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Research article

Precise large deviations for aggregate claims in a two-dimensional compound dependent risk model

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Abstract: This paper considers a two-dimensional compound risk model. We mainly investigate the claim sizes and inter-arrival times are size-dependent. When the claim sizes have consistently varying tails, we obtain the precise large deviations for aggregate amount of claims in the above dependent compound risk model.

Keywords: precise large deviations; two-dimensional compound risk model; size-dependence; consistently varying distribution
Mathematics Subject Classification: 60F10, 91B05, 91G05

1. Introduction

This paper will investigate a two-dimensional compound risk model. In this risk model, an insurance company has two dependent classes of business sharing a common claim-number process, which is a compound renewal counting process. Let the inter-arrival times of events $\{\theta_k, k \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) nonnegative random variables (r.v.s) with finite mean $\beta^{-1} > 0$. Let Z_k be the number of claims caused by the kth ($k \geq 1$) event. Suppose that $\{Z_k, k \geq 1\}$ are i.i.d. positive integer r.v.s with finite mean μ_Z and independent of $\{\theta_k, k \geq 1\}$. Then the number of events up to time $t \geq 0$ is denoted by

$$N(t) = \sup\left\{n \geq 1, \sum_{k=1}^{n} \theta_k \leq t\right\}$$

and the number of claims up to time $t \geq 0$ is denoted by

$$\Lambda(t) = \sum_{k=1}^{N(t)} Z_k,$$

which is a compound renewal counting process. Set $\theta(t) = E(N(t))$ and $\lambda(t) = E(\Lambda(t))$, $t \geq 0$, then $\theta(t)/t \to \beta$ as $t \to \infty$ and $\lambda(t) = \mu_Z\theta(t)$, $t \geq 0$. The claim-amount vectors $\vec{X}_k = (X_{1k}, X_{2k})^T, k \geq 1$ are i.i.d. copies of $\vec{X} = (X_1, X_2)^T$ with finite mean vector $\vec{\mu} = (\mu_1, \mu_2)^T$. Assume that X_1 and X_2 are nonnegative r.v.s with distributions F_1 and F_2, respectively. Their joint distribution is denoted by $F_{12}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$ and their joint survival function is $\overline{F_{12}}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$. Then the aggregate amount of claims up to time $t \geq 0$ is expressed as

$$\vec{S}(t) = \sum_{k=1}^{\Lambda(t)} \vec{X}_k. \tag{1.1}$$

This paper will investigate the precise large deviations of $\vec{S}(t), t \geq 0$.

In this paper, we assume that $\{Z_k, k \geq 1\}$ are independent of $\{\vec{X}_k, k \geq 1\}$ and $\{(\vec{X}_k, \theta_k), k \geq 1\}$ are i.i.d. random vectors with generic pair $(\vec{X}, \theta)$. This paper mainly considers for each $k \geq 1$, X_{1k}, X_{2k} and θ_k may be dependent and the claims have heavy-tailed distributions. In the following section some heavy-tailed distribution classes will be given.

Without special statement, in this paper a limit is taken as $t \to \infty$. For a real-valued number a, let $a^+ = \max\{0, a\}$ and $a^- = -\min\{0, a\}$. Denote $[a]$ by the large integer that does not exceed a. For two vectors $\vec{y} = (y_1, y_2)^T$ and $\vec{z} = (z_1, z_2)^T$, $\vec{y} > \vec{z}$ (or $\geq$) means $y_i > z_i$ (or $\geq$), $i = 1, 2$. For two nonnegative functions $a(\cdot)$ and $b(\cdot)$, we write $a(t) \lesssim b(t)$ if $\limsup a(t)/b(t) \leq 1$, write $a(t) \gtrsim b(t)$ if $\liminf a(t)/b(t) \geq 1$, write $a(t) \sim b(t)$ if $\lim a(t)/b(t) = 1$, and write $a(t) = o(b(t))$ if $\lim a(t)/b(t) = 0$. For two positive bivariate functions $g(\cdot, \cdot)$ and $h(\cdot, \cdot)$, we write $g(x, t) \lesssim h(x, t)$, as $t \to \infty$, holds uniformly for $x \in \Delta \neq \phi$, if

$$\limsup_{t\to\infty} \sup_{x\in\Delta} \frac{g(x, t)}{h(x, t)} \leq 1.$$

We write $g(x, t) \gtrsim h(x, t)$, as $t \to \infty$, holds uniformly for $x \in \Delta \neq \phi$, if

$$\liminf_{t\to\infty} \inf_{x\in\Delta} \frac{g(x, t)}{h(x, t)} \geq 1.$$

In the following, we give some heavy-tailed distribution classes. For a proper distribution V on $(-\infty, \infty)$, let $\overline{V} = 1 - V$ be the tail of V. Say that a distribution V on $(-\infty, \infty)$ is heavy-tailed, if for any $s > 0$,

$$\int_{-\infty}^{\infty} e^{su} V(\mathrm{d}u) = \infty.$$

Otherwise, say that V is light-tailed. The dominated variation distribution class $\mathscr{D}$ is an important class of heavy-tailed distributions. Say that a distribution V on $(-\infty, \infty)$ belongs to the class $\mathscr{D}$, if for any $y \in (0, 1)$,

$$\limsup_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} < \infty.$$

The slightly smaller class is the class $\mathscr{C}$, which consists of all distributions with consistently varying tails. Say that a distribution V on $(-\infty, \infty)$ belongs to the class $\mathscr{C}$ if

$$\lim_{y\searrow 1} \liminf_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} = \lim_{y\nearrow 1} \limsup_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} = 1.$$

Another class is the long-tailed distribution class $\mathscr{L}$. Say that a distribution V on $(-\infty, \infty)$ belongs to the class $\mathscr{L}$ if for any $y > 0$,

$$\lim_{x\to\infty} \frac{\overline{V}(x-y)}{\overline{V}(x)} = 1.$$

It is well known that these distribution classes have the following relationships:

$$\mathscr{C} \subset \mathscr{L} \cap \mathscr{D} \subset \mathscr{L}$$

(see, e.g., Cline and Samorodnitsky [5], Embrechts et al. [7]).

For a distribution V on $(-\infty, \infty)$, let

$$J_V^+ = \inf\left\{-\frac{\log \overline{V}_*(y)}{\log y}, y \geq 1\right\} \quad \text{with} \quad \overline{V}_*(y) = \liminf_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)}, y \geq 1.$$

We call J_V^+ the upper Matuszewska index of V. For the details of the Matuszewska index one can see Bingham et al. [2].

In recent years, more and more researchers pay attention to multi-dimensional risk models and study the precise large deviations of aggregate amount of claims, see e.g. Wang and Wang [19], Wang and Wang [20], Lu [12], Tian and Shen [14] and so on. Recently, Fu et al. [8] studied the precise large deviations of $S_{N(t)} = \sum_{k=1}^{N(t)} X_k$, $t \geq 0$ under the following assumptions.

Assumption 1.1. *The random vector* (X_1, X_2) *has a survival copula* $\hat{C}(\cdot, \cdot)$ *satisfying*

$$\hat{C}\left(\overline{F}_1(x_1), \overline{F}_2(x_2)\right) \leq g_u(2)\overline{F}_1(x_1)\overline{F}_2(x_2)$$

where $g_u(\cdot)$ *is a finite positive function.*

Definition 2.2.2 of Nelsen [13] gave the definition of copula. A copula is a function C from $[0,1] \times [0,1] \to [0,1]$ with the following properties:
(1) For every $u, v \in [0,1]$, $C(u,0) = C(0,v) = 0$, $C(u,1) = u$ and $C(1,v) = v$.
(2) For every $u_1, u_2, v_1, v_2 \in [0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

The Sklar's theorem (i.e. Theorem 2.3.3 of Nelsen [13]) states that for the r.v.s X_1 and X_2 in Assumption 1.1, there exists a copula C such that for all $x_i \in (-\infty, \infty), i = 1, 2$,

$$F_{12}(x_1, x_2) = C(F_1(x_1), F_2(x_2)).$$

Let $\hat{C}(u, v) = u + v - 1 + C(1-u, 1-v), u, v \in [0,1]$, then for all $x_i \in (-\infty, \infty), i = 1, 2$,

$$\overline{F_{12}}(x_1, x_2) = \hat{C}(\overline{F_1}(x_1), \overline{F_2}(x_2)).$$

We call $\hat{C}$ as the survival copula of X_1 and X_2 (see (2.6.1) and (2.6.2) of Nelsen [13]).

Assumption 1.2. *There exists a nonnegative random variable* θ^* *with finite mean such that* θ *conditional on* $(X_i > x_i), i = 1, 2$, *is stochastically bounded by* θ^* *for all large* x_1 *and* x_2; *i.e., there exists some* $\vec{x}_0 = (x_{10}, x_{20})^T$ *such that it holds for all* $\vec{x} = (x_1, x_2)^T > \vec{x}_0$ *and* $t \in [0, \infty)$ *that*

$$\mathrm{P}(\theta > t \mid X_i > x_i) \leq \mathrm{P}(\theta^* > t), \quad i = 1, 2.$$

This paper still uses the above two assumptions. We will investigate the precise large deviations of the aggregate amount of claims in a two-dimensional compound risk model. For the one-dimensional compound risk model, there are many papers studying the aggregate amount of claims, such as Tang et al. [15], Aleškevičienė et al. [1], Konstantinides and Loukissas [11], Yang et al. [22], Guo et al. [9], Wang and Chen [18], Yang et al. [23], Wang et al. [17], Xun et al. [21] and so on. For a two-dimensional compound risk model researchers mainly studied the ruin probabilities, such as Cai and Li [4], Delsing et al. [6] and so on. This paper will consider the precise large deviations of compound sum (1.1) in a two-dimensional compound risk model. The following is the main result of this paper.

Theorem 1.1. *Consider the model (1.1). Suppose that Assumptions 1.1 and 1.2 are satisfied, $F_i \in \mathscr{C}$, $i = 1, 2$ and there exists a constant $\alpha_Z > 2\max\{J^+_{F_1}, J^+_{F_2}\} + 4$ such that $EZ_1^{\alpha_Z} < \infty$. Then for any $\vec{\gamma} = (\gamma_1, \gamma_2)^T > \vec{0}$,*

$$P(\vec{S}(t) - \vec{\mu}\lambda(t) > \vec{x}) \sim (\lambda(t))^2 \overline{F}_1(x_1)\, \overline{F}_2(x_2),$$

holds uniformly for all $\vec{x} \geq \vec{\gamma}\lambda(t)$.

Remark 1.1. *In the two-dimensional compound renewal risk model (1.1), if $Z_k \equiv 1, k \geq 1$, then model (1.1) degenerates into the classic two-dimensional renewal risk model. In the classic two-dimensional renewal risk model, suppose that $F_i \in \mathscr{C}$, i=1,2 and Assumptions 1.1 and 1.2 are satisfied. Then from Theorem 1.1 the main result of Fu et al. [8] can be obtained.*

The proof of Theorem 1.1 will be given in the following section.

2. Proof of the main result

By Assumption 1.2, we introduce two independent nonnegative r.v.s θ_1^{**} and θ_2^{**}, which have the same distributions as θ conditional on $\{X_1 > x_1\}$ and $\{X_2 > x_2\}$, respectively. Assume that θ_1^{**} and θ_2^{**} are independent of all other r.v.s. Let $\tau_1^{**} = \theta_1^{**}, \tau_2^{**} = \theta_1^{**} + \theta_2^{**}, \tau_n^{**} = \theta_1^{**} + \theta_2^{**} + \sum_{i=3}^n \theta_i$, $n \geq 3$, and define

$$N^{**}(t) = \sup\{n \geq 1 : \tau_n^{**} \leq t\}, \quad t \geq 0.$$

Set $\Lambda^{**}(t) = \sum_{k=1}^{N^{**}(t)} Z_k$, $t \geq 0$. The following relation implies that for each $t \geq 0$, $\Lambda^{**}(t)$ is also identically distributed as $\Lambda(t)$ conditional on $\{X_1 > x_1, X_2 > x_2\}$. In fact, noticing the independence assumption between $\{Z_k, k \geq 1\}$ and $(\vec{X}, \theta)$, it holds for $t \geq 0$, $n \geq 1$ and $x_1, x_2 \geq 0$ that

$$\begin{aligned}
& P(\Lambda(t) = n \mid X_1 > x_1, X_2 > x_2) \\
= & \sum_{k=1}^{\infty} P\left(\sum_{i=1}^{k} Z_i = n \mid X_1 > x_1, X_2 > x_2, N(t) = k\right) P(N(t) = k \mid X_1 > x_1, X_2 > x_2) \\
= & \sum_{k=1}^{\infty} P\left(\sum_{i=1}^{k} Z_i = n\right) P(N(t) = k \mid X_1 > x_1, X_2 > x_2) \\
= & \sum_{k=1}^{\infty} P\left(\sum_{i=1}^{k} Z_i = n\right) P\left(N^{**}(t) = k\right) \\
= & P\left(\Lambda^{**}(t) = n\right).
\end{aligned} \tag{2.1}$$

Before giving the proof of Theorem 1.1, we first give some lemmas. The first lemma gives a property about $\Lambda^{**}(t), t \geq 0$.

Lemma 2.1. *In addition to Assumption 1.2, assume that* $\operatorname{Var}\theta < \infty$. *Then it holds for every* $0 < \delta < \beta$ *and every functions* $a(t)$ *and* $b(t)$ *that*

$$\lim_{t\to\infty} \sup_{\substack{x_1 \geq a(t) \\ x_2 \geq b(t)}} P\left(\left|\frac{\Lambda^{**}(t) - \lambda(t)}{t}\right| > \delta\right) = 0, \tag{2.2}$$

where $a(\cdot) : [0, \infty) \to (0, \infty)$ *with* $a(t) \uparrow \infty$ *and* $b(\cdot) : [0, \infty) \to (0, \infty)$ *with* $b(t) \uparrow \infty$.

Proof. Using the same method of the proof of Lemma 3.4 of Bi and Zhang [3], we can get that

$$\lim_{t\to\infty} \sup_{\substack{x_1 \geq a(t) \\ x_2 \geq b(t)}} P\left(\left|\frac{N^{**}(t)}{t} - \beta\right| > \delta\right) = 0. \tag{2.3}$$

In the following we will prove for any $\epsilon > 0$

$$\lim_{t\to\infty} \sup_{\substack{x_1 \geq a(t) \\ x_2 \geq b(t)}} P\left(\left|\frac{\sum_{k=1}^{N^{**}(t)} Z_k}{N^{**}(t)} - \mu_Z\right| > \epsilon\right) = 0. \tag{2.4}$$

For the above $\epsilon > 0$, by (2.3) and the law of large number for i.i.d r.v.s, it holds uniformly for $x_1 \geq a(t)$ and $x_2 \geq b(t)$ that

$$\begin{aligned}
P\left(\frac{\sum_{k=1}^{N^{**}(t)} Z_k}{N^{**}(t)} - \mu_Z > \epsilon\right) &= P\left(\frac{\sum_{k=1}^{N^{**}(t)} Z_k}{N^{**}(t)} > \epsilon + \mu_Z, N^{**}(t) < (\beta - \delta)t\right) \\
&\quad + P\left(\frac{\sum_{k=1}^{N^{**}(t)} Z_k}{N^{**}(t)} > \epsilon + \mu_Z, N^{**}(t) > (\beta + \delta)t\right) \\
&\quad + P\left(\frac{\sum_{k=1}^{N^{**}(t)} Z_k}{N^{**}(t)} > \epsilon + \mu_Z, (\beta - \delta)t \leq N^{**}(t) \leq (\beta + \delta)t\right) \\
&\leq P\left(\left|\frac{N^{**}(t)}{t} - \beta\right| > \delta\right) + P\left(\frac{\sum_{k=1}^{(\beta+\delta)t} Z_k}{(\beta - \delta)t} > \mu_Z + \epsilon\right) \\
&\to 0
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
P\left(\frac{\sum_{k=1}^{N^{**}(t)} Z_k}{N^{**}(t)} - \mu_Z < -\epsilon\right) &\leq P\left(\left|\frac{N^{**}(t)}{t} - \beta\right| > \delta\right) + P\left(\frac{\sum_{k=1}^{(\beta-\delta)t} Z_k}{(\beta + \delta)t} < \mu_Z - \epsilon\right) \\
&\to 0.
\end{aligned}$$

In the following, we prove (2.2). Since $\lambda(t) \sim \mu_Z\beta t$, it holds for any $0 < \epsilon < \delta(\mu_Z\beta)^{-1}$ that $(1 - \epsilon)\mu_Z\beta t \leq \lambda(t) \leq (1 + \epsilon)\mu_Z\beta t$. Thus by (2.3) and (2.4), it holds uniformly for $x_1 \geq a(t)$ and $x_2 \geq b(t)$ that

$$P\left(\Lambda^{**}(t) > \delta t + \lambda(t)\right) = P\left(\frac{\sum_{k=1}^{N^{**}(t)} Z_k}{N^{**}(t)\mu_Z} \cdot \frac{N^{**}(t)}{\beta t} > \frac{\delta}{\mu_Z\beta} + \frac{\lambda(t)}{\mu_Z\beta t}\right)$$

$$\begin{aligned} &\le P\left(\frac{\sum_{k=1}^{N^{**}(t)} Z_k}{N^{**}(t)\mu_Z}\cdot\frac{N^{**}(t)}{\beta t} > 1+\frac{\delta}{\mu_Z\beta}-\epsilon\right) \\ &\to 0. \end{aligned} \tag{2.6}$$

Similarly, it holds uniformly for $x_1 \ge a(t)$ and $x_2 \ge b(t)$ that

$$P(\Lambda^{**}(t) < \lambda(t) - \delta t) \to 0,$$

which combining with (2.6) yields that (2.2) holds. □

The following lemma is Lemma 3.2 of Fu et al. [8].

Lemma 2.2. *Let $\left\{\vec{X}_k, k \ge 1\right\}$ be a sequence of i.i.d. random vectors with finite mean vector $\vec{\mu}$. In addition to Assumptions 1.1 and 1.2, suppose that $F_i \in \mathscr{C}$, $i = 1, 2$. Then for any $\vec{\gamma} = (\gamma_1, \gamma_2)^T > \overrightarrow{0}$, it holds uniformly for all $\vec{x} = (x_1, x_2)^T \ge \vec{\gamma}n$ that*

$$P\left(\vec{S}_n - n\vec{\mu} > \vec{x}\right) \sim n^2\overline{F}_1(x_1)\overline{F}_2(x_2), \tag{2.7}$$

as $n \to \infty$, where $\vec{S}_n = (S_{1n}, S_{2n})^T = \sum_{k=1}^{n} \vec{X}_k$.

From Proposition 2.2.1 of Bingham et al. [2], we obtain

Lemma 2.3. *If $V \in \mathscr{D}$ then for every $p > J_V^+$, there are positive constants C and x_0 such that*

$$\frac{\overline{V}(x)}{\overline{V}(xy)} \le Cy^p$$

holds for all $xy \ge x \ge x_0$.

The next lemma comes from Lemma 1(i) of Kočetova et al. [10].

Lemma 2.4. *Let the inter-arrival times $\{\theta_k, k \ge 1\}$ form a sequence of i.i.d. nonnegative r.v.s with common mean $\beta^{-1} \in (0, \infty)$. Then it holds for every $a > \beta$ and some $b > 1$ that*

$$\lim_{t\to\infty} \sum_{n>at} b^n P\left(\sum_{j=1}^{n} \theta_j \le t\right) = 0.$$

The last lemma is a restatement of Lemma 2.3 of Tang [16].

Lemma 2.5. *Let $\{\xi_k, k \ge 1\}$ be i.i.d. real-valued r.v.s with common distribution V and mean 0 satisfying $E\left(\xi_1^+\right)^r < \infty$ for some $r > 1$. Then for each fixed $\gamma > 0$ and $p > 0$, there exist positive numbers v and $C = C(v, \gamma)$ irrespective to x and n such that for all $x \ge \gamma n$ and $n \ge 1$*

$$P\left(\sum_{k=1}^{n} \xi_k \ge x\right) \le n\overline{V}(vx) + Cx^{-p}.$$

Proof of Theorem 1.1: Without special statement, in this proof a limit relation is understood as valid uniformly for all $\vec{x} \geq \vec{\gamma}\lambda(t)$ as $t \to \infty$. We will show the following two relations

$$\mathrm{P}(\vec{S}(t) - \vec{\mu}\lambda(t) > \vec{x}) \lesssim (\lambda(t))^2 \overline{F}_1(x_1)\overline{F}_2(x_2) \tag{2.8}$$

and

$$\mathrm{P}(\vec{S}(t) - \vec{\mu}\lambda(t) > \vec{x}) \gtrsim (\lambda(t))^2 \overline{F}_1(x_1)\overline{F}_2(x_2). \tag{2.9}$$

We first prove (2.8). For any $0 < \delta < 1$, it holds that for $x_i > 0, i = 1, 2$ and $t > 0$

$$\begin{aligned} \mathrm{P}(\vec{S}(t) - \vec{\mu}\lambda(t) > \vec{x}) &= \mathrm{P}(\vec{S}(t) - \vec{\mu}\lambda(t) > \vec{x}, \Lambda(t) \leq (1+\delta)\lambda(t)) \\ &\quad + \mathrm{P}(\vec{S}(t) - \vec{\mu}\lambda(t) > \vec{x}, \Lambda(t) > (1+\delta)\lambda(t)) \\ &=: I_1 + I_2. \end{aligned} \tag{2.10}$$

For I_1, by Lemma 2.2 it holds that

$$\begin{aligned} I_1 &\leq \mathrm{P}\left(\vec{S}_{[(1+\delta)\lambda t]} - \vec{\mu}\lambda(t) > \vec{x}\right) \\ &= \mathrm{P}\left(\vec{S}_{[(1+\delta)\lambda t]} - \vec{\mu}[(1+\delta)\lambda(t)] > \vec{x} + \vec{\mu}\lambda(t) - \vec{\mu}[(1+\delta)\lambda(t)]\right) \\ &\lesssim [(1+\delta)\lambda(t)]^2 \overline{F}_1(x_1 + \mu_1\lambda(t) - \mu_1[(1+\delta)\lambda(t)])\, \overline{F}_2(x_2 + \mu_2\lambda(t) - \mu_2[(1+\delta)\lambda(t)]) \\ &\leq [(1+\delta)\lambda(t)]^2 \overline{F}_1\left(\left(1 - \delta\mu_1\gamma_1^{-1}\right)x_1\right) \overline{F}_2\left(\left(1 - \delta\mu_2\gamma_2^{-1}\right)x_2\right) \end{aligned} \tag{2.11}$$

where in the third step Lemma 2.2 is used, which is due to the fact that for small δ such that $\gamma_i - \mu_i\delta > 0$, and for any $0 < \gamma_i' < \frac{\gamma_i - \mu_i\delta}{1+\delta}$, it holds that $x_i + \mu_i\lambda(t) - \mu_i[(1+\delta)\lambda(t)] \geq \gamma_i'[(1+\delta)\lambda(t)]$ for $x_i \geq \gamma_i\lambda(t), i = 1, 2$. By $F_i \in \mathscr{C}, i = 1, 2$, we have

$$\lim_{\delta \downarrow 0} \limsup_{t \to \infty} \sup_{\vec{x} \geq \vec{\gamma}\lambda(t)} \frac{I_1}{(\lambda(t))^2 \overline{F}_1(x_1)\overline{F}_2(x_2)} \leq 1. \tag{2.12}$$

For I_2, take any $0 < \varepsilon < \frac{\delta\mu_Z\beta}{\delta+\beta+1}$ we have

$$\begin{aligned} I_2 &\leq \sum_{n > (1+\delta)\lambda(t)} P(S_{1n} > x_1, S_{2n} > x_2, \Lambda(t) = n) \\ &\leq \sum_{n > (1+\delta)\lambda(t)} \left[P\left(S_{1n} > x_1, S_{2n} > x_2, \sum_{j=1}^{\Theta(t)} Z_j = n, \Theta(t) > \frac{n}{\varepsilon + \mu_Z}\right) \right. \\ &\qquad \left. + P\left(S_{1n} > x_1, S_{2n} > x_2, \sum_{j=1}^{\Theta(t)} Z_j = n, \Theta(t) \leq \frac{n}{\varepsilon + \mu_Z}\right) \right] \\ &=: \sum_{n > (1+\delta)\lambda(t)} (K_1 + K_2). \end{aligned} \tag{2.13}$$

We first estimate K_1. Letting $p > \max\{J_{F_1}^+, J_{F_2}^+\}$, it follows from Assumption 1.1 and Lemma 2.3 that there exists some positive constant C such that for any $0 < \varepsilon < \mu_Z\beta$

$$K_1 = P\left(S_{1n} > x_1, S_{2n} > x_2, \sum_{j=1}^{\Theta(t)} Z_j = n, \Theta(t) > \frac{n}{\varepsilon + \mu_Z}\right)$$

$$
\begin{aligned}
&\le \sum_{m>\frac{n}{\varepsilon+\mu_Z}} \left(\bigcup_{i=1}^{n}\{X_{1i} > x_1/n\}, \bigcup_{j=1}^{n}\{X_{2j} > x_2/n\}, \Theta(t) = m \right) \\
&\le \sum_{m>\frac{n}{\varepsilon+\mu_Z}} \sum_{1\le i,j\le n} P\left(X_{1i} > x_1/n, X_{2j} > x_2/n, \sum_{k=1}^{m} \theta_k \le t \right) \\
&= \sum_{m>\frac{n}{\varepsilon+\mu_Z}} \left(\sum_{1\le i\ne j\le n} + \sum_{1\le i=j\le n} \right) P\left(X_{1i} > x_1/n, X_{2j} > x_2/n, \sum_{k=1}^{m} \theta_k \le t \right) \\
&\le \sum_{m>\frac{n}{\varepsilon+\mu_Z}} \sum_{1\le i\ne j\le n} P\left(X_{1i} > x_1/n, X_{2j} > x_2/n, \sum_{k=1,k\ne i,j}^{m} \theta_k \le t \right) \\
&\quad + \sum_{m>\frac{n}{\varepsilon+\mu_Z}} \sum_{1\le i=j\le n} P\left(X_{1i} > x_1/n, X_{2j} > x_2/n, \sum_{k=1,k\ne i}^{m} \theta_k \le t \right) \\
&= \sum_{m>\frac{n}{\varepsilon+\mu_Z}} n(n-1)P(X_{11} > x_1/n)(X_{21} > x_2/n)P\left(\sum_{k=3}^{m} \theta_k \le t \right) \\
&\quad + \sum_{m>\frac{n}{\varepsilon+\mu_Z}} nP(X_{11} > x_1/n, X_{21} > x_2/n)P\left(\sum_{k=2}^{m} \theta_k \le t \right) \\
&\le C \sum_{m>\frac{n}{\varepsilon+\mu_Z}} n^{2p+1}(n-1)\overline{F_1}(x_1)\overline{F_2}(x_2)P\left(\sum_{k=3}^{m} \theta_k \le t \right) \\
&\quad + C \sum_{m>\frac{n}{\varepsilon+\mu_Z}} n^{2p+1}\overline{F_1}(x_1)\overline{F_2}(x_2)P\left(\sum_{k=2}^{m} \theta_k \le t \right) \\
&\le C\overline{F_1}(x_1)\overline{F_2}(x_2) \sum_{m>\frac{n}{\varepsilon+\mu_Z}} n^{2p+2}P\left(\sum_{k=3}^{m} \theta_k \le t \right). \qquad (2.14)
\end{aligned}
$$

In the following, interchanging the order of sums yields that

$$
\begin{aligned}
\sum_{n>(1+\delta)\lambda(t)} K_1 &\le \sum_{m>\frac{(1+\delta)\lambda(t)}{\varepsilon+\mu_Z}} \sum_{(1+\delta)\lambda(t)<n<(\varepsilon+\mu_Z)m} Cn^{2p+2}\overline{F_1}(x_1)\overline{F_2}(x_2)P\left(\sum_{k=3}^{m} \theta_k \le t \right) \\
&\le C(\varepsilon+\mu_Z)^{2p+2}\overline{F_1}(x_1)\overline{F_2}(x_2) \sum_{m>\frac{(1+\delta)\lambda(t)}{\varepsilon+\mu_Z}} m^{2p+2}P\left(\sum_{k=3}^{m} \theta_k \le t \right). \qquad (2.15)
\end{aligned}
$$

Since $\lambda(t) \sim \mu_Z\beta t$, for sufficiently large t,

$$
\sum_{m>\frac{(1+\delta)\lambda(t)}{\varepsilon+\mu_Z}} m^{2p+2}P\left(\sum_{k=3}^{m} \theta_k \le t \right) \le \sum_{m>\frac{(1+\delta)(\mu_Z\beta-\varepsilon)t}{\varepsilon+\mu_Z}} m^{2p+2}P\left(\sum_{k=3}^{m} \theta_k \le t \right). \qquad (2.16)
$$

Since $\frac{(1+\delta)(\mu_Z\beta-\varepsilon)}{\varepsilon+\mu_Z} > \beta$, by (2.16) and Lemma 2.4 it holds that

$$\sum_{n>(1+\delta)\lambda)(t)} K_1 = o(\overline{F_1}(x_1)\overline{F_2}(x_2)). \tag{2.17}$$

We continue to deal with K_2. As K_1, by Assumption 1.1 there exists positive constant C such that

$$\begin{aligned} K_2 &\leq P\left(S_{1n} > x_1, S_{2n} > x_2, \sum_{j\leq \frac{n}{\varepsilon+\mu_Z}} Z_j \geq n\right) \\ &\leq Cn^{2p+2}\overline{F_1}(x_1)\overline{F_2}(x_2)P\left(\sum_{j\leq\frac{n}{\varepsilon+\mu_Z}} (Z_j-\mu_Z) \geq \frac{\varepsilon n}{\varepsilon+\mu_Z}\right). \end{aligned} \tag{2.18}$$

By Lemma 2.5, for fixed $\tilde{\gamma} > 0$ and $\tilde{p} > 0$ there exist some positive v and C_1 such that

$$K_2 \leq Cn^{2p+2}\overline{F_1}(x_1)\overline{F_2}(x_2)\left[\frac{n\varepsilon}{\varepsilon+\mu_Z}\overline{F_Z}\left(\frac{\varepsilon v n}{\varepsilon+\mu_Z}\right) + C_1\left(\frac{\varepsilon n}{\varepsilon+\mu_Z}\right)^{-\tilde{p}}\right], \tag{2.19}$$

where by taking $\tilde{\gamma} = \varepsilon$ and $\tilde{p} > 2p+3$, Markov's inequality and (2.19) it holds that

$$\begin{aligned} &\sum_{n>(1+\delta)\lambda(t)} K_2 \\ \leq\ & C\overline{F_1}(x_1)\overline{F_2}(x_2)\sum_{n>(1+\delta)\lambda(t)}\left[\frac{(\varepsilon+\mu_Z)^{\alpha_Z-1}\,EZ_1^{\alpha_Z}}{(\varepsilon v)^{\alpha_Z}}n^{-(\alpha_Z-2p-3)} + \frac{C_1\,(\varepsilon+\mu_Z)^{\tilde{p}}}{\varepsilon^{\tilde{p}}}n^{-(\tilde{p}-2p-2)}\right] \\ =\ & o\left(\overline{F_1}(x_1)\overline{F_2}(x_2)\right), \end{aligned} \tag{2.20}$$

where the last step is due to $\alpha_Z - 2p - 3 > 1$ and $\tilde{p} - 2p - 2 > 1$.

By (2.13), (2.17) and (2.20) it holds that

$$I_2 = o\left(\overline{F_1}(x_1)\overline{F_2}(x_2)\right). \tag{2.21}$$

By (2.12) and (2.21), we get (2.8) holds.

In the following we prove (2.9). For small enough $0 < \delta < 1$ and $v > 1$,

$$\begin{aligned} & P(\vec{S}(t) - \vec{\mu}\lambda(t) > \vec{x}) \\ \geq\ & \sum_{n=(1-\delta)\lambda(t)}^{(1+\delta)\lambda(t)} P\left(\vec{S}_n - \vec{\mu}\lambda(t) > \vec{x}, \Lambda(t) = n\right) \\ \geq\ & \sum_{n=(1-\delta)\lambda(t)}^{(1+\delta)\lambda(t)} P\Big(S_{1n} - \mu_1\lambda(t) > x_1, S_{2n} - \mu_2\lambda(t) > x_2, \Lambda(t) = n, \max_{1\leq i\leq n} X_{1i} > vx_1, \\ & \max_{1\leq j\leq n} X_{2j} > vx_2\Big) \end{aligned}$$

$$\geq \sum_{n=(1-\delta)\lambda(t)}^{(1+\delta)\lambda(t)} \sum_{1\leq i,j\leq n} P\Big(S_{1n}-\mu_1\lambda(t) > x_1, S_{2n}-\mu_2\lambda(t) > x_2, \Lambda(t)=n, X_{1i} > \nu x_1, X_{2j} > \nu x_2\Big)$$
$$- \sum_{n=(1-\delta)\lambda(t)}^{(1+\delta)\lambda(t)} \sum_{i=1}^{n} \sum_{j_1\neq j_2} P\Big(\Lambda(t)=n, X_{1i} > \nu x_1, X_{2j_1} > \nu x_2, X_{2j_2} > \nu x_2\Big)$$
$$- \sum_{n=(1-\delta)\lambda(t)}^{(1+\delta)\lambda(t)} \sum_{i_1\neq i_2} \sum_{j=1}^{n} P\Big(\Lambda(t)=n, X_{1i_1} > \nu x_1, X_{1i_2} > \nu x_1, X_{2j} > \nu x_2\Big)$$
$$=: P_1 - P_2 - P_3. \tag{2.22}$$

To estimate P_1. Similarly to (2.1), we can check that $N^{**}(t)$ is also identically distributed as $N(t)$ conditional on $\{X_{1i} > x_1, X_{2j} > x_2\}$. Following the similar method of (3.7) in Fu et al. [8] only by replacing event $\{N(t) = n\}$ with event $\{\Lambda(t) = n\}$, together with Lemma 2.1, we can get

$$P_1 \geq (1-\delta)\lambda(t)((1-\delta)\lambda(t)-1)\overline{F}_1(\nu x_1)\overline{F}_2(\nu x_2).$$

Hence, for $F_i \in \mathscr{C}, i = 1, 2$, we have

$$\lim_{\delta\downarrow 0}\lim_{\nu\downarrow 1}\liminf_{t\to\infty}\inf_{\vec{x}\geq\vec{\gamma}\lambda(t)} \frac{P_1}{(\lambda(t))^2\overline{F}_1(x_1)\overline{F}_2(x_2)} \geq 1. \tag{2.23}$$

As for P_2 and P_3, following the similar argument as Fu et al. (2021) we can get

$$\limsup_{t\to\infty}\sup_{\vec{x}\geq\vec{\gamma}\lambda(t)} \frac{P_2}{(\lambda(t))^2\overline{F}_1(x_1)\overline{F}_2(x_2)} = 0 \tag{2.24}$$

and

$$\limsup_{t\to\infty}\sup_{\vec{x}\geq\vec{\gamma}\lambda(t)} \frac{P_3}{(\lambda(t))^2\overline{F}_1(x_1)\overline{F}_2(x_2)} = 0. \tag{2.25}$$

By (2.22)–(2.25) we get (2.9) holds. □

3. Conclusions

This paper studies a dependent two-dimensional compound risk model with heavy-tailed claims. We mainly investigate the case that there exists a size-dependent structure between the claim sizes and inter-arrival times. Using the probability limiting theory we give the precise large deviations for aggregate amount of claims in the compound risk model.

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Conflict of interest

The authors declare no conflicts of interest.

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