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Research article

Medical diagnosis for the problem of Chikungunya disease using soft rough sets

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Abstract: One of the most difficulties that doctors face when diagnosing a disease is making an accurate decision to correctly determine the nature of the injury. This is attributable to the similarity of symptoms for different diseases. The current work is devoted to proposing new mathematical methodologies to help in precise decision-making in the medical diagnosis of the problem of Chikungunya virus disease through the use of soft rough sets. In fact, we introduce some improvements for soft rough sets (given by Feng et al.). We suggest a new approach to studying roughness through the use of soft sets to find approximations of any set, i.e., so-called "soft δ-rough sets". To illustrate this approach, we compare it with the previous studies and prove that the proposed approach is more accurate than the previous works. The proposed approach is more accurate than Feng et al. approach and extends the scope of applications because the problem of soft upper approximation is solved. The main characterizations of the presented technique are elucidated. Some important relations related to soft δ-rough approximations (such as soft δ-memberships, soft δ-equality and soft δ-inclusion) are provided and their properties are examined. In addition, an important medical application in the diagnosis of the problem of Chikungunya virus using soft δ-rough sets is provided with two algorithms. These algorithms were tested on fictitious data in order to compare them to existing methods which represent simple techniques to use in MATLAB. Additionally, we examine the benefits and weaknesses of the proposed approach and present a plan for some upcoming work.

Keywords: soft sets; soft rough sets; soft δ-rough sets; Chikungunya virus; decision-making
Mathematics Subject Classification: 54A10, 68W25, 68U01, 68U35, 92C50

List of symbols and abbreviations

CHIKV	Chikungunya virus	$\underline{S}(\mathcal{M})$	A soft lower approximation of $\mathcal{M} \subseteq \Theta$
Θ	The universal set	$\overline{S}(\mathcal{M})$	A soft upper approximation of $\mathcal{M} \subseteq \Theta$
Φ	The empty set	$\widetilde{bnd}_{\mathcal{A}_S}(\mathcal{M})$	A soft boundary of $\mathcal{M} \subseteq \Theta$
Ω	An equivalence relation	$\widetilde{\mu}_{\mathcal{A}_S}(\mathcal{M})$	A soft accuracy of approximations of $\mathcal{M} \subseteq \Theta$
$\mathcal{S}$	A soft set	$\underline{S}_{\delta}(\mathcal{M})$	A soft δ-lower approximation of $\mathcal{M} \subseteq \Theta$
κ_{ε}	A mapping of parameters	$\overline{S}_{\delta}(\mathcal{M})$	A soft δ-upper approximation of $\mathcal{M} \subseteq \Theta$
$\mathcal{E}$	A set of parameters	$\widetilde{bnd}_{\delta}(\mathcal{M})$	A soft boundary of $\mathcal{M} \subseteq \Theta$
$\mathcal{A}_S$	A soft approximation space	$\widetilde{\mu}_{\delta}(\mathcal{M})$	A soft δ-accuracy of the approximations of $\mathcal{M} \subseteq \Theta$

1. Introduction

The problem with the medical diagnosis of many diseases is the great similarity between their symptoms. Therefore, many doctors resort to doing a lot of different medical tests to make sure that the patient has a specific disease. There is no doubt that these examinations cause a big problem for some patients due to the high financial cost of these examinations. On the other hand, these tests may fail to identify the disease with high accuracy. Hence, many approaches, such as the theory of rough sets [1], the theory of fuzzy sets [2], the theory of soft sets [3] and soft rough theory [4], represent accurate tools for dealing with medical problems from many perspectives, like decision-making problems or attributes reduction, for instance [5–9]. Topological structures are interesting tools in various real-life applications such as in decision-making problems [12–14], medical applications [15, 16], economic fields [17], topological reductions of attributes for predicting COVID-19 [18], and biology [20]. Topology has been applied in the generalization of rough sets [21, 22], and in rough fuzzy approximation operators [23]. Pawlak proposed the theory of rough sets [1]. It is an extension of set theory that studies ill-posed data or data with incomplete information. Any subset of the universe is described in rough set theory by two approximations, named lower and upper approximations. Researchers have been interested in the theoretical enrichment and applications of rough set theory since its inception for instance, see [24–29]). As a generalized form of rough set, the decision-theoretic rough set was suggested to properly explain (see [30–33]), and it has been used in three-way decision-making [34–36].

Molodtsov presented the theory of soft sets [3]. That theory represents a newer approach to discussing vagueness and has many applications in decision-making (see [9, 37–42]). Rough set theory and soft set theory are two approaches to dealing with uncertainty. Although there appears to be no direct connection between these two theories, efforts have been made to establish some types of linkage. Feng et al. [4] went a long way toward making parameterization tools available in rough sets. They defined soft rough sets as parameterized subsets of a set that are used to find lower and upper approximations of a subset rather than equivalence classes. Some unusual situations may arise as a result of this. For instance, for any nonempty set X, its upper approximation may be empty or might not contain the set X. In classical rough set theory, these situations do not exist. In addition, Example 3.3 in [38] and Examples 1 and 2 in [43] demonstrated that the definition proposed by Feng et al. never meets the exact criteria of rough sets presented by Pawlak. El-Bably et al. [16] have succeeded in suggesting new topological approaches to solve these problems. Moreover, they illustrated that their approaches represented generalizations for soft rough sets [4]. There are some important models to generalize soft rough sets including the idea of N-soft sets (for example [44–47]). On the other hand, Alcantud and colleagues applied soft sets in many fields, as described in [48–51]. In 2014, Das and Acharjya [52] proposed an interesting work that uses soft sets and rough sets in the decision-making model as an application on fuzzy approximation spaces. Moreover, many scholars have used these models to apply in medical applications, such as the diagnosis of chronic liver disease [53], the hybridization and extent of applications [54] and several other applications [55–57]. Liu et al. [58] presented new approaches to m-polar Diophantine fuzzy N-soft sets with applications. Also, Hameed et al. used the concept of N-soft sets to introduce Pythagorean fuzzy N-soft groups (see [59]). In addition, Gulzar et al. introduced complex intuitionistic fuzzy sets of subrings and applied them in group theory (see [60, 61]). A new group decision-making technique under picture fuzzy soft expert information was studied by Tchier et al. [62].

So, the aim of the current paper is to make some improvements and modifications to the approaches proposed by Feng and colleagues. The proposed approach is more accurate and extends the scope of applications because the problem of soft upper approximation is solved. Mathematically, these approaches are called soft δ-rough sets. Some of their properties are studied and their connections with other methods are examined. We will prove that the suggested techniques improve and solve some problems in [4] by providing many results and illustrative examples. Some important relations associated with soft δ-approximations (such as soft δ-memberships, soft δ-equality, and soft δ-inclusion) are superimposed and their properties are examined.

According to the World Health Organization WHO [63], Chikungunya (CHIKV, for short) is a virus that is spread to humans by infected mosquitoes. Fever and severe joint pain are symptoms of CHIKV infection. The disease has some medical symptoms in common with dengue and Zika and can be misdiagnosed in areas where both are prevalent. Because an accurate diagnosis of CHIKV is difficult, there is no reliable estimate of the number of people affected by the disease globally on an annual basis. Therefore, we need accurate tools to identify the virus and hence the accurate diagnoses of CHIKV which can help treat and prevent disease outbreaks. Consequently, we apply the proposed techniques to this problem and prove that the accuracy of the proposed approaches is better than that of the others (see [4, 26, 43]) and is thus suitable for the diagnoses of CHIKV. Besides, we use soft δ-rough sets to make a topological reduction of the provided information table to identify the factors that affect the decision of CHIKV infection. Additionally, we examine the benefits and weaknesses of the proposed approach and present a plan for some upcoming work.

So, the main objectives of the manuscript are as follows:

- Suggesting some improvements of soft rough sets [4].
- Introducing some soft rough relations, such as soft δ-memberships, soft δ-equality, and soft δ-inclusion.
- Investigating some comparisons with the other methods in the literature.
- Applying the suggested techniques in the medical diagnosis of CHIKV.
- Proposing two algorithms for the introduced methods by using MATLAB and frameworks for decision-making problems.

2. Preliminaries

Herein, we recall some basic notions and results, which were given in [1, 3, 4].

2.1. Pawlak rough set theory

Definition 2.1. *[1] Let Θ be a finite set (named, universe), and Ω be an equivalence relation on Θ. If Θ/Ω represents a family of all equivalence classes of Ω and $[\mathfrak{m}]_\Omega$ denotes an equivalence class of Ω, the pair (Θ, Ω) is called an approximation space for any $\mathcal{M} \subseteq \Theta$; hence, the lower and upper approximations of $\mathcal{M}$ are $\underline{\Omega}(\mathcal{M}) = \{\varkappa \in \Theta : [\mathfrak{m}]_\Omega \subseteq \mathcal{M}\}$ and $\overline{\Omega}(\mathcal{M}) = \{\mathfrak{m} \in \Theta : [\mathfrak{m}]_\Omega \cap \mathcal{M} \neq \Phi\}$, respectively. In addition, the following holds true:*

- *The boundary region is $\widetilde{bnd}_\Omega(\mathcal{M}) = \overline{\Omega}(\mathcal{M}) - \underline{\Omega}(\mathcal{M})$.*
- *The positive region is $\widetilde{pos}_\Omega(\mathcal{M}) = \underline{\Omega}(\mathcal{M})$.*
- *The negative region is $\widetilde{neg}_\Omega(\mathcal{M}) = \Theta - \overline{\Omega}(\mathcal{M})$.*
- *The accuracy of approximations is $\widetilde{\mu}_\Omega(\mathcal{M}) = \frac{|\underline{\Omega}(\mathcal{M})|}{|\overline{\Omega}(\mathcal{M})|}$, where $\left|\overline{\Omega}(\mathcal{M})\right| \neq \Phi$.*
- *$\mathcal{M}$ is called a rough set if $\underline{\Omega}(\mathcal{M}) \neq \overline{\Omega}(\mathcal{M})$, i.e., $\widetilde{bnd}_\Omega(\mathcal{M}) = \Phi$ and $\widetilde{\mu}_\Omega(\mathcal{M}) \neq 1$. Otherwise, $\mathcal{M}$ is an exact set.*

For further details about Pawlak rough sets, see [1].

2.2. Soft sets and soft rough sets

Definition 2.2. *[3] Suppose that $\kappa_\varepsilon : \breve{A} \to P(\Theta)$ is a mapping, and that the soft set $\mathcal{S}$ over Θ is a pair $(\kappa_\varepsilon, \breve{A})$, where $\breve{A} \subseteq \mathcal{E}$ and $\mathcal{E}$ represents a set of parameters. So, we can say that $(\kappa_\varepsilon, \breve{A})$ represents a parameterized family of subsets of Θ such that for each $\varepsilon \in \breve{A}$, $\kappa_\varepsilon(\varepsilon)$ is considered as a set of ε-approximate elements of $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$.*

Definition 2.3. *[4] Let $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ be a soft set over Θ. Thus, a pair $\mathcal{A}_\mathcal{S} = (\Theta, \mathcal{S})$ is named a soft approximation space. Based on a soft approximation space $\mathcal{A}_\mathcal{S}$, Feng et al. proposed the following:*

- *The soft $\mathcal{A}_\mathcal{S}$-lower and soft $\mathcal{A}_\mathcal{S}$-upper approximations of $\mathcal{M} \subseteq \Theta$ defined by $\underline{\mathcal{S}}(\mathcal{M}) = \{\mathfrak{m} \in \Theta : \exists \varepsilon \in \breve{A}, [\mathfrak{m} \in \kappa_\varepsilon(\varepsilon) \subseteq \mathcal{M}]\}$ and $\overline{\mathcal{S}}(\mathcal{M}) = \{\mathfrak{m} \in \Theta : \exists \varepsilon \in \breve{A}, [\mathfrak{m} \in \kappa_\varepsilon(\varepsilon), \kappa_\varepsilon(\varepsilon) \cap \mathcal{M} \neq \Phi]\}$, respectively.*
- *The soft $\mathcal{A}_\mathcal{S}$-positive, soft $\mathcal{A}_\mathcal{S}$-negative, and soft $\mathcal{A}_\mathcal{S}$-boundary regions are $\widetilde{pos}_{\mathcal{A}_\mathcal{S}}(\mathcal{M}) = \underline{\mathcal{S}}(\mathcal{M})$, $\widetilde{neg}_{\mathcal{A}_\mathcal{S}}(\mathcal{M}) = \Theta - \overline{\mathcal{S}}(\mathcal{M}) = (\overline{\mathcal{S}}(\mathcal{M}))^c$, and $\widetilde{bnd}_{\mathcal{A}_\mathcal{S}}(\mathcal{M}) = \overline{\mathcal{S}}(\mathcal{M}) - \underline{\mathcal{S}}(\mathcal{M})$, respectively.*
- *The accuracy of approximations is $\widetilde{\mu}_{\mathcal{A}_\mathcal{S}}(\mathcal{M}) = \frac{|\underline{\mathcal{S}}(\mathcal{M})|}{|\overline{\mathcal{S}}(\mathcal{M})|}$, where $\overline{\mathcal{S}}(\mathcal{M}) \neq \Phi$ and we call $\widetilde{\mu}_{\mathcal{A}_\mathcal{S}}(\mathcal{M})$ "soft $\mathcal{A}_\mathcal{S}$-accuracy".*
- *A subset $\mathcal{M}$ is called a "soft $\mathcal{A}_\mathcal{S}$-definable" (soft $\mathcal{A}_\mathcal{S}$-exact) set if $\overline{\mathcal{S}}(\mathcal{M}) = \underline{\mathcal{S}}(\mathcal{M}) = \mathcal{M}$, i.e., if $\widetilde{bnd}_{\mathcal{A}_\mathcal{S}}(\mathcal{M}) = \Phi$. Otherwise, $\mathcal{M}$ is called a "soft $\mathcal{A}_\mathcal{S}$-rough" set.*

Proposition 2.1. *[4] Suppose that $\mathcal{A}_\mathcal{S} = (\Theta, \mathcal{S})$ represents a soft approximation space such that $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ is a soft set over Θ. Then, for each $\mathcal{M} \subseteq \Theta$,*

$$\underline{\mathcal{S}}(\mathcal{M}) = \bigcup_{\varepsilon \in \breve{A}} \{\kappa_\varepsilon(\varepsilon) : \kappa_\varepsilon(\varepsilon) \subseteq \mathcal{M}\} \text{ and } \overline{\mathcal{S}}(\mathcal{M}) = \bigcup_{\varepsilon \in \breve{A}} \{\kappa_\varepsilon(\varepsilon) : \kappa_\varepsilon(\varepsilon) \cap \mathcal{M} \neq \Phi\}.$$

Proposition 2.2. *[4] Let $\mathcal{A}_\mathcal{S} = (\Theta, \mathcal{S})$ represent a soft approximation space such that $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ represents a soft set over Θ. Then, the soft $\mathcal{A}_\mathcal{S}$-lower, and $\mathcal{A}_\mathcal{S}$-upper approximations of $\mathcal{M}, \mathcal{N} \subseteq \Theta$ satisfy the following:*

(1) $\underline{S}(\Phi) = \overline{S}(\Phi) = \Phi$.

(2) $\underline{S}(\Theta) = \overline{S}(\Theta) = \bigcup_{\varepsilon\in\breve{A}} \kappa_\varepsilon(\varepsilon)$.

(3) $\underline{S}(\mathcal{M}\cap\mathcal{N}) \subseteq \underline{S}(\mathcal{M})\cap\underline{S}(\mathcal{N})$.

(4) $\underline{S}(\mathcal{M}\cup\mathcal{N}) \supseteq \underline{S}(\mathcal{M})\cup\underline{S}(\mathcal{N})$.

(5) $\overline{S}(\mathcal{M}\cup\mathcal{N}) = \overline{S}(\mathcal{M})\cup\overline{S}(\mathcal{N})$.

(6) $\overline{S}(\mathcal{M}\cap\mathcal{N}) \subseteq \overline{S}(\mathcal{M})\cap\overline{S}(\mathcal{N})$.

(7) If $\mathcal{M}\subseteq\mathcal{N}$*, then* $\underline{S}(\mathcal{M}) \subseteq \underline{S}(\mathcal{N})$.

(8) If $\mathcal{M}\subseteq\mathcal{N}$*, then* $\overline{S}(\mathcal{M}) \subseteq \overline{S}(\mathcal{N})$.

Proposition 2.3. *[4] Suppose that* $\mathcal{A}_\mathcal{S} = (\Theta,\mathcal{S})$ *is a soft approximation space such that* $\mathcal{S} = (\kappa_\varepsilon,\breve{A})$ *is a soft set over* Θ*. Then, for each* $\mathcal{M}\subseteq\Theta$*,*

(1) $\underline{S}(\underline{S}(\mathcal{M})) = \underline{S}(\mathcal{M})$.

(2) $\overline{S}(\overline{S}(\mathcal{M}))\supseteq\overline{S}(\mathcal{M})$.

(3) $\underline{S}(\mathcal{M}) \subseteq \overline{S}(\underline{S}(\mathcal{M}))$.

(4) $\underline{S}(\overline{S}(\mathcal{M})) = \overline{S}(\mathcal{M})$.

Definition 2.4. *[4] If* $\mathcal{A}_\mathcal{S} = (\Theta,\mathcal{S})$ *represents a soft approximation space such that* $\mathcal{S} = (\kappa_\varepsilon,\breve{A})$ *is a soft set over* Θ*,* $\mathcal{S}$ *is said to be a "full soft set" if* $\Theta = \bigcup_{\varepsilon\in\breve{A}} \kappa_\varepsilon(\varepsilon)$*. Therefore, the full soft set* $\mathcal{S}$ *means that* $\forall \mathfrak{m}\in\Theta$ *and* $\exists\varepsilon\in\breve{A}$ *such that* $\mathfrak{m}\in\kappa_\varepsilon(\varepsilon)$*.*

Proposition 2.4. *[4] Suppose that* $\mathcal{A}_\mathcal{S} = (\Theta,\mathcal{S})$ *represents a soft approximation space such that* $\mathcal{S} = (\kappa_\varepsilon,\breve{A})$ *is a a full soft set. The next axioms are true:*

(1) $\underline{S}(\Theta) = \overline{S}(\Theta) = \Theta$.

(2) $\mathcal{M}\subseteq\overline{S}(\mathcal{M}),\ \forall\mathcal{M}\subseteq\Theta$.

(3) $\overline{S}(\{\mathfrak{m}\}) \neq \Phi,\ \forall\mathfrak{m}\in\Theta$.

Definition 2.5. *[4] Suppose that* $\mathcal{A}_\mathcal{S} = (\Theta,\mathcal{S})$ *represents a soft approximation space with the full soft set* $\mathcal{S} = (\kappa_\varepsilon,\breve{A})$ *and* $\mathcal{Z}\subseteq\Theta$*. Therefore, the following holds true:*

(1) If $\underline{S}(\mathcal{Z}) \neq \Phi$ and $\overline{S}(\mathcal{Z}) \neq \Theta$, then $\mathcal{Z}$ is roughly soft $\mathcal{A}_\mathcal{S}$-definable.

(2) If $\underline{S}(\mathcal{Z}) = \Phi$ and $\overline{S}(\mathcal{Z}) \neq \Theta$, then $\mathcal{Z}$ is internally soft $\mathcal{A}_\mathcal{S}$-indefinable.

(3) If $\underline{S}(\mathcal{Z}) \neq \Phi$ and $\overline{S}(\mathcal{Z}) = \Theta$, then $\mathcal{Z}$ is externally soft.

(4) If $\underline{S}(\mathcal{Z}) = \Phi$ and $\overline{S}(\mathcal{Z}) = \Theta$, then $\mathcal{Z}$ is totally soft $\mathcal{A}_\mathcal{S}$-indefinable.

3. Soft δ-rough approximations

In this section, a novel generalization of soft rough approximations called "soft δ-rough approximations" is proposed as an improvement to the approach proposed by Feng et al. in [4]. The properties of these approaches are superimposed. The proposed approaches are linked to the previous technique described in Feng et al. model and, hence, improves it. Numerous examples and counterexamples are provided.

Definition 3.1. *If $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represents a soft approximation space with a soft set $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ on Θ. Then, for each $\mathcal{M} \subseteq \Theta$, the soft δ-lower, and δ-upper approximations are given by $\underline{\mathcal{S}}_\delta(\mathcal{M}) = \mathcal{M} \cap \overline{\mathcal{S}}(\underline{\mathcal{S}}(\mathcal{M}))$ and $\overline{\mathcal{S}}_\delta(\mathcal{M}) = \mathcal{M} \cup \overline{\mathcal{S}}(\underline{\mathcal{S}}(\mathcal{M}))$, respectively. Generally, it is referred to $\underline{\mathcal{S}}_\delta(\mathcal{M})$ and $\overline{\mathcal{S}}_\delta(\mathcal{M})$ as "soft δ-rough approximations" of $\mathcal{M} \subseteq \Theta$, as associated with $\mathcal{A}_{\mathcal{S}}$. Additionally, we define the following:*

(1) The soft δ-positive, δ-negative and δ-boundary regions are
$\widetilde{pos}_\delta(\mathcal{M}) = \underline{\mathcal{S}}_\delta(\mathcal{M})$, $\widetilde{neg}_\delta(\mathcal{M}) = \Theta - \overline{\mathcal{S}}_\delta(\mathcal{M})$ *and* $\widetilde{bnd}_\delta(\mathcal{M}) = \overline{\mathcal{S}}_\delta(\mathcal{M}) - \underline{\mathcal{S}}_\delta(\mathcal{M})$, *respectively.*

(2) The soft δ-accuracy of soft δ-approximations is defined as follows:
$\widetilde{\mu}_\delta(\mathcal{M}) = \frac{|\underline{\mathcal{S}}_\delta(\mathcal{M})|}{|\overline{\mathcal{S}}_\delta(\mathcal{M})|}$, *where* $\overline{\mathcal{S}}_\delta(\mathcal{M}) \neq \Phi$.

(3) If $\underline{\mathcal{S}}_\delta(\mathcal{M}) = \overline{\mathcal{S}}_\delta(\mathcal{M})$, i.e., if $\widetilde{bnd}_\delta(\mathcal{M}) = \Phi$ and $\widetilde{\mu}_\delta(\mathcal{M}) = 1$, we call the subset $\mathcal{M}$ as a "soft δ-exact" set. Otherwise, it is a "soft δ-rough" set.

In the next results, we examine the basic characteristics of the soft δ-approximations.

Proposition 3.1. *If $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represents a soft approximation space with a soft set $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ on Θ, then, for each $\mathcal{M}, \mathcal{N} \subseteq \Theta$, the following holds true:*

(1) $\underline{\mathcal{S}}_\delta(\Phi) = \overline{\mathcal{S}}_\delta(\Phi) = \Phi$.

(2) $\underline{\mathcal{S}}_\delta(\Theta) = \bigcup_{\varepsilon \in \breve{A}} \kappa_\varepsilon(\varepsilon)$ *and* $\overline{\mathcal{S}}_\delta(\Theta) = \Theta$.

(3) $\underline{\mathcal{S}}_\delta(\mathcal{M}) \subseteq \mathcal{M} \subseteq \overline{\mathcal{S}}_\delta(\mathcal{M})$.

(4) If $\mathcal{M} \subseteq \mathcal{N}$, then $\underline{\mathcal{S}}_\delta(\mathcal{M}) \subseteq \underline{\mathcal{S}}_\delta(\mathcal{N})$.

(5) If $\mathcal{M} \subseteq \mathcal{N}$, then $\overline{\mathcal{S}}_\delta(\mathcal{M}) \subseteq \overline{\mathcal{S}}_\delta(\mathcal{N})$.

(6) $\underline{\mathcal{S}}_\delta(\mathcal{M} \cap \mathcal{N}) \subseteq \underline{\mathcal{S}}_\delta(\mathcal{M}) \cap \underline{\mathcal{S}}_\delta(\mathcal{N})$.

(7) $\underline{\mathcal{S}}_\delta(\mathcal{M} \cup \mathcal{N}) \supseteq \underline{\mathcal{S}}_\delta(\mathcal{M}) \cup \underline{\mathcal{S}}_\delta(\mathcal{N})$.

(8) $\overline{\mathcal{S}}_\delta(\mathcal{M} \cap \mathcal{N}) \subseteq \overline{\mathcal{S}}_\delta(\mathcal{M}) \cap \overline{\mathcal{S}}_\delta(\mathcal{N})$.

(9) $\overline{\mathcal{S}}_\delta(\mathcal{M} \cup \mathcal{N}) \supseteq \overline{\mathcal{S}}_\delta(\mathcal{M}) \cup \overline{\mathcal{S}}_\delta(\mathcal{N})$.

Proof. By using Proposition 2.2, we get the following:

(1) As $\underline{\mathcal{S}}(\Phi) = \overline{\mathcal{S}}(\Phi) = \Phi$, $\underline{\mathcal{S}}_\delta(\Phi)=\Phi \cap \overline{\mathcal{S}}(\underline{\mathcal{S}}(\Phi)) = \Phi$ and, similarly, $\overline{\mathcal{S}}_\delta(\Phi) = \Phi$.

(2) Since $\underline{\mathcal{S}}(\Theta) = \overline{\mathcal{S}}(\Theta) = \bigcup_{\varepsilon \in \breve{A}} \kappa_\varepsilon(\varepsilon)$, $\underline{\mathcal{S}}_\delta(\Theta) = \Theta \cap \overline{\mathcal{S}}(\underline{\mathcal{S}}(\Theta)) = \Theta \cap (\bigcup_{\varepsilon \in \breve{A}} \kappa_\varepsilon(\varepsilon)) = \bigcup_{\varepsilon \in \breve{A}} \kappa_\varepsilon(\varepsilon)$. On the other hand, $\overline{\mathcal{S}}_\delta(\Theta) = \Theta \cup \overline{\mathcal{S}}(\underline{\mathcal{S}}(\Theta)) = \Theta$.

(3) By Definition 3.1, the proof is obvious.

(4) Since $\underline{\mathcal{S}}(\mathcal{M}) \subseteq \underline{\mathcal{S}}(\mathcal{N})$, for each $\mathcal{M} \subseteq \mathcal{N}$, for each $\mathcal{M} \subseteq \mathcal{N}$, $\underline{\mathcal{S}}_{\delta}(\mathcal{M}) = \mathcal{M} \cap \overline{\mathcal{S}}(\underline{\mathcal{S}}(\mathcal{M})) \subseteq \mathcal{N} \cap \overline{\mathcal{S}}(\underline{\mathcal{S}}(\mathcal{N})) = \underline{\mathcal{S}}_{\delta}(\mathcal{N})$.

(5) Follow a similar path as (3).

(6) $\underline{\mathcal{S}}_{\delta}(\mathcal{M} \cap \mathcal{N}) = (\mathcal{M} \cap \mathcal{N}) \cap \overline{\mathcal{S}}(\underline{\mathcal{S}}(\mathcal{M} \cap \mathcal{N}) \subseteq (\mathcal{M} \cap \mathcal{N}) \cap \overline{\mathcal{S}}[\underline{\mathcal{S}}(\mathcal{M}) \cap \underline{\mathcal{S}}(\mathcal{N})]$, and this implies that $\underline{\mathcal{S}}_{\delta}(\mathcal{M} \cap \mathcal{N}) \subseteq (\mathcal{M} \cap \mathcal{N}) \cap [\overline{\mathcal{S}}(\underline{\mathcal{S}}(\mathcal{M})) \cap \overline{\mathcal{S}}(\underline{\mathcal{S}}(\mathcal{N}))] = [\mathcal{M} \cap \overline{\mathcal{S}}(\underline{\mathcal{S}}(\mathcal{M})] \cap [\mathcal{N} \cap \overline{\mathcal{S}}(\underline{\mathcal{S}}(\mathcal{N})] = \underline{\mathcal{S}}_{\delta}(\mathcal{M}) \cap \underline{\mathcal{S}}_{\delta}(\mathcal{N})$.

(7) Follow a similar path as (5).

(8) Follow a similar path as (5).

(9) By a similar way like (5).

□

Proposition 3.2. *Let $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represent a soft approximation space with a soft set $\mathcal{S} = (\kappa_{\varepsilon}, \breve{A})$ on Θ. Then, for each $\mathcal{M} \subseteq \Theta$, the following holds true:*

(1) $\underline{\mathcal{S}}_{\delta}(\underline{\mathcal{S}}_{\delta}(\mathcal{M})) = \underline{\mathcal{S}}_{\delta}(\mathcal{M})$.

(2) $\underline{\mathcal{S}}_{\delta}(\overline{\mathcal{S}}_{\delta}(\mathcal{M})) \subseteq \overline{\mathcal{S}}_{\delta}(\mathcal{M})$.

(3) $\underline{\mathcal{S}}_{\delta}(\mathcal{M}) \subseteq \overline{\mathcal{S}}_{\delta}(\underline{\mathcal{S}}_{\delta}(\mathcal{M}))$.

(4) $\overline{\mathcal{S}}_{\delta}(\mathcal{M}) \subseteq \overline{\mathcal{S}}_{\delta}(\overline{\mathcal{S}}_{\delta}(\mathcal{M}))$.

Proof. The first item will be proved, and the others similarly.

(1) First, according to Proposition 3.1, since $\underline{\mathcal{S}}_{\delta}(\mathcal{M}) \subseteq \mathcal{M}$ for each $\mathcal{M} \subseteq \Theta$, $\underline{\mathcal{S}}_{\delta}(\underline{\mathcal{S}}_{\delta}(\mathcal{M})) \subseteq \underline{\mathcal{S}}_{\delta}(\mathcal{M})$.
Now, we need to prove that $\underline{\mathcal{S}}_{\delta}(\mathcal{M}) \subseteq \underline{\mathcal{S}}_{\delta}(\underline{\mathcal{S}}_{\delta}(\mathcal{M}))$ as follows:
Assume that $\varpi \notin \underline{\mathcal{S}}_{\delta}(\underline{\mathcal{S}}_{\delta}(\mathcal{M}))$; thus, $\varpi \notin [\underline{\mathcal{S}}_{\delta}(\mathcal{M}) \cap \overline{\mathcal{S}}(\underline{\mathcal{S}}(\underline{\mathcal{S}}_{\delta}(\mathcal{M})))] \subseteq \overline{\mathcal{S}}_{\delta}(\mathcal{M})$. Consequently, using Proposition 2.2, $\varpi \notin \underline{\mathcal{S}}_{\delta}(\mathcal{M})$ which implies that $\underline{\mathcal{S}}_{\delta}(\mathcal{M}) \subseteq \underline{\mathcal{S}}_{\delta}(\underline{\mathcal{S}}_{\delta}(\mathcal{M}))$. Therefore, $\underline{\mathcal{S}}_{\delta}(\mathcal{M}) = \underline{\mathcal{S}}_{\delta}(\underline{\mathcal{S}}_{\delta}(\mathcal{M}))$.

□

As shown in Examples 3.1 and 3.2, the inclusion relations in Proposition 3.2 can be strict.

Example 3.1. *Assume that $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represents a soft approximation space with a soft set $\mathcal{S} = (\kappa_{\varepsilon}, \breve{A})$ on Θ, where $\Theta = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, ..., \widetilde{x}_6\}$, $\mathcal{E} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_6\}$ and $\breve{A} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \subseteq \mathcal{E}$ such that $(\kappa_{\varepsilon}, \breve{A}) = \{(\varepsilon_1, \{\widetilde{x}_1, \widetilde{x}_6\}), (\varepsilon_2, \{\widetilde{x}_3\}), (\varepsilon_3, \Phi), (\varepsilon_4, \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_5\})\}$. If $\mathcal{M} = \{\widetilde{x}_1, \widetilde{x}_6\}$, and $\mathcal{N} = \{\widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5\}$, then we get $\underline{\mathcal{S}}_{\delta}(\mathcal{M}) = \{\widetilde{x}_1, \widetilde{x}_6\}$ which implies that $\overline{\mathcal{S}}_{\delta}(\underline{\mathcal{S}}_{\delta}(\mathcal{M})) = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_5, \widetilde{x}_6\}$. Hence, $\underline{\mathcal{S}}_{\delta}(\mathcal{M}) \neq \overline{\mathcal{S}}_{\delta}(\underline{\mathcal{S}}_{\delta}(\mathcal{M}))$. Also, $\overline{\mathcal{S}}_{\delta}(\mathcal{N}) = \{\widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5\}$ and this means that $\underline{\mathcal{S}}_{\delta}(\overline{\mathcal{S}}_{\delta}(\mathcal{N})) = \{\widetilde{x}_3\}$. Hence, $\underline{\mathcal{S}}_{\delta}(\overline{\mathcal{S}}_{\delta}(\mathcal{N})) \neq \overline{\mathcal{S}}_{\delta}(\mathcal{N})$.*

Example 3.2. *Suppose that $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represents a soft approximation space with a soft set $\mathcal{S} = (\kappa_{\varepsilon}, \breve{A})$ on Θ, where $\Theta = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4\}$, $\mathcal{E} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, ..., \varepsilon_6\}$ and $\breve{A} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \subseteq \mathcal{E}$ such that $(\kappa_{\varepsilon}, \breve{A}) = \{(\varepsilon_1, \{\widetilde{x}_1, \widetilde{x}_4\}), (\varepsilon_2, \{\widetilde{x}_3\}), (\varepsilon_3, \{\widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4\})\}$. If $\mathcal{M} = \{\widetilde{x}_2, \widetilde{x}_3\}$, then we get $\overline{\mathcal{S}}_{\delta}(\mathcal{M}) = \{\widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4\}$ which implies that $\overline{\mathcal{S}}_{\delta}(\overline{\mathcal{S}}_{\delta}(\mathcal{M})) = \Theta$. Hence, $\overline{\mathcal{S}}_{\delta}(\mathcal{M}) \neq \overline{\mathcal{S}}_{\delta}(\overline{\mathcal{S}}_{\delta}(\mathcal{M}))$.*

Example 3.3. *Consider that $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represents a soft approximation space such that $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ is a soft set on Θ, where $\Theta = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, ..., \widetilde{x}_6\}$, $\mathcal{E} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, ..., \varepsilon_6\}$ and $\breve{A} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \subseteq \mathcal{E}$ such that $(\kappa_\varepsilon, \breve{A}) = \{(\varepsilon_1, \{\widetilde{x}_1, \widetilde{x}_3\}), (\varepsilon_2, \{\widetilde{x}_4\}), (\varepsilon_3, \{\widetilde{x}_2, \widetilde{x}_4, \widetilde{x}_5\}), (\varepsilon_4, \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3\})\}$.*

To illustrate the importance of the suggested method, we compare the current method with the Feng et al. [4] method as shown in Table 1 for some subsets.

Theorem 3.1. *If $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ is a soft approximation space with a soft set $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ on Θ. Then, for each $\mathcal{X} \subseteq \Theta$, $\underline{\mathcal{S}}(\mathcal{X}) \subseteq \underline{\mathcal{S}}_\delta(\mathcal{X})$.*

Proof. First, from Definition 2.3 and Proposition 2.3 for each $\mathcal{X} \subseteq \Theta$, $\underline{\mathcal{S}}(\mathcal{X}) \subseteq \mathcal{M}$ and $\underline{\mathcal{S}}(\mathcal{X}) \subseteq \overline{\mathcal{S}}(\underline{\mathcal{S}}(\mathcal{X}))$, it holds that $\underline{\mathcal{S}}(\mathcal{X}) \subseteq \mathcal{X} \cap \overline{\mathcal{S}}(\underline{\mathcal{S}}(\mathcal{X})) = \underline{\mathcal{S}}_\delta(\mathcal{X})$. □

Table 1. Comparison between Feng et al. method and current method.

	Feng et al. method [4]		**Current method**	
	$\widetilde{bnd}_{\mathcal{A}_{\mathcal{S}}}(\mathcal{M})$	$\widetilde{\mu}_{\mathcal{A}_{\mathcal{S}}}(\mathcal{M})$	$\widetilde{bnd}_\delta(\mathcal{M})$	$\widetilde{\mu}_\delta(\mathcal{M})$
$\{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_4, \widetilde{x}_5\}$	$\{\widetilde{x}_1, \widetilde{x}_3\}$	3/5	$\{\widetilde{x}_3\}$	4/5
$\{\widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4\}$	$\{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_5\}$	1/5	$\{\widetilde{x}_3, \widetilde{x}_5\}$	1/2
$\{\widetilde{x}_1, \widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5\}$	$\{\widetilde{x}_2, \widetilde{x}_5\}$	3/5	$\{\widetilde{x}_2\}$	4/5
$\{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_4, \widetilde{x}_5\}$	$\{\widetilde{x}_1, \widetilde{x}_3\}$	3/5	$\{\widetilde{x}_1\}$	3/4
$\{\widetilde{x}_1, \widetilde{x}_3, \widetilde{x}_4\}$	$\{\widetilde{x}_2, \widetilde{x}_5\}$	3/5	$\{\widetilde{x}_2, \widetilde{x}_5\}$	3/5

Theorem 3.2. *If $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ is a soft approximation space with a full soft set $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ on Θ. Then, for each $\mathcal{X} \subseteq \Theta$, $\overline{\mathcal{S}}_\delta(\mathcal{X}) \subseteq \overline{\mathcal{S}}(\mathcal{X})$.*

Proof. From Proposition 2.4, let $\mathcal{S}$ be a full soft set. Thus, $\mathcal{X} \subseteq \overline{\mathcal{S}}(\mathcal{X})$ for each $\mathcal{X} \subseteq \Theta$. Now, $\underline{\mathcal{S}}(\mathcal{X}) \subseteq \mathcal{X}$. Thus, $\overline{\mathcal{S}}(\underline{\mathcal{S}}(\mathcal{X})) \subseteq \overline{\mathcal{S}}(\mathcal{X})$. Therefore, $\mathcal{X} \cup \overline{\mathcal{S}}(\underline{\mathcal{S}}(\mathcal{X})) \subseteq \overline{\mathcal{S}}(\mathcal{X})$. □

Corollary 3.1. *If $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ is a soft approximation space with a full soft set $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ on Θ. Then, for each $\mathcal{X} \subseteq \Theta$, the following holds true:*

(1) $\widetilde{bnd}_\delta(\mathcal{X}) \subseteq \widetilde{bnd}_{\mathcal{A}_{\mathcal{S}}}(\mathcal{X})$.

(2) $\widetilde{\mu}_{\mathcal{A}_{\mathcal{S}}}(\mathcal{X}) \leq \widetilde{\mu}_\delta(\mathcal{X})$.

Corollary 3.2. *Suppose that $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represents a soft approximation space with a full soft set $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ on Θ. If the subset $\mathcal{M}$ is a soft $\mathcal{A}_{\mathcal{S}}$-exact set, then it is a soft δ-exact set.*

Example 3.4. *Let $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ be a soft approximation space with a soft set $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ on Θ, where $\Theta = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, ..., \widetilde{x}_6\}$, $\mathcal{E} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, ..., \varepsilon_6\}$ and $\breve{A} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \subseteq \mathcal{E}$ such that $(\kappa_\varepsilon, \breve{A}) = \{(\varepsilon_1, \{\widetilde{x}_1, \widetilde{x}_6\}), (\acute{e}_2, \{\widetilde{x}_3\}), (\varepsilon_3, \{\widetilde{x}_1, \widetilde{x}_4\})\}$ let $\mathcal{B} = \{\widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5\}$ and $\mathcal{D} = \{\widetilde{x}_1, \widetilde{x}_4, \widetilde{x}_6\}$. Thus $\underline{\mathcal{S}}(\mathcal{B}) = \{\widetilde{x}_3\}$ and $\overline{\mathcal{S}}(\mathcal{B}) = \{\widetilde{x}_1, \widetilde{x}_3, \widetilde{x}_4\}$. But $\underline{\mathcal{S}}_\delta(\mathcal{B}) = \{\widetilde{x}_3\}$ and $\overline{\mathcal{S}}_\delta(\mathcal{B}) = \{\widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5\}$. It is clear that $\mathcal{B} \not\subset \overline{\mathcal{S}}(\mathcal{B})$, but $\underline{\mathcal{S}}_\delta(\mathcal{B}) \subseteq \mathcal{B} \subseteq \overline{\mathcal{S}}_\delta(\mathcal{B})$. Similarly, $\underline{\mathcal{S}}(\mathcal{D}) = \overline{\mathcal{S}}(\mathcal{D}) = \mathcal{D}$ which implies that $\mathcal{D}$ is a soft exact set (according to Feng et al. [4]) . On the other hand, we get $\underline{\mathcal{S}}_\delta(\mathcal{D}) = \overline{\mathcal{S}}_\delta(\mathcal{D}) = \mathcal{D}$ and then $\widetilde{bnd}_\delta(\mathcal{D}) = \phi$ and $\widetilde{\mu}_\delta(\mathcal{D}) = 1$. Accordingly, $\mathcal{D}$ is also soft δ-exact (definable) in our approach.*

Remark 3.1. *According to Theorem 3.1 and its corollaries, It should be noted that the proposed approach (soft δ-approximations) improves the soft rough approximations [4]. Theorem 3.1 proved that soft δ-lower approximation is bigger than soft lower approximation. Moreover, the suggested soft δ-upper approximation improves soft upper approximation since $\mathcal{X} \subseteq \overline{\mathcal{S}}(\mathcal{X})$ does not hold in general. But $\mathcal{X} \subseteq \overline{\mathcal{S}}_\delta(\mathcal{X})$, for each $\mathcal{X} \subseteq \Theta$. Examples 3.3 and 3.4 illustrated this remark.*

Definition 3.2. *Suppose that $\mathcal{A}_\mathcal{S} = (\Theta, \mathcal{S})$ is a soft approximation space such that $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ is a soft set over Θ, and that $\mathcal{X} \subseteq \Theta$. Thus, the following holds true:*

(1) If $\underline{\mathcal{S}}_\delta(\mathcal{X}) \neq \Phi$ and $\overline{\mathcal{S}}_\delta(\mathcal{X}) \neq \Theta$, then $\mathcal{X}$ is roughly soft δ-definable.

(2) If $\underline{\mathcal{S}}_\delta(\mathcal{X}) = \Phi$ and $\overline{\mathcal{S}}_\delta(\mathcal{X}) \neq \Theta$, then $\mathcal{X}$ is internally soft δ-indefinable.

(3) If $\underline{\mathcal{S}}_\delta(\mathcal{X}) \neq \Phi$ and $\overline{\mathcal{S}}_\delta(\mathcal{X}) = \Theta$, then $\mathcal{X}$ is externally soft δ-indefinable.

(4) If $\underline{\mathcal{S}}_\delta(\mathcal{X}) = \Phi$ and $\overline{\mathcal{S}}_\delta(\mathcal{X}) = \Theta$, then $\mathcal{X}$ is totally soft δ-indefinable.

The above classifications can be interpreted as follows:

- This case is interpreted as an exact decision for some members of Θ which belong to $\mathcal{X}$, and for other elements of Θ which exist in $\mathcal{X}^c$, by utilizing existing knowledge from $\mathcal{A}_\mathcal{S}$.
- The internally soft δ-indefinable case is interpreted as a defined decision for some elements of Θ that belong to $\mathcal{X}^c$, but we cannot determine for any element of Θ that it belongs to $\mathcal{X}$ based on the existing knowledge of $\mathcal{A}_\mathcal{S}$.
- For an externally soft δ-indefinable set $\mathcal{X}$, this case is interpreted as a well-defined decision for some elements of Θ that belong to $\mathcal{X}$, but we cannot determine for any element of Θ that it belongs to $\mathcal{X}^c$ based on the current knowledge of $\mathcal{A}_\mathcal{S}$.
- For a totally soft δ-indefinable set $\mathcal{X}$, this case is interpreted as a clear decision for any element of Θ, in terms of whether it belongs to $\mathcal{X}$ or $\mathcal{X}^c$ based on the present knowledge of $\mathcal{A}_\mathcal{S}$.

Based on Theorem 3.1, it is easy to prove the following result. So, the proof is omitted.

Theorem 3.3. *Suppose that $\mathcal{A}_\mathcal{S} = (\Theta, \mathcal{S})$ represents a soft approximation space with a soft set $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ on Θ and $\mathcal{X} \subseteq \Theta$:*

(1) $\mathcal{X}$ is a roughly soft δ-definable set, which implies that it is roughly soft $\mathcal{A}_\mathcal{S}$-definable.

(2) $\mathcal{X}$ is an internally soft δ-indefinable set, implying that it is internally soft $\mathcal{A}_\mathcal{S}$-indefinable.

(3) $\mathcal{X}$ is an externally soft δ-indefinable set, implying that it is externally soft $\mathcal{A}_\mathcal{S}$-indefinable.

(4) $\mathcal{X}$ is a totally soft δ-indefinable set, implying that it is totally soft $\mathcal{A}_\mathcal{S}$-indefinable.

Remark 3.2. *(1) According to Theorem 3.2, we conclude that the suggested approximations (soft δ-rough approximations) improved soft rough approximations [4]; for example, let $\mathcal{X}$ be totally soft $\mathcal{A}_\mathcal{S}$-indefinable which implies that $\underline{\mathcal{S}}(\mathcal{X}) = \Phi$ and $\overline{\mathcal{S}}(\mathcal{X}) = \Theta$; thus, we cannot decide for any element of Θ whether it belongs to $\mathcal{X}$ or $\mathcal{X}^c$. On the other hand,* *according to the soft δ-rough approximations,* $\underline{\mathcal{S}}_{\delta}(\mathcal{X}) \neq \Phi$ *and* $\overline{\mathcal{S}}_{\delta}(\mathcal{X}) \neq \Theta$*; then,* $\mathcal{X}$ *can be a roughly soft δ-definable set which means that we can determine for some elements of* Θ *that they belong to* $\mathcal{X}$*; meanwhile, for some elements of* Θ*, we can decide that they belong to* $\mathcal{X}^c$*, by using available knowledge from the soft approximation space* $\mathcal{A}_{\mathcal{S}}$ *(Examples 3.3, and 3.4 illustrate this remark).*

(2) The reverse of Theorem 3.2 is not valid in general, as demonstrated by Examples 3.3 and 3.4.

4. Some relations generated by soft δ-approximations

In ordinary set theory, some notions (namely, memberships, equality, and inclusion relations) differ essentially from soft rough set theory. So, in the present section, we introduce and study these relations in soft rough sets by using the suggested approaches (soft δ-rough sets).

Definition 4.1. *Suppose that* $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ *represents a soft approximation space with a soft set* $\mathcal{S} = (\kappa_{\varepsilon}, \breve{A})$ *on* Θ*. Thus, for any* $\varpi \in \Theta$*, we define two soft δ-membership relations as follows:*

(1) The soft δ-lower belonging (symbolized by $\varpi \underline{\in}_{\delta} \mathcal{M}$ *) if and only if* $\varpi \in \underline{\mathcal{S}}_{\delta}(\mathcal{M})$ *such that* $\mathcal{M} \subseteq \Theta$*.*

(2) The soft δ-upper belonging (symbolized by $\varpi \overline{\in}_{\delta} \mathcal{M}$ *) if and only if* $\varpi \in \overline{\mathcal{S}}_{\delta}(\mathcal{M})$ *such that* $\mathcal{M} \subseteq \Theta$*.*

Proposition 4.1. *Suppose that* $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ *represents a soft approximation space with a soft set* $\mathcal{S} = (\kappa_{\varepsilon}, \breve{A})$ *on* Θ*. Then, for each* $\mathcal{M}, \mathcal{N} \subseteq \Theta$*, the following holds true:*

(1) If $\mathcal{M} \subseteq \mathcal{N}$ *and* $\varpi \underline{\in}_{\delta} \mathcal{M}$*, then* $\varpi \underline{\in}_{\delta} \mathcal{N}$*. Also, if* $\varpi \overline{\in}_{\delta} \mathcal{M}$*, then* $\varpi \overline{\in}_{\delta} \mathcal{N}$*.*

(2) If $\varpi \underline{\in}_{\delta} (\mathcal{M} \cap \mathcal{N})$*, then* $\varpi \underline{\in}_{\delta} \mathcal{M}$ *and* $\varpi \underline{\in}_{\delta} \mathcal{N}$*.*

(3) $\varpi \overline{\in}_{\delta} (\mathcal{M} \cup \mathcal{N})$ *if and only if* $\varpi \overline{\in}_{\delta} \mathcal{M}$ *or* $\varpi \overline{\in}_{\delta} \mathcal{N}$*.*

(4) If $\varpi \overline{\in}_{\delta} (\mathcal{M} \cap \mathcal{N})$*, then* $\varpi \overline{\in}_{\delta} \mathcal{M}$ *and* $\varpi \overline{\in}_{\delta} \mathcal{N}$*.*

Proof. We will prove the first item and the others easily in a same way.

(1) First, let $\mathcal{M} \subseteq \mathcal{N}$. Then $\underline{\mathcal{S}}_{\delta}(\mathcal{M}) \subseteq \underline{\mathcal{S}}_{\delta}(\mathcal{N})$ and $\overline{\mathcal{S}}_{\delta}(\mathcal{M}) \subseteq \overline{\mathcal{S}}_{\delta}(\mathcal{M})$. Accordingly, if $\varpi \underline{\in}_{\delta} \mathcal{M}$ then $\varpi \in \underline{\mathcal{S}}_{\delta}(\mathcal{M}) \subseteq \underline{\mathcal{S}}_{\delta}(\mathcal{N})$. Therefore, $\varpi \underline{\in}_{\delta} \mathcal{N}$. Similarly, if $\varpi \overline{\in}_{\delta} \mathcal{M}$ then $\varpi \in \overline{\mathcal{S}}_{\delta}(\mathcal{M}) \subseteq \bar{\acute{\mathcal{S}}}_{\delta}(\mathcal{N})$. Therefore, $\varpi \overline{\in}_{\delta} \mathcal{N}$.

□

In ordinary set theory, the concept of set equality means that two sets are equal if they have the same elements. This concept is fundamentally different from soft rough sets. As a result, another concept of set equality is required in our approach (namely, approximate or rough equality). Thus, in set theory, the two sets can be unequal (i.e., have different elements), but in our approach, they can be roughly equal. In what follows, we define new concepts for equality (namely, soft-δ-lower equal, soft-δ-upper equal).

Definition 4.2. *Suppose that $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represents a soft approximation space with a soft set $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ on Θ. Then, for each $\mathcal{M}, \mathcal{N} \subseteq \Theta$, the following holds true:*

(1) $\mathcal{M}$ and $\mathcal{N}$ are soft δ-lower equal (briefly, $\mathcal{M} \simeq_\delta \mathcal{N}i$) if and only if $\underline{\mathcal{S}}_\delta(\mathcal{N}) = \underline{\mathcal{S}}_\delta(\mathcal{N})$.

(2) $\mathcal{M}$ and $\mathcal{N}$ are soft δ-upper equal (briefly, $\mathcal{M} \eqsim_\delta \mathcal{N}$) if and only if $\overline{\mathcal{S}}_\delta(\mathcal{M}) = \overline{\mathcal{S}}_\delta(\mathcal{N})$.

(3) $\mathcal{M}$ and $\mathcal{N}$ are soft δ-equal (briefly, $\mathcal{M} \approx_\delta \mathcal{N}$) if $\mathcal{M} \simeq_\delta \mathcal{N}$ and $\mathcal{M} \eqsim_\delta \mathcal{N}$.

Example 4.1. *Suppose that $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represents a soft approximation space with a soft set $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ on Θ, where $\Theta = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, ..., \widetilde{x}_6\}$, $\acute{E} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, ..., \varepsilon_6\}$ and $\breve{A} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \subseteq \acute{E}$ such that $(\kappa_\varepsilon, \breve{A}) = \{(\varepsilon_1, \{\widetilde{x}_1, \widetilde{x}_6\}), (\varepsilon_2, \{\widetilde{x}_3\}), (\varepsilon_3, \Phi), (\varepsilon_4, \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_5\})\}$. Now, let $\mathcal{B} = \{\widetilde{x}_3, \widetilde{x}_4\}$ and $\mathcal{D} = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3\}$. Then, we get $\underline{\mathcal{S}}_\delta(\mathcal{B}) = \{\widetilde{x}_3\} = \underline{\mathcal{S}}_\delta(\mathcal{D}) = \{\widetilde{x}_3\}$, which implies that $\mathcal{B} \simeq_\delta \mathcal{D}$. Also, set $\mathcal{W} = \{\widetilde{x}_1, \widetilde{x}_3, \widetilde{x}_6\}$ and $\mathcal{V} = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_6\}$. Then, we obtain $\overline{\mathcal{S}}_\delta(\mathcal{W}) = \overline{\mathcal{S}}_\delta(\mathcal{V}) = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_5, \widetilde{x}_6\}$ and thus $\mathcal{W} \eqsim_\delta \mathcal{V}$.*

Proposition 4.2. *Suppose that $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represents a soft approximation space with a soft set $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ on Θ. Then, for each $\mathcal{M}, \mathcal{N}, \mathcal{W}, \mathcal{V} \subseteq \Theta$, the following holds true:*

(1) If $\mathcal{M} \eqsim_\delta \mathcal{W}$ and $\mathcal{N} \eqsim_\delta \mathcal{V}$, then $\mathcal{M} \cup \mathcal{N} \eqsim_\delta \mathcal{W} \cup \mathcal{V}$.

(2) $\mathcal{M} \eqsim_\delta \mathcal{N}$ if and only if $\mathcal{M} \eqsim_\delta (\mathcal{M} \cup \mathcal{N}) \eqsim_\delta \mathcal{N}$.

(3) If $\mathcal{M} \eqsim_\delta \mathcal{N}$, then $\mathcal{M} \cup \mathcal{N}^c \eqsim_\delta \Theta$.

(4) If $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{N} \eqsim_\delta \Phi$, then $\mathcal{M} \eqsim_\delta \Phi$.

(5) If $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M} \eqsim_\delta \Theta$, then $\mathcal{N} \eqsim_\delta \Theta$.

Proof. (1) Assume that $\mathcal{M} \eqsim_\delta \mathcal{W}$ and $\mathcal{N} \eqsim_\delta \mathcal{V}$; then, $\overline{\mathcal{S}}_\delta(\mathcal{M}) = \overline{\mathcal{S}}_\delta(\mathcal{W})$ and $\overline{\mathcal{S}}_\delta(\mathcal{N}) = \overline{\mathcal{S}}_\delta(\mathcal{V})$. Thus, $\overline{\mathcal{S}}_\delta(\mathcal{M}) \cup \overline{\mathcal{S}}_\delta(\mathcal{N}) = \overline{\mathcal{S}}_\delta(\mathcal{W}) \cup \overline{\mathcal{S}}_\delta(\mathcal{V})$, which implies that $\overline{\mathcal{S}}_\delta(\mathcal{M} \cup \mathcal{N}) = \overline{\mathcal{S}}_\delta(\mathcal{W} \cup \mathcal{V})$. Hence, $\mathcal{M} \cup \mathcal{N} \eqsim_\delta \mathcal{W} \cup \mathcal{V}$.

(2) Follow as similar path (1).

(3) Let $\mathcal{M} \eqsim_\delta \mathcal{N}$; then, $\overline{\mathcal{S}}_\delta(\mathcal{M}) = \overline{\mathcal{S}}_\delta(\mathcal{N})$, which implies that $\overline{\mathcal{S}}_\delta(\mathcal{M} \cup \mathcal{N}^c) = \overline{\mathcal{S}}_\delta(\mathcal{M}) \cup \overline{\mathcal{S}}_\delta(\mathcal{N}^c) = \overline{\mathcal{S}}_\delta(\mathcal{N}) \cup \overline{\mathcal{S}}_\delta(\mathcal{N}^c)$. But, $\overline{\mathcal{S}}_\delta(\mathcal{N}) \cup \overline{\mathcal{S}}_\delta(\mathcal{N}^c) = \overline{\mathcal{S}}_\delta(\mathcal{N} \cup \mathcal{N}^c) = \overline{\mathcal{S}}_\delta(\Theta)$, which means that $\overline{\mathcal{S}}_\delta(\mathcal{M} \cup \mathcal{N}^c) = \overline{\mathcal{S}}_\delta(\Theta)$ and, hence, $\mathcal{M} \cup \mathcal{N}^c \eqsim_\delta \Theta$.

(4) Follow a similar path (3).

(5) By using (3) and (4), the proof is obvious.

□

Definition 4.3. *Suppose that $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represents a soft approximation space with a soft set $\mathcal{S} = (\kappa_\varepsilon, \breve{A})$ on Θ. Then, for each $\mathcal{M}, \mathcal{N} \subseteq \Theta$, the following holds true:*

(1) The set $\mathcal{M}$ is a soft δ-lower subset of $\mathcal{N}$ (written, $\mathcal{M} \underset{\sim\delta}{\sqsubset} \mathcal{N}$) if and only if $\underline{\mathcal{S}}_\delta(\mathcal{M}) \subseteq \underline{\mathcal{S}}_\delta(\mathcal{N})$.

(2) The set $\mathcal{M}$ is a soft δ-upper subset of $\mathcal{N}$ (written, $\mathcal{M} \overset{\sim}{\sqsubset}_\delta \mathcal{N}$) if and only if $\overline{\mathcal{S}}_\delta(\mathcal{M}) \subseteq \overline{\mathcal{S}}_\delta(\mathcal{N})$.

(3) The set $\mathcal{M}$ is a soft δ-roughly subset in $\mathcal{N}$ (written, $\mathcal{M} \underset{\sim}{\overset{\sim}{\sqsubset}}_\delta \mathcal{N}$) if and only if $\underline{\mathcal{S}}_\delta(\mathcal{M}) \subseteq \underline{\mathcal{S}}_\delta(\mathcal{N})$ and $\overline{\mathcal{S}}_\delta(\mathcal{M}) \subseteq \overline{\mathcal{S}}_\delta(\mathcal{N})$.

Remark 4.1. *It must be noticed that the soft δ-rough inclusion of sets does not imply ordinary inclusion. Example 4.2 demonstrates this note:*

Example 4.2. *Consider Example 4.1, $\mathcal{M} = \{\widetilde{x}_1\}$ and $\xi = \{\widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5\}$. Hence, we get $\underline{\mathcal{S}}_{\delta}(\mathcal{M}) = \Phi$ and $\underline{\mathcal{S}}_{\delta}(\mathcal{N}) = \{\widetilde{x}_1\}$, which implies, $\mathcal{M} \underset{\smile\delta}{\sqsubset} \mathcal{N}$ although $\mathcal{M} \nsubseteq \mathcal{N}$.*

Proposition 4.3. *Suppose that $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represents a soft approximation space with a soft set $\acute{\mathcal{S}} = (\kappa_{\varepsilon}, \breve{A})$ on Θ. Then, for each $\mathcal{M}, \mathcal{N} \subseteq \Theta$, the following holds true:*

(1) If $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} \underset{\smile\delta}{\sqsubset} \mathcal{N}$, $\mathcal{M} \overset{\frown}{\sqsubset}_{\delta} \mathcal{N}$ and $\mathcal{M} \underset{\smile}{\overset{\frown}{\sqsubset}}_{\delta} \mathcal{N}$.

(2) If $\mathcal{M} \underset{\smile\delta}{\sqsubset} \mathcal{N}$ and $\mathcal{N} \underset{\smile\delta}{\sqsubset} \mathcal{M}$, then $\mathcal{M} \simeq_{\delta} \mathcal{N}$.

(3) If $\mathcal{M} \overset{\frown}{\sqsubset}_{\delta} \mathcal{N}$ and $\mathcal{N} \overset{\frown}{\sqsubset}_{\delta} \mathcal{M}$, then $\mathcal{M} \eqsim_{\delta} \mathcal{N}$.

(4) If $\mathcal{M} \underset{\smile}{\overset{\frown}{\sqsubset}}_{\delta} \mathcal{N}$ and $\mathcal{N} \underset{\smile}{\overset{\frown}{\sqsubset}}_{\delta} \mathcal{M}$, then $\mathcal{M} \approx_{\delta} \mathcal{N}$.

Proof. (1) It is obvious.

(2) Let $\mathcal{M} \underset{\smile\delta}{\sqsubset} \mathcal{N}$ and $\mathcal{N} \underset{\smile\delta}{\sqsubset} \mathcal{M}$. Then, $\underline{\mathcal{S}}_{\delta}(\mathcal{M}) \subseteq \underline{\mathcal{S}}_{\delta}(\mathcal{N})$ and $\underline{\mathcal{S}}_{\delta}(\mathcal{N}) \subseteq \underline{\mathcal{S}}_{\delta}(\mathcal{M})$ which implies $\underline{\mathcal{S}}_{\delta}(\mathcal{M}) = \underline{\mathcal{S}}_{\delta}(\mathcal{N})$. Accordingly, $\mathcal{M} \simeq_{\delta} \mathcal{N}$.

(3) Follow a similar path as (2).

(4) Follow a similar path as (2).

□

Based on Proposition 4.3, it is easy to prove the next proposition. Therefore, the proof is omitted.

Proposition 4.4. *Suppose that $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represents a soft approximation space with a soft set $\mathcal{S} = (\kappa_{\varepsilon}, \breve{A})$ on Θ. Then, for each $\mathcal{M}, \mathcal{N}, \mathcal{W}, \mathcal{V} \subseteq \Theta$, the following holds true:*

(1) $\mathcal{M} \overset{\frown}{\sqsubset}_{\delta} \mathcal{N}$ if and only if $\mathcal{M} \cup \mathcal{N} \eqsim_{\delta} \mathcal{N}$.

(2) $\mathcal{M} \cap \mathcal{N} \underset{\smile\delta}{\sqsubset} \mathcal{M} \overset{\frown}{\sqsubset}_{\delta} \mathcal{M} \cup \mathcal{N}$.

(3) If $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M} \simeq_{\delta} \mathcal{W}$ and $\mathcal{M} \simeq_{\delta} \mathcal{V}$, then $\mathcal{W} \underset{\smile\delta}{\sqsubset} \mathcal{V}$.

(4) If $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M} \eqsim_{\delta} \mathcal{W}$ and $\mathcal{N} \eqsim_{\delta} \mathcal{V}$, then $\mathcal{W} \overset{\frown}{\sqsubset}_{\delta} \mathcal{V}$.

(5) If $\mathcal{M} \overset{\frown}{\sqsubset}_{\delta} \mathcal{N}$ and $\mathcal{W} \overset{\frown}{\sqsubset}_{\delta} \mathcal{V}$, then $\mathcal{M} \cup \mathcal{W} \overset{\frown}{\sqsubset}_{\delta} \mathcal{N} \cup \mathcal{V}$.

(6) If $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M} \approx_{\delta} \mathcal{W}$ and $\mathcal{N} \approx_{\delta} \mathcal{V}$, then $\mathcal{W} \underset{\smile}{\overset{\frown}{\sqsubset}}_{\delta} \mathcal{V}$.

5. Medical application for diagnosis of the Chikungunya virus using soft δ-rough sets

The problem of Chikungunya virus (CHIKV), which is transmitted to humans by virus-carrying Aedes mosquitos, is addressed here. CHIKV outbreaks have recently been linked to severe illness. It causes a high fever and excruciating joint pain. Muscle pain, headache, and nausea are some of the other symptoms. The initial symptoms are similar to those of dengue fever. It is not usually life-threatening. However, joint pain can last for a long time and complete recovery can take months. They usually occur as a lifelong immunity to infection. Therefore, an accurate diagnosis of the disease can limit the spread of infection and thus eliminate the disease. In this section, we present a medical application for CHIKV. We try to help the doctor in two ways. The first is help with making an accurate medical decision about the diagnosis of the disease. The other is by enabling data reduction to try to reduce the number of medical tests required to diagnose the disease, which saves time and effort in addition to saving money. Suppose the following information table has data about eight patients as in Table 2, which represents a decision table.

Example 5.1. *A set valued information system is presented in Table 2, where* $\Theta = \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, ..., \tilde{p}_8\}$ *for eight patients,* $\breve{A} = \{\tilde{e}_1 =$ *Joint pain,* $\tilde{e}_2 =$ *Headache,* $\tilde{e}_3 =$ *Nausea,* $\tilde{e}_4 =$ *Temperature*$\}$ *is a set of parameters which illustrate symptoms for patients. Suppose that a soft set* $\mathcal{S} = (\kappa_{\tilde{e}}, \breve{A})$ *which describes the patients having or not having CHIKV as in Table 2, where* $\mathcal{S} = (\kappa_{\tilde{e}}, \breve{A}) = \{(\tilde{e}_1, \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_6, \tilde{p}_8\}), (\tilde{e}_2, \{\tilde{p}_1, \tilde{p}_3, \tilde{p}_6\}), (\tilde{e}_3, \{\tilde{p}_1, \tilde{p}_4, \tilde{p}_7\}), (\tilde{e}_4, \{\tilde{p}_1, \tilde{p}_3, \tilde{p}_4, \tilde{p}_6, \tilde{p}_8\})\}$.

5.1. Clinical decision-making of CHIKV using soft δ-rough sets

First, we illustrate that the proposed method, in the presented paper, is more accurate and valuable in decision making than the previous one [4, 26, 43]. We will apply the two methods in the existing decision table Table 2 as follows:

According to Table 2, consider that $\tilde{X} = \{\tilde{p}_1, \tilde{p}_3, \tilde{p}_6, \tilde{p}_8\}$ represents a set of patients with CHIKV. So, we will compute the approximations of it by using the previous method [4] and via the suggested approaches as follows:

- **Soft rough sets** [4]:
  $\underline{\mathcal{S}}(\tilde{X}) = \{\tilde{p}_1, \tilde{p}_3, \tilde{p}_6\} \subset \tilde{X}$ and $\overline{\mathcal{S}}(\tilde{X}) = \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_6, \tilde{p}_7, \tilde{p}_8\}$. Accordingly, we obtain $\widetilde{bnd}_{\mathcal{A}_S}(\tilde{X}) = \{\tilde{p}_2, \tilde{p}_4, \tilde{p}_7, \tilde{p}_8\}$ and $\widetilde{\mu}_{\mathcal{A}_S}(\tilde{X}) = 3/7 \cong 42\%$.
- **Soft δ-rough sets:**
  $\underline{\mathcal{S}}_\delta(\tilde{X}) = \tilde{X}$ and $\overline{\mathcal{S}}_\delta(\tilde{X}) = \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_6, \tilde{p}_7, \tilde{p}_8\}$. Accordingly, we obtain $\widetilde{bnd}_\delta(\tilde{X}) = \{\tilde{p}_2, \tilde{p}_4, \tilde{p}_7\}$ and $\widetilde{\mu}_\delta(\tilde{X}) = 4/7 \cong 57\%$.

Observation: According to the above comparison, the following is noticed:

According to the Feng et al. method, we cannot determine whether the patient $\tilde{p}_8$ has CHIKV infection or not, because that patient is located in the boundary region although this patient surely has the disease according to Table 2.

On the other hand, by using the proposed approaches, we surely identify the patients that have CHIKV since the positive region is $\underline{\mathcal{S}}_\delta(\tilde{X}) = \tilde{X}$ which is interpreted as the only patients with CHIKV are $\tilde{p}_1$, $\tilde{p}_3$, $\tilde{p}_6$ and $\tilde{p}_8$ which is identical to the decision table (Table 2). So, accurate decision-making is critical in our daily lives, and this technique produces the greatest choice from a set of alternatives. Therefore, we can say that the suggested techniques in this paper may be valuable in medical diagnosis which always requires an accurate decision, such as in the cases of COVID-19, Omicron and SARS.

Because decision-making represents an important base in everyday life, especially in medical diagnosis, we present Algorithm 1, which may be used to make an accurate decision for any information system using soft δ-approximations.

Table 2. Decision table for CHIKV.

patients	$\tilde{e}_1$	$\tilde{e}_2$	$\tilde{e}_3$	$\tilde{e}_4$	CHIKV
$\tilde{p}_1$	1	1	1	1	1
$\tilde{p}_2$	1	0	0	0	0
$\tilde{p}_3$	0	1	0	1	1
$\tilde{p}_4$	0	0	1	1	0
$\tilde{p}_5$	0	0	0	0	0
$\tilde{p}_6$	1	1	0	1	1
$\tilde{p}_7$	0	0	1	0	0
$\tilde{p}_8$	1	0	0	1	1

Algorithm 1: Decision-making via soft δ-approximations

Input: An information system of given medical data in a tabular form.
Output: Exact decision of whether patients have CHIKV or not.

Step 1: Construct a soft set $\mathcal{S} = (\kappa_{\tilde{e}}, \breve{A})$, where $\breve{A}$ represents a set of parameters (symptoms) from the tabular form of given data.
Step 2: Determine the set $\tilde{X}$ of choice patients under consideration for diagnosis.
Step 3: Compute a soft δ-lower approximation, $\underline{\mathcal{S}}_\delta(\tilde{X})$, and a soft δ-upper approximation, $\overline{\mathcal{S}}_\delta(\tilde{X})$, for $\tilde{X}$ by using Definition 3.1.
Step 4: Evaluate a soft δ-boundary region, $\widetilde{bnd}_\delta(\tilde{X})$, and a soft δ-accuracy $\widetilde{\mu}_\delta(\tilde{X})$ of $\tilde{X}$ from Step 2.
Step 5:

- **If** $\widetilde{bnd}_\delta(\tilde{X}) = \Phi$ and $\widetilde{\mu}_\delta(\tilde{X}) = 1$, then, $\tilde{X}$ is a soft δ-exact set; hence, these patients surely have CHIKV.
- **Else**, $\tilde{X}$ is a soft δ-rough set and hence the patients in the soft δ-positive region $\widetilde{pos}_\delta(\tilde{X}) = \underline{\mathcal{S}}_\delta(\tilde{X})$ surely have CHIKV. And the patients in the soft δ-negative region $\widetilde{neg}_\delta(\tilde{X}) = [\overline{\mathcal{S}}_\delta(\tilde{X})]^c$ surely do not have CHIKV.

End

Figure 1 represents a simple framework to make an accurate medical-decision using the proposed techniques (soft δ-approximations) in MATLAB.

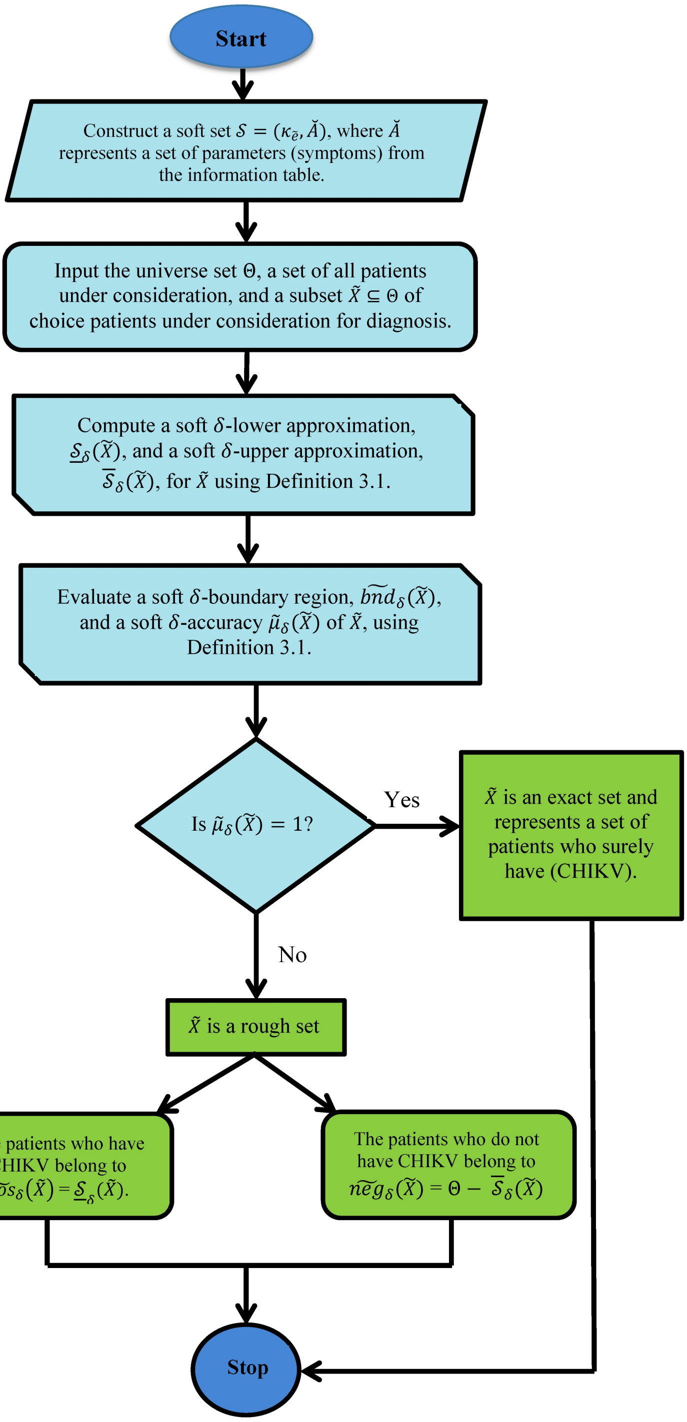

Figure 1. Simple flowchart for medical decision-making via soft δ-approximations.

5.2. Some comparisons with the previous methods

In addition to comparing the current methods with the Feng et al. method (to prove that the proposed method improves that approach), we present in this section some comparisons with some previous generalizations of soft rough sets to emphasize the significance of the proposed approach in making decisions for life applications. To illustrate this fact, we have applied the previous two methods [26, 43] in the suggested medical application (i.e., the problem of CHIKV [63, 64]). And then we prove that these methods are not suitable for solving this medical problem and fail to give a correct decision that is useful in the medical diagnosis.

- **Shabir et al. technique "modified soft rough sets (MSR-sets)" [43] :**
  Shabir et al. proposed a new approach to studying roughness through the use of soft sets in order to find the approximations of a set. This new model is known as modified soft rough sets (MSR-sets). Thus, we will apply this technique in the applied Example 5.1.

Definition 5.1. *Suppose that $\mathcal{S} = (\kappa_{\tilde{e}}, \breve{A})$ is a soft set over Θ, where $\kappa_{\tilde{e}}$ is a map $\kappa_{\tilde{e}} : \breve{A} \longrightarrow P(\Theta)$. If $\sigma_{\grave{u}} : \Theta \longrightarrow P(\breve{A})$ is another map defined as $\sigma_{\grave{u}}(\grave{u}) = \{\tilde{e} \in \breve{A} : \grave{u} \in \kappa_{\tilde{e}}(\tilde{e})\}$, then the pair $(\Theta, \sigma_{\grave{u}})$ is called the modified soft rough approximation space (briefly, MSR-approximation space) and for each $\mathcal{M} \subseteq \Theta$, the lower MSR-approximation and upper MSR-approximation are given by:*

$$\underline{\sigma_{\grave{u}}}(\mathcal{M}) = \{\grave{u} \in \mathcal{M} : \sigma_{\grave{u}}(\grave{u}) \neq \sigma_{\grave{u}}(\grave{v}), \textit{for all } \grave{v} \in \mathcal{M}^c\} \textit{ and}$$
$$\overline{\sigma_{\grave{u}}}(\mathcal{M}) = \{\grave{u} \in \Theta : \sigma_{\grave{u}}(\grave{u}) = \sigma_{\grave{u}}(\grave{v}), \textit{for some } \grave{v} \in \mathcal{M}\}, \textit{respectively.}$$

Therefore, from Table 2, we get the following $\sigma_{\grave{u}}(\tilde{p}_1) = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$, $\sigma_{\grave{u}}(\tilde{p}_2) = \{\tilde{e}_1\}$, $\sigma_{\grave{u}}(\tilde{p}_3) = \{\tilde{e}_2, \tilde{e}_4\}$, $\sigma_{\grave{u}}(\tilde{p}_4) = \{\tilde{e}_3, \tilde{e}_4\}$, $\sigma_{\grave{u}}(\tilde{p}_5) = \Phi$, $\sigma_{\grave{u}}(\tilde{p}_6) = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_4\}$, $\sigma_{\grave{u}}(\tilde{p}_7) = \{\tilde{e}_3\}$ and $\sigma_{\grave{u}}(\tilde{p}_8) = \{\tilde{e}_1, \tilde{e}_4\}$. As a result, these sets are distinct; thus, all subsets of Θ are definable; this means that, in the proposed application example, that method fails to present different lower and upper approximations (Example 5.1).

- **Ali technique [26]:** Based on a soft set, Ali succeeded in proving that every soft set gives rise to an approximation space in Pawlak's sense. In fact, he suggested a new technique for roughness associated with soft sets.

Definition 5.2. *Suppose that $\mathcal{S} = (\kappa_{\tilde{e}}, \breve{A})$ is a soft set over Θ. Thus, $(\kappa_{\tilde{e}}, \breve{A})$ is said to be a soft binary relation on Θ. Obviously, $(\kappa_{\tilde{e}}, \breve{A})$ represents a parameterized collection of binary relations on Θ. Therefore, for each parameter $\tilde{e} \in \breve{A}$, we obtain a binary relation $(\kappa_{\tilde{e}}, \breve{A})$ on Θ.*

Definition 5.3. *A soft binary relation $(\kappa_{\tilde{e}}, \breve{A})$ on Θ, is said to be a soft equivalence relation, if $\kappa_{\tilde{e}}(\tilde{e}) \neq \Phi$ is an equivalence relation on Θ for all $\tilde{e} \in \breve{A}$.*

Remark 5.1. *It is well known that each equivalence relation on a set partitions the set into disjoint classes , and that each partition of the set offers us an equivalence relation on the set. So, a soft equivalence relation over Θ, provides us a parameterized collection of partitions of Θ.*

Now, from Table 2, the classes for each parameter are as follows:

The classes of $\tilde{e}_1$ are $\{\tilde{p}_1, \tilde{p}_2, \tilde{p}_6, \tilde{p}_8\}$, $\{\tilde{p}_3, \tilde{p}_4, \tilde{p}_5, \tilde{p}_7\}$.
The classes of $\tilde{e}_2$ are $\{\tilde{p}_1, \tilde{p}_3, \tilde{p}_6\}$, $\{\tilde{p}_2, \tilde{p}_4, \tilde{p}_5, \tilde{p}_7, \tilde{p}_8\}$.
The classes of $\tilde{e}_3$ are $\{\tilde{p}_1, \tilde{p}_4, \tilde{p}_7\}$, $\{\tilde{p}_2, \tilde{p}_3, \tilde{p}_5, \tilde{p}_6, \tilde{p}_8\}$.
The classes of $\tilde{e}_4$ are $\{\tilde{p}_1, \tilde{p}_3, \tilde{p}_4, \tilde{p}_6, \tilde{p}_8\}$, $\{\tilde{p}_2, \tilde{p}_5, \tilde{p}_7\}$.

Accordingly, the intersection of each of these classes is $\{\tilde{p}_1\}$, $\{\tilde{p}_2\}$, $\{\tilde{p}_3\}$, $\{\tilde{p}_4\}$, $\{\tilde{p}_5\}$, $\{\tilde{p}_6\}$, $\{\tilde{p}_7\}$, $\{\tilde{p}_8\}$. As a result, the intersection of all equivalence relations results in the identity relation $\acute{R} = \{(\tilde{p}_1, \tilde{p}_1), (\tilde{p}_2, \tilde{p}_2), (\tilde{p}_3, \tilde{p}_3), (\tilde{p}_4, \tilde{p}_4), (\tilde{p}_5, \tilde{p}_5), (\tilde{p}_6, \tilde{p}_6), (\tilde{p}_7, \tilde{p}_7), (\tilde{p}_8, \tilde{p}_8)\}$. As a result, every subset is definable in the approximation space $(\Theta, acuteR)$. Accordingly, in the given example, the method discussed in [26] fails to produce separated lower and upper approximations.

5.3. Soft δ-reduction of attributes

One of the important goals of medical diagnosis is to determine the most important factors affecting the development of the disease. On the other hand, reducing the number of medical examinations is one of the goals that save time, effort and money in an accurate medical diagnosis. So, we are trying to help the doctor here by doing data scraping to try to reduce the number of medical tests to diagnose the disease, thereby saving time and effort, as well as money.

Soft δ-rough sets serve an important purpose in data reduction by eliminating redundant attributes in an information table. As a result, the current subsection is dedicated to making a topological reduction of an information system of Table 2. First, we use soft δ-rough sets to identify the key factors of CHIKV, extending the concept of "nano-topology" proposed by Thivagar and Richard [65].

First, let us extend the definition of "nano-topology" by using soft δ-rough sets.

Definition 5.4. *Consider the soft approximation space $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ with a soft set $\mathcal{S} = (\kappa_{\tilde{e}}, \breve{A})$ on Θ, as well as $\mathcal{M} \subseteq \Theta$. Therefore, the collection $\tau_\delta = \{\Theta, \Phi, \underline{\mathcal{S}}_\delta(\mathcal{M}), \overline{\mathcal{S}}_\delta(\mathcal{M}), \widetilde{bnd}_\delta(\mathcal{M})\}$ is called a "δ-nano topology" and it is a general topology produced by the soft δ-rough set $\mathcal{M} \subseteq \Theta$. In addition, the basis of this topology is given by $\beta_\delta = \{\Theta, \underline{\mathcal{S}}_\delta(\mathcal{M}), \widetilde{bnd}_\delta(\mathcal{M})\}$.*

Definition 5.5. *Suppose that $\mathcal{A}_{\mathcal{S}} = (\Theta, \mathcal{S})$ represents a soft approximation space with a soft set $\mathcal{S} = (\kappa_{\tilde{e}}, \breve{A})$ on Θ, and that τ_δ is a δ-nano topology on Θ with a basis β_δ. Thus, the following holds true:*
(i) If $\beta_{\delta-\tilde{e}_k} = \beta_\delta$, then $\tilde{e}_k$ is called a "dispensable" attribute.
(ii) If $\beta_{\delta-\tilde{e}_k} \neq \beta_\delta$, then $\tilde{e}_k$ is a "non-dispensable" attribute.
Furthermore, $CORE = \{\tilde{e}_k\}$ is defined as a core attributes that represents the common part of the reduction.

Now, using soft δ-rough sets, the topological reduction of Table 2 to identify the key factors of CHIKV is superimposed as follows:
First, we can evaluate the δ-nano topology for decision-making by using two sets of patients:

- **Case 1:** $\mathcal{M} = \{\tilde{p}_1, \tilde{p}_3, \tilde{p}_6, \tilde{p}_8\}$ which represents a set of patients that have CHIKV, and
- **Case 2:** $\mathcal{N} = \{\tilde{p}_2, \tilde{p}_4, \tilde{p}_5, \tilde{p}_7\}$ which represents a set of patients that do not have CHIKV.

Thus, we use $\mathcal{M}$ to perform a topological reduction and $\mathcal{N}$ in the same way.
Case 1: (Patients with CHIKV)
According to Table 2, we obtain $\underline{\mathcal{S}}_\delta(\mathcal{M}) = \{\tilde{p}_1, \tilde{p}_3, \tilde{p}_6, \tilde{p}_8\}$, $\overline{\mathcal{S}}_\delta(\mathcal{M}) = \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_6, \tilde{p}_7, \tilde{p}_8\}$ and hence $\widetilde{bnd}_\delta(\mathcal{M}) = \{\tilde{p}_2, \tilde{p}_4, \tilde{p}_7\}$. Thus, the basis of δ-nano topology generated by $\mathcal{M}$ is:

$$\beta_\delta = \{\Theta, \{\tilde{p}_2, \tilde{p}_4, \tilde{p}_7\}, \{\tilde{p}_1, \tilde{p}_3, \tilde{p}_6, \tilde{p}_8\}\}.$$

Step 1: If we omit "$\tilde{e}_1$ = Joint pain", then the soft δ-approximations of $\mathcal{M}$ in this case are

$\underline{\mathcal{S}}_{\delta-\tilde{e}_1}(\mathcal{M}) = \{\tilde{p}_1, \tilde{p}_3, \tilde{p}_6, \tilde{p}_8\}$, $\overline{\mathcal{S}}_{\delta-\tilde{e}_1}(\mathcal{M}) = \{\tilde{p}_1, \tilde{p}_3, \tilde{p}_4, \tilde{p}_6, \tilde{p}_7, \tilde{p}_8\}$ and $\widetilde{bnd}_{\delta-\tilde{e}_1}(\mathcal{M}) = \{\tilde{p}_4, \tilde{p}_7\}$.
So, the basis of δ-nano topology generated by $\mathcal{M}$ is

$$\beta_{\delta-\tilde{e}_1} = \{\Theta, \{\tilde{p}_4, \tilde{p}_7\}, \{\tilde{p}_1, \tilde{p}_3, \tilde{p}_6, \tilde{p}_8\}\} \neq \beta_\delta.$$

Step 2: If we omit "$\tilde{e}_2$ = Headache", then by the same method as in Step 1, we obtain

$$\beta_{\delta-\tilde{e}_2} = \{\Theta, \Phi, \{\tilde{p}_1, \tilde{p}_3, \tilde{p}_6, \tilde{p}_8\}\} \neq \beta_\delta.$$

Step 3: If we omit "$\tilde{e}_3$ = Nausea", then by the same method as in Step 1, we obtain

$$\beta_{\delta-\tilde{e}_3} = \{\Theta, \{\tilde{p}_2, \tilde{p}_4\}, \{\tilde{p}_1, \tilde{p}_3, \tilde{p}_6, \tilde{p}_8\}\} \neq \beta_\delta.$$

Step 4: If we omit "$\tilde{e}_4$ = Temperature", then by the same method as in Step 1, we obtain

$$\beta_{\delta-\tilde{e}_4} = \{\Theta, \{\tilde{p}_2, \tilde{p}_4, \tilde{p}_7\}, \{\tilde{p}_1, \tilde{p}_3, \tilde{p}_6, \tilde{p}_8\}\} = \beta_\delta.$$

Therefore, we get that the attribute $\tilde{e}_4$ is dispensable while the attributes $\tilde{e}_1$, $\tilde{e}_2$ and $\tilde{e}_3$ are not dispensable. Thus, the core of attributes is $CORE = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$. That is, joint pain, headache, and nausea are the main symptoms associated with the CHIKV.
Algorithm 2 represents a simple tool for the topological reduction of attributes for an information system via soft δ-approximations.
Algorithm 2: Topological reduction of attributes via soft δ-rough approximations

Input: An information system of given medical data in a tabular form.
Output: The core of attributes for deciding the CHIKV infection.

Step 1: Construct the soft set $\mathcal{S} = (\kappa_{\tilde{e}}, \breve{A})$, where $\breve{A}$ represents a set of parameters (symptoms) from the tabular form of given data.
Step 2: Input the universe set Θ, a set of all patients under consideration, and a subset $\tilde{X} \subset \Theta$ of some infected patients with CHIKV infection.
Step 3: Evaluate a soft δ-lower approximation, $\underline{\mathcal{S}}_\delta(\tilde{X})$, a soft δ-upper approximation, $\overline{\mathcal{S}}_\delta(\tilde{X})$, and a soft δ-boundary, $\widetilde{bnd}_\delta(\tilde{X})$, for $\tilde{X}$ by using Definition 3.1.
Step 4: Produce β_δ as a basis of a δ-nano topology of $\tilde{X}$ by applying $\beta_\delta = \{\Theta, \underline{\mathcal{S}}_\delta(\tilde{X}), \widetilde{bnd}_\delta(\tilde{X})\}$ via Step 2.
Step 5: Remove the first attribute $\tilde{e}_1$ from the attributes $\breve{A}$ and then compute the soft δ-approximations and soft δ-boundary for $\tilde{X}$.
Step 6: By using Step 5, generate $\beta_{\delta-\tilde{e}_1}$ as a basis of a δ-nano topology of $\tilde{X}$ by:

$$\beta_{\delta-\tilde{e}_1} = \{\Theta, \underline{\mathcal{S}}_{\delta-\tilde{e}_1}(\tilde{X}), \widetilde{bnd}_{\delta-\tilde{e}_1}(\tilde{X})\}.$$

Step 7:

- **If** $\beta_{\delta-\tilde{e}_1} \neq \beta_\delta$, then the attribute $\tilde{e}_1$ is not dispensable.
- **Else** the attribute $\tilde{e}_1$ is dispensable; hence, it can be eliminate from examinations.

Step 8: Repeat Steps 5–7 for all attributes and then decide the core as follows:

$$CORE = \{\tilde{e}_k : \beta_{\delta-\tilde{e}_k} \neq \beta_\delta, \forall k = 1, 2, 3, ...\}.$$

End

Figure 2 represents a simple framework to make an accurate medical decision by using the proposed technique (soft δ-approximations) in MATLAB.

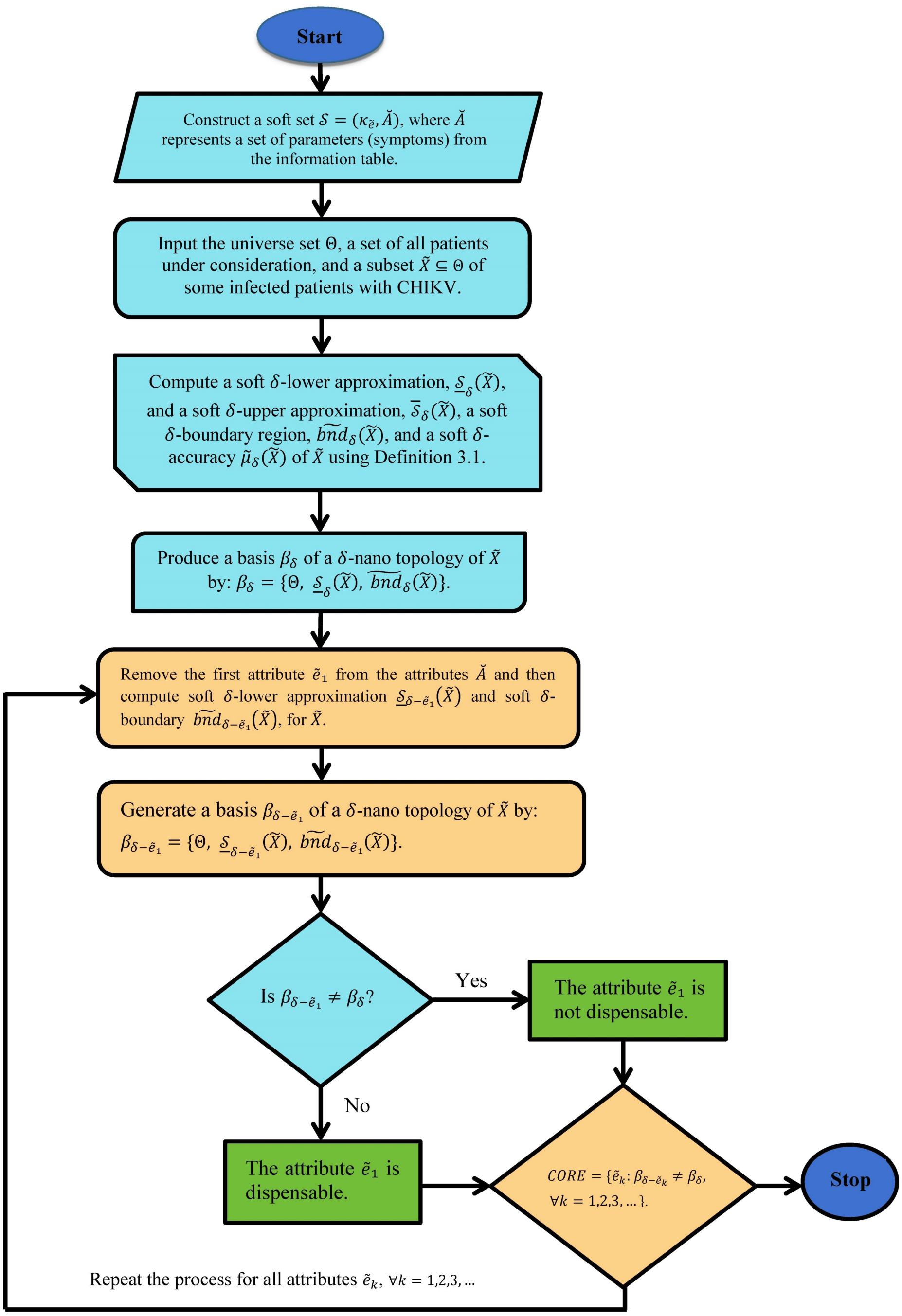

Figure 2. Flowchart for a topological reduction of attributes via soft δ-approximations.

6. Conclusions and discussions

In this manuscript, the idea of "soft δ-rough sets" was suggested as a modification and improvement to the soft rough sets approach proposed by Feng et al. [4]. Some of their characteristics were assumed. Theorems 3.1–3.3 and their corollaries proved that soft δ-rough approaches improved the soft rough set models. On the other hand, this approach is more accurate than the Feng et al. technique and extends the scope of applications because the problem of soft upper approximation is solved. The main virtues of the current methodology are that it yields the highest accuracy values as compared with some of the other approaches given in the published literature, and also that it preserves the monotonicity property of the accuracy and roughness measures. In addition, some illustrative examples and medical applications with results demonstrated that the accuracy of the proposed method is better than that of other techniques [4, 26, 43]. Some important relations related to soft δ-rough approximations (such as soft δ-memberships, soft δ-equality, and soft δ-inclusion) have been provided and their properties have been examined. One of the difficulties of medical diagnosis is making an accurate decision about patients; so, we have presented medical applications related to making decisions for medical diagnosis for CHIKV problems [63, 64]; thus, we demonstrated the importance of current methods in real-life applications as compared with the previous approaches. In fact, we proved that the previous techniques [4, 26, 43] are not suitable for solving this medical problem and fail to give a correct decision in the medical diagnosis. Furthermore, to help the doctor in a medical diagnosis of CHIKV, we have extended the notion of nano-topology (given by [65]) using soft δ-rough approximations; hence, we have proposed a topological reduction for existing decision tables. Accordingly, we have recognized the key factors in CHIKV diagnosis and thus can help the doctor to reduce the number of medical tests and save time and effort as well as save money. In addition, we have obtained two algorithms for the suggested methods for decision-making problems. We hope that the presented framework will be useful in future studies of COVID-19 and other diseases.

In the future, we are going to study the following subjects:

(I) Discussing the soft δ-rough models in the topological approach [16].

(II) Form a new frame consisting of an ideal structure and topology to improve the approximation operators and accuracy values given in the present manuscript and applied in [19].

(III) Investigating the suggested work for some other applications such as economic applications [17].

(IV) Emphasis on the soft δ-rough concepts in some other frames such as N-soft sets [44], and fuzzy rough sets [39].

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Conflict of interest

All authors declare that there is no conflict of interest regarding the publication of this manuscript.

References

1. Z. Pawlak, Rough sets, *Int. J. Inform. Comput. Sci.*, **11** (1982), 341–356. https://doi.org/10.1007/BF01001956
2. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
3. D. A. Molodtsov, Soft set theory-first results , *Comput. Math. Appl.*, **37** (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5
4. F. Feng, X. Liu, V. Leoreanu-Fotea, Y. B. Jun, Soft sets and soft rough sets, *Inform. Sci.*, **181** (2011), 1125–1137. https://doi.org/10.1016/j.ins.2010.11.004
5. D. P. Acharjya, D. Arya, *Multicriteria decision-making using interval valued neutrosophic soft set*, Artificial Intelligence and Global Society, Chapman and Hall/CRC, New York, 2021, 275. https://doi.org/10.1201/9781003006602
6. N. Kumari, D. P. Acharjya, Data classification using rough set and bioinspired computing in healthcare applications-an extensive review, *Multimed. Tools Appl.*, 2022, 1–27. https://doi.org/10.1007/s11042-022-13776-1
7. N. Kumari, D. P. Acharjya, A decision support system for diagnosis of hepatitis disease using an integrated rough set and fish swarm algorithm, *Concurr. Comp.-Pract. E.*, **34** (2022), e7107. https://doi.org/10.1002/cpe.7107
8. M. K. El-Bably, E. A. Abo-Tabl, A topological reduction for predicting of a lung cancer disease based on generalized rough sets, *J. Intell. Fuzzy Syst.*, **41** (2021), 335–346. https://doi.org/10.1155/2021/2559495
9. M. K. El-Bably, A. A. El Atik, Soft β-rough sets and its application to determine COVID-19, *Turk. J. Math.*, **45** (2021), 1133–1148. https://doi.org/10.3906/mat-2008-93
10. E. A. Marei, *Neighborhood system and decision making*, Master's Thesis, Zagazig University, Zagazig, Egypt, 2007.
11. E. A. Marei, Generalized soft rough approach with a medical decision making problem, *Eur. J. Sci. Res.*, **133** (2015), 49–65.
12. R. Abu-Gdairi, M. A. El-Gayar, T. M. Al-shami, A. S. Nawar, M. K. El-Bably, Some topological approaches for generalized rough sets and their decision-making applications, *Symmetry*, **14** (2022). https://doi.org/10.3390/sym14010095
13. M. A. El-Gayar, A. A. El Atik, Topological models of rough sets and decision making of COVID-19, *Complexity*, **2022** (2022), 2989236. https://doi.org/10.1155/2021/2989236
14. H. Lu, A. M. Khalil, W. Alharbi, M. A. El-Gayar, A new type of generalized picture fuzzy soft set and its application in decision making , *J. Intell. Fuzzy Syst.*, **40** (2021), 12459–12475. https://doi.org/10.3233/JIFS-201706
15. E. A. Abo-Tabl, M. K. El-Bably, Rough topological structure based on reflexivity with some applications, *AIMS Math.*, **7** (2022), 9911–9925. https://doi.org/10.3934/math.2022553
16. M. K. El-Bably, M. I. Ali, E. A. Abo-Tabl, New topological approaches to generalized soft rough approximations with medical applications, *J. Math.*, **2021** (2021), 2559495. https://doi.org/10.1155/2021/2559495
17. M. K. El-Bably, M. El-Sayed, Three methods to generalize Pawlak approximations via simply open concepts with economic applications, *Soft Comput.*, **26** (2022), 4685–4700. https://doi.org/10.1007/s00500-022-06816-3

18. M. El Sayed, M. A. El Safty, M. K. El-Bably, Topological approach for decision-making of COVID-19 infection via a nano-topology model, *AIMS Math.*, **6** (2021), 7872–7894. https://doi.org/10.3934/math.2021457

19. M. M. El-Sharkasy, Topological model for recombination of DNA and RNA, *Int. J. Biomath.*, **11** (2018). https://doi.org/10.1142/S1793524518500973

20. A. S. Nawar, M. A. El-Gayar, M. K. El-Bably, R. A. Hosny, $\theta\beta$-ideal approximation spaces and their applications, *AIMS Math.*, **7** (2022), 2479–2497. https://doi.org/10.3934/math.2022139

21. R. Abu-Gdairi, M. A. El-Gayar, M. K. El-Bably, K. K. Fleifel, Two views for generalized rough sets with applications, *Mathematics*, **18** (2021), 2275. https://doi.org/10.3390/math9182275

22. Z. Li, T. Xie, Q. Li, Topological structure of generalized rough sets, *Comput. Math. Appl.*, **63** (2021), 1066–1071. https://doi.org/10.1016/j.camwa.2011.12.011

23. M. E. Abd El-Monsef, M. A. EL-Gayar, R. M. Aqeel, On relationships between revised rough fuzzy approximation operators and fuzzy topological spaces, *Int. J. Granul. Comput. Rough Set. Intel. Syst.*, **3** (2014), 257–271. https://doi.org/10.1504/IJGCRSIS.2014.068022

24. M. K. El-Bably, T. M. Al-shami, Different kinds of generalized rough sets based on neighborhoods with a medical application, *Int. J. Biomath.*, **14** (2021), 2150086. https://doi.org/10.1142/S1793524521500868

25. M. E. A. El-Monsef, M. A. El-Gayar, R. M. Aqeel, A comparison of three types of rough fuzzy sets based on two universal sets, *Int. J. Mach. Learn. Cyb.*, **8** (2017), 343–353. https://doi.org/10.1007/s13042-015-0327-8

26. M. I. Ali, A note on soft sets, rough soft sets and fuzzy soft sets, *Appl. Soft Comput.*, **11** (2011), 3329–3332. https://doi.org/10.1016/j.asoc.2011.01.003

27. E. A. Abo-Tabl, A comparison of two kinds of definitions of rough approximations based on a similarity relation, *Inform. Sci.*, **181** (2011), 2587–2596. https://doi.org/10.1016/j.ins.2011.01.007

28. A. A. Allam, M. Y. Bakeir, E. A. Abo-Tabl, *New approach for basic rough set concepts*, In: Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing, Part of the Lecture Notes in Artificial Intelligence, Springer, Berlin, Heidelberg, 2005, 64–73.

29. Y. Yao, *Generalized rough set models*, In: Rough Sets in knowledge Discovery, Polkowski, Physica Verlag, Heidelberg 1998, 286–318.

30. P. Zhang, T. Li, C. Luo, G. Wang, AMG-DTRS: Adaptive multi-granulation decision-theoretic rough sets, *Int. J. Approx. Reason.*, **140** (2022), 7–30. https://doi.org/10.1016/j.ijar.2021.09.017

31. M. Suo, L. Tao, B. Zhu, X. Miao, Z. Liang, Y. Ding, et al., Single-parameter decision-theoretic rough set, *Inform. Sci.*, **539** (2020), 49–80. https://doi.org/10.1016/j.ins.2020.05.124

32. H. Dou, X. Yang, X. Song, H. Yu, W. Z. Wu, J. Yang, Decision-theoretic rough set: A multicost strategy, *Knowl.-Based Syst.*, **91** (2016), 71–83. https://doi.org/10.1016/j.knosys.2015.09.011

33. Y. Yao, Three-way decisions with probabilistic rough sets, *Inform. Sci.*, **180** (2010), 341–353. https://doi.org/10.1016/j.ins.2009.09.021

34. Y. Yao, J. Yang, Granular rough sets and granular shadowed sets: Three-way approximations in Pawlak approximation spaces, *Int. J. Approx. Reason.*, **142** (2022), 231−247. https://doi.org/10.1016/j.ijar.2021.11.012
35. Y. Yao, Three-way granular computing, rough sets, and formal concept analysis, *Int. J. Approx. Reason.*, **116** (2022), 106−125. https://doi.org/10.3917/empa.125.0116
36. J. Yang, Y. Yao, Semantics of soft sets and three-way decision with soft sets , *Knowl.-Based Syst.*, **194** (2020), 105538. https://doi.org/10.1016/j.knosys.2020.105538
37. Z. Liu, J. C. R. Alcantud, K. Qin, L. Xiong, The soft sets and fuzzy sets-based neural networks and application, *IEEE Access*, **8** (2020), 41615−41625. https://doi.org/10.1109/ACCESS.2020.2976731
38. M. El Sayed, A. A. Q. Al Qubati, M. K. El-Bably, Soft pre-rough sets and its applications in decision making, *Math. Biosci. Eng.*, **17** (2021), 6045−6063. https://doi.org/10.3934/mbe.2020321
39. A. R. Roy, P. K. Maji, Fuzzy soft set theoretic approach to decision making problems, *J. Comput. Appl. Math.*, **203** (2007), 412−1418. https://doi.org/10.1016/j.cam.2006.04.008
40. P. K. Maji, A. R. Roy, R. Biswas, An application of soft sets in a decision making problem, *Comput. Math. Appl.*, **44** (2002), 1077−1083. https://doi.org/10.1016/S0898-1221(02)00216-X
41. O. Dalkılıç, N. Demirta, Algorithms for Covid-19 outbreak using soft set theory: Estimation and application, *Soft Comput.*, 2022. https://doi.org/10.1007/s00500-022-07519-5
42. O. Dalkılıç, N. Demirta, Decision analysis review on the concept of class for bipolar soft set theory, *Comput. Appl. Math.*, **41** (2022), 205. https://doi.org/10.1007/s40314-022-01922-2
43. M. Shabir, M. I. Ali, T. Shaheen, Another approach to soft rough sets, *Knowl.-Based Syst.*, **40** (2013), 72−80. https://doi.org/10.1016/j.knosys.2012.11.012
44. J. C. R. Alcantud, The semantics of N-soft sets, their applications, and a coda about three-way decision, *Inform. Sci.*, **606** (2022), 837−852. https://doi.org/10.1016/j.ins.2022.05.084
45. J. C. R. Alcantud, G. Santos-García, M. Akram, OWA aggregation operators and multi-agent decisions with N-soft sets, *Expert Syst. Appl.*, **203** (2022), 117430. https://doi.org/10.1016/j.eswa.2022.117430
46. M. Akram, U. Amjad, J. C. R. Alcantud, G. Santos-García, Complex fermatean fuzzy N-soft sets: A new hybrid model with applications, *J. Amb. Intel. Hum. Comput.*, 2022. https://doi.org/10.1007/s12652-021-03629-4
47. J. C. R. Alcantud, J. Zhan, Convex rough sets on finite domains, *Inform. Sci.*, **611** (2022), 81−94. https://doi.org/10.1016/j.ins.2022.08.013
48. J. C. R. Alcantud, The relationship between fuzzy soft and soft topologies, *Int. J. Fuzzy Syst.*, **24** (2022), 1653–1668. https://doi.org/10.1007/s40815-021-01225-4
49. J. C. R. Alcantud, J. Zhan, Multi-granular soft rough covering sets, *Soft Comput.*, **24** (2020), 9391–9402. https://doi.org/10.1007/s00500-020-04987-5
50. J. C. R. Alcantud, F. Feng, R. R. Yager, An N-soft set approach to rough sets, *IEEE T. Fuzzy Syst.*, **28** (2020), 2996−3007. https://doi.org/10.1109/TFUZZ.2019.2946526
51. J. C. R. Alcantud, Some formal relationships among soft sets, fuzzy sets, and their extensions, *Int. J. Approx. Reason.*, **68** (2016), 45−53. https://doi.org/10.1016/j.ijar.2015.10.004

52. T. K. Das, D. P. Acharjya, A decision making model using soft set and rough set on fuzzy approximation spaces, *Int. J. Intell. Syst. Technol. Appl.*, **13** (2014), 170–186. https://doi.org/10.1504/IJISTA.2014.065172
53. D. P. Acharjya, P. K. Ahmed, A hybridized rough set and bat-inspired algorithm for knowledge inferencing in the diagnosis of chronic liver disease, *Multimed. Tools Appl.*, **81** (2022), 13489–13512. https://doi.org/10.1007/s11042-021-11495-7
54. D. P. Acharjya, A. Abraham, Rough computing—A review of abstraction, hybridization and extent of applications, *Eng. Appl. Artif. Intel.*, **96** (2020), 103924. https://doi.org/10.1016/j.engappai.2020.103924
55. D. P. Acharjya, Knowledge inferencing using artificial bee colony and rough set for diagnosis of hepatitis disease, *Int. J. Healthc. Inf. Sy.*, **16** (2021), 49–72. https://doi.org/10.1353/aph.2021.0086
56. D. P. Acharjya, A hybrid scheme for heart disease diagnosis using rough set and cuckoo search technique, *J. Med. Syst.*, **44** (2020). https://doi.org/10.1007/s10916-019-1497-9
57. X. Yang, D. Yu, J. Yang, C. Wu, *Generalization of soft set theory: From crisp to fuzzy case*, Fuzzy Information and Engineering, Springer, Berlin, Heidelberg, **40** (2007), 345–354. https://doi.org/10.1007/BF03215615
58. J.-B. Liu, S. Ali, M. K. Mahmood, M. H. Mateen, On m-polar diophantine fuzzy N-soft set with applications, *Comb. Chem. High T. Scr.*, **25** (2022), 536–546. https://doi.org/10.2174/1386207323666201230092354
59. M. S. Hameed, S. Mukhtar, H. N. Khan, S. Ali, M. H. Mateen, M. Gulzar, Pythagorean fuzzy N-soft groups, *Indones. J. Electr. Eng. Comput. Sci.*, **21** (2021), 1030–1038. http://dx.doi.org/10.11591/ijeecs.v21i2.pp1030-1038
60. M. Gulzar, M. H. Mateen, Y. M. Chu, D. Alghazzawi, G. Abbas, Generalized direct product of complex intuitionistic fuzzy subrings, *Int. J. Comput. Intell. Syst.*, **14** (2021), 582–593. https://doi.org/10.2991/ijcis.d.210106.001
61. M. Gulzar, M. H. Mateen, D. Alghazzawi, N. Kausar, A novel applications of complex intuitionistic fuzzy sets in group theory, *IEEE Access*, **8** (2021), 196075–196085. https://doi.org/10.1109/ACCESS.2020.3034626
62. F. Tchier, G. Ali, M. Gulzar, D. Pamučar, G. Ghorai, A new group decision-making technique under picture fuzzy soft expert information, *Entropy*, **23** (2021), 1176. https://doi.org/10.3390/e23091176
63. *Chikungunya, World Health Organization*, Geneva, Switzerland. Available from: `https://www.who.int/news-room/fact-sheets/detail/chikungunya.`
64. M. L. Thivagar, C. Richard, N. R. Paul, Mathematical innovations of a modern topology in medical events, *Int. J. Inform. Sci.*, **2** (2012), 33–36. https://doi.org/10.25291/VR/36-VR-33
65. M. L. Thivagar, C. Richard, On nano forms of weakly open sets, *Int. J. Math. Stat. Invent.*, **1** (2013), 31–37.