AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 9146–9165.
DOI: 10.3934/math.2023458
Received: 27 December 2022
Revised: 05 February 2023
Accepted: 08 February 2023
Published: 13 February 2023

Research article

Nontrivial solutions for a fourth-order Riemann-Stieltjes integral boundary value problem

Keyu Zhang[1], **Yaohong Li**[2], **Jiafa Xu**[3,*] **and Donal O'Regan**[4]

[1] School of Mathematics, Qilu Normal University, Jinan 250013, China
[2] School of Mathematics and Statistics, Suzhou University, Suzhou 234000, China
[3] School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China
[4] School of Mathematical and Statistical Sciences, University of Galway, Galway, Ireland

* **Correspondence:** Email: xujiafa292@sina.com; Tel: +8618202339305; Fax: +8618202339305.

Abstract: In this paper we study a fourth-order differential equation with Riemann-Stieltjes integral boundary conditions. We consider two cases, namely when the nonlinearity satisfies superlinear growth conditions (we use topological degree to obtain an existence theorem on nontrivial solutions), when the nonlinearity satisfies a one-sided Lipschitz condition (we use the method of upper-lower solutions to obtain extremal solutions).

Keywords: fourth-order differential equation; integral boundary value problem; topological degree; upper-lower solution; nontrivial solutions; extremal solutions
Mathematics Subject Classification: 34B10, 34B15, 34B18

---

1. Introduction

In this paper we study the existence of solutions for the following integral boundary value problem of the fourth-order differential equation

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), 0 < t < 1, \\ u(0) = u''(0) = u''(1) = 0, \ u(1) = \int_0^1 u(t)d\alpha(t), \end{cases} \tag{1.1}$$

where f is a continuous function on $[0, 1] \times \mathbb{R}$, $\int_0^1 u(t)d\alpha(t)$ denotes the Riemann-Stieltjes integral, α is a function of bounded variation and satisfies the condition

(H1) $\alpha(t) \geq 0, t \in [0, 1]$ with $\int_0^1 td\alpha(t) \in [0, 1)$.

Boundary value problems can describe many phenomena in the applied sciences such as nonlinear diffusion, thermal ignition of gases and concentration in chemical or biological problems. There are many papers in the literature considering the existence of solutions using Leray-Schauder degree, the method of upper-lower solutions and the Guo-Krasnoselskii fixed point theorem in cones; we refer the reader to [1–32] and the references cited therein. In [4] the authors used the Guo-Krasnoselskii fixed point theorem to study the existence of positive solutions of the fourth-order integral boundary value problem

$$\begin{cases} u^{(4)}(t) + Mu(t) = f\left(t, u(t), u''(t)\right), \quad t \in (0,1), \\ u(1) = u'(0) = u'(1) = 0, u(0) = \lambda \displaystyle\int_0^1 u(s)v(s)ds, \end{cases}$$

and in [13] the authors investigated monotone positive solutions for the nonlinear fourth-order boundary value problem with integral and multi-point boundary conditions

$$\begin{cases} u^{(4)}(t) + f\left(t, u(t), u'(t)\right) = 0, t \in (0,1), \\ u'(0) = u'(1) = u''(0) = 0, u(0) = \alpha \displaystyle\int_v^\xi u(s)ds + \sum_{i=1}^n \beta_i u'\left(\eta_i\right), \end{cases}$$

where $f \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ satisfies some superlinear and sublinear growth conditions. In [14] the authors studied the existence and uniqueness of positive solutions for the fourth-order m-point boundary value problem

$$\begin{cases} u^{(4)}(t) + \alpha u'' - \beta u = f(t, u), 0 < t < 1, \\ u(0) = \sum_{i=1}^{m-2} a_i u\left(\xi_i\right), u(1) = \sum_{i=1}^{m-2} b_i u\left(\xi_i\right), \\ u''(0) = \sum_{i=1}^{m-2} a_i u''\left(\xi_i\right), u''(1) = \sum_{i=1}^{m-2} b_i u''\left(\xi_i\right), \end{cases} \tag{1.2}$$

where $f \in C\left([0,1] \times \mathbb{R}^+, \mathbb{R}^+\right)$ satisfies the following conditions:

$(H)_{\text{Hao1}}$ $\lim_{u\to\infty} \inf \min_{t\in[0,1]} \frac{f(t,u)}{u} > \lambda^*$, $\lim_{u\to 0^+} \sup \max_{t\in[0,1]} \frac{f(t,u)}{u} < \lambda^*$,

and

$(H)_{\text{Hao2}}$ $\lim_{u\to 0^+} \inf \min_{t\in[0,1]} \frac{f(t,u)}{u} > \lambda^*$, $\lim_{u\to\infty} \sup \max_{t\in[0,1]} \frac{f(t,u)}{u} < \lambda^*$,

where λ^* is the first eigenvalue of the eigenvalue problem

$$u^{(4)}(t) + \alpha u'' - \beta u = \lambda u$$

with the boundary conditions in (1.2).

Note all integral boundary conditions include the two-point, three-point and multi-point boundary conditions as special cases and naturally this kind of problem has interested researchers; see for example [1,2,4,8,9,11,13,19,22,24–28,30,31] and the references cited therein. In [11] the author studied the following nonlocal fractional boundary value problem with a Riemann-Stieltjes integral boundary condition

$$\begin{cases} D^\alpha u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u''(0) = 0, u(1) = \mu u(\eta) + \beta\gamma[u], \end{cases}$$

where D^α is the standard Caputo derivative, $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $\gamma[u] = \int_0^1 u(s)dA(s)$ and in [31] the authors studied the eigenvalue problem for a class of singular p-Laplacian fractional differential equations involving the Riemann-Stieltjes integral boundary condition

$$\begin{cases} -D_t^{\beta}\left(\varphi_p\left(D_t^{\alpha}x\right)\right)(t) = \lambda f(t, x(t)),\ t \in (0,1), \\ x(0) = 0,\ D_t^{\alpha}x(0) = 0,\ x(1) = \int_0^1 x(s)dA(s), \end{cases}$$

where D_t^{β} and D_t^{α} are the standard Riemann-Liouville derivatives, and $f(t,x) : (0,1) \times (0,+\infty) \to \mathbb{R}^+$ is continuous.

As is well known, due to the non-locality of fractional calculus, more and more problems in physics, electromagnetism, electrochemistry, diffusion and general transport theory can be described by the fractional calculus approach. As a new modeling tool, it has a wide range of applications in many fields. However, in the process of research, more and more scholars have found that a variety of important dynamical problems exhibit fractional-order behavior that may vary with time, space or other conditions. This phenomenon indicates that variable-order fractional calculus is a natural choice, which provides an effective mathematical framework for the description of complex mathematics. For more definitions of fractional derivatives and physical understandings, we refer the reader to [3,33–35].

Motivated by the aforementioned works, in this paper we use topological degree and the method of upper-lower solutions to study the fourth-order Riemann-Stieltjes integral boundary value problem (1.1), and obtain existence theorems for nontrivial solutions and extremal solutions. Moreover, we note that the conditions in this paper are more general than $(\mathrm{H})_{\mathrm{Hao1}}$ and $(\mathrm{H})_{\mathrm{Hao2}}$. Finally, some appropriate examples to illustrate our main results are given.

2. Preliminaries

In this section motivated by the variational iteration method (see [13, Lemma 1]), we first obtain an equivalent integral equation for our problem (1.1). Let

$$u(t) = \int_0^t \frac{1}{6}(t-s)^3 f(s,u(s))ds + c_0 + c_1 t + c_2 t^2 + c_3 t^3, \text{ for some } c_i \in \mathbb{R}, i = 0,1,2,3.$$

Then we have

$$u(0) = c_0 = 0,\ u(1) = \int_0^1 \frac{1}{6}(1-s)^3 f(s,u(s))ds + c_1 + c_2 + c_3 = \int_0^1 u(t)d\alpha(t),$$

and

$$u''(t) = \int_0^t (t-s)f(s,u(s))ds + 2c_2 + 6c_3 t.$$

By using $u''(0) = u''(1) = 0$ we obtain

$$u''(0) = 2c_2 = 0,\ u''(1) = \int_0^1 (1-s)f(s,u(s))ds + 6c_3 = 0,$$

and

$$c_2 = 0, c_3 = -\frac{1}{6}\int_0^1 (1-s)f(s,u(s))ds.$$

Note that

$$u^{(4)}(t) = f(t,u(t)) \text{ and } \int_0^1 \frac{1}{6}(1-s)^3 f(s,u(s))ds + c_1 - \frac{1}{6}\int_0^1 (1-s)f(s,u(s))ds = \int_0^1 u(t)d\alpha(t),$$

and hence

$$c_1 = \frac{1}{6}\int_0^1 (1-s)f(s,u(s))ds - \int_0^1 \frac{1}{6}(1-s)^3 f(s,u(s))ds + \int_0^1 u(t)d\alpha(t).$$

Therefore, we obtain

$$\begin{aligned} u(t) &= \int_0^t \frac{1}{6}(t-s)^3 f(s,u(s))ds + \int_0^1 \frac{1}{6}t(1-s)f(s,u(s))ds - \int_0^1 \frac{1}{6}t(1-s)^3 f(s,u(s))ds + t\int_0^1 u(t)d\alpha(t) \\ &\quad - \int_0^1 \frac{1}{6}t^3(1-s)f(s,u(s))ds \\ &= \int_0^1 K(t,s)f(s,u(s))ds + t\int_0^1 u(t)d\alpha(t), \end{aligned} \tag{2.1}$$

where

$$K(t,s) = \frac{1}{6}\begin{cases} (t-s)^3 + t(1-s) - t(1-s)^3 - t^3(1-s), 0 \le s \le t \le 1, \\ t(1-s) - t(1-s)^3 - t^3(1-s), 0 \le t \le s \le 1. \end{cases}$$

We multiply both sides of (2.1) by $d\alpha(t)$ and integrate over $[0,1]$, then (note (H1))

$$\int_0^1 u(t)d\alpha(t) = \int_0^1\int_0^1 K(t,s)f(s,u(s))dsd\alpha(t) + \int_0^1 td\alpha(t)\int_0^1 u(t)d\alpha(t),$$

and

$$\int_0^1 u(t)d\alpha(t) = \frac{1}{1-\int_0^1 td\alpha(t)}\int_0^1\int_0^1 K(t,s)f(s,u(s))dsd\alpha(t).$$

Consequently, we have

$$\begin{aligned} u(t) &= \int_0^1 K(t,s)f(s,u(s))ds + \frac{t}{1-\int_0^1 td\alpha(t)}\int_0^1\int_0^1 K(t,s)f(s,u(s))dsd\alpha(t) \\ &= \int_0^1 \Theta(t,s)f(s,u(s))ds, \end{aligned}$$

where

$$\Theta(t,s) = K(t,s) + \frac{t}{1-\int_0^1 td\alpha(t)}\int_0^1 K(t,s)d\alpha(t).$$

Lemma 2.1. $K(t,s)$ has the following properties:

(i) $K(t,s) = \int_0^1 H(t,\tau)H(\tau,s)d\tau$, where

$$H(t,s) = \begin{cases} t(1-s), 0 \le t \le s \le 1, \\ s(1-t), 0 \le s \le t \le 1; \end{cases}$$

(ii) $K(t,s)>0$ for $t,s\in(0,1)$;
(iii) $\frac{1}{30}t(1-t)s(1-s)\le K(t,s)\le\frac{1}{6}s(1-s)$ for $t,s\in[0,1]$;
(iv) $K(t,s)\le\frac{1}{6}t(1-t)$ for $t,s\in[0,1]$.

By simple calculations we obtain Lemma 2.1(i). Moreover, note that H satisfies $t(1-t)s(1-s)\le H(t,s)\le s(1-s)$ and $H(t,s)\le t(1-t)$ for $t,s\in[0,1]$, so we can easily obtain Lemma 2.1 (iii)–(iv), so we here omit their proofs.

Lemma 2.2. $\Theta(t,s)$ has the following properties:
(i) $\Theta(t,s)>0$ for $t,s\in(0,1)$;
(ii) $\Theta(t,s)\ge\frac{\int_0^1 t(1-t)d\alpha(t)}{30[1-\int_0^1 td\alpha(t)]}ts(1-s)$ for $t,s\in[0,1]$;
(iii) $\Theta(t,s)\le\frac{1}{6}\left[1+\frac{\alpha(1)}{1-\int_0^1 td\alpha(t)}\right]s(1-s)$ for $t,s\in[0,1]$;
(iv) $\Theta(t,s)\le\frac{1}{6}t\left[1+\frac{\int_0^1 t(1-t)d\alpha(t)}{1-\int_0^1 td\alpha(t)}\right]$ for $t,s\in[0,1]$.

These conclusions can be obtained from Lemma 2.1.

Let $E:=C[0,1],\|u\|:=\max_{t\in[0,1]}|u(t)|,P:=\{u\in E:u(t)\ge0,\forall t\in[0,1]\}$. Then $(E,\|\cdot\|)$ is a real Banach space and P a cone on E. Define a linear operator:

$$(Bu)(t):=\int_0^1 K(t,s)u(s)ds,\ u\in E.$$

Then $B:E\to E$ is a completely continuous, positive, linear operator, and its spectral radius, denoted by $r(B)$, is $\frac{1}{\pi^4}$. Let an operator $L_\xi(\xi>0)$ be given by

$$\left(L_\xi u\right)(t):=\xi\int_0^1 K(t,s)u(s)ds+t\int_0^1 u(t)d\alpha(t),\ \xi>0.$$

Now $L_\xi:P\to P$ is a completely continuous, linear, positive operator. Note that the spectral radius $r\left(L_\xi\right)\ge\xi r(B)>0$. Then the Krein-Rutman theorem [17] implies that there exists $\varphi_\xi\in P\backslash\{0\}$ such that

$$L_\xi\varphi_\xi=r\left(L_\xi\right)\varphi_\xi. \tag{2.2}$$

Define an operator $A:C[0,1]\to C[0,1]$ as

$$(Au)(t):=\int_0^1 K(t,s)f(s,u(s))ds+t\int_0^1 u(t)d\alpha(t).$$

It is clear that u^* is a solution of (1.1) if and only if $Au^*=u^*$, i.e.,

$$\int_0^1 K(t,s)f(s,u^*(s))ds+t\int_0^1 u^*(t)d\alpha(t)=u^*(t),$$

and (H1) implies that

$$u^*(t)=\int_0^1\Theta(t,s)f(s,u^*(s))ds.$$

Therefore, the operator A can also be expressed as

$$(Au)(t)=\int_0^1\Theta(t,s)f(s,u(s))ds,u\in E,t\in[0,1].$$

Lemma 2.3. Let $(L_\Theta u)(t) = \int_0^1 \Theta(t,s)u(s)ds$. Then $L_\Theta(P) \subset P_{01}$, where

$$P_{01} = \left\{ u \in P : u(t) \geq t\frac{\int_0^1 t(1-t)d\alpha(t)}{5[1+\int_0^1 (1-t)d\alpha(t)]}\|u\|, t \in [0,1] \right\}.$$

Proof. If $u \in P$, from Lemma 2.2(iii)–(iv) we have

$$(L_\Theta u)(t) = \int_0^1 \frac{1}{6}\left[1 + \frac{\alpha(1)}{1-\int_0^1 td\alpha(t)}\right] s(1-s)u(s)ds,$$

and

$$\begin{aligned}(L_\Theta u)(t) &\geq \int_0^1 \frac{\int_0^1 t(1-t)d\alpha(t)}{30[1-\int_0^1 td\alpha(t)]} ts(1-s)u(s)ds \\ &= t\frac{\int_0^1 t(1-t)d\alpha(t)}{5[1+\int_0^1 (1-t)d\alpha(t)]}\int_0^1 \frac{1}{6}\left[1+\frac{\alpha(1)}{1-\int_0^1 td\alpha(t)}\right] s(1-s)u(s)ds \\ &\geq t\frac{\int_0^1 t(1-t)d\alpha(t)}{5[1+\int_0^1 (1-t)d\alpha(t)]}\|L_\Theta u\|.\end{aligned}$$

This completes the proof. □

Lemma 2.4. (*see [10]*) Let Ω be a bounded open set in a Banach space E, and $T : \Omega \to E$ a continuous compact operator. If there exists $x_0 \in E\backslash\{0\}$ such that

$$x - Tx \neq \mu x_0, \forall x \in \partial\Omega, \mu \geq 0,$$

then the topological degree $\deg(I-T,\Omega,0) = 0$.

Lemma 2.5. (*see [10]*) Let Ω be a bounded open set in a Banach space E with $0 \in \Omega$, and $T : \Omega \to E$ a continuous compact operator. If

$$Tx \neq \mu x, \forall x \in \partial\Omega, \mu \geq 1,$$

then the topological degree $\deg(I-T,\Omega,0) = 1$.

3. Nontrivial solutions for (1.1)

In this section, we assume that the nonlinearity f satisfies the conditions:

(H2) $f \in C([0,1]\times\mathbb{R},\mathbb{R})$. Moreover, there exist three functions $\gamma_i \in C([0,1],\mathbb{R}^+), i = 1,2$, and $\mathcal{M} \in C(\mathbb{R},\mathbb{R}^+)$ with $\gamma_2(t) \not\equiv 0, t \in [0,1]$ such that

$$f(t,x) \geq -\gamma_1(t) - \gamma_2(t)\mathcal{M}(x), \forall x \in \mathbb{R}, t \in [0,1].$$

(H3) $\lim_{|x|\to+\infty} \frac{\mathcal{M}(x)}{|x|} = 0$.

(H4) There exists $\xi_1 > 0$ such that $r(L_{\xi_1}) \geq 1$ and

$$\liminf_{|x|\to+\infty} \frac{f(t,x)}{|x|} > \xi_1, \text{ uniformly for } t \in [0,1],$$

(H5) There exists $\xi_2 > 0$ such that $r(L_{\xi_2}) < 1$ and

$$\limsup_{|x|\to 0} \frac{|f(t,x)|}{|x|} \le \xi_2, \text{ uniformly for } t \in [0,1].$$

Theorem 3.1. Suppose that (H1)–(H5) hold. Then (1.1) has at least one nontrivial solution.
Proof. From (2.2) there exists $\varphi_{\xi_1} \in P\backslash\{0\}$ such that $L_{\xi_1}\varphi_{\xi_1} = r(L_{\xi_1})\varphi_{\xi_1}$, i.e.,

$$\left(L_{\xi_1}\varphi_{\xi_1}\right)(t) = \xi_1 \int_0^1 K(t,s)\varphi_{\xi_1}(s)ds + t\int_0^1 \varphi_{\xi_1}(t)d\alpha(t) = r(L_{\xi_1})\varphi_{\xi_1}(t), t \in [0,1]. \tag{3.1}$$

Note that $r(L_{\xi_1}) \ge 1$. We multiply both sides of the above equation by $d\alpha(t)$ and integrate over $[0,1]$ (note (H1)) so we obtain

$$\int_0^1 \xi_1 \int_0^1 K(t,s)\varphi_{\xi_1}(s)dsd\alpha(t) + \int_0^1 td\alpha(t)\int_0^1 \varphi_{\xi_1}(t)d\alpha(t) = r(L_{\xi_1})\int_0^1 \varphi_{\xi_1}(t)d\alpha(t),$$

and

$$\int_0^1 \varphi_{\xi_1}(t)d\alpha(t) = \frac{1}{r(L_{\xi_1}) - \int_0^1 td\alpha(t)}\int_0^1 \xi_1\int_0^1 K(t,s)\varphi_{\xi_1}(s)dsd\alpha(t).$$

Consequently, we have

$$\begin{aligned}\varphi_{\xi_1}(t) &= \frac{\xi_1}{r(L_{\xi_1})}\int_0^1 K(t,s)\varphi_{\xi_1}(s)ds + \frac{t}{r(L_{\xi_1})}\int_0^1 \varphi_{\xi_1}(t)d\alpha(t)\\ &= \frac{\xi_1}{r(L_{\xi_1})}\int_0^1 K(t,s)\varphi_{\xi_1}(s)ds + \frac{t}{r(L_{\xi_1})}\frac{1}{r(L_{\xi_1}) - \int_0^1 td\alpha(t)}\int_0^1 \xi_1\int_0^1 K(t,s)\varphi_{\xi_1}(s)dsd\alpha(t)\\ &= \frac{\xi_1}{r(L_{\xi_1})}\int_0^1 \Lambda(t,s)\varphi_{\xi_1}(s)ds,\end{aligned}$$

where

$$\Lambda(t,s) = K(t,s) + \frac{t}{r(L_{\xi_1}) - \int_0^1 td\alpha(t)}\int_0^1 K(t,s)d\alpha(t).$$

Let

$$P_{02} = \left\{u \in P : u(t) \ge t\frac{\int_0^1 t(1-t)d\alpha(t)}{5[r(L_{\xi_1}) + \int_0^1 (1-t)d\alpha(t)]}\|u\|, t \in [0,1]\right\}.$$

Now

$$\varphi_{\xi_1} \in P_{02}. \tag{3.2}$$

Indeed, from Lemma 2.1(iii) we have

$$\varphi_{\xi_1}(t) \le \frac{\xi_1}{r(L_{\xi_1})}\int_0^1 \frac{1}{6}s(1-s)\left[1 + \frac{\alpha(1)}{r(L_{\xi_1}) - \int_0^1 td\alpha(t)}\right]\varphi_{\xi_1}(s)ds,$$

and

$$\begin{aligned}\varphi_{\xi_1}(t) &\geq \frac{\xi_1}{r(L_{\xi_1})}\frac{t}{r(L_{\xi_1})-\int_0^1 td\alpha(t)}\int_0^1\int_0^1 \frac{1}{30}t(1-t)s(1-s)d\alpha(t)\varphi_{\xi_1}(s)ds\\ &= \frac{\xi_1}{r(L_{\xi_1})}\frac{t\int_0^1 t(1-t)d\alpha(t)}{5[r(L_{\xi_1})+\int_0^1(1-t)d\alpha(t)]}\int_0^1 \frac{1}{6}s(1-s)\left[1+\frac{\alpha(1)}{r(L_{\xi_1})-\int_0^1 td\alpha(t)}\right]\varphi_{\xi_1}(s)ds\\ &\geq \frac{t\int_0^1 t(1-t)d\alpha(t)}{5[r(L_{\xi_1})+\int_0^1(1-t)d\alpha(t)]}\|\varphi_{\xi_1}\|.\end{aligned}$$

Note that

$$\frac{\int_0^1 t(1-t)d\alpha(t)}{5[1+\int_0^1(1-t)d\alpha(t)]} \geq \frac{\int_0^1 t(1-t)d\alpha(t)}{5[r(L_{\xi_1})+\int_0^1(1-t)d\alpha(t)]}.$$

Therefore, by Lemma 2.3 we obtain

$$L_\Theta(P) \subset P_{02}. \tag{3.3}$$

By (H4), there exist $\varepsilon_0 > 0$ and $X_0 > 0$ such that

$$f(t,x) \geq (\xi_1+\varepsilon_0)|x|, \text{ for } |x| > X_0, t \in [0,1].$$

For any fixed ε with $\varepsilon_0 - \|\gamma_2\|\varepsilon > 0$, from (H3) there exists $X_1 > X_0$ such that

$$\mathcal{M}(x) \leq \varepsilon|x|, \text{ for } |x| > X_1.$$

Note from (H2), we also obtain

$$f(t,x) \geq (\xi_1+\varepsilon_0)|x| - \gamma_1(t) - \gamma_2(t)\mathcal{M}(x) \geq (\xi_1+\varepsilon_0-\varepsilon\|\gamma_2\|)|x| - \gamma_1(t), t \in [0,1], |x| > X_1.$$

Let $C_{X_1} = (\xi_1+\varepsilon_0-\varepsilon\|\gamma_2\|)X_1 + \max_{t\in[0,1],|x|\leq X_1}|f(t,x)|$, $\mathcal{M}^* = \max_{|x|\leq X_1}\mathcal{M}(x)$, and we have

$$f(t,x) \geq (\xi_1+\varepsilon_0-\varepsilon\|\gamma_2\|)|x| - \gamma_1(t) - C_{X_1},\ \mathcal{M}(x) \leq \varepsilon|x| + \mathcal{M}^*,\ t \in [0,1], x \in \mathbb{R}. \tag{3.4}$$

Note that ε can be chosen arbitrarily small, and we let

$$\mathcal{R}_1 > \max\left\{\frac{\|\gamma_1\|+\|\gamma_2\|\mathcal{M}^*+C_{X_1}}{\mathcal{N}_2^{-1}-\varepsilon\|\gamma_2\|}, \frac{[\|\gamma_1\|+\|\gamma_2\|\mathcal{M}^*+C_{X_1}]\left[(\varepsilon_0-\varepsilon\|\gamma_2\|)\mathcal{N}_1\mathcal{N}_2+(\xi_1+\varepsilon_0-\varepsilon\|\gamma_2\|)\mathcal{N}_3\right]}{(\varepsilon_0-\varepsilon\|\gamma_2\|)\mathcal{N}_1(1-\varepsilon\|\gamma_2\|\mathcal{N}_2)-\varepsilon\|\gamma_2\|(\xi_1+\varepsilon_0-\varepsilon\|\gamma_2\|)\mathcal{N}_3}\right\}, \tag{3.5}$$

where

$$\mathcal{N}_1 = \frac{\int_0^1 t(1-t)d\alpha(t)}{5[r(L_{\xi_1})+\int_0^1(1-t)d\alpha(t)]}, \mathcal{N}_2 = \frac{1}{36}\left[1+\frac{\alpha(1)}{1-\int_0^1 td\alpha(t)}\right], \mathcal{N}_3 = \frac{1}{6}\left[1+\frac{\int_0^1 t(1-t)d\alpha(t)}{1-\int_0^1 td\alpha(t)}\right].$$

In what follows, we prove that

$$u - Au \neq \mu\varphi_{\xi_1}, \text{ for } u \in \partial B_{\mathcal{R}_1}, \mu \geq 0, \tag{3.6}$$

where φ_{ξ_1} is defined in (3.1), and $B_{\mathcal{R}_1} = \{u \in E : \|u\| < \mathcal{R}_1\}$. Suppose the contrary. Then there exist $u_1 \in \partial B_{\mathcal{R}_1}$ and $\mu_1 \geq 0$ such that

$$u_1 - Au_1 = \mu_1 \varphi_{\xi_1}. \tag{3.7}$$

Note that $\mu_1 \neq 0$ (otherwise, u_1 is a solution for (1.1) and the theorem is proved). Let

$$\widetilde{u}_1(t) = \int_0^1 K(t,s)[\gamma_1(s) + \gamma_2(s)\mathcal{M}(u_1(s)) + C_{X_1}]ds + t\int_0^1 \widetilde{u}_1(t)d\alpha(t), t \in [0,1].$$

Then (H1) implies that

$$\widetilde{u}_1(t) = \int_0^1 \Theta(t,s)[\gamma_1(s) + \gamma_2(s)\mathcal{M}(u_1(s)) + C_{X_1}]ds.$$

Note that $\gamma_1(s) + \gamma_2(s)\mathcal{M}(u_1(s)) + C_{X_1} \geq 0, s \in [0,1]$, and by (3.3) we have

$$\widetilde{u}_1 \in P_{02}.$$

Moreover, from (3.7) we have

$$u_1(t) + \widetilde{u}_1(t) = (Au_1)(t) + \widetilde{u}_1(t) + \mu_1\varphi_{\xi_1}(t),$$

i.e.,

$$u_1(t)+\widetilde{u}_1(t) = \int_0^1 K(t,s)[f(s,u_1(s))+\gamma_1(s)+\gamma_2(s)\mathcal{M}(u_1(s))+C_{X_1}]ds+t\int_0^1 [u_1(t)+\widetilde{u}_1(t)]d\alpha(t)+\mu_1\varphi_{\xi_1}(t).$$

From (H1) we get

$$\begin{aligned}\int_0^1 [u_1(t) + \widetilde{u}_1(t)]d\alpha(t) = &\frac{1}{1-\int_0^1 td\alpha(t)}\int_0^1\int_0^1 K(t,s)[f(s,u_1(s)) + \gamma_1(s) + \gamma_2(s)\mathcal{M}(u_1(s)) + C_{X_1}]dsd\alpha(t)\\ &+ \frac{\mu_1}{1-\int_0^1 td\alpha(t)}\int_0^1 \varphi_{\xi_1}(t)d\alpha(t).\end{aligned}$$

Hence, we have

$$\begin{aligned}u_1(t) + \widetilde{u}_1(t) = &\int_0^1 \Theta(t,s)[f(s,u_1(s)) + \gamma_1(s) + \gamma_2(s)\mathcal{M}(u_1(s)) + C_{X_1}]ds\\ &+ \frac{\mu_1 t}{1-\int_0^1 td\alpha(t)}\int_0^1 \varphi_{\xi_1}(t)d\alpha(t) + \mu_1\varphi_{\xi_1}(t).\end{aligned}$$

Note that $f(s,u_1(s)) + \gamma_1(s) + \gamma_2(s)\mathcal{M}(u_1(s)) + C_{X_1} \geq 0, s \in [0,1]$. Then (3.2) and (3.3) imply that

$$\frac{\mu_1 t}{1-\int_0^1 td\alpha(t)}\int_0^1 \varphi_{\xi_1}(t)d\alpha(t) \geq t\frac{\int_0^1 t(1-t)d\alpha(t)}{5[r(L_{\xi_1}) + \int_0^1 (1-t)d\alpha(t)]}\left\|\frac{\mu_1 t}{1-\int_0^1 td\alpha(t)}\int_0^1 \varphi_{\xi_1}(t)d\alpha(t)\right\|$$

implies that

$$u_1 + \widetilde{u}_1 \in P_{02}. \tag{3.8}$$

Now, we estimate the norm of $\widetilde{u}_1$. Note that (3.5) and $\|u_1\| = \mathcal{R}_1$, from Lemma 2.2 (iii) and (3.4) we have

$$\begin{aligned}
\widetilde{u}_1(t) &\le \int_0^1 \Theta(t,s)[\gamma_1(s) + \gamma_2(s)\mathcal{M}(u_1(s)) + C_{X_1}]ds \\
&\le \frac{1}{6}\left[1 + \frac{\alpha(1)}{1 - \int_0^1 t d\alpha(t)}\right]\int_0^1 s(1-s)[\gamma_1(s) + \gamma_2(s)(\varepsilon|u_1(s)| + \mathcal{M}^*) + C_{X_1}]ds \\
&\le \frac{1}{36}\left[1 + \frac{\alpha(1)}{1 - \int_0^1 t d\alpha(t)}\right][\|\gamma_1\| + \|\gamma_2\|(\varepsilon\|u_1\| + \mathcal{M}^*) + C_{X_1}] \\
&< \mathcal{R}_1.
\end{aligned}$$

From (3.8) we have $u_1(t)+\widetilde{u}_1(t) \ge t\frac{\int_0^1 t(1-t)d\alpha(t)}{5[r(L_{\xi_1})+\int_0^1(1-t)d\alpha(t)]}\|u_1+\widetilde{u}_1\| \ge t\frac{\int_0^1 t(1-t)d\alpha(t)}{5[r(L_{\xi_1})+\int_0^1(1-t)d\alpha(t)]}(\|u_1\|-\|\widetilde{u}_1\|), t \in [0,1]$. Note (3.5), and

$$\begin{aligned}
&(\varepsilon_0 - \varepsilon\|\gamma_2\|)\frac{\int_0^1 t(1-t)d\alpha(t)}{5[r(L_{\xi_1}) + \int_0^1(1-t)d\alpha(t)]}(\mathcal{R}_1 - \|\widetilde{u}_1\|) \\
&- (\xi_1 + \varepsilon_0 - \varepsilon\|\gamma_2\|)\int_0^1 \frac{1}{6}\left[1 + \frac{\int_0^1 t(1-t)d\alpha(t)}{1 - \int_0^1 t d\alpha(t)}\right][\gamma_1(\tau) + \gamma_2(\tau)\mathcal{M}(u_1(\tau)) + C_{X_1}]d\tau \\
&\ge (\varepsilon_0 - \varepsilon\|\gamma_2\|)\frac{\int_0^1 t(1-t)d\alpha(t)}{5[r(L_{\xi_1}) + \int_0^1(1-t)d\alpha(t)]}\left(\mathcal{R}_1 - \frac{1}{36}\left[1 + \frac{\alpha(1)}{1 - \int_0^1 t d\alpha(t)}\right][\|\gamma_1\| + \|\gamma_2\|(\varepsilon\mathcal{R}_1 + \mathcal{M}^*) + C_{X_1}]\right) \\
&- \frac{\xi_1 + \varepsilon_0 - \varepsilon\|\gamma_2\|}{6}\left[1 + \frac{\int_0^1 t(1-t)d\alpha(t)}{1 - \int_0^1 t d\alpha(t)}\right][\|\gamma_1\| + \|\gamma_2\|(\varepsilon\mathcal{R}_1 + \mathcal{M}^*) + C_{X_1}] \\
&\ge 0.
\end{aligned}$$

Then Lemma 2.2 (iv) implies that

$$\begin{aligned}
&(\varepsilon_0 - \varepsilon\|\gamma_2\|)\int_0^1 K(t,s)[u_1(s) + \widetilde{u}_1(s)]ds - (\xi_1 + \varepsilon_0 - \varepsilon\|\gamma_2\|)\int_0^1 K(t,s)\widetilde{u}_1(s)ds \\
&\ge (\varepsilon_0 - \varepsilon\|\gamma_2\|)\int_0^1 K(t,s)s\frac{\int_0^1 t(1-t)d\alpha(t)}{5[r(L_{\xi_1}) + \int_0^1(1-t)d\alpha(t)]}(\mathcal{R}_1 - \|\widetilde{u}_1\|)ds \\
&\quad - (\xi_1 + \varepsilon_0 - \varepsilon\|\gamma_2\|)\int_0^1 K(t,s)\int_0^1 \Theta(s,\tau)[\gamma_1(\tau) + \gamma_2(\tau)\mathcal{M}(u_1(\tau)) + C_{X_1}]d\tau ds \\
&\ge (\varepsilon_0 - \varepsilon\|\gamma_2\|)\int_0^1 K(t,s)s\frac{\int_0^1 t(1-t)d\alpha(t)}{5[r(L_{\xi_1}) + \int_0^1(1-t)d\alpha(t)]}(\mathcal{R}_1 - \|\widetilde{u}_1\|)ds \\
&\quad - (\xi_1 + \varepsilon_0 - \varepsilon\|\gamma_2\|)\int_0^1 K(t,s)\int_0^1 \frac{1}{6}s\left[1 + \frac{\int_0^1 t(1-t)d\alpha(t)}{1 - \int_0^1 t d\alpha(t)}\right][\gamma_1(\tau) + \gamma_2(\tau)\mathcal{M}(u_1(\tau)) + C_{X_1}]d\tau ds \\
&\ge 0, t \in [0,1]. \qquad (3.9)
\end{aligned}$$

Therefore, from (3.4) we have

$$
\begin{aligned}
&(Au_1)(t)+\widetilde{u}_1(t)=\int_0^1 K(t,s)[f(s,u_1(s))+\gamma_1(s)+\gamma_2(s)\mathcal{M}(u_1(s))+C_{X_1}]ds+t\int_0^1[u_1(t)+\widetilde{u}_1(t)]d\alpha(t)\\
&\geq\int_0^1 K(t,s)[(\xi_1+\varepsilon_0-\varepsilon\|\gamma_2\|)|u_1(s)|-\gamma_1(s)-C_{X_1}+\gamma_1(s)+C_{X_1}]ds+t\int_0^1[u_1(t)+\widetilde{u}_1(t)]d\alpha(t)\\
&\geq(\xi_1+\varepsilon_0-\varepsilon\|\gamma_2\|)\int_0^1 K(t,s)[u_1(s)+\widetilde{u}_1(s)]ds+t\int_0^1[u_1(t)+\widetilde{u}_1(t)]d\alpha(t)\\
&\quad-(\xi_1+\varepsilon_0-\varepsilon\|\gamma_2\|)\int_0^1 K(t,s)\widetilde{u}_1(s)ds\\
&\geq\xi_1\int_0^1 K(t,s)[u_1(s)+\widetilde{u}_1(s)]ds+t\int_0^1[u_1(t)+\widetilde{u}_1(t)]d\alpha(t).
\end{aligned}
\tag{3.10}
$$

Together with (3.7), we have

$$u_1(t)+\widetilde{u}_1(t)=(Au_1)(t)+\widetilde{u}_1(t)+\mu_1\varphi_{\xi_1}(t)\geq(L_{\xi_1}(u_1+\widetilde{u}_1))(t)+\mu_1\varphi_{\xi_1}(t)\geq\mu_1\varphi_{\xi_1}(t),t\in[0,1].$$

Define

$$\mu^*=\sup\{\mu>0:u_1+\widetilde{u}_1\geq\mu\varphi_{\xi_1}\}.$$

Clearly, $\mu^*\geq\mu_1$, and $u_1+\widetilde{u}_1\geq\mu^*\varphi_{\xi_1}$. Note that $L_{\xi_1}\varphi_{\xi_1}=r(L_{\xi_1})\varphi_{\xi_1}$ and we have

$$u_1(t)+\widetilde{u}_1(t)\geq(L_{\xi_1}(u_1+\widetilde{u}_1))(t)+\mu_1\varphi_{\xi_1}(t)\geq(L_{\xi_1}\mu^*\varphi_{\xi_1})(t)+\mu_1\varphi_{\xi_1}(t)=(\mu^*r(L_{\xi_1})+\mu_1)\varphi_{\xi_1}(t),$$

which contradicts the definition of $\mu^*(r(L_{\xi_1})\geq 1)$. Therefore, (3.6) holds, and from Lemma 2.4 we obtain

$$\deg(I-A,B_{\mathcal{R}_1},0)=0.\tag{3.11}$$

From (H5) there exists $r_1\in(0,\mathcal{R}_1)$ such that

$$|f(t,x)|\leq\xi_2|x|,\ \text{for}\ |x|\leq r_1,t\in[0,1].$$

Now for this r_1, we prove that

$$Au\neq\mu u,\quad\forall u\in\partial B_{r_1},\ \mu\geq 1.\tag{3.12}$$

Suppose the contrary. Then there exist $u_2\in\partial B_{r_1}$ and $\mu_2\geq 1$ such that

$$Au_2=\mu_2u_2,$$

where $B_{r_1}=\{u\in E:\|u\|<r_1\}$. Consequently, we have

$$
\begin{aligned}
|u_2(t)|&\leq\frac{1}{\mu_2}|(Au_2)(t)|\\
&\leq\int_0^1 K(t,s)|f(s,u_2(s))|ds+t\int_0^1|u_2(t)|d\alpha(t)\\
&\leq\xi_2\int_0^1 K(t,s)|u_2(s)|ds+t\int_0^1|u_2(t)|d\alpha(t).
\end{aligned}
$$

Let $v_2(t) = |u_2(t)| \in P, t \in [0,1]$. Then we have

$$v_2(t) \leq \xi_2 \int_0^1 K(t,s)v_2(s)ds + t\int_0^1 v_2(t)d\alpha(t) = (L_{\xi_2}v_2)(t), t \in [0,1].$$

Note that $r(L_{\xi_2}) < 1$, which implies that $\left(I - L_{\xi_2}\right)^{-1}$ exists, and

$$\left(I - L_{\xi_2}\right)^{-1} = I + L_{\xi_2} + L_{\xi_2}^2 + \cdots + L_{\xi_2}^n + \cdots .$$

Consequently, note that $\left(I - L_{\xi_2}\right)^{-1} : P \to P$, and we have

$$((I - L_{\xi_2})v_2)(t) \leq 0 \Rightarrow \|v_2\| \leq \|\left(I - L_{\xi_2}\right)^{-1} 0\| = 0.$$

Hence, $\|v_2\| = 0 \Rightarrow \|u_2\| = 0$, and this contradicts $u_2 \in \partial B_{r_1}$. Thus, (3.12) holds, and Lemma 2.5 implies that

$$\deg(I - A, B_{r_1}, 0) = 1.$$

Combining this with (3.11) we have

$$\deg\left(I - A, B_{\mathcal{R}_1}\backslash\overline{B}_{r_1}, 0\right) = \deg(I - A, B_{\mathcal{R}_1}, 0) - \deg(I - A, B_{r_1}, 0) = -1.$$

Therefore the operator A has at least one fixed point in $B_{\mathcal{R}_1}\backslash\overline{B}_{r_1}$. Equivalently, (1.1) has at least one nontrivial solution. This completes the proof.

4. Extremal solutions for (1.1)

In this section we use the method of upper-lower solutions to study the existence of extremal solutions for (1.1). We first provide the definitions of upper and lower solutions.

Definition 4.1. We say that $u \in E$ is an upper solution of (1.1) if

$$\begin{cases} u^{(4)}(t) \geq f(t, u(t)), 0 < t < 1, \\ u(0) = u''(0) = u''(1) = 0,\ u(1) \geq \displaystyle\int_0^1 u(t)d\alpha(t). \end{cases}$$

Definition 4.2. We say that $u \in E$ is a lower solution of (1.1) if

$$\begin{cases} u^{(4)}(t) \leq f(t, u(t)), 0 < t < 1, \\ u(0) = u''(0) = u''(1) = 0,\ u(1) \leq \displaystyle\int_0^1 u(t)d\alpha(t). \end{cases}$$

Lemma 4.3. Suppose that (H1) holds. Let $u \in E$ satisfy

$$\begin{cases} u^{(4)}(t) + c(t)u(t) \geq 0,\ t \in (0,1), \\ u(0) = u''(0) = u''(1) = 0,\ u(1) \geq \displaystyle\int_0^1 u(t)d\alpha(t). \end{cases} \tag{4.1}$$

Then $u(t) \geq 0, t \in [0,1]$; here $c(t)$ satisfies the condition

(H6) $-\pi^4 < c(t) < c_0$, $t \in [0,1]$, and $c_0 := 4k_0^4$ with k_0 being the smallest positive solution of the equation $\tan k = \tanh k$ (i.e., $k_0 \approx 3.9266$ and $c_0 \approx 950.8843$).

Proof. From [6, 7, 32] we introduce a result. Let $L_c : W \to C[0,1]$ be defined by $L_c u = u^{(4)} + c(t)u$. Then by (H6), L_c has a positive inverse, where $W = \left\{u \in C^4([0,1]) : u(0) = u(1) = u''(0) = u''(1) = 0\right\}$. In (4.1) let $u^{(4)}(t) + c(t)u(t) = z(t) \geq 0$ and $\chi_1 = u(1) - \int_0^1 u(t)d\alpha(t) \geq 0$, then we have

$$\begin{cases} u^{(4)}(t) + c(t)u(t) = z(t), 0 < t < 1, \\ u(0) = u''(0) = u''(1) = 0, u(1) = \chi_1 + \displaystyle\int_0^1 u(t)d\alpha(t) \end{cases} \tag{4.2}$$

is equivalent to

$$u(t) = \int_0^1 G(t,s)z(s)ds + t\left(\chi_1 + \int_0^1 u(t)d\alpha(t)\right), \tag{4.3}$$

where G is defined in [32, Lemma 2.1].

We multiply both sides of (4.3) by $d\alpha(t)$ and integrate over $[0,1]$, then (H1) enables us to obtain

$$\int_0^1 u(t)d\alpha(t) = \int_0^1 \int_0^1 G(t,s)z(s)dsd\alpha(t) + \int_0^1 td\alpha(t)\left(\chi_1 + \int_0^1 u(t)d\alpha(t)\right)$$

and

$$\int_0^1 u(t)d\alpha(t) = \frac{1}{1 - \int_0^1 td\alpha(t)} \int_0^1 \int_0^1 G(t,s)z(s)dsd\alpha(t) + \frac{\chi_1}{1 - \int_0^1 td\alpha(t)} \int_0^1 td\alpha(t).$$

Therefore, we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)z(s)ds + \chi_1 t + \frac{t}{1 - \int_0^1 td\alpha(t)} \int_0^1 \int_0^1 G(t,s)z(s)dsd\alpha(t) + \frac{\chi_1 t}{1 - \int_0^1 td\alpha(t)} \int_0^1 td\alpha(t) \\ &= \int_0^1 K_G(t,s)z(s)ds + \frac{\chi_1 t}{1 - \int_0^1 td\alpha(t)}, \end{aligned}$$

where

$$K_G(t,s) = G(t,s) + \frac{t}{1 - \int_0^1 td\alpha(t)} \int_0^1 G(t,s)d\alpha(t), t \in [0,1].$$

Note that $G(t,s) \geq 0, t, s \in [0,1]$. Then, (H1) implies that

$$u(t) \geq 0, t \in [0,1].$$

This completes the proof.

For $v_0, w_0 \in E$ with $v_0(t) \leq w_0(t)$ for $t \in [0,1]$, we denote an ordered interval:

$$[v_0, w_0] = \{u \in E : v_0(t) \leq u(t) \leq w_0(t),\ t \in [0,1]\}.$$

Also, we list our other assumptions in this section.

(H7) There exist $w_0, v_0 \in E$ which are the upper and lower solutions of problem (1.1), respectively, and $v_0(t) \leq w_0(t), t \in [0,1]$.

(H8) $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$ and

$$f(t,w) - f(t,v) \geq -c(t)(w-v) \text{ for } v_0(t) \leq v \leq w \leq w_0(t), t \in [0,1].$$

Theorem 4.4. Suppose that (H1) and (H6)–(H8) hold. Then there exist monotone iterative sequences $\{v_n\}, \{w_n\} \subset [v_0, w_0]$ such that $v_n \to v^*, w_n \to w^*$ as $n \to \infty$ uniformly in $[v_0, w_0]$, and v^*, w^* are the minimal and the maximal solution of (1.1) in $[v_0, w_0]$, respectively.

Proof. We define two sequences $\{w_n\}, \{v_n\} \subset E$ satisfying the following boundary value problems

$$\begin{cases} v_n^{(4)}(t) + c(t)v_n(t) = f(t, v_{n-1}) + c(t)v_{n-1}(t),\ 0 < t < 1, n = 1, 2, \cdots, \\ v_n(0) = v_n''(0) = v_n''(1) = 0,\ v_n(1) = \displaystyle\int_0^1 v_n(t)d\alpha(t), \end{cases} \tag{4.4}$$

and

$$\begin{cases} w_n^{(4)}(t) + c(t)w_n(t) = f(t, w_{n-1}) + c(t)w_{n-1}(t),\ 0 < t < 1, n = 1, 2, \cdots, \\ w_n(0) = w_n''(0) = w_n''(1) = 0,\ w_n(1) = \displaystyle\int_0^1 w_n(t)d\alpha(t). \end{cases} \tag{4.5}$$

Step 1. We prove

$$v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t), t \in [0,1]. \tag{4.6}$$

Let $x(t) = v_1(t) - v_0(t)$. Then we have

$$\begin{cases} x^{(4)}(t) + c(t)x(t) = v_1^{(4)}(t) - v_0^{(4)}(t) + c(t)v_1(t) - c(t)v_0(t) \\ \qquad \geq f(t, v_0) + c(t)v_0(t) - f(t, v_0) - c(t)v_0(t) = 0,\ 0 < t < 1, \\ x(0) = v_1(0) - v_0(0) = 0, x''(0) = v_1''(0) - v_0''(0) = 0, x''(1) = v_1''(1) - v_0''(1) = 0, \\ x(1) = v_1(1) - v_0(1) \geq \displaystyle\int_0^1 v_1(t)d\alpha(t) - \int_0^1 v_0(t)d\alpha(t) = \int_0^1 x(t)d\alpha(t). \end{cases} \tag{4.7}$$

From Lemma 4.3, $x(t) \geq 0$, i.e., $v_1(t) \geq v_0(t), t \in [0,1]$.

Let $y(t) = w_0(t) - w_1(t)$. Then we obtain

$$\begin{cases} y^{(4)}(t) + c(t)y(t) = w_0^{(4)}(t) - w_1^{(4)}(t) + c(t)w_0(t) - c(t)w_1(t) \\ \qquad \geq f(t, w_0) + c(t)w_0(t) - c(t)w_0(t) - f(t, w_0) = 0,\ 0 < t < 1, \\ y(0) = w_0(0) - w_1(0) = 0, y''(0) = w_0''(0) - w_1''(0) = 0, y''(1) = w_0''(1) - w_1''(1) = 0, \\ y(1) = w_0(1) - w_1(1) \geq \displaystyle\int_0^1 w_0(t)d\alpha(t) - \int_0^1 w_1(t)d\alpha(t) = \int_0^1 y(t)d\alpha(t). \end{cases} \tag{4.8}$$

Lemma 4.3 implies that $y(t) \geq 0$, i.e., $w_0(t) \geq w_1(t), t \in [0,1]$.

Let $h(t) = w_1(t) - v_1(t)$. Then we have

$$\begin{cases} h^{(4)}(t) = w_1^{(4)}(t) - v_1^{(4)}(t) = f(t, w_0) + c(t)w_0(t) - c(t)w_1(t) + c(t)v_1(t) - c(t)v_0(t) + f(t, v_0) \\ \qquad \geq c(t)w_0(t) - c(t)w_1(t) + c(t)v_1(t) - c(t)v_0(t) - c(t)(w_0(t) - v_0(t)),\ 0 < t < 1, \\ h(0) = w_1(0) - v_1(0) = 0, h''(0) = w_1''(0) - v_1''(0) = 0, h''(1) = w_1''(1) - v_1''(1) = 0, \\ h(1) = w_1(1) - v_1(1) = \displaystyle\int_0^1 w_1(t)d\alpha(t) - \int_0^1 v_1(t)d\alpha(t) = \int_0^1 h(t)d\alpha(t), \end{cases} \tag{4.9}$$

and thus

$$\begin{cases} h^{(4)}(t) + c(t)h(t) \geq 0, \\ h(0) = h''(0) = h''(1) = 0, \ h(1) = \int_0^1 h(t)d\alpha(t). \end{cases} \tag{4.10}$$

Lemma 4.3 enable us to obtain $h(t) \geq 0$, i.e., $w_1(t) \geq v_1(t), t \in [0, 1]$.

As a result, (4.6) holds.

Step 2. We prove that w_1, v_1 are upper and lower solutions of problem (1.1), respectively.

From (H8) and (4.4) we have

$$\begin{aligned} v_1^{(4)}(t) &= f(t, v_0) + c(t)v_0(t) - c(t)v_1(t) \\ &= f(t, v_0) + c(t)v_0(t) - c(t)v_1(t) - f(t, v_1) + f(t, v_1) \\ &\leq c(t)(v_1(t) - v_0(t)) + c(t)v_0(t) - c(t)v_1(t) + f(t, v_1) \\ &= f(t, v_1), \end{aligned}$$

and note that

$$v_1(0) = v_1''(0) = v_1''(1) = 0, \ v_1(1) = \int_0^1 v_1(t)d\alpha(t).$$

From Definition 4.2, v_1 is a lower solution for (1.1).

From (H8) and (4.5) we have

$$\begin{aligned} w_1^{(4)}(t) &= f(t, w_0) + c(t)w_0(t) - c(t)w_1(t) \\ &= f(t, w_0) + c(t)w_0(t) - c(t)w_1(t) - f(t, w_1) + f(t, w_1) \\ &\geq -c(t)(w_0(t) - w_1(t)) + c(t)w_0(t) - c(t)w_1(t) + f(t, w_1) \\ &= f(t, w_1), \end{aligned}$$

and

$$w_1(0) = w_1''(0) = w_1''(1) = 0, \ w_1(1) = \int_0^1 w_1(t)d\alpha(t).$$

From Definition 4.1, w_1 is an upper solution for (1.1).

Therefore, for $v_{n-1}, v_n, w_{n-1}, w_n$ we can use the method in Steps 1 and 2 to obtain

$$v_{n-1}(t) \leq v_n(t) \leq w_n(t) \leq w_{n-1}(t), t \in [0, 1], n = 1, 2, \cdots, \tag{4.11}$$

and $w_n, v_n \in E$ are upper and lower solutions of problem (1.1), respectively.

Using mathematical induction, it is easy to verify that

$$v_0(t) \leq v_1(t) \leq \cdots \leq v_n(t) \leq \cdots \leq w_n(t) \leq \cdots \leq w_1(t) \leq w_0(t), t \in [0, 1].$$

It is easy to conclude that $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are uniformly bounded in E, and from the monotone bounded theorem we have

$$\lim_{n\to\infty} v_n(t) = v^*(t), \lim_{n\to\infty} w_n(t) = w^*(t), t \in [0, 1].$$

Step 3. We prove (1.1) has solutions.

Note that (4.4) and (4.5) are respectively equivalent to the following integral equations

$$v_n(t) = \int_0^1 G(t,s)[f(s, v_{n-1}(s)) + c(s)v_{n-1}(s)]ds + t\int_0^1 v_n(t)d\alpha(t),$$

and

$$w_n(t) = \int_0^1 G(t,s)[f(s, w_{n-1}(s)) + c(s)w_{n-1}(s)]ds + t\int_0^1 w_n(t)d\alpha(t).$$

Let $n \to \infty$ and we have

$$v^*(t) = \int_0^1 G(t,s)[f(s, v^*(s)) + c(s)v^*(s)]ds + t\int_0^1 v^*(t)d\alpha(t),$$

and

$$w^*(t) = \int_0^1 G(t,s)[f(s, w^*(s)) + c(s)w^*(s)]ds + t\int_0^1 w^*(t)d\alpha(\tau).$$

These two integral equations can be transformed into the following boundary value problems

$$\begin{cases} [v^*(t)]^{(4)} + \cancel{c(t)v^*(t)} = f(t, v^*) + \cancel{c(t)v^*(t)},\ 0 < t < 1, \\ v^*(0) = [v^*]''(0) = [v^*]''(1) = 0,\ v^*(1) = \int_0^1 v^*(t)d\alpha(t), \end{cases} \tag{4.12}$$

and

$$\begin{cases} [w^*(t)]^{(4)} + \cancel{c(t)w^*(t)} = f(t, w^*) + \cancel{c(t)w^*(t)},\ 0 < t < 1, \\ w^*(0) = [w^*]''(0) = [w^*]''(1) = 0,\ w^*(1) = \int_0^1 w^*(t)d\alpha(t), \end{cases} \tag{4.13}$$

i.e., v^*, w^* are solutions for (1.1).

Step 4. We prove that v^* and w^* are extremal solutions for (1.1) in $[v_0, w_0]$.

Let $u \in [v_0, w_0]$ be any solution for (1.1). We assume that $v_m(t) \le u(t) \le w_m(t), t \in [0,1]$ for some m. Let $p(t) = u(t) - v_{m+1}(t), q(t) = w_{m+1}(t) - u(t)$. Then from (1.1), (4.4) and (H8) we have

$$\begin{cases} p^{(4)}(t) = u^{(4)}(t) - v_{m+1}^{(4)}(t) \ge f(t,u) - f(t, v_{m+1}) \ge -c(t)(u(t) - v_{m+1}(t)), t \in [0,1], \\ u(0) - v_{m+1}(0) = u''(0) - v''_{m+1}(0) = u''(1) - v''_{m+1}(1) = 0, \\ u(1) - v_{m+1}(1) \ge \int_0^1 u(t)d\alpha(t) - \int_0^1 v_{m+1}(t)d\alpha(t), \end{cases}$$

and this leads to the following boundary value problem

$$\begin{cases} p^{(4)}(t) + c(t)p(t) \ge 0, t \in [0,1], \\ p(0) = p''(0) = p''(1) = 0, p(1) \ge \int_0^1 p(t)d\alpha(t). \end{cases}$$

Lemma 4.3 implies that $p(t) \ge 0$, i.e., $u(t) \ge v_{m+1}(t), t \in [0,1]$.

By (1.1), (4.5) and (H8) we have

$$\begin{cases} q^{(4)}(t) = w_{m+1}^{(4)}(t) - u^{(4)}(t) \geq f(t, w_{m+1}) - f(t, u) \geq -c(t)(w_{m+1}(t) - u(t)), t \in [0, 1], \\ w_{m+1}(0) - u(0) = w''_{m+1}(0) - u''(0) = w''_{m+1}(1) - u''(1) = 0, \\ w_{m+1}(1) - u(1) \geq \int_0^1 w_{m+1}(t)d\alpha(t) - \int_0^1 u(t)d\alpha(t), \end{cases}$$

and this leads to the following boundary value problem

$$\begin{cases} q^{(4)}(t) + c(t)q(t) \geq 0, t \in [0, 1], \\ q(0) = q''(0) = q''(1) = 0, q(1) \geq \int_0^1 q(t)d\alpha(t). \end{cases}$$

Lemma 4.3 implies that $q(t) \geq 0$, i.e., $w_{m+1}(t) \geq u(t), t \in [0, 1]$.

Combining the above two cases, we have

$$v_{m+1}(t) \leq u(t) \leq w_{m+1}(t),\ t \in [0, 1].$$

Applying mathematical induction, we obtain $v_n(t) \leq u(t) \leq w_n(t)$ on $[0, 1]$ for any n. Taking the limit, we conclude $v^*(t) \leq u(t) \leq w^*(t), t \in [0, 1]$. This completes the proof.

Remark 4.1. As noted in [32], the Green's function G in (4.2) has no explicit expression, but this does not affect our result. In our study we only use its positiveness and continuity.

5. Examples

Now, we provide some examples to illustrate our main results.

Example 5.1. From (2.2) we have

$$\frac{\xi}{\pi^4} \leq r\left(L_\xi\right) \leq \frac{\xi}{36} + \alpha(1), \xi > 0.$$

Let $\alpha(t) = \frac{1}{2}t, t \in [0, 1]$. Then we can choose $\xi_1 \geq \pi^4, \xi_2 \in (0, 18)$ such that

$$r(L_{\xi_1}) \geq 1,\ r(L_{\xi_2}) < 1.$$

Let $\gamma_1(t) \equiv \zeta_1 \in (\xi_1, +\infty), \gamma_2(t) \equiv \zeta_2 \in (0, \zeta_1 + \xi_2]$, and $f(t, x) = \zeta_1|x| - \zeta_2\mathcal{M}(x), \mathcal{M}(x) = \ln(|x| + 1), x \in \mathbb{R}, t \in [0, 1]$. Then $\lim_{|x|\to+\infty} \frac{\mathcal{M}(x)}{|x|} = 0$, and $\lim_{|x|\to+\infty} \frac{\zeta_1|x|-\zeta_2\mathcal{M}(x)}{|x|} = \zeta_1 > \xi_1, \lim_{|x|\to 0} \frac{|\zeta_1|x|-\zeta_2\mathcal{M}(x)|}{|x|} = |\zeta_1 - \zeta_2| \leq \xi_2$. Therefore, (H1)–(H5) hold. From Theorem 3.1, (1.1) has a nontrivial solution.

Example 5.2. Let $\alpha(t) = \frac{1}{2}t$, and $v_0(t) = -t^4 + 2t^3 - 5t, w_0(t) = t^4 - 2t^3 + 5t, f(t, u) = 5tu(t), t \in [0, 1]$. Then we have

$$\begin{cases} [w_0(t)]^{(4)} = 24 \geq 5tw_0(t) = f(t, w_0(t)),\ 0 < t < 1, \\ w_0(0) = w_0''(0) = w_0''(1) = 0,\ w_0(1) = 4 \geq 1.1 = \int_0^1 (t^4 - 2t^3 + 5t)d\frac{1}{2}t, \end{cases}$$

and

$$\begin{cases} [v_0(t)]^{(4)} = -24 \leq 5tv_0(t) = f(t, v_0(t)),\ 0 < t < 1, \\ v_0(0) = v_0''(0) = v_0''(1) = 0,\ v_0(1) = -4 \leq -1.1 = \int_0^1 (-t^4 + 2t^3 - 5t)d\frac{1}{2}t. \end{cases}$$

Moreover,

$$f(t,w) - f(t,v) = 5t(w-v),\ t \in [0,1].$$

Then (H1) and (H6)–(H8) hold. From Theorem 4.4, (1.1) has two extremal solutions.

6. Conclusions

In this paper we use topological degree and the method of upper-lower solutions to study the existence of solutions for (1.1). When the nonlinearity satisfies some superlinear growth conditions involving the first eigenvalue corresponding to the relevant linear operator we obtain nontrivial solutions. Also, when the nonlinearity satisfies a one-sided Lipschitz condition, we use the method of upper-lower solutions to obtain extremal solutions. We also provide two iterative sequences for these solutions.

Acknowledgments

This research was supported by the Nature Science Foundation of Anhui Provincial Education Department (Grant Nos. KJ2020A0735, KJ2021ZD0136), the Foundation of Suzhou University (Grant Nos. 2019XJZY02, szxy2020xxkc03).

Conflict of interest

The authors declare no conflict of interest.

References

1. B. Ahmad, J. J. Nieto, A. Alsaedi, H. Al-Hutami, Boundary value problems of nonlinear fractional q-difference (integral) equations with two fractional orders and four-point nonlocal integral boundary conditions, *Filomat*, **28** (2014), 1719–1736. https://doi.org/10.2298/FIL1408719A
2. A. Alsaedi, B. Ahmad, Y. Alruwaily, S. K. Ntouyas, On a coupled system of higher order nonlinear Caputo fractional differential equations with coupled Riemann-Stieltjes type integro-multipoint boundary conditions, *Adv. Differ. Equ.*, **2019** (2019), 474. https://doi.org/10.1186/s13662-019-2412-x
3. N. Anjum, C. He, J. He, Two-scale fractal theory for the population dynamics, *Fractals*, **29** (2021), 2150182. https://doi.org/10.1142/S0218348X21501826
4. A. Cabada, R. Jebari, Existence results for a clamped beam equation with integral boundary conditions, *Electron. J. Qual. Theory Differ. Equ.*, **2020** (2020), 70. https://doi.org/10.14232/ejqtde.2020.1.70
5. E. Cancès, B. Mennucci, J. Tomasi, A new integral equation formalism for the polarizable continuum model: Theoretical background and applications to isotropic and anisotropic dielectrics, *J. Chem. Phys.*, **107** (1997), 3032. https://doi.org/10.1063/1.474659
6. P. Drábek, G. Holubová, On the maximum and antimaximum principles for the beam equation, *Appl. Math. Lett.*, **56** (2016), 29–33. https://doi.org/10.1016/j.aml.2015.12.009

7. P. Drábek, G. Holubová, Positive and negative solutions of one-dimensional beam equation, *Appl. Math. Lett.*, **51** (2016), 1–7. https://doi.org/10.1016/j.aml.2015.06.019
8. M. Feng, J. Qiu, Multi-parameter fourth order impulsive integral boundary value problems with one-dimensional *m*-Laplacian and deviating arguments, *J. Inequal. Appl.*, **2015** (2015), 64. https://doi.org/10.1186/s13660-015-0587-6
9. Z. Fu, S. Bai, D. O'Regan, J. Xu, Nontrivial solutions for an integral boundary value problem involving Riemann-Liouville fractional derivatives, *J. Inequal. Appl.*, **2019** (2019), 104. https://doi.org/10.1186/s13660-019-2058-y
10. D. Guo, V. Lakshmikantham, *Nonlinear problems in abstract cones*, Academic Press, 1988.
11. F. Haddouchi, Positive solutions of nonlocal fractional boundary value problem involving Riemann-Stieltjes integral condition, *J. Appl. Math. Comput.*, **64** (2020), 487–502. https://doi.org/10.1007/s12190-020-01365-0
12. F. Haddouchi, C. Guendouz, S. Benaicha, Existence and multiplicity of positive solutions to a fourth-order multi-point boundary value problem, *Mat. Vesn.*, **73** (2021), 25–36.
13. F. Haddouchi, N. Houari, Monotone positive solution of fourth order boundary value problem with mixed integral and multi-point boundary conditions, *J. Appl. Math. Comput.*, **66** (2021), 87–109. https://doi.org/10.1007/s12190-020-01426-4
14. X. Hao, N. Xu, L. Liu, Existence and uniqueness of positive solutions for fourth-order *m*-point boundary value problems with two parameters, *Rocky Mountain J. Math.*, **43** (2013), 1161–1180. https://doi.org/10.1216/RMJ-2013-43-4-1161
15. J. H. He, A short review on analytical methods for a fully fourth-order nonlinear integral boundary value problem with fractal derivatives, *Int. J. Numer. Method. H.*, **30** (2020), 4933–4943. https://doi.org/10.1108/HFF-01-2020-0060
16. J. H. He, M. H. Taha, M. A. Ramadan, G. M. Moatimid, A combination of bernstein and improved block-pulse functions for solving a system of linear fredholm integral equations, *Math. Probl. Eng.*, **2022** (2022), 6870751. https://doi.org/10.1155/2022/6870751
17. M. G. Kreĭn, M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, New York: American Mathematical Society, 1950.
18. C. Lanczos, An iteration method for the solution of the eigenvalue problem of linear differential and integral operators, *J. Res. Natl. Bur. Stand.*, **45** (1950), 255–282. https://doi.org/10.6028/jres.045.026
19. B. Liu, J. Li, L. Liu, Nontrivial solutions for a boundary value problem with integral boundary conditions, *Bound. Value Probl.*, **2014** (2014), 15. https://doi.org/10.1186/1687-2770-2014-15
20. J. H. He, M. H. Taha, M. A. Ramadan, G. M. Moatimid, Improved block-pulse functions for numerical solution of mixed volterra-fredholm integral equations, *Axioms*, **10** (2021), 200. https://doi.org/10.3390/axioms10030200
21. A. Ramazanova, Y. Mehraliyev, On solvability of inverse problem for one equation of fourth order, *Turkish J. Math.*, **44** (2020), 611–621. https://doi.org/10.3906/mat-1912-51
22. F. T. Fen, I. Y. Karaca, Existence of positive solutions for fourth-order impulsive integral boundary value problems on time scales, *Math. Method. Appl. Sci.*, **40** (2017), 5727–5741. https://doi.org/10.1002/mma.4420

23. R. Vrabel, On the lower and upper solutions method for the problem of elastic beam with hinged ends, *J. Math. Anal. Appl.*, **421** (2015), 1455–14685. https://doi.org/10.1016/j.jmaa.2014.08.004
24. F. Wang, L. Liu, Y. Wu, Iterative unique positive solutions for a new class of nonlinear singular higher order fractional differential equations with mixed-type boundary value conditions, *J. Inequal. Appl.*, **2019** (2019), 210. https://doi.org/10.1186/s13660-019-2164-x
25. W. Wang, J. Ye, J. Xu, D. O'Regan, Positive solutions for a high-order Riemann-Liouville type fractional integral boundary value problem involving fractional derivatives, *Symmetry*, **14** (2022), 2320. https://doi.org/10.3390/sym14112320
26. J. R. L. Webb, Positive solutions of nonlinear differential equations with Riemann-Stieltjes boundary conditions, *Electron. J. Qual. Theory Differ. Equ.*, **2016** (2016), 86. https://doi.org/10.14232/ejqtde.2016.1.86
27. J. Xu, D. O'Regan, Z. Yang, Positive solutions for a nth-order impulsive differential equation with integral boundary conditions, *Differ. Equ. Dyn. Syst.*, **22** (2014), 427–439. https://doi.org/10.1007/s12591-013-0176-4
28. C. Zhai, Y. Ma, H. Li, Unique positive solution for a p-Laplacian fractional differential boundary value problem involving Riemann-Stieltjes integral, *AIMS Math.*, **5** (2020), 4754–4769. https://doi.org/10.3934/math.2020304
29. G. Zhang, Positive solutions to three classes of non-local fourth-order problems with derivative-dependent nonlinearities, *Electron. J. Qual. Theory Differ. Equ.*, **2022** (2022), 11. https://doi.org/10.14232/ejqtde.2022.1.11
30. X. Zhang, X. Liu, M. Jia, H. Chen, The positive solutions of fractional differential equation with Riemann-Stieltjes integral boundary conditions, *Filomat*, **32** (2018), 2383–2394. https://doi.org/10.2298/FIL1807383Z
31. X. Zhang, L. Liu, B. Wiwatanapataphee, Y. Wu, The eigenvalue for a class of singular p-Laplacian fractional differential equations involving the Riemann-Stieltjes integral boundary condition, *Appl. Math. Comput.*, **235** (2014), 412–422. https://doi.org/10.1016/j.amc.2014.02.062
32. Y. Zhang, L. Chen, Positive solution for a class of nonlinear fourth-order boundary value problem, *AIMS Math.*, **8** (2023), 1014–1021. https://doi.org/10.3934/math.2023049
33. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 73–85. https://doi.org/10.12785/pfda/010201
34. M. Al-Refai, A. M. Jarrah, Fundamental results on weighted Caputo-Fabrizio fractional derivative, *Chaos Soliton. Fract.*, **126** (2019), 7–11. https://doi.org/10.1016/J.CHAOS.2019.05.035
35. A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel: Theory and applications to heat transfer model, *Therm. Sci.*, **20** (2016), 763–769. https://doi.org/10.2298/TSCI160111018A