c})^2} g(\mathfrak{L}) + \frac{\delta(\delta+1)}{(\mathfrak{L}-\mathfrak{c})^2} \int_0^1 v^{\delta-1}\, g(v\mathfrak{L}+(1-v)\mathfrak{c})\, dv\\
&= \frac{g'(\mathfrak{L})}{\mathfrak{L}-\mathfrak{c}} - \frac{\delta+1}{(\mathfrak{L}-\mathfrak{c})^2} g(\mathfrak{L}) + \frac{M(\delta)\Gamma(\delta+2)}{\delta(\mathfrak{L}-\mathfrak{c})^{\delta+2}}\left\{ {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\, g(\mathfrak{c}) - \frac{1-\delta}{M(\delta)} g(\mathfrak{L}) \right\}
\end{aligned}$$

and similarly

$$I_2 = \int_0^1 v^{\delta+1} g''(v\mathfrak{L}+(1-v)\mathfrak{d})\, dv$$

$$= \frac{v^{\delta+1} g'(v\mathfrak{L} + (1-v)\mathfrak{d})}{\mathfrak{L} - \mathfrak{d}}\Big|_0^1 - \int_0^1 \frac{(\delta+1) v^{\delta} g'(v\mathfrak{L} + (1-v)\mathfrak{d})}{\mathfrak{L} - \mathfrak{d}} dv$$
$$= -\frac{g'(\mathfrak{L})}{\mathfrak{d} - \mathfrak{L}} - \frac{\delta+1}{\mathfrak{L} - \mathfrak{c}} \int_0^1 v^{\delta} g'(v\mathfrak{L} + (1-v)\mathfrak{d}) dv$$
$$= -\frac{g'(\mathfrak{L})}{\mathfrak{d} - \mathfrak{L}} - \frac{\delta+1}{(\mathfrak{d} - \mathfrak{L})^2} g(\mathfrak{L}) + \frac{\delta(\delta+1)}{(\mathfrak{d} - \mathfrak{L})^2} \int_0^1 v^{\delta-1} g(v\mathfrak{L} + (1-v)\mathfrak{d}) dv$$
$$= -\frac{g'(\mathfrak{L})}{\mathfrak{d} - \mathfrak{L}} - \frac{\delta+1}{(\mathfrak{d} - \mathfrak{L})^2} g(\mathfrak{L}) + \frac{M(\delta)\Gamma(\delta+2)}{\delta(\mathfrak{L} - \mathfrak{c})^{\delta+2}} \left\{ {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\, g(\mathfrak{d}) - \frac{1-\delta}{M(\delta)} g(\mathfrak{L}) \right\}.$$

using I_1 and I_2 with (2.2), we obtain (2.4). □

Remark 1. *If we set $\delta = 1$ in Lemma 1, we get (Lemma 1 in [8]).*

Theorem 1. *Suppose a mapping $g : \mathcal{J} \subset [0, \infty) \to \mathfrak{R}$ is twice differentiable on $(\mathfrak{c}, \mathfrak{d})$ with $\mathfrak{c} < \mathfrak{d}$ such that $g'' \in \mathcal{L}_1[\mathfrak{c}, \mathfrak{d}]$. If $|g''|$ is convex function on $[\mathfrak{c}, \mathfrak{d}]$, then for all $\delta \in (0, 1]$, we get for AB-fractional integrals inequality*

$$\begin{aligned} &\Bigg| \frac{(\mathfrak{L} - \mathfrak{c})^{\delta+1} - (\mathfrak{d} - \mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d} - \mathfrak{c})} g'(\mathfrak{L}) - \frac{(\mathfrak{L} - \mathfrak{c})^{\delta} - (\mathfrak{d} - \mathfrak{L})^{\delta}}{(\mathfrak{d} - \mathfrak{c})} g(\mathfrak{L}) \\ &- \frac{M(\delta)\Gamma(\delta)}{\mathfrak{d} - \mathfrak{c}} \left\{ {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\, g(\mathfrak{c}) + {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\, g(\mathfrak{d}) \right\} - \frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d} - \mathfrak{c}} g(\mathfrak{L}) \Bigg| \\ &\leq \frac{(\mathfrak{L} - \mathfrak{c})^{\delta+2}}{(\delta+1)(\delta+3)(\mathfrak{d} - \mathfrak{c})} \left\{ |g''(\mathfrak{L})| + |g''(\mathfrak{c})| \frac{1}{\delta+2} \right\} \\ &+ \frac{(\mathfrak{d} - \mathfrak{L})^{\delta+2}}{(\delta+1)(\delta+3)(\mathfrak{d} - \mathfrak{c})} \left\{ |g''(\mathfrak{L})| + |g''(\mathfrak{d})| \frac{1}{\delta+2} \right\}, \end{aligned} \tag{2.3}$$

for $v \in [0, 1]$.

Proof. From Lemma 1 and since $|g''|$ is convex function on $[\mathfrak{c}, \mathfrak{d}]$, we obtain

$$\begin{aligned} &\Bigg| \frac{(\mathfrak{L} - \mathfrak{c})^{\delta+1} - (\mathfrak{d} - \mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d} - \mathfrak{c})} g'(\mathfrak{L}) - \frac{(\mathfrak{L} - \mathfrak{c})^{\delta} - (\mathfrak{d} - \mathfrak{L})^{\delta}}{(\mathfrak{d} - \mathfrak{c})} g(\mathfrak{L}) \\ &- \frac{M(\delta)\Gamma(\delta)}{\mathfrak{d} - \mathfrak{c}} \left\{ {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\, g(\mathfrak{c}) + {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\, g(\mathfrak{d}) \right\} - \frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d} - \mathfrak{c}} g(\mathfrak{L}) \Bigg| \\ &\leq \frac{(\mathfrak{L} - \mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d} - \mathfrak{c})} \int_0^1 v^{\delta+1} |g''(v\mathfrak{L} + (1-v)\mathfrak{c})| dv \\ &+ \frac{(\mathfrak{d} - \mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d} - \mathfrak{c})} \int_0^1 v^{\delta+1} |g''(v\mathfrak{L} + (1-v)\mathfrak{d})| dv \\ &\leq \frac{(\mathfrak{L} - \mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d} - \mathfrak{c})} \int_0^1 v^{\delta+1} \left\{ v|g''(\mathfrak{L})| + (1-v)|g''(\mathfrak{c})| \right\} dv \end{aligned}$$

$$\begin{aligned} &+ \frac{(\mathfrak{d} - \mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d} - \mathfrak{c})} \int_0^1 v^{\delta+1} \left\{ v|g''(\mathfrak{L})| + (1-v)|g''(\mathfrak{d})| \right\} dv \\ &\leq \frac{(\mathfrak{L} - \mathfrak{c})^{\delta+2}}{(\delta+1)(\delta+3)(\mathfrak{d} - \mathfrak{c})} \left\{ |g''(\mathfrak{L})| + |g''(\mathfrak{c})| \frac{1}{\delta+2} \right\} \end{aligned}$$

$$+\ \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\delta+3)(\mathfrak{d}-\mathfrak{c})}\left\{|g''(\mathfrak{L})|\ +|g''(\mathfrak{d})|\frac{1}{\delta+2}\right\}.$$

Which completes the proof. □

Remark 2. *If we set $\delta = 1$, then from Theorem 1, we get (Theorem 4 in [8]) that yields the same result with $s = 1$.*

Corollary 1. *By using Theorem 1 with $|g''| \leq M$, we get the following inequality*

$$\begin{aligned}&\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L})-\frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L})\\&-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{{}_{\mathfrak{L}}^{AB}I_{\mathfrak{c}}^{\delta}\,g(\mathfrak{c})+{}_{\mathfrak{L}}^{AB}I_{\mathfrak{d}}^{\delta}\,g(\mathfrak{d})\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\Bigg|\\&\leq M\left(\frac{1}{(\delta+1)(\delta+2)(\mathfrak{d}-\mathfrak{c})}\right)\left[\ (\mathfrak{L}-\mathfrak{c})^{\delta+2}\ +\ (\mathfrak{d}-\mathfrak{L})^{\delta+2}\right].\end{aligned}$$

Remark 3. *If we set $\delta = 1$, then from Corollary 1, we getting (Theorem 2.1, [9]).*

Theorem 2. *Suppose a mapping $g : \mathcal{J} \subset [0,\infty) \to \mathfrak{R}$ is twice differentiable on $(\mathfrak{c},\mathfrak{d})$ with $\mathfrak{c} < \mathfrak{d}$ such that $g'' \in \mathcal{L}_1[\mathfrak{c},\mathfrak{d}]$. If $|g''|^q$ is convex function on $[\mathfrak{c},\mathfrak{d}]$, $q > 1$, then for all $\delta \in (0,1]$, we get the AB-fractional integrals inequality*

$$\begin{aligned}&\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L})-\frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L})\\&-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{{}_{\mathfrak{L}}^{AB}I_{\mathfrak{c}}^{\delta}\,g(\mathfrak{c})+{}_{\mathfrak{L}}^{AB}I_{\mathfrak{d}}^{\delta}\,g(\mathfrak{d})\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\Bigg|\\&\leq\left(\frac{1}{(\delta+1)p+1}\right)^{\frac{1}{p}}\\&\times\left[\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left(\frac{|g''(\mathfrak{L})|^q+|g''(\mathfrak{c})|^q}{2}\right)^{\frac{1}{q}}+\frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left(\frac{|g''(\mathfrak{L})|^q+|g''(\mathfrak{d})|^q}{2}\right)^{\frac{1}{q}}\right],\end{aligned}\tag{2.4}$$

for $v \in [0,1]$, where $q^{-1} + p^{-1} = 1$.

Proof. Suppose that $q > 1$. From using the Lemma 1, by using the well-known Hölder integral inequality and the convexity of $|g''|^q$, we obtain

$$\begin{aligned}&\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L})-\frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L})\\&-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{{}_{\mathfrak{L}}^{AB}I_{\mathfrak{c}}^{\delta}\,g(\mathfrak{c})+{}_{\mathfrak{L}}^{AB}I_{\mathfrak{d}}^{\delta}\,g(\mathfrak{d})\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\Bigg|\\&\leq\ \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\int_0^1 v^{\delta+1}\,|g''(v\mathfrak{L}+(1-v)\mathfrak{c})|\,dv\\&+\ \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\int_0^1 v^{\delta+1}\,|g''(v\mathfrak{L}+(1-v)\mathfrak{d})|\,dv\end{aligned}\tag{2.5}$$

$$\leq \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left(\int_0^1 v^{(\delta+1)p}dv\right)^{\frac{1}{p}}\left(\int_0^1 |g''(v\mathfrak{L}+(1-v)\mathfrak{c})|^q\,dv\right)^{\frac{1}{q}}$$
$$+\frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left(\int_0^1 v^{(\delta+1)p}dv\right)^{\frac{1}{p}}\left(\int_0^1 |g''(v\mathfrak{L}+(1-v)\mathfrak{d})|^q\,dv\right)^{\frac{1}{q}}. \tag{2.6}$$

Since $|g''|^q$ is convexity on $[\mathfrak{c},\mathfrak{d}]$, we obtain

$$\int_0^1 |g''(v\mathfrak{L}+(1-v)\mathfrak{c})|^q\,dv \leq \int_0^1 \left\{v\,|g''(\mathfrak{L})|^q+(1-v)\,|g''(\mathfrak{c})|^q\right\}dv \tag{2.7}$$
$$=\frac{|g''(\mathfrak{L})|^q+|g''(\mathfrak{c})|^q}{2}$$

and

$$\int_0^1 |g''(v\,\mathfrak{L}+(1-v)\,\mathfrak{d})|^q\,dv \leq \int_0^1 \left\{v\,|g''(\mathfrak{L})|^q+(1-v)\,|g''(\mathfrak{d})|^q\right\}dv \tag{2.8}$$
$$=\frac{|g''(\mathfrak{L})|^q+|g''(\mathfrak{d})|^q}{2}.$$

By using (2.7) and (2.8) with (2.6), we obtain

$$\left|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L})-\frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L})\right.$$
$$\left.-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\,g(\mathfrak{c})+{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\,g(\mathfrak{d})\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\right|$$
$$\leq\left(\frac{1}{(\delta+1)p+1}\right)^{\frac{1}{p}}$$
$$\times\left[\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left(\frac{|g''(\mathfrak{L})|^q+|g''(\mathfrak{c})|^q}{2}\right)^{\frac{1}{q}}+\frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left(\frac{|g''(\mathfrak{L})|^q+|g''(\mathfrak{d})|^q}{2}\right)^{\frac{1}{q}}\right].$$

Which completes the proof. □

Remark 4. *If we set $\delta = 1$, then from Theorem 2, we get (Theorem 5, [8]) that yields the same result with $s = 1$.*

Corollary 2. *Using Theorem 2 with $|g''| \leq M$, we get*

$$\left|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L})-\frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L})\right.$$
$$\left.-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\,g(\mathfrak{c})+{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\,g(\mathfrak{d})\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\right|$$
$$\leq M\left(\frac{1}{(\delta+1)p+1}\right)^{\frac{1}{p}}\left[\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}+\frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\right].$$

Theorem 3. *Suppose a mapping* $g : \mathcal{J} \subset [0, \infty) \to \Re$ *is twice differentiable on* $(\mathfrak{c}, \mathfrak{d})$ *with* $\mathfrak{c} < \mathfrak{d}$ *such that* $g'' \in \mathcal{L}_1[\mathfrak{c}, \mathfrak{d}]$. *If* $|g''|^q$ *is convex function on* $[\mathfrak{c}, \mathfrak{d}]$, $q \geq 1$, *then for all* $\delta \in (0, 1]$, *the inequality for AB-fractional integrals*

$$\begin{aligned}
&\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L})-\frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L}) \\
&-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\,g(\mathfrak{c})+{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\,g(\mathfrak{d})\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\Bigg| \\
&\leq\left(\frac{1}{\delta+2}\right)^{1-\frac{1}{q}}\Bigg[\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left(\frac{|g''(\mathfrak{L})|^q}{\delta+3}+\frac{|g''(\mathfrak{c})|^q}{(\delta+2)(\delta+3)}\right)^{\frac{1}{q}} \\
&+\frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left(\frac{|g''(\mathfrak{L})|^q}{\delta+3}+\frac{|g''(\mathfrak{d})|^q}{(\delta+2)(\delta+3)}\right)^{\frac{1}{q}}\Bigg],
\end{aligned} \tag{2.9}$$

holds for $v \in [0, 1]$.

Proof. Suppose that $q \geq 1$. From Lemma 1, by using the power-mean integral inequality and convexity of $|g''|^q$, we obtain

$$\begin{aligned}
&\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L})-\frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L}) \\
&-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\,g(\mathfrak{c})+{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\,g(\mathfrak{d})\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\Bigg| \\
&\leq\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\int_0^1 v^{\delta+1}\,|g''(v\mathfrak{L}+(1-v)\mathfrak{c})|\,dv \\
&+\frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\int_0^1 v^{\delta+1}\,|g''(v\mathfrak{L}+(1-v)\mathfrak{d})|\,dv
\end{aligned}$$

$$\begin{aligned}
&\leq\frac{(\mathfrak{L}-a)^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left(\int_0^1 v^{\delta+1}dv\right)^{1-\frac{1}{q}}\left(\int_0^1|g''(v\mathfrak{L}+(1-v)\mathfrak{c})|^q\,dv\right)^{\frac{1}{q}} \\
&+\frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left(\int_0^1 v^{\delta+1}dv\right)^{1-\frac{1}{q}}\left(\int_0^1|g''(v\mathfrak{L}+(1-v)\mathfrak{d})|^q\,dv\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.10}$$

Since $|g''|^q$ is convexity on $[\mathfrak{c}, \mathfrak{d}]$, we obtain

$$\begin{aligned}
&\int_0^1 v^{\delta+1}\;|g''(v\mathfrak{L}+(1-v)\mathfrak{c})|^q\,dv\leq\int_0^1 v^{\delta+1}\left\{v\,|g''(\mathfrak{L})|^q+(1-v)\;|g''(\mathfrak{c})|^q\right\}dv \\
&=\frac{|g''(\mathfrak{L})|^q}{\delta+3}+\frac{|g''(\mathfrak{c})|^q}{(\delta+2)(\delta+3)}
\end{aligned} \tag{2.11}$$

and

$$\int_0^1 v^{\delta+1}\,|g''(v\mathfrak{L}+(1-v)\mathfrak{d})|^q\,dv\leq\int_0^1 v^{\delta+1}\left\{v\,|g''(\mathfrak{L})|^q+(1-v)\,|g''(\mathfrak{d})|^q\right\}dv \tag{2.12}$$

$$= \frac{|g''(\mathfrak{L})|^q}{\delta+3} + \frac{|g''(\mathfrak{d})|^q}{(\delta+2)(\delta+3)}.$$

By using (2.11) and (2.12) with (2.10), we obtain

$$\begin{aligned}
&\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L}) - \frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L}) \\
&- \frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{ {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\, g(\mathfrak{c}) + {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\, g(\mathfrak{d})\right\} - \frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\Bigg| \\
&\leq \left(\frac{1}{\delta+2}\right)^{1-\frac{1}{q}} \left[\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left(\frac{|g''(\mathfrak{L})|^q}{\delta+3} + \frac{|g''(\mathfrak{c})|^q}{(\delta+2)(\delta+3)}\right)^{\frac{1}{q}}\right. \\
&\left. + \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left(\frac{|g''(\mathfrak{L})|^q}{\delta+3} + \frac{|g''(\mathfrak{d})|^q}{(\delta+2)(\delta+3)}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Which completes the proof. □

Remark 5. *If we set $\delta = 1$, then from Theorem 3, we get (Theorem 6, [8]) that yields the same result with $s = 1$.*

Corollary 3. *Under the same assumptions in Theorem 3 with $|g''| \leq M$, we get the following inequality*

$$\begin{aligned}
&\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L}) - \frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L}) \\
&- \frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{ {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\, g(\mathfrak{c}) + {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\, g(\mathfrak{d})\right\} - \frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\Bigg| \\
&\leq M\left(\frac{1}{(\delta+1)(\delta+2)(\mathfrak{d}-\mathfrak{c})}\right)\left[(\mathfrak{L}-\mathfrak{c})^{\delta+2} + (\mathfrak{d}-\mathfrak{L})^{\delta+2}\right].
\end{aligned}$$

Theorem 4. *Suppose a mapping $g : \mathcal{J} \subset [0,\infty) \to \mathfrak{R}$ is twice differentiable on $(\mathfrak{c},\mathfrak{d})$ with $\mathfrak{c} < \mathfrak{d}$ such that $g'' \in \mathcal{L}_1[\mathfrak{c},\mathfrak{d}]$. If $|g''|^q$ is convex function on $[\mathfrak{c},\mathfrak{d}]$, $q > 1$, then for all $\delta \in (0,1]$, we get the following inequality*

$$\begin{aligned}
&\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L}) - \frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L}) \\
&- \frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{ {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\, g(\mathfrak{c}) + {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\, g(\mathfrak{d})\right\} - \frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\Bigg| \\
&\leq \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left\{\frac{1}{((\delta+1)p+1)p} + \frac{|g''(\mathfrak{L})|^q + |g''(\mathfrak{c})|^q}{2q}\right\} \\
&+ \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left\{\frac{1}{((\delta+1)p+1)p} + \frac{|g''(\mathfrak{L})|^q + |g''(\mathfrak{d})|^q}{2q}\right\},
\end{aligned} \tag{2.13}$$

holds for $v \in [0,1]$.

Proof. From Lemma 1, we obtain

$$\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L}) - \frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L})$$

$$
\begin{aligned}
&-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{ {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\, g(\mathfrak{c}) + {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\, g(\mathfrak{d})\right\} - \frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}} g(\mathfrak{L})\Bigg| \\
&\le \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} \int_0^1 v^{\delta+1} \left|g''(v\mathfrak{L}+(1-v)\mathfrak{c})\right| dv \\
&+ \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} \int_0^1 v^{\delta+1} \left|g''(v\mathfrak{L}+(1-v)\mathfrak{d})\right| dv,
\end{aligned}
$$

By using the Young's inequality as

$$UV \le \frac{1}{p}U^p + \frac{1}{q}V^q.$$

$$
\begin{aligned}
&\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} g'(\mathfrak{L}) - \frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})} g(\mathfrak{L}) \\
&+ \frac{\Gamma(\delta+1)}{(\mathfrak{d}-\mathfrak{c})}\left\{ J^{\delta}_{\mathfrak{L}^-} g(\mathfrak{c}) + J^{\delta}_{\mathfrak{L}^+} g(\mathfrak{d})\right\}\Bigg| \\
&\le \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left\{\frac{1}{p}\int_0^1 v^{(\delta+1)p} dv + \frac{1}{q}\int_0^1 \left|g''(v\mathfrak{L}+(1-v)\mathfrak{c})\right|^q dv\right\} \\
&+ \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left\{\frac{1}{p}\int_0^1 v^{(\delta+1)p} dv + \frac{1}{q}\int_0^1 \left|g''(v\mathfrak{L}+(1-v)\mathfrak{d})\right|^q dv\right\} \\
&\le \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left\{\frac{1}{p}\int_0^1 v^{(\delta+1)p} dv + \frac{1}{q}\int_0^1 \left\{v\left|g''(\mathfrak{L})\right|^q + (1-v)\left|g''(\mathfrak{c})\right|^q\right\}\right\} dv \\
&+ \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left\{\frac{1}{p}\int_0^1 v^{(\delta+1)p} dv + \frac{1}{q}\int_0^1 \left\{v\left|g''(\mathfrak{L})\right|^q + (1-v)\left|g''(\mathfrak{d})\right|^q\right\}\right\} dv \\
&\le \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left\{\frac{1}{((\delta+1)p+1)p} + \frac{\left|g''(\mathfrak{L})\right|^q + \left|g''(\mathfrak{c})\right|^q}{2q}\right\} \\
&+ \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\left\{\frac{1}{((\delta+1)p+1)p} + \frac{\left|g''(\mathfrak{L})\right|^q + \left|g''(\mathfrak{d})\right|^q}{2q}\right\}.
\end{aligned}
$$

Which completes the proof. □

Theorem 5. *Suppose a mapping $g : \mathcal{J} \subset [0, \infty) \to \Re$ is twice differentiable on $(\mathfrak{c}, \mathfrak{d})$ with $\mathfrak{c} < \mathfrak{d}$ such that $g'' \in \mathcal{L}_1[\mathfrak{c}, \mathfrak{d}]$. If $|g''|^q$ is convex function on $[\mathfrak{c}, \mathfrak{d}]$, $q > 1$, then for all $\delta \in (0, 1]$, for AB-fractional integrals inequality*

$$
\begin{aligned}
&\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} g'(\mathfrak{L}) - \frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})} g(\mathfrak{L}) \\
&-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{ {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\, g(\mathfrak{c}) + {}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\, g(\mathfrak{d})\right\} - \frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}} g(\mathfrak{L})\Bigg|
\end{aligned}
\tag{2.14}
$$

$$
\le \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\Bigg[\left(\frac{1}{(\delta p+p+1)(\delta p+p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{6}\left|g''(\mathfrak{L})\right|^q + \frac{1}{3}\left|g''(\mathfrak{c})\right|^q\right)^{\frac{1}{q}}
$$

$$+ \left(\frac{1}{(\delta+1)p+2}\right)^{\frac{1}{p}} \left(\frac{1}{3}|g''(\mathfrak{L})|^q + \frac{1}{6}|g''(\mathfrak{c})|^q\right)^{\frac{1}{q}} \Bigg]$$
$$+ \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} \Bigg[\left(\frac{1}{(\delta p+p+1)(\delta p+p+2)}\right)^{\frac{1}{p}} \left(\frac{1}{6}|g''(\mathfrak{L})|^q + \frac{1}{3}|g''(\mathfrak{d})|^q\right)^{\frac{1}{q}}$$
$$+ \left(\frac{1}{(\delta+1)p+2}\right)^{\frac{1}{p}} \left(\frac{1}{3}|g''(\mathfrak{L})|^q + \frac{1}{6}|g''(\mathfrak{d})|^q\right)^{\frac{1}{q}} \Bigg],$$

holds for $v \in [0,1]$, *where* $q^{-1} + p^{-1} = 1$.

Proof. From Lemma 1, by using the Hölder-İşcan integral inequality (see in [10]) and the convexity of $|g''|^q$, we obtain

$$\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1} - (\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} g'(\mathfrak{L}) - \frac{(\mathfrak{L}-\mathfrak{c})^{\delta} - (\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})} g(\mathfrak{L})$$
$$+ \frac{\Gamma(\delta+1)}{(\mathfrak{d}-\mathfrak{c})}\left\{J^{\delta}_{\mathfrak{L}^-} g(\mathfrak{c}) + J^{\delta}_{\mathfrak{L}^+} g(\mathfrak{d})\right\}\Bigg|$$
$$\leq \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} \int_0^1 v^{\delta+1} |g''(v\mathfrak{L} + (1-v)\mathfrak{c})|\, dv$$

$$+ \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} \int_0^1 v^{\delta+1} |g''(v\mathfrak{L} + (1-v)\mathfrak{d})|\, dv$$
$$\leq \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} \Bigg[\left(\int_0^1 (1-v) v^{(\delta+1)p} dv\right)^{\frac{1}{p}} \left(\int_0^1 (1-v) |g''(v\mathfrak{L} + (1-v)\mathfrak{c})|^q\, dv\right)^{\frac{1}{q}}$$
$$+ \left(\int_0^1 v^{(\delta+1)p+1} dv\right)^{\frac{1}{p}} \left(\int_0^1 v |g''(v\mathfrak{L} + (1-v)\mathfrak{c})|^q\, dv\right)^{\frac{1}{q}} \Bigg]$$
$$+ \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} \Bigg[\left(\int_0^1 (1-v) v^{(\delta+1)p} dv\right)^{\frac{1}{p}} \left(\int_0^1 (1-v) |g''(v\mathfrak{L} + (1-v)\mathfrak{d})|^q\, dv\right)^{\frac{1}{q}}$$

$$+ \left(\int_0^1 v^{(\delta+1)p+1} dv\right)^{\frac{1}{p}} \left(\int_0^1 v |g''(v\mathfrak{L} + (1-v)\mathfrak{d})|^q\, dv\right)^{\frac{1}{q}} \Bigg]$$
$$\leq \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} \Bigg[\left(\int_0^1 (1-v) v^{(\delta+1)p} dv\right)^{\frac{1}{p}} \left(\int_0^1 (1-v) \{v |g''(\mathfrak{L})|^q + (1-v) |g''(\mathfrak{c})|^q\} dv\right)^{\frac{1}{q}}$$

$$+ \left(\int_0^1 v^{(\delta+1)p+1} dv\right)^{\frac{1}{p}} \left(\int_0^1 v\{v |g''(\mathfrak{L})|^q + (1-v) |g''(\mathfrak{c})|^q\} dv\right)^{\frac{1}{q}} \Bigg]$$
$$+ \frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})} \Bigg[\left(\int_0^1 (1-v) v^{(\delta+1)p} dv\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 (1-v) \{v |g''(\mathfrak{L})|^q + (1-v) |g''(\mathfrak{d})|^q\} dv\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 v^{(\delta+1)p+1}dv\right)^{\frac{1}{p}}\left(\int_0^1 v\{v\left|g''(\mathfrak{L})\right|^q+(1-v)\left|g''(\mathfrak{d})\right|^q\}dv\right)^{\frac{1}{q}}\Bigg].$$

After getting the simplification, we get (2.14). Which completes the proof. □

Corollary 4. *If we set $\delta = 1$ in Theorem 5, we get the following inequality*

$$\begin{aligned}&\left|\frac{1}{\mathfrak{d}-\mathfrak{c}}\int_{\mathfrak{c}}^{\mathfrak{d}} g(u)\,du-g(\mathfrak{L})+\left(\mathfrak{L}-\frac{\mathfrak{c}+\mathfrak{d}}{2}\right)g'(\mathfrak{L})\right|\\
\leq\ &\frac{(\mathfrak{L}-\mathfrak{c})^3}{2(\mathfrak{d}-\mathfrak{c})}\Bigg[\left(\frac{1}{(2p+1)(2p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{6}\left|g''(\mathfrak{L})\right|^q+\frac{1}{3}\left|g''(\mathfrak{c})\right|^q\right)^{\frac{1}{q}}\\
+\ &\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}\left(\frac{1}{3}\left|g''(\mathfrak{L})\right|^q+\frac{1}{6}\left|g''(\mathfrak{c})\right|^q\right)^{\frac{1}{q}}\Bigg]\\
+\ &\frac{(\mathfrak{d}-\mathfrak{L})^3}{2(\mathfrak{d}-\mathfrak{c})}\Bigg[\left(\frac{1}{(2p+1)(2p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{6}\left|g''(\mathfrak{L})\right|^q+\frac{1}{3}\left|g''(\mathfrak{d})\right|^q\right)^{\frac{1}{q}}\\
+\ &\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}\left(\frac{1}{3}\left|g''(\mathfrak{L})\right|^q+\frac{1}{6}\left|g''(\mathfrak{d})\right|^q\right)^{\frac{1}{q}}\Bigg].\end{aligned}$$

Corollary 5. *Using the same assumptions in Theorem 5 with $|g''| \leq M$, we get*

$$\begin{aligned}&\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L})-\frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L})\\
&-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\,g(\mathfrak{c})+{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\,g(\mathfrak{d})\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\Bigg|\\
&\leq\frac{M}{2^{\frac{1}{q}}(\delta+1)(\mathfrak{d}-\mathfrak{c})}\Bigg[\left(\frac{1}{(\delta p+p+1)(\delta p+p+2)}\right)^{\frac{1}{p}}+\left(\frac{1}{(\delta+1)p+2}\right)^{\frac{1}{p}}\Bigg]\\
&\times\left[(\mathfrak{L}-\mathfrak{c})^{\delta+2}+(\mathfrak{d}-\mathfrak{L})^{\delta+2}\right].\end{aligned}$$

Theorem 6. *Suppose a mapping $g : \mathcal{J} \subset [0,\infty) \to \mathfrak{R}$ is twice differentiable on $(\mathfrak{c},\mathfrak{d})$ with $\mathfrak{c} < \mathfrak{d}$ such that $g'' \in \mathcal{L}_1[\mathfrak{c},\mathfrak{d}]$. If $|g''|^q$ is convex function on $[\mathfrak{c},\mathfrak{d}]$, $q \geq 1$, then for all $\delta \in (0,1]$, we get the following inequality*

$$\begin{aligned}&\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L})-\frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L})\\
&-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\,g(\mathfrak{c})+{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\,g(\mathfrak{d})\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\Bigg|\\
\leq\ &\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\Bigg[\left(\frac{1}{(\delta+2)(\delta+3)}\right)^{1-\frac{1}{q}}\left(\frac{\left|g''(\mathfrak{L})\right|^q}{(\delta+3)(\delta+4)}+\frac{2\left|g''(\mathfrak{c})\right|^q}{(\delta+2)(\delta+3)(\delta+4)}\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{\delta+3}\right)^{1-\frac{1}{q}}\left(\frac{\left|g''(\mathfrak{L})\right|^q}{\delta+4}+\frac{\left|g''(\mathfrak{c})\right|^q}{(\delta+3)(\delta+4)}\right)^{\frac{1}{q}}\Bigg]\end{aligned}\tag{2.15}$$

$$+\frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\Bigg[\left(\frac{1}{(\delta+2)(\delta+3)}\right)^{1-\frac{1}{q}}\left(\frac{|g''(\mathfrak{L})|^{q}}{(\delta+3)(\delta+4)}+\frac{2|g''(\mathfrak{d})|^{q}}{(\delta+2)(\delta+3)(\delta+4)}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\delta+3}\right)^{1-\frac{1}{q}}\left(\frac{|g''(\mathfrak{L})|^{q}}{\delta+4}+\frac{|g''(\mathfrak{d})|^{q}}{(\delta+3)(\delta+4)}\right)^{\frac{1}{q}}\Bigg],$$

holds for $v\in[0,1]$, *where* $q^{-1}+p^{-1}=1$.

Proof. From Lemma 1, improved power-mean integral inequality (see in [10]) and the convexity of $|g''|^{q}$, we obtain

$$\begin{aligned}
&\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L})-\frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L})\\
&-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\,g(\mathfrak{c})+{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\,g(\mathfrak{d})\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\Bigg|\\
&\leq\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\int_0^1 v^{\delta+1}|g''(v\mathfrak{L}+(1-v)\mathfrak{c})|\,dv\\
&+\frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\int_0^1 v^{\delta+1}|g''(v\mathfrak{L}+(1-v)\mathfrak{d})|\,dv\\
&\leq\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\Bigg[\left(\int_0^1(1-v)v^{\delta+1}dv\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1(1-v)v^{\delta+1}|g''(v\mathfrak{L}+(1-v)\mathfrak{c})|^{q}\,dv\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 v^{\delta+2}dv\right)^{1-\frac{1}{q}}\left(\int_0^1 v^{\delta+2}|g''(v\mathfrak{L}+(1-v)\mathfrak{c})|^{q}\,dv\right)^{\frac{1}{q}}\Bigg]
\end{aligned}$$

$$\begin{aligned}
&+\frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\Bigg[\left(\int_0^1(1-v)v^{\delta+1}dv\right)^{1-\frac{1}{q}}\left(\int_0^1(1-v)v^{\delta+1}|g''(v\mathfrak{L}+(1-v)\mathfrak{d})|^{q}\,dv\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 v^{\delta+2}dv\right)^{1-\frac{1}{q}}\left(\int_0^1 v^{\delta+2}|g''(v\mathfrak{L}+(1-v)\mathfrak{d})|^{q}\,dv\right)^{\frac{1}{q}}\Bigg]\\
&\leq\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\Bigg[\left(\int_0^1(1-v)v^{\delta+1}dv\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1(1-v)v^{\delta+1}\{v|g''(\mathfrak{L})|^{q}+(1-v)|g''(\mathfrak{c})|^{q}\}dv\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 v^{\delta+2}dv\right)^{1-\frac{1}{q}}\left(\int_0^1 v^{\delta+2}\{v|g''(\mathfrak{L})|^{q}+(1-v)|g''(\mathfrak{c})|^{q}\}dv\right)^{\frac{1}{q}}\Bigg]\\
&+\frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\\
&\times\Bigg[\left(\int_0^1(1-v)v^{\delta+1}dv\right)^{1-\frac{1}{q}}\left(\int_0^1(1-v)v^{\delta+1}\{v|g''(\mathfrak{L})|^{q}+(1-v)|g''(\mathfrak{d})|^{q}\}dv\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^1 v^{\delta+2}dv\right)^{1-\frac{1}{q}}\left(\int_0^1 v^{\delta+2}\{v\left|g''(\mathfrak{L})\right|^q+(1-v)\left|g''(\mathfrak{d})\right|^q\}dv\right)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{(\mathfrak{L}-\mathfrak{c})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\Bigg[\left(\frac{1}{(\delta+2)(\delta+3)}\right)^{1-\frac{1}{q}}\left(\frac{\left|g''(\mathfrak{L})\right|^q}{(\delta+3)(\delta+4)}+\frac{2\left|g''(\mathfrak{c})\right|^q}{(\delta+2)(\delta+3)(\delta+4)}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\delta+3}\right)^{1-\frac{1}{q}}\left(\frac{\left|g''(\mathfrak{L})\right|^q}{\delta+4}+\frac{\left|g''(\mathfrak{c})\right|^q}{(\delta+3)(\delta+4)}\right)^{\frac{1}{q}}\Bigg]$$
$$+\frac{(\mathfrak{d}-\mathfrak{L})^{\delta+2}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}\Bigg[\left(\frac{1}{(\delta+2)(\delta+3)}\right)^{1-\frac{1}{q}}\left(\frac{\left|g''(\mathfrak{L})\right|^q}{(\delta+3)(\delta+4)}+\frac{2\left|g''(\mathfrak{d})\right|^q}{(\delta+2)(\delta+3)(\delta+4)}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\delta+3}\right)^{1-\frac{1}{q}}\left(\frac{\left|g''(\mathfrak{L})\right|^q}{\delta+4}+\frac{\left|g''(\mathfrak{d})\right|^q}{(\delta+3)(\delta+4)}\right)^{\frac{1}{q}}\Bigg].$$

This completes the proof. □

Corollary 6. *If we set $\delta=1$ in Theorem 6, we get the following inequality*

$$\left|\frac{1}{\mathfrak{d}-\mathfrak{c}}\int_{\mathfrak{c}}^{\mathfrak{d}} g(u)\,du-g(\mathfrak{L})+\left(\mathfrak{L}-\frac{\mathfrak{c}+\mathfrak{d}}{2}\right)g'(\mathfrak{L})\right|$$
$$\leq \frac{(\mathfrak{L}-\mathfrak{c})^3}{2(\mathfrak{d}-\mathfrak{c})}\Bigg[\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\frac{\left|g''(\mathfrak{L})\right|^q}{20}+\frac{\left|g''(\mathfrak{c})\right|^q}{30}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{\left|g''(\mathfrak{L})\right|^q}{5}+\frac{\left|g''(\mathfrak{c})\right|^q}{20}\right)^{\frac{1}{q}}\Bigg]$$
$$+\frac{(\mathfrak{d}-\mathfrak{L})^3}{2(\mathfrak{d}-\mathfrak{c})}\Bigg[\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left(\frac{\left|g''(\mathfrak{L})\right|^q}{20}+\frac{\left|g''(\mathfrak{d})\right|^q}{60}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{\left|g''(\mathfrak{L})\right|^q}{5}+\frac{\left|g''(\mathfrak{d})\right|^q}{20}\right)^{\frac{1}{q}}\Bigg].$$

Corollary 7. *Using the same assumption of Theorem 6 with $|g''|\leq M$, we get*

$$\Bigg|\frac{(\mathfrak{L}-\mathfrak{c})^{\delta+1}-(\mathfrak{d}-\mathfrak{L})^{\delta+1}}{(\delta+1)(\mathfrak{d}-\mathfrak{c})}g'(\mathfrak{L})-\frac{(\mathfrak{L}-\mathfrak{c})^{\delta}-(\mathfrak{d}-\mathfrak{L})^{\delta}}{(\mathfrak{d}-\mathfrak{c})}g(\mathfrak{L})$$
$$-\frac{M(\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}\left\{{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{c}}\,g(\mathfrak{c})+{}^{AB}_{\mathfrak{L}}I^{\delta}_{\mathfrak{d}}\,g(\mathfrak{d})\right\}-\frac{2(1-\delta)\Gamma(\delta)}{\mathfrak{d}-\mathfrak{c}}g(\mathfrak{L})\Bigg|$$
$$\leq \frac{M}{(\delta+1)(\delta+2)(\mathfrak{d}-\mathfrak{c})}\left[(\mathfrak{L}-\mathfrak{c})^{\delta+2}+(\mathfrak{d}-\mathfrak{L})^{\delta+2}\right].$$

3. Applications to bivariate numbers

Let's consider the following special means for real numbers $\mathfrak{c},\mathfrak{d}$ such that $\mathfrak{c}\neq\mathfrak{d}$.

The arithmetic mean:

$$A(\mathfrak{c},\mathfrak{d}) = \frac{\mathfrak{c}+\mathfrak{d}}{2}.$$

The logarithmic mean:

$$L(\mathfrak{c},\mathfrak{d}) = \frac{\mathfrak{d}-\mathfrak{c}}{log\mathfrak{d} - log\mathfrak{c}}.$$

The generalized logarithmic mean:

$$L_c(\mathfrak{c},\mathfrak{d}) = \left[\frac{\mathfrak{d}^{c+1}-\mathfrak{c}^{c+1}}{(c+1)(\mathfrak{d}-\mathfrak{c})}\right]^{\frac{1}{c}}; \quad c \in \mathfrak{R} \setminus \{-1,0\}.$$

Proposition 1. *Suppose* $k \in Z\backslash\{-1,0\}$ *and* $\mathfrak{c},\mathfrak{d} \in \mathfrak{R}$ *such that* $0 < \mathfrak{c} < \mathfrak{d}$, *then the following inequality*

$$\begin{aligned}
&\left|L_k^k(\mathfrak{c},\mathfrak{d}) - u^k + k\left(x - \frac{\mathfrak{c}+\mathfrak{d}}{2}\right)u^{k-1}\right| \\
&\leq \frac{k(k-1)(\mathfrak{L}-\mathfrak{c})^3}{6^{\frac{1}{q}}.2(\mathfrak{d}-\mathfrak{c})} \\
&\times\left[\left(\frac{1}{(2p+1)(2p+2)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(|\mathfrak{L}|^{(k-2)q}, 2|\mathfrak{c}|^{(k-2)q}\right)\right. \\
&\left.+\left(\frac{1}{2p+2}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(2|\mathfrak{L}|^{(k-2)q}, |\mathfrak{c}|^{(k-2)q}\right)\right] \\
&+\frac{k(k-1)(\mathfrak{d}-\mathfrak{L})^3}{6^{\frac{1}{q}}.2(\mathfrak{d}-\mathfrak{c})} \\
&\times\left[\left(\frac{1}{(2p+1)(2p+2)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(|\mathfrak{L}|^{(k-2)q}, 2|\mathfrak{d}|^{(k-2)q}\right)+\right. \\
&\left.\left(\frac{1}{2p+2}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(2|\mathfrak{L}|^{(k-2)q}, |\mathfrak{d}|^{(k-2)q}\right)\right]
\end{aligned}$$

satisfies.

Proof. The assertion follows from Corollary 4 for the function $g(u) = u^k$ and k as specified above. □

Proposition 2. *Suppose* $r \geq 1$ *and* $\mathfrak{c},\mathfrak{d} \in \mathfrak{R}$ *such that* $0 < \mathfrak{c} < \mathfrak{d}$, *then the following inequality*

$$\begin{aligned}
&\left|L^{-1}(\mathfrak{c},\mathfrak{d}) - u^{-1} - \left(\mathfrak{L} - \frac{\mathfrak{c}+\mathfrak{d}}{2}\right)u^{-2}\right| \\
&\leq \frac{(\mathfrak{L}-\mathfrak{c})^3}{3^{\frac{1}{q}}2(\mathfrak{d}-\mathfrak{c})}\left[\left(\frac{1}{(2p+1)(2p+2)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(|x|^{-3q}, 2|\mathfrak{c}|^{-3q}\right)\right. \\
&\left.+\left(\frac{1}{2p+2}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(2|\mathfrak{L}|^{-3q}, |\mathfrak{c}|^{-3q}\right)\right] \\
&+\frac{(\mathfrak{d}-\mathfrak{L})^3}{3^{\frac{1}{q}}.2(\mathfrak{d}-\mathfrak{c})}\left[\left(\frac{1}{(2p+1)(2p+2)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(|\mathfrak{L}|^{-3q}, 2|\mathfrak{d}|^{-3q}\right)\right.
\end{aligned}$$

$$+\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}A^{\frac{1}{q}}\left(2\left|\mathfrak{L}\right|^{-3q},\left|\mathfrak{d}\right|^{-3q}\right)\Big]$$

satisfies.

Proof. The assertion follows from Corollary 4 for the function $g(u)=\frac{1}{u}$. □

4. Modified Bessel function

We recall the first kind modified Bessel function $\mathfrak{I}_{\wp}$, which has the series representation (see [17], p.77)

$$\mathfrak{I}_{\wp}(\zeta)=\Sigma_{n\geq0}\frac{\left(\frac{\zeta}{2}\right)^{\wp+2n}}{n!\Gamma\left(\wp+n+1\right)},$$

where $\zeta\in\mathfrak{R}$ and $\wp>-1$, while the second kind modified Bessel function $g_{\wp}$ (see [17], p.78, [18]) is usually defined as

$$g_{\wp}\left(\zeta\right)=\frac{\pi}{2}\frac{\mathfrak{I}_{-\wp}\left(\zeta\right)-\mathfrak{I}_{\wp}\left(\zeta\right)}{\sin\wp\pi}.$$

Consider the function $\Omega_{\wp}\left(\zeta\right):\mathfrak{R}\to[1,\infty)$ defined by

$$\Omega_{\wp}\left(\zeta\right)=2^{\wp}\Gamma\left(\wp+1\right)\zeta^{-\wp}g_{\wp}\left(\zeta\right),$$

where Γ is the gamma function.
The first order derivative formula of $\Omega_{\wp}\left(\zeta\right)$ is given by [17]:

$$\Omega_{\wp}'\left(\zeta\right)=\frac{\zeta}{2\left(\wp+1\right)}\Omega_{\wp+1}\left(\zeta\right) \tag{4.1}$$

and the second derivative can be easily calculated from (4.1) as

$$\Omega_{\wp}''\left(\zeta\right)=\frac{\zeta^{2}}{4\left(\wp+1\right)\left(\wp+2\right)}\Omega_{\wp+2}\left(\zeta\right)+\frac{1}{2\left(\wp+1\right)}\Omega_{\wp+1}\left(\zeta\right). \tag{4.2}$$

and the third derivative can be easily calculated from (4.2) as

$$\Omega_{\wp}'''\left(\zeta\right)=\frac{\zeta^{3}}{4\left(\wp+1\right)\left(\wp+2\right)\left(\wp+3\right)}\Omega_{\wp+3}\left(\zeta\right)+\frac{3\zeta}{4\left(\wp+1\right)\left(\wp+2\right)}\Omega_{\wp+2}\left(\zeta\right). \tag{4.3}$$

Proposition 3. *Suppose that* $\wp>-1$ *and* $0<\mathfrak{c}<\mathfrak{d},q>1$. *Then we have*

$$\Bigg|\frac{\Omega_{\wp}\left(\mathfrak{d}\right)-\Omega_{\wp}\left(\mathfrak{c}\right)}{\mathfrak{d}-\mathfrak{c}}-\frac{\mathfrak{L}}{2\left(\wp+1\right)}\Omega_{\wp+1}\left(\mathfrak{L}\right)+\left(\mathfrak{L}-\frac{\mathfrak{c}+\mathfrak{d}}{2}\right)$$
$$\times\left\{\frac{\mathfrak{L}^{2}}{4\left(\wp+1\right)\left(\wp+2\right)}\Omega_{\wp+2}\left(\mathfrak{L}\right)+\frac{1}{2\left(\wp+1\right)}\Omega_{\wp+1}\left(\mathfrak{L}\right)\right\}\Bigg|$$
$$\leq\frac{\left(\mathfrak{L}-\mathfrak{c}\right)^{3}}{2\left(\mathfrak{d}-\mathfrak{c}\right)}\Bigg[\left(\frac{1}{\left(2p+1\right)\left(2p+2\right)}\right)^{\frac{1}{p}}$$

$$
\begin{aligned}
&\times\left\{\frac{1}{6}\left(\frac{\mathfrak{L}^3}{4(\wp+1)(\wp+2)(\wp+3)}\Omega_{\wp+3}(\mathfrak{L})+\frac{3\mathfrak{L}}{4(\wp+1)(\wp+2)}\Omega_{\wp+2}(\mathfrak{L})\right)^q\right.\\
&+\left.\frac{1}{3}\left(\frac{\mathfrak{c}^3}{4(\wp+1)(\wp+2)(\wp+3)}\Omega_{\wp+3}(\mathfrak{c})+\frac{3\mathfrak{c}}{4(\wp+1)(\wp+2)}\Omega_{\wp+2}(\mathfrak{c})\right)^q\right\}^{\frac{1}{q}}\\
&+\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}\left\{\frac{1}{3}\left(\frac{\mathfrak{L}^3}{4(\wp+1)(\wp+2)(\wp+3)}\Omega_{\wp+3}(\mathfrak{L})+\frac{3\mathfrak{L}}{4(\wp+1)(\wp+2)}\Omega_{\wp+2}(\mathfrak{L})\right)^q\right.\\
&+\left.\left.\frac{1}{6}\left(\frac{\mathfrak{c}^3}{4(\wp+1)(\wp+2)(\wp+3)}\Omega_{m+3}(\mathfrak{c})+\frac{3\mathfrak{c}}{4(\wp+1)(m+2)}\Omega_{\wp+2}(\mathfrak{c})\right)^q\right\}^{\frac{1}{q}}\right]\\
&+\frac{(\mathfrak{d}-\mathfrak{L})^3}{2(\mathfrak{d}-\mathfrak{c})}\left[\left(\frac{1}{(2p+1)(2p+2)}\right)^{\frac{1}{p}}\right.\\
&\times\left\{\frac{1}{6}\left(\frac{\mathfrak{L}^3}{4(\wp+1)(\wp+2)(\wp+3)}\Omega_{\wp+3}(\mathfrak{L})+\frac{3\mathfrak{L}}{4(\wp+1)(\wp+2)}\Omega_{\wp+2}(\mathfrak{L})\right)^q\right.\\
&+\left.\frac{1}{3}\left(\frac{\mathfrak{d}^3}{4(\wp+1)(\wp+2)(\wp+3)}\Omega_{\wp+3}(\mathfrak{d})+\frac{3\mathfrak{d}}{4(\wp+1)(\wp+2)}\Omega_{\wp+2}(\mathfrak{d})\right)^q\right\}^{\frac{1}{q}}\\
&+\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}\\
&\times\left\{\frac{1}{3}\left(\frac{\mathfrak{L}^3}{4(\wp+1)(\wp+2)(\wp+3)}\Omega_{\wp+3}(\mathfrak{L})+\frac{3\mathfrak{L}}{4(\wp+1)(\wp+2)}\Omega_{\wp+2}(\mathfrak{L})\right)^q\right.\\
&+\left.\left.\frac{1}{6}\left(\frac{\mathfrak{d}^3}{4(\wp+1)(\wp+2)(\wp+3)}\Omega_{\wp+3}(\mathfrak{d})+\frac{3\mathfrak{d}}{4(\wp+1)(\wp+2)}\Omega_{\wp+2}(\mathfrak{d})\right)^q\right\}^{\frac{1}{q}}\right].
\end{aligned}
$$

Proof. The assertion follows immediately from Corollary 4 using $g(\zeta) = \Omega'_{\wp}(\zeta), \zeta > 0$ and the identities (4.2) and (4.3). □

5. Conclusions

We defined the concept of fractional integral inequalities with convex second derivatives in this study. In addition, we investigated and demonstrated a novel lemma for the AB-fractional integral operator's second derivatives. The ideas presented in this study, we believe, will inspire scholars in functional analysis, information theory, and statistical theory. By using generalized convexities, it is possible to consider Ostrowski versions for generalized integral operators using Mittag-Leffler operators, for example. The findings reported in this article may encourage scholars to investigate comparable and more generic integral inequalities for a variety of other problems.

Conflict of interest

There is no conflict of interest among the all authors.

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