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Brief report

Finite groups all of whose proper subgroups have few character values

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Abstract: In this paper, the structures of non-solvable groups whose all proper subgroups have at most seven character values are identified.

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1. Introduction

We always think that groups under consideration are all finite. Let G be a group and $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of a group G. Let g be an element of a group G. Then denote by $\mathrm{cv}(G) = \{\chi(g) : \chi \in \mathrm{Irr}(G), g \in G\}$, the set of character values of G, so $\mathrm{cd}(G) \subseteq \mathrm{cv}(G)$ where $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$ is the set of character degrees of a group G. We will use these symbols in this paper:

E_{p^n}: the elementary abelian p-group of order p^n;
C_n: the cyclic group of order n;
Q_8: the quaternion group of order 8;
D_{2n}: the dihedral group of order $2n$.

Character values of finite groups have large influence on their structures; see [1,2,23,26]. Recently, Madanha [17] and Sakural [24] studied the influence of character values in a character table on the structure of a group respectively. In particular, Madanha in [17] showed that if a non-solvable group G with $|\mathrm{cv}(G)| = 8$, then $G \cong \mathrm{PSL}_2(5)$ or $\mathrm{PGL}_2(5)$.

Some scholars are of interest in the set $\mathrm{cd}(G)$. For example, the structure of finite groups are determined if the degrees of finite groups are either prime powers [13,19,20] or divisible by two prime divisors [21] or the direct product of at most two primes [9, 16] or square-free [12] or p'-numbers [6] or are consecutive [8, 14, 15, 22].

In this paper, we go on the subgroup's character values and group structure, namely, we replace the condition "the number of character values of a group is small" with the condition "the number of character values of each proper subgroup of a group is small". For convenient arguments, we introduce the following definition.

Definition 1.1. *Let $\sum G$ be the set of the proper subgroups of a group G, and n a positive integer. A group is called a* $\mathbf{pcv}_n$*-group if for each $H \in \sum G$, $|\mathrm{cv}(H)| \leq n$.*

We mainly show the following.

Theorem 1.2. *If G is a* $\mathbf{pcv}_n$*-group with $n \leq 5$, then G is solvable.*

Theorem 1.2 is corresponding to [10, Theorem 12.15] or [17, Theorem 1.1] which says that a finite group with $|\mathrm{cv}(G)| \leq 7$ or $|\mathrm{cd}(G)| \leq 3$ is solvable. Here we use the structure of a minimal simple group to prove Theorem 1.2.

We also obtain the following result which is corresponding to [17, Theorem 1.2] or [18, Theorem 2.2].

Theorem 1.3. *Let G be a non-solvable* $\mathbf{pcv}_n$*-group.*
(1) If $n = 6$, then G is isomorphic to A_5.
(2) If $n = 7$, then G is isomorphic to $\mathrm{PSL}_2(q)$ with $q \in \{5, 7\}$.

The structure of this short paper is as follows. In Section 2, some basic results are given, and in Section 3, the structures of non-solvable $\mathbf{pcv}_6$- and $\mathbf{pcv}_7$-groups are identified respectively. For the other notions and symbols are standard, please see [5, 10].

2. Minimal simple groups

In this section, we assemble some results needed. First result is due to Madanha.

Lemma 2.1. *[17, Theorem 1.1] If $|\mathrm{cv}(G)| \leq 7$, then G is solvable.*

A ***minimal simple group*** is a simple group of composite order all of whose proper subgroups are solvable.

Lemma 2.2. *[25, Corollary 1] Every minimal simple group is isomorphic to one of the following minimal simple groups:*

(1) $\mathrm{PSL}_2(2^p)$ *for p a prime;*
(2) $\mathrm{PSL}_2(3^p)$ *for p an odd prime;*
(3) $\mathrm{PSL}_2(p)$*, for p any prime exceeding 3 such that $p^2 + 1 \equiv 0 \pmod 5$;*
(4) $Sz(2^p)$ *for p an odd prime;*
(5) $\mathrm{PSL}_3(3)$.

Let A be a group, and let $\exp A$ be a number which is minimal such that the order of all elements from A divides $\exp A$. The following two lemmas are given because the subgroups of a group can control the structure of a group.

Lemma 2.3. *Let G be a dihedral group D_{2n} of order $2n$. Then $|\mathrm{cv}(G)| \leq n+1$.*

Proof. Obviously $\exp A \leq n$ with equality when A is cyclic and so by [17, Lemma 2.2], $|\mathrm{cv}(A)| \leq n$, and so $|\mathrm{cv}(G)| \leq n+1$. □

Remark 2.4. *In Lemma 2.3 we cannot replace $\leq$ with $=$. For instance, Let $n = 16$, then by [4],* $\mathrm{cv}(D_{32}) = \{1, 2, -1, -2, 0, A, -A, B, -B, C, -C\}$. *Now* $|\mathrm{cv}(D_{32})| = 11 < 16$.

Remark 2.5. *The result of Lemma 2.3 is mostly possible. For example, let $n = 5$. Then by [4],* $\mathrm{cv}(D_{10}) = \{1, 0, -1, 2, A, A^*\}$. *Now* $|\mathrm{cv}(D_{10})| = 6 = 5 + 1$.

Lemma 2.6. *Let G be a Frobenius group with the form $E_{p^n} : C_{p^n-1}$ where $n \geq 1$ is a positive integer. Then $|\mathrm{cv}(G)| = p+1$. In particular, if $E_{p^n} : C_k$ is a Frobenius subgroup of $E_{p^n} : C_{p^n-1}$, then $|\mathrm{cv}(E_{p^n} : C_k)| \leq p+1$.*

Proof. We see $\exp E_{p^n} = p$, so Lemma 2.2 of [17] forces $\mathrm{cv}(E_{p^n}) = p$. We know that C_{p^n-1}, acts fixed-point-freely on E_{p^n} so by Theorem 18.7 of [8], $|\mathrm{cv}(G)| = p+1$. □

Lemma 2.7. *(1) G is a $\mathbf{pcv}_n$-group G if and only if for $H \in \sum G$, $|\mathrm{cv}(H)| \leq n$.*

(2) Let N be a normal subgroup of a $\mathbf{pcv}_n$-group G, then, both N and G/N are $\mathbf{pcv}_n$-groups.

Proof. We conclude the two results from the definition of a $\mathbf{pcv}_n$-group. □

3. Simple $\mathbf{pcv}_n$-groups and Solvable $\mathbf{pcv}_5$-groups

In this section we will first determine the structure of simple $\mathbf{pcv}_n$-groups for $n = 6, 7$ by using Lemma 2.2 and then show the solvability of $\mathbf{pcv}_n$-groups when $n \leq 5$. For easy reading, we rewrite Theorem 1.3 here.

Theorem 3.1. *Let G be a non-abelian simple $\mathbf{pcv}_n$-group with $n \leq 7$. Then*

(1) if $n = 6$, G is isomorphic to A_5;

(2) if $n = 7$, G is isomorphic to A_5 or $\mathrm{PSL}_2(7)$.

Proof. We know that for each $H \in \sum G$, $|\mathrm{cv}(H)| \leq 7$, H is solvable by Lemma 2.1. It follows that G is a group whose proper subgroups are all solvable, so we can assume that G is a minimal simple group. Thus G is isomorphic to $\mathrm{PSL}_2(2^p)$ for p a prime; $\mathrm{PSL}_2(3^p)$ for p an odd prime; $\mathrm{PSL}_2(p)$, for p any prime exceeding 3 such that $p^2 + 1 \equiv 0 \pmod 5$; $Sz(2^p)$ for p an odd prime; $\mathrm{PSL}_3(3)$; see Lemma 2.2. So in what follows, these cases are considered.

Case 1: $\mathrm{PSL}_2(q)$ for certain q.

By Table 1, $D_{2(q+1)/k} \in \max \mathrm{PSL}_2(q)$ and so by Lemma 2.3, $\frac{q+1}{2} \leq 7$ when q is odd or $q+1 \leq 7$ when q is even. It follows that q is equal to 4, 5, 7, 9, 11 or 13. Note that $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong A_5$, so two subcases are dealt with.

Subcase 1: $q \in \{5, 7\}$.

Let $q = 5$. Then by [5, p. 2], $\max A_5 = \{A_4, D_{10}, S_3\}$ and by [4], $\mathrm{cv}(A_4) = \{-1, 0, 1, A, A^*\}$, $\mathrm{cv}(D_{10}) = \{-1, 0, 1, 2, A, A^*\}$ and $\mathrm{cv}(S_3) = \{-1, 0, 1, 2\}$. It follows that for each $H \in \max A_5$, $|\mathrm{cv}(H)| \leq 6$, so G is isomorphic to A_5.

If $q = 7$, then $\max \mathrm{PSL}_2(7) = \{S_4, 7 : 3\}$ and by [4], $\mathrm{cv}(7 : 3) = \{1, 3, A, A^*, B, B^*, 0\}$, $\mathrm{cv}(S_4) = \{1, 2, 3, -1, 0\}$, so $|\mathrm{cv}(7 : 3)| = 7$ and $|\mathrm{cv}(S_4)| = 5$. Assumption shows that G is isomorphic to $\mathrm{PSL}_2(7)$ as desired.

Subcase 2: $q \in \{9, 11, 13\}$.

By Table 1, $A_5 \in \max \mathrm{PSL}_2(q)$ when $q = 9$ or 11 and $|\mathrm{cv}(A_5)| = 8$ by [17, Theorem 1.2] or [5, p. 2].

If $q = 13$, then $13 : 6 \in \max \mathrm{PSL}_2(13)$ and $\mathrm{cv}(13 : 6) = \{1, 6, A, A^*, B, -B, B^*, -B^*, -1, 0\}$ by [4], so $|\mathrm{cv}(13 : 6)| = 10$, a contradiction.

It follows that $\mathrm{PSL}_2(q)$ for $q \in \{9, 11, 13\}$ is not a $\mathbf{pcv}_n$-group with $n \leq 7$.

Table 1. $\mathrm{PSL}_2(q)$, $q \geq 5$ [11, p. 191].

	$\max(\mathrm{PSL}_2(q))$	Condition
C_1	$E_q : C_{(q-1)/k}$	$k = \gcd(q - 1, 2)$
C_2	$D_{2(q-1)/k}$	$q \notin \{5, 7, 9, 11\}$
C_3	$D_{2(q+1)/k}$	$q \notin \{7, 9\}$
C_5	$\mathrm{PSL}_2(q_0).(k, b)$	$q = q_0^b$, b a prime, $q_0 \neq 2$
C_6	S_4	$q = p \equiv \pm 1 \pmod 8$
	A_4	$q = p \equiv 3, 5, 13, 27, 37 \pmod{40}$
$\mathcal{S}$	A_5	$q \equiv \pm 1 \pmod{10}$, $F_q = F_p[\sqrt{5}]$

Case 2: $Sz(2^p)$ for p an odd prime.

In this case, by Table 2, $D_{2(q-1)} \in \max Sz(q)$, so by Lemma 2.3, $q - 1 \leq 7$, so $q = 8$. Now $2^{3+3} : 7 \in \max Sz(8)$ and by [4], $\mathrm{cv}(2^{3+3} : 7) = \{1, 7, 14, -2, -1, A, -A, B, B^*, C, C^*, D, D^*, 0\}$. Thus $|\mathrm{cv}(2^{3+3} : 7)| = 14$, a contradiction.

Table 2. $Sz(q)$, $q = 2^{2m+1} \geq 8$ [3, p. 385].

$\max({}^2B_2(q))$	Condition
$E_q^{1+1} : C_{q-1}$	
$D_{2(q-1)}$	
$C_{q+\sqrt{2q}+1}.C_4$	
$C_{q-\sqrt{2q}+1}.C_4$	
${}^2B_2(q_0)$	$q = q_0^a$, a prime, $q_0 \geq 8$

Case 3: $\mathrm{PSL}_3(3)$.

By [5, pp. 13], $13 : 3 \in \mathrm{PSL}_3(3)$ and so by [4], $\mathrm{cv}(13 : 3) = \{0, 1, 3, A, A^*, B, B^*, C, C^*\}$. Now $|\mathrm{cv}(13 : 3)| = 9 \nleq 7$, a contradiction. □

Now we can prove Theorem 1.2. For reader's convenience, we rewrite it here.

Theorem 3.2. *If G is a* $\mathbf{pcv}_n$*-group with $n \leq 5$, then G is solvable.*

Proof. By hypothesis, we know that every proper subgroup of G is solvable. If G is non-solvable, then we can assume that G is simple. Now as the proof of Theorem 3.1 we obtain a contradiction. Thus G is solvable. □

4. Non-solvable $\mathbf{pcv}_n$-groups

In this section, we first show the structure of non-solvable $\mathbf{pcv}_6$-groups and then the structures of non-solvable $\mathbf{pcv}_7$-groups are determined. For convenient reading, we rewrite Theorem 1.3 here.

Theorem 4.1. *Let G be a non-solvable* $\mathbf{pcv}_n$*-group.*
(1) If $n = 6$*, then G is isomorphic to* A_5
(2) If $n = 7$*, then G is isomorphic to* $\mathrm{PSL}_2(q)$ *with* $q \in \{5, 7\}$.

Proof. The non-solvability of G shows that G has a normal sequel $1 \leq H \leq K \leq G$ such that K/H is isomorphic to a direct product of isomorphic simple groups and that $|G/K|$ divides $|\mathrm{Out}(K/H)|$, where $\mathrm{Out}(A)$ denotes the outer-automorphism group of a group A; see [27].

Now we have that

$$K/H \text{ is isomorphic to } \underbrace{S \times S \times \cdots \times S}_{m \text{ times}}$$

where $S \cong A_5$ when $n = 6$ and $S \cong A_5$ or $\mathrm{PSL}_2(7)$ when $n = 7$; see Theorem 3.1. By Lemma 2.7, K/H is a $\mathbf{pcv}_n$-group, so let $H \in \max S$, now

$$H \times \underbrace{S \times \cdots \times S}_{m-1 \text{ times}} \text{ is a maximal subgroup of } \underbrace{S \times S \times \cdots \times S}_{m \text{ times}}.$$

If $m \geq 2$, then $|\mathrm{cv}(A_5)| = 8$ and $|\mathrm{cv}(\mathrm{PSL}_2(7))| = 10$ as $\mathrm{cv}(A_5) = \{1, 3, 4, 5, -1, 0, A, A^*\}$ and $\mathrm{cv}(\mathrm{PSL}_2(7)) = \{1, 3, 6, 7, 8, -1, 2, 0, A, A^*\}$ by [5, p. 2-3], so $|\mathrm{cv}(H \times S \times \cdots \times S)| \geq 8$, a contradiction. Thus $m = 1$ and for any $S < H \leq \mathrm{Aut}(S)$, $|\mathrm{cv}(S)| \geq 8$ shows that H is not a $\mathbf{pcv}_n$-group with $n = 6, 7$; see [17, Theorem 1.2]. Now $G/H \cong A_5$ or $\mathrm{PSL}_2(7)$ and G is not an almost simple group. It follows that

$$G'/H \cong \mathrm{PSL}_2(q) \text{ or } \mathrm{SL}_2(q) \text{ with } q \in \{5, 7\};$$

see [7, Chap 2, Theorem 6.10].

So in the following two cases are done with.

Where N1 maximal under $< \delta >$ with $|\delta| = (q - 1, 2)$; N2 maximal under subgroups not contained in $< \varphi >$ with $|\varphi| = e$, $q = p^e$, p a prime.

Case 1: $G'/H \cong A_5$ or $\mathrm{SL}_2(5)$.

By Table 3, $2.A_4 \in \max \mathrm{SL}_2(5)$ and by [4], $\mathrm{cv}(2.A_4) = \{1, 3, 2, -2, 0, A, A^*, -A^*, -A\}$. It follows that $|\mathrm{cv}(2.A_4)| = 9 > 7$, so $\mathrm{SL}_2(5)$ is not a $\mathbf{pcv}_n$-group with $n = 6, 7$. Thus $G'/H \cong \mathrm{SL}_2(5)$ is impossible. Now $G'/H \cong A_5$ and $G' \cap H = 1$, so $[G', H] \leq G' \cap H = 1$. Thus,

$$G' \cong \tfrac{G'}{G' \cap H} \cong \tfrac{G'}{H} \cong A_5.$$

It follows that $G \cong H \times A_5$. If $H \neq 1$, then $A_5 \in \sum G$. Observe that $|\mathrm{cv}(A_5)| = 8$, so in this case G is a non-$\mathbf{pcv}_n$-group with $n = 6, 7$. Therefore G is isomorphic to A_5, the desired result.

Table 3. $\mathrm{SL}_2(q)$, $q \geq 4$([3, p. 377]).

$\max \mathrm{SL}_2(q)$	Condition
$E_q : C_{q-1}$	
$Q_{2(q-1)}$	$q \neq 5, 7, 9, 11$; q odd
	N1 if $q = 7, 11$; N2 if $q = 9$
$D_{2(q-1)}$	q even
$Q_{2(q+1)}$	$q \neq 7, 9$; q odd
	N1 if $q = 7$; N2 if $q = 9$
$D_{2(q+1)}$	q even
$\mathrm{SL}_2(q_0).2$	$q = q_0^2$, q odd
$\mathrm{SL}_2(q_0)$	$q = q_0^r$, q odd, r odd prime
$\mathrm{PSL}_2(q_0)$	$q = q_0^r$, q even, $q_0 \neq 2$, r prime
$2.S_4$	$q = p \equiv \pm 1 \pmod 8$
$2.A_4$	$q = p \equiv \pm 3, 5, \pm 13 \pmod{40}$
	N1 if $q = p \equiv \pm 11, \pm 19 \pmod{40}$
$2.A_5$	$q = p \equiv \pm 1 \pmod{10}$
	$q = p^2$, $p \equiv \pm 3 \pmod{10}$

Case 2: $G'/H \cong \mathrm{PSL}_2(7)$ or $\mathrm{SL}_2(7)$.

In this case $n = 7$. Table 3 gives that $E_7 : C_6 \in \max \mathrm{SL}_2(7)$ and by [4], $\mathrm{cv}(E_7 : C_6) = \{1, 6, -1, 0, A, -A, A^*, -A^*\}$. It follows that $|\mathrm{cv}(E_7 : C_6)| = 8$ (this result can be gotten from Lemma 2.6), so $\mathrm{SL}_2(7)$ is not a $\mathbf{pcv}_7$-group. Thus $G'/H \cong \mathrm{SL}_2(7)$ is not possible. Now consider when $G'/H \cong \mathrm{PSL}_2(7)$. Then $G' \cap H = 1$, $[G', H] = 1$ and $G \cong H \times \mathrm{PSL}_2(7)$ too. If $H \neq 1$, then $\mathrm{PSL}_2(7)$ is not a $\mathbf{pcv}_7$-group, so $H = 1$ and G is isomorphic to $\mathrm{PSL}_2(7)$, the wanted result. □

Proposition 4.2. *Let G be a $\mathbf{pcv}_n$-group with $n \leq 7$. Assume that G has no section isomorphic to* $\mathrm{PSL}_2(q)$ *for $q \in \{5, 7\}$, then G is solvable.*

Proof. By Theorems 1.2 and 1.3, we can get the desired result. □

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Conflict of interest

The authors declare that they have no conflicts of interest.

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