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Research article

Precise asymptotics for complete integral convergence in the law of the logarithm under the sub-linear expectations

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Abstract: The aim of this paper is to study and establish precise asymptotics for complete integral convergence in the law of the logarithm under the sub-linear expectation space. The methods and tools in this paper are different from those used to study precise asymptotics theorems in probability space. We extend precise asymptotics for complete integral convergence from the classical probability space to sub-linear expectation space. Our results generalize corresponding results obtained by Fu and Yang [13]. We further extend the limit theorems in classical probability space.

Keywords: precise asymptotics; complete integral convergence; the law of the logarithm; sub-linear expectations
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1. Introduction

It is well known that complete convergence plays an important role in probability limit theory. Since the concept of complete convergence was introduced by Hsu and Robbins [1], there have been several directions of extension. One important topic of them is to discuss the precise rate, which is much more accurate than complete convergence. Heyde [2] proved for the first time the precise rate of sequences of independentand identically distributed random variables, and he got the following result:

$$\lim_{\varepsilon\to 0}\varepsilon^2\sum_{n=1}^{\infty}P(|S_n|\geq\varepsilon n)=EX^2,$$

under the conditions $EX = 0$ and $EX^2 < \infty$. Results of this kind are frequently called precise asymptotics. For more results on the precise asymptotics, see Chen [3], Spătaru [4], Gut and Spătaru [5, 6], Gut and Steinebach [7], He and Xie [8], etc. Liu and Lin [9] achieved the precise asymptotics for complete moment convergence. From then until now, the study on the precise asymptotics, no matter for complete convergence or complete moment convergence, is still a hot issue. For example, Zhao [10] established precise rates in complete moment convergence for ρ-mixing sequences; Zhang,Yang and Dong [11] discussed a general law of precise asymptotics for the complete moment convergence; Lin and Zhou [12] investigated precise asymptotics of complete moment convergence on moving average; Fu and Yang [13] offered moment convergence rates in the law of the logarithm for dependent sequences.

The results above are based on the additivity of the probability measures and mathematical expectations, which are built on the distribution-certainty or model-certainty. However, with the continuous progress of social economy, many uncertain phenomena gradually appear in financial insurance, statistical forecasting and other industrial problems, and these uncertain problems cannot be simulated by additive probability and expectation, such as risk measurement and super hedging in the field of mathematical finance. For the relevant references, we can refer to El Karoui et al. [36], Peng [37], Chen and Epstein [38], and so forth. Therefore, inspired by the desire to simulate uncertain models, academician Peng [16, 17] is the first one to introduce a notion of sublinear expectation. In the framework of sublinear expectation, we have recently seen a lot of limit theorems, including the classical (weighted) central limit theory (see Peng [17], Fang et al. [18], Zhang and Chen [19], Li [20], Guo and Zhang [21], Blessing and Kupper [33]), strong law of large numbers (SLLN) (see Chen [22], Wu and Jiang [23], Yang and Xiao [24], Zhan and Wu [25], Ma and Wu [26]), weak LLN (see Chen et al. [27], Hu [28]), Marcinkiewicz-Zygmund LLN (see Hu [29]), and so forth. In addition, there are some extensions of the precise asymptotics theorems under sub-linear expectations, For example: Wu [14] obtained precise asymptotics for complete integral convergence under sub-linear expectations; Ding [15] proved a general form for precise asymptotics for complete convergence under sub-linear expectations; Wu and Wang [32] investigated general results on precise asymptotics under sub-linear expectations. Further, since Peng introduced the nonlinear expectation, the theory and application of the nonlinear expectation have been well developed in financial risk measurement and control. For instance: Peng [17] established multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation; Marinacci [39] obtained limit laws for non-additive probabilities and their frequentist interpretation; Xi et al. [40] offered complete convergence for arrays of rowwise END random variables and its statistical applications under sub-linear expectations. For more relevant results, see Denis and Martini [41], Chen and Epstein [42], and so forth.

Motivated by the topic of volatility uncertainty, the theory of the nonlinear expectation and its applications, we concentrate on precise asymptotics theorems under the sub-linear expectations. However, many basic properties or tools for classical probability theory are no longer available under sublinear expectations, the study on limit theorems under sublinear expectations is much more complex and difficult. The methods and tools in this paper are different from those used to study precise asymptotics theorems in probability space. We have obtained the precise asymptotics for complete integral convergence in the law of the logarithm under the sub-linear expectation space. As a result, the corresponding results obtained by Fu and Yang [13] have been generalized to the sublinear expectation space context.

The paper is organized as follows: In Section 2, we summarize some of the basic concepts, definitions, and related properties under the sub-linear expectations . Not only that, we have also enumerated some important lemmas that are useful to prove the main results. In Section 3, we establish the main results of this paper. The proofs of main results are presented in Sections 4. The conclusion part is listed in Section 5.

2. Preliminaries

We use the framework and notions of Peng [16]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq c(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \ \forall \ \mathbf{x}, \mathbf{y} \in \mathbb{R}_n,$$

for some $c > 0$, $m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variables. In this case we denote $X \in \mathcal{H}$.

In addition, Denk et al. [34] established that nonlinear expectations can always extend nonlinear expectations from a certain subset $\mathcal{H}$ of bounded measurable functions to the space of all bounded suitably measurable functions. Following this work, we could directly work on the space of all bounded measurable functions.

Definition 2.1. (Peng [16]) A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ satisfying the following properties: For all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$, then $\hat{\mathbb{E}}(X) \geq \hat{\mathbb{E}}(Y)$;

(b) Constant preserving: $\hat{\mathbb{E}}(c) = c$;

(c) Sub-additivity: $\hat{\mathbb{E}}(X + Y) \leq \hat{\mathbb{E}}(X) + \hat{\mathbb{E}}(Y)$; whenever $\hat{\mathbb{E}}(X) + \hat{\mathbb{E}}(Y)$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}(X), \ \lambda \geq 0$.

Here $\bar{\mathbb{R}} := [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space.

Give a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ by

$$\hat{\varepsilon}(X) := -\hat{\mathbb{E}}(-X), \ \forall X \in \mathcal{H}.$$

From the definition, it is easily shown that for all $X, Y \in \mathcal{H}$

$$\hat{\varepsilon}(X) \leq \hat{\mathbb{E}}(X), \ \hat{\mathbb{E}}(X + c) = \hat{\mathbb{E}}(X) + c, \ \hat{\mathbb{E}}(X - Y) \geq \hat{\mathbb{E}}(X) - \hat{\mathbb{E}}(Y),$$

and

$$|\hat{\mathbb{E}}X - \hat{\mathbb{E}}Y| \leq \hat{\mathbb{E}}(|X - Y|). \tag{2.1}$$

If $\hat{\mathbb{E}}(Y) = \hat{\varepsilon}(Y)$, then $\hat{\mathbb{E}}(X + aY) = \hat{\mathbb{E}}(X) + a\hat{\mathbb{E}}(Y)$ for any $a \in \mathbb{R}$.

Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, \ V(\Omega) = 1 \text{ and } V(A) \leq V(B) \text{ , for } \forall A \subseteq B, \ A, B \in \mathcal{G}.$$

$I(A)$ denotes the indicator function of A, $A \in \mathcal{G}$, $I(A) \in \mathcal{H}$.

It is called to be sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$. In the sub-linear space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, we denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\hat{\mathbb{E}}(\xi); I(A) \leq \xi, \xi \in \mathcal{H}\},\ \mathcal{V}(A) := 1 - \mathbb{V}(A^c),\ \forall A \in \mathcal{F},$$

where $\mathbb{V}(A^c)$ is the complement set of A. It is obvious that $\mathbb{V}$ is sub-additive, and

$$\mathcal{V}(A) \leq \mathbb{V}(A),\ \forall A \in \mathcal{F};\ \mathbb{V}(A) = \hat{\mathbb{E}}(I(A)),\ \mathcal{V}(A) = \hat{\varepsilon}(I(A)),\ \text{if } I(A) \in \mathcal{H}.$$

Property 2.1. *For all $B \in \mathcal{F}$, if $\eta \leq I(B) \leq \xi$, $\eta, \xi \in \mathcal{H}$, then*

$$\hat{\mathbb{E}}(\eta) \leq \mathbb{V}(B) \leq \hat{\mathbb{E}}(\xi). \tag{2.2}$$

Remark 2.1. *From* (2.2)*, for all $X \in \mathcal{H}$, $y > 0$, $\gamma > 0$, it emerges that $\mathbb{V}(|X| \geq y) \leq \hat{\mathbb{E}}(|X|^\gamma)/y^\gamma$, which is the well-known Markov's inequality.*

Remark 2.2. *Mathematical expectation corresponds to the integral in $(\Omega, \mathcal{A}, P)$, where the integral depends on a probability. In $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, capacity is an alternative to probability, so what is the relationship between the capacity and integral? The following is the definition of the upper integral.*

Definition 2.2. For all $|X| \in \mathcal{H}$, define

$$C_{\mathbb{V}}(|X|) := \int_0^\infty \mathbb{V}\left(|X| > x\right) \mathrm{d}x.$$

From the above definition, we cannot help but think of the definition of mathematical expectation in probability space, $E(|X|) := \int_0^\infty P\left(|X| > x\right) \mathrm{d}x$. In $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, $\hat{\mathbb{E}}(|X|)$ and $C_{\mathbb{V}}(|X|)$ are not related in the general situations. From Zhang [30], we can learn that $\hat{\mathbb{E}}(|X|) \leq C_{\mathbb{V}}(|X|)$ if one of the following three circumstances is satisfied: (i) $\hat{\mathbb{E}}$ is countably sub-additive; (ii) $\hat{\mathbb{E}}(|X| - d)I(|X| > d) \to 0$, as $d \to \infty$; (iii) $|X|$ is bounded.

Definition 2.3. (Peng [16] and Peng [35])

(i) Identical distribution: Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two n-dimensional random vectors defined, respectively, in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed if

$$\hat{\mathbb{E}}_1(\varphi(\mathbf{X}_1)) = \hat{\mathbb{E}}_2(\varphi(\mathbf{X}_2)),\ \forall \varphi \in C_{l,Lip}(\mathbb{R}_n),$$

whenever the subexpectations are finite. A sequence $\{X_n; n \geq 1\}$ of random variables is said to be identically distributed if for each $i \geq 1$, X_i and X_1 are identically distributed.

(ii) Independence: In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $\mathbf{Y} = \{Y_1, ..., Y_n\}$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $\mathbf{X} = \{X_1, ..., X_m\}$, $X_i \in \mathcal{H}$, under $\hat{\mathbb{E}}$ if for each test function $\varphi \in C_{l,Lip}(\mathbb{R}_m \times \mathbb{R}_n)$, we have $\hat{\mathbb{E}}(\varphi(\mathbf{X}, \mathbf{Y})) = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}(\varphi(\mathrm{x}, \mathbf{Y}))\ |_{\mathrm{x}=\mathbf{X}}\right]$,whenever $\bar{\varphi}(\mathrm{x}) := \hat{\mathbb{E}}(|\varphi(\mathrm{x}, \mathbf{Y})|) < \infty$ for all x and $\hat{\mathbb{E}}(|\bar{\varphi}(\mathbf{X})|) < \infty$.

(iii) IID random variables: A sequence of random variables $\{X_n; n \geq 1\}$ is said to be independent, if X_{i+1} is independent of $(X_1, ..., X_i)$ for each $i \geq 1$. It is said to be identically distributed, if $X_i \overset{d}{=} X_1$ for each $i \geq 1$.

In the following, let $\{X_n; n \geq 1\}$ be a sequence of random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, $S_n = \sum_{i=1}^{n} X_i$. The symbol c stands for a generic positive constant which may differ from one place to another. Let a_x and b_x be positive numbers, $a_x \sim b_x$ denotes $\lim_{x\to\infty} a_x/b_x = 1$, $a_x \ll b_x$ denotes that there exists a constant $c > 0$ such that $a_x \leq c b_x$ for sufficiently large x, and $I(\cdot)$ denotes an indicator function.

To prove our results, we need the following four lemmas.

Lemma 2.1. *(Zhang [31]). Let $\{Z_{n,k}; k = 1, ..., k_n\}$ be an array of independent random variables such that $\hat{\mathbb{E}}(Z_{n,k}) \leq 0$, and $\hat{\mathbb{E}}(Z_{n,k}^2) < \infty, k = 1, ..., k_n$. Then for all $x, y > 0$*

$$\begin{aligned} &\mathbb{V}\left(\max_{m\leq k_n} \sum_{k=1}^{m} Z_{n,k} \geq x\right) \\ \leq &\mathbb{V}\left(\max_{k\leq k_n} Z_{n,k} \geq y\right) + \exp\left\{\frac{x}{y} - \frac{x}{y}(\frac{B_n}{xy} + 1)\ln(1 + \frac{xy}{B_n})\right\}, \end{aligned} \tag{2.3}$$

where $B_n = \sum_{k=1}^{k_n} \hat{\mathbb{E}}\left(Z_{n,k}^2\right)$.

Lemma 2.2. *Let $\{X_k; k \geq 1\}$ be a sequence of independent random variables with $\hat{\mathbb{E}}(X_k) = \hat{\mathbb{E}}(-X_k) = 0$ in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Then there exists a constant $c > 0$ such that for any $x > 0$,*

$$\mathbb{V}(|S_n| \geq x) \leq c\frac{\sum_{k=1}^{n} \hat{\mathbb{E}}\left(X_k^2\right)}{x^2}. \tag{2.4}$$

Proof. It follows from Theorem 3.1 in Zhang [30] that: Let $\{X_k; k \geq 1\}$ be a sequence of independent random variables with $\hat{\mathbb{E}}(X_k) \leq 0$ in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, then

$$\mathbb{V}(S_n \geq x) \leq c\frac{\sum_{k=1}^{n} \hat{\mathbb{E}}(X_k^2)}{x^2}. \tag{2.5}$$

By $\hat{\mathbb{E}}(-X_k) = 0$, then, $\{-X, -X_i\}$ also satisfies the conditions of Theorem 3.1 in Zhang [30], we replace the $\{X, X_i\}$ with the $\{-X, -X_i\}$ in the upper form:

$$\mathbb{V}(-S_n \geq x) \leq c\frac{\sum_{k=1}^{n} \hat{\mathbb{E}}\left(X_k^2\right)}{x^2}. \tag{2.6}$$

Therefore, combining with (2.5) and (2.6), we obtain

$$\mathbb{V}(|S_n| \geq x) \leq \mathbb{V}(S_n \geq x) + \mathbb{V}(-S_n \geq x) \leq c\frac{\sum_{k=1}^{n} \hat{\mathbb{E}}\left(X_k^2\right)}{x^2}.$$

Hence, we get that (2.4) in Lemma 2.2 is established.

Lemma 2.3. *(Wu [14]). Suppose that $\{X_n; n \geq 1\}$ is a sequence of independent and identically distributed random variables with $\hat{\mathbb{E}}(X) = \hat{\mathbb{E}}(-X) = 0$ and $\hat{\mathbb{E}}$ is continuous, set $\Delta_n(x) := \mathbb{V}((|S_n| / \sqrt{n}) \geq x) - \mathbb{V}(|\xi| \geq x)$, then,*

$$\Delta_n := \sup_{x\geq 0} |\Delta_n(x)| \longrightarrow 0, \ as\ n \longrightarrow \infty, \tag{2.7}$$

where $\xi \sim \mathcal{N}\left(0, \left[\underline{\sigma}^2, \overline{\sigma}^2\right]\right)$, and $\overline{\sigma}^2 = \hat{\mathbb{E}}(X^2)$, $\underline{\sigma}^2 = \hat{\varepsilon}(X^2)$.

Lemma 2.4. *Suppose that the conditions of Lemma 2.3 hold and $C_{\mathbb{V}}\left(X^2\right) < \infty$, for $\varphi \in C_b(\mathbb{R})$, $C_b(\mathbb{R})$ denotes the space of bounded continuous functions, then*

$$\lim_{n\to\infty} \hat{\mathbb{E}}\left(\varphi\left(\frac{S_n}{\sqrt{n}}\right)\right) = \hat{\mathbb{E}}\left(\varphi(\xi)\right), \tag{2.8}$$

where $\xi \sim \mathcal{N}\left(0, \left[\underline{\sigma}^2, \overline{\sigma}^2\right]\right)$ under $\hat{\mathbb{E}}$, $\mathcal{N}$ is G-normal random variable, $\overline{\sigma}^2 = \hat{\mathbb{E}}(X^2)$, $\underline{\sigma}^2 = \hat{\varepsilon}(X^2)$.

And

$$\lim_{n\to\infty} \mathbb{V}\left(\max_{k\le n} |S_k| > x\sqrt{n}\right) = 2G(x),\ \ x > 0, \tag{2.9}$$

where $2G(x) = 2\sum_{i=0}^{\infty}(-1)^i P\left\{|N| \ge (2i+1)x\right\} = \mathbb{V}(\max_{0\le t\le 1}|W(t)| \ge x)$, $W(t)$ is a G-Brownian motion with $W(1) \sim \mathcal{N}\left(0, \left[\underline{\sigma}^2, \overline{\sigma}^2\right]\right)$.

Proof. In order to facilitate the proof of the theorems in this paper, we use the conditions of the central limit theorem in Zhang [32] to prove our Lemma 2.4. Next, we prove that the conditions of Lemma 2.4 satisfy the conditions of Lemma 2.4 (i)–(iii) in Zhang [31], let $X^{(c)} := (-c) \vee (X \wedge c)$.

(i) By the condition of $C_{\mathbb{V}}\left(X^2\right) < \infty$ in Lemma 2.4, $\hat{\mathbb{E}}$ is continuous, then

$$\hat{\mathbb{E}}\left(X^2 \wedge c\right) \le C_{\mathbb{V}}\left(X^2 \wedge c\right) \le C_{\mathbb{V}}\left(X^2\right) < \infty.$$

and $\hat{\mathbb{E}}\left(X^2 \wedge c\right) \uparrow c$, hence

$$\lim_{c\to\infty} \hat{\mathbb{E}}(X^2 \wedge c)\ is\ finite. \tag{2.10}$$

(ii) By the condition of $C_{\mathbb{V}}\left(X^2\right) < \infty$ in Lemma 2.4, then

$$C_{\mathbb{V}}\left(X^2\right) = \int_0^\infty \mathbb{V}(|X|^2 > x)\mathrm{d}x < \infty.$$

and combining with $\mathbb{V}(|X|^2 > x) \downarrow x$, hence

$$\lim_{x\to\infty} x\mathbb{V}(|X|^2 > x) = 0 \Leftrightarrow \lim_{x\to\infty} x^2\mathbb{V}(|X| > x) = 0. \tag{2.11}$$

(iii) By the condition of $C_{\mathbb{V}}\left(X^2\right) < \infty$ in Lemma 2.4, $\hat{\mathbb{E}}$ is continuous, then

$$\hat{\mathbb{E}}\left(X^2\right) \le C_{\mathbb{V}}\left(X^2\right) < \infty.$$

Since,

$$\begin{aligned}\hat{\mathbb{E}}\left(|X| - c\right)^+ &= \hat{\mathbb{E}}\left(|X| - c\right) I_{(|X|>c)} \\ &\le \hat{\mathbb{E}}\left(|X| - c\right)\left(\frac{|X|}{c}\right) I_{(|X|>c)} \\ &\le \frac{\hat{\mathbb{E}}X^2}{c} \longrightarrow 0, c \longrightarrow \infty.\end{aligned}$$

Hence,

$$\lim_{c\to\infty} \hat{\mathbb{E}}\,(|X|-c)^+ = 0. \tag{2.12}$$

Combining with (2.1), (2.12) and $\hat{\mathbb{E}}\,(|X|) \le C_{\mathbb{V}}\,(|X|) < \infty$, we get

$$\begin{aligned}\left|\hat{\mathbb{E}}(\pm X)^{(c)} - \hat{\mathbb{E}}(\pm X)\right| &\le \hat{\mathbb{E}}\left|(\pm X)^{(c)} - (\pm X)\right| \\ &= \hat{\mathbb{E}}\,(|X|-c)\,I_{(|X|>c)} \\ &= \hat{\mathbb{E}}\,(|X|-c)^+ \longrightarrow 0, c \longrightarrow \infty.\end{aligned}$$

Therefore,

$$\lim_{c\to\infty} \hat{\mathbb{E}}(\pm X)^{(c)} = \hat{\mathbb{E}}\,(\pm X) = 0. \tag{2.13}$$

By $X^2 = X^2 \wedge c + \left(X^2 - c\right) I_{(X^2>c)}$, we get

$$\hat{\mathbb{E}}(X^2 \wedge c) \le \hat{\mathbb{E}}(X^2) \le \hat{\mathbb{E}}(X^2 \wedge c) + \hat{\mathbb{E}}(X^2 - c)I_{(X^2>c)}.$$

Since,

$$\begin{aligned}\hat{\mathbb{E}}(X^2 - c)I_{(X^2>c)} &\le C_{\mathbb{V}}\left((X^2-c)I_{(X^2>c)}\right) \\ &= \int_0^\infty \mathbb{V}\left((X^2-c)I_{(X^2>c)} > t\right)\mathrm{d}t \\ &= \int_0^\infty \mathbb{V}(X^2 > c+t)\mathrm{d}t \\ &= \int_c^\infty \mathbb{V}(X^2 > y)\mathrm{d}y \qquad (y = c+t) \\ &\longrightarrow 0, c \longrightarrow \infty.\end{aligned}$$

Hence,

$$\lim_{c\to\infty} \hat{\mathbb{E}}(X^2 \wedge c) = \hat{\mathbb{E}}(X^2). \tag{2.14}$$

Therefore, (2.10), (2.11) and (2.13) corresponds to the conditions of Lemma 2.4 (i)–(iii) in Zhang [31], respectively. By (2.14), we obtain $\overline{\sigma}^2 = \lim_{c\to\infty} \hat{\mathbb{E}}(X^2\wedge c) = \hat{\mathbb{E}}(X^2)$, $\underline{\sigma}^2 = \lim_{c\to\infty} \hat{\varepsilon}(X^2\wedge c) = \hat{\varepsilon}(X^2)$. Hence, we get that (2.8) and (2.9) are established.

For better understanding Lemma 2.4, we need to review some central limit theorems under the sub-linear expectations. Peng [35,43] initially proved that: If $\hat{\mathbb{E}}(X_1) = \hat{\mathbb{E}}(-X_1) = 0$ and $\hat{\mathbb{E}}\left(|X_1|^{2+\alpha}\right) < \infty$, for some $\alpha > 0$, then $\lim_{n\to\infty} \hat{\mathbb{E}}\left(\varphi\left(\frac{S_n}{\sqrt{n}}\right)\right) = \hat{\mathbb{E}}\,(\varphi(\xi))$, *where* $\xi \sim \mathcal{N}\left(0, \left[\underline{\sigma}^2, \overline{\sigma}^2\right]\right)$, $\overline{\sigma}^2 = \hat{\mathbb{E}}(X_1^2)$, $\underline{\sigma}^2 = \hat{\varepsilon}(X_1^2)$. On the basis of Peng [35,43], Zhang [30] showed that the moment condition that $\hat{\mathbb{E}}\left(|X_1|^{2+\alpha}\right) < \infty$ can be weakened to $\hat{\mathbb{E}}\left[\left(|X_1|^2 - c\right)^+\right] \longrightarrow 0$ as $c \longrightarrow \infty$. Further, Zhang [44] proved the sufficient and necessary conditions for the central limit theorem : (i) $\lim_{c\to\infty} \hat{\mathbb{E}}(X_1^2 \wedge c)$ is finite; (ii) $x^2\mathbb{V}(|X_1| \ge x) \longrightarrow 0$ as $c \longrightarrow \infty$; (iii) $\lim_{c\to\infty} \hat{\mathbb{E}}\,[-c \vee X_1 \wedge c] = \lim_{c\to\infty} \hat{\mathbb{E}}\,[-c \vee (-X_1) \wedge c] = 0$. Under the sufficient and necessary conditions in Zhang [44], Zhang [31] further extended the central limit theorem for maximum value. On the basis of Zhang [31], we use the conditions of the central limit theorem in Zhang [31] to prove our Lemma 2.4 in this paper.

3. Main results

In this subsection, we first recall some notations which will be used in the main results.

If $\hat{\mathbb{E}}\left(|X|^{2+\delta}\right) < \infty$ ($\delta > 0$), combining with Markov's inequality, it is easy to derive that

$$\begin{aligned} C_{\mathbb{V}}(X^2) &= \int_0^\infty \mathbb{V}(X^2 > x)\mathrm{d}x \\ &= \int_0^\infty \mathbb{V}(|X| > x^{1/2})\mathrm{d}x \\ &\leq 1 + \int_1^\infty \frac{\hat{\mathbb{E}}(|X|^{2+\delta})}{x^{1+\delta/2}}\mathrm{d}x < \infty. \end{aligned}$$

Therefore, for any $\delta > 0$, we know that $\hat{\mathbb{E}}\left(|X|^{2+\delta}\right) < \infty$ implies $C_{\mathbb{V}}(X^2) < \infty$.

Now we present our results, which are stated as follows.

Theorem 3.1. *Let $\{X, X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}(X) = \hat{\mathbb{E}}(-X) = 0$, there exists a constant $\delta > 0$ such that $\hat{\mathbb{E}}\left(|X|^{2+\delta}\right) < \infty$. We assume that $\hat{\mathbb{E}}$ is continuous, then for any $b > -1/2$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{2b+1} \sum_{n=2}^\infty \frac{(\ln n)^{b-1/2}}{n^{3/2}} C_{\mathbb{V}}\left(|S_n|\, I(|S_n| \geq \varepsilon\sqrt{n \ln n})\right) = \frac{C_{\mathbb{V}}\left(|\xi|^{2b+2}\right)}{(b+1)(2b+1)}, \tag{3.1}$$

where $\xi \sim \mathcal{N}\left(0, \left[\underline{\sigma}^2, \overline{\sigma}^2\right]\right)$, and $\overline{\sigma}^2 = \hat{\mathbb{E}}(X^2)$, $\underline{\sigma}^2 = \hat{\varepsilon}(X^2)$.

Remark 3.1. *Under the moment condition with $C_{\mathbb{V}}\left(|X|^{2\vee\frac{1}{s}}\right)$ in Theorem 2.2 of Wu and Wang [32], we take $s = \frac{1}{2b+2}$, $p = 1$ and $g(x) = (\ln x)^{b+1}$ in Theorem 2.2 of Wu and Wang [32], then for $0 \leq p < \frac{1}{s}$, we have* (3.1). *However, if $b > 0$, the moment condition of Theorem 3.1 is weaker than that of Theorem 2.2 in Wu and Wang [32].*

For further investigation, Theorem 3.2 further studies the maximum value of partial sums.

Theorem 3.2. *Let $M_n = \max_{k\leq n} |S_k|$. Under the conditions of Theorem 3.1, we have*

$$\begin{aligned} &\lim_{\varepsilon\to 0} \varepsilon^{2b+1} \sum_{n=2}^\infty \frac{(\ln n)^{b-1/2}}{n^{3/2}} C_{\mathbb{V}}\left(M_n I(M_n \geq \varepsilon\sqrt{n \ln n})\right) \\ &= \frac{2E\,|N|^{2(b+1)}}{(b+1)(2b+1)} \sum_{i=0}^\infty \frac{(-1)^i}{(2i+1)^{2(b+1)}}, \end{aligned} \tag{3.2}$$

where N is a standard normal random variable.

Remark 3.2. *Theorems 3.1 and 3.2 extend the corresponding results obtained by Fu and Yang [13] from the probability space to sublinear expectation space.*

4. Proof of main results

4.1. Proof of Theorem 3.1

Note that

$$\begin{aligned}
&\varepsilon^{2b+1}\sum_{n=2}^{\infty}\frac{(\ln n)^{b-1/2}}{n^{3/2}}C_{\mathbb{V}}\left(|S_n|\,I(|S_n|\geq\varepsilon\sqrt{n\ln n})\right)\\
=&\varepsilon^{2b+1}\sum_{n=2}^{\infty}\frac{(\ln n)^{b-1/2}}{n}C_{\mathbb{V}}\left(|\xi|\,I(|\xi|\geq\varepsilon\sqrt{\ln n})\right)\\
&+\varepsilon^{2b+1}\sum_{n=2}^{\infty}\frac{(\ln n)^{b-1/2}}{n}\left\{n^{-1/2}C_{\mathbb{V}}\left(|S_n|\,I(|S_n|\geq\varepsilon\sqrt{n\ln n})\right)\right.\\
&\left.-C_{\mathbb{V}}\left(|\xi|\,I(|\xi|\geq\varepsilon\sqrt{\ln n})\right)\right\}\\
:=&I_1(\varepsilon)+I_2(\varepsilon).
\end{aligned}$$

Hence, in order to establish (3.1), it suffices to prove that

$$\lim_{\varepsilon\to 0}I_1(\varepsilon)=\frac{C_{\mathbb{V}}\left(|\xi|^{2b+2}\right)}{(b+1)(2b+1)},\tag{4.1}$$

and

$$\lim_{\varepsilon\to 0}I_2(\varepsilon)=0.\tag{4.2}$$

We first prove (4.1). For any $b>-1/2$, then

$$\begin{aligned}
\lim_{\varepsilon\to 0}I_1(\varepsilon)&=\lim_{\varepsilon\to 0}\varepsilon^{2b+1}\sum_{n=2}^{\infty}\frac{(\ln n)^{b-1/2}}{n}\int_0^{\infty}\mathbb{V}\left(|\xi|\,I(|\xi|\geq\varepsilon\sqrt{\ln n})>x\right)\mathrm{d}x\\
&=\lim_{\varepsilon\to 0}\varepsilon^{2b+1}\sum_{n=2}^{\infty}\frac{(\ln n)^{b-1/2}}{n}\int_{\varepsilon\sqrt{\ln n}}^{\infty}\mathbb{V}(|\xi|\geq x)\mathrm{d}x\\
&=\lim_{\varepsilon\to 0}\varepsilon^{2b+1}\int_2^{\infty}\frac{(\ln y)^{b-1/2}}{y}\mathrm{d}y\int_{\varepsilon\sqrt{\ln y}}^{\infty}\mathbb{V}(|\xi|\geq x)\mathrm{d}x\\
&=2\lim_{\varepsilon\to 0}\int_{\varepsilon\sqrt{\ln 2}}^{\infty}t^{2b}\mathrm{d}t\int_t^{\infty}\mathbb{V}(|\xi|\geq x)\mathrm{d}x\qquad\left(t=\varepsilon\sqrt{\ln y}\right)\\
&=2\lim_{\varepsilon\to 0}\int_{\varepsilon\sqrt{\ln 2}}^{\infty}\mathbb{V}(|\xi|\geq x)\mathrm{d}x\int_{\varepsilon\sqrt{\ln 2}}^{x}t^{2b}\mathrm{d}t\\
&=\frac{2}{2b+1}\lim_{\varepsilon\to 0}\int_{\varepsilon\sqrt{\ln 2}}^{\infty}\mathbb{V}(|\xi|\geq x)\left(x^{2b+1}-(\varepsilon\sqrt{\ln 2})^{2b+1}\right)\mathrm{d}x\\
&=\frac{2}{2b+1}\int_0^{\infty}x^{2b+1}\mathbb{V}(|\xi|\geq x)\mathrm{d}x-\frac{2}{2b+1}\lim_{\varepsilon\to 0}\left(\varepsilon\sqrt{\ln 2}\right)^{2b+1}\int_{\varepsilon\sqrt{\ln 2}}^{\infty}\mathbb{V}(|\xi|\geq x)\mathrm{d}x\\
&=\frac{C_{\mathbb{V}}\left(|\xi|^{2b+2}\right)}{(b+1)(2b+1)}.
\end{aligned}$$

Next, we prove (4.2). Without loss of generality, here and later, we assume that $\hat{\mathbb{E}}(X^2) = 1$. Let $b(M,\varepsilon) = \exp(M/\varepsilon^2)$, $M > 6$. Note that

$$\begin{aligned}|I_2(\varepsilon)| \leq & \varepsilon^{2b+1} \sum_{n\leq b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}}{n} \left| n^{-1/2} C_{\mathbb{V}}\left(|S_n| I(|S_n| \geq \varepsilon\sqrt{n\ln n})\right) - C_{\mathbb{V}}\left(|\xi| I(|\xi| \geq \varepsilon\sqrt{\ln n})\right)\right| \\ & + \varepsilon^{2b+1} \sum_{n>b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}}{n^{3/2}} C_{\mathbb{V}}\left(|S_n| I(|S_n| \geq \varepsilon\sqrt{n\ln n})\right) \\ & + \varepsilon^{2b+1} \sum_{n>b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}}{n} C_{\mathbb{V}}\left(|\xi| I(|\xi| \geq \varepsilon\sqrt{\ln n})\right) \\ := & I_{21}(\varepsilon) + I_{22}(\varepsilon) + I_{23}(\varepsilon).\end{aligned} \tag{4.3}$$

Hence, in order to establish (4.2), it suffices to prove that

$$\lim_{\varepsilon\to 0} I_{21}(\varepsilon) = 0, \tag{4.4}$$

and

$$\lim_{M\longrightarrow\infty} I_{22}(\varepsilon) = 0, \quad \lim_{M\longrightarrow\infty} I_{23}(\varepsilon) = 0, \tag{4.5}$$

uniformly for $0 < \varepsilon < 1/4$.

We first prove (4.4). Let $\Gamma_n = (\ln n)^{-1/2}\Delta_n^{-1/2}$ and $\Delta_n = \sup_{x\geq 0}\left|\mathbb{V}(|S_n| \geq x\sqrt{n}) - \mathbb{V}(|\xi| \geq x)\right|$. It follows from Lemma 4 in Wu [14] that $\hat{\mathbb{E}}\xi^2 < \infty$, by (2.7) in Lemma 2.3 and (2.4) in Lemma 2.2, Markov's inequality, we get

$$\begin{aligned}I_{21}(\varepsilon) = & \varepsilon^{2b+1} \sum_{n\leq b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}}{n} \left| n^{-\frac{1}{2}} \int_0^\infty \mathbb{V}(|S_n| \geq x + \varepsilon\sqrt{n\ln n})\mathrm{d}x - \int_0^\infty \mathbb{V}(|\xi| \geq x + \varepsilon\sqrt{\ln n})\mathrm{d}x\right| \\ \leq & \varepsilon^{2b+1} \sum_{n\leq b(M,\varepsilon)} \frac{(\ln n)^b}{n} \int_0^\infty \left|\mathbb{V}(|S_n| \geq (x+\varepsilon)\sqrt{n\ln n}) - \mathbb{V}(|\xi| \geq (x+\varepsilon)\sqrt{\ln n})\right| \mathrm{d}x \\ \leq & \varepsilon^{2b+1} \sum_{n\leq b(M,\varepsilon)} \frac{(\ln n)^b}{n} \int_0^{\Gamma_n} \left|\mathbb{V}(|S_n| \geq (x+\varepsilon)\sqrt{n\ln n}) - \mathbb{V}(|\xi| \geq (x+\varepsilon)\sqrt{\ln n})\right| \mathrm{d}x \\ & + \varepsilon^{2b+1} \sum_{n\leq b(M,\varepsilon)} \frac{(\ln n)^b}{n} \int_{\Gamma_n}^\infty \left(\mathbb{V}(|S_n| \geq (x+\varepsilon)\sqrt{n\ln n}) + \mathbb{V}(|\xi| \geq (x+\varepsilon)\sqrt{\ln n})\right) \mathrm{d}x \\ \leq & \varepsilon^{2b+1} \sum_{n\leq b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}\Delta_n^{1/2}}{n} + \varepsilon^{2b+1} \sum_{n\leq b(M,\varepsilon)} \frac{(\ln n)^b}{n} \int_{\Gamma_n}^\infty \left(\frac{n\hat{\mathbb{E}}X^2}{(x+\varepsilon)^2 n\ln n} + \frac{\hat{\mathbb{E}}\xi^2}{(x+\varepsilon)^2\ln n}\right)\mathrm{d}x \\ \ll & \varepsilon^{2b+1} \sum_{n\leq b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}\Delta_n^{1/2}}{n} + \varepsilon^{2b+1} \sum_{n\leq b(M,\varepsilon)} \frac{(\ln n)^{b-1}}{n} \int_{\Gamma_n}^\infty \frac{1}{(x+\varepsilon)^2}\mathrm{d}x \\ \ll & \varepsilon^{2b+1} \sum_{n\leq b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}\Delta_n^{1/2}}{n}.\end{aligned}$$

By

$$\sum_{n\leq b(M,\varepsilon)} \left((\ln n)^{b-1/2}/n\right) = O\left(\ln\left(b(M,\varepsilon)\right)\right)^{b+1/2} = O\left(\varepsilon^{-2b-1}\right) \longrightarrow \infty, \varepsilon \longrightarrow 0,$$

using Lemma 2.3, $\Delta_n \longrightarrow 0$ as $n \longrightarrow \infty$, and combining with Toeplitz's lemma: If $x_n \longrightarrow x, \omega_i \geq 0$, and $\sum_{i=1}^{n}\omega_i \longrightarrow \infty$, then $(\sum_{i=1}^{n}\omega_i x_i/\sum_{i=1}^{n}\omega_i) \longrightarrow x$, we obtain

$$\begin{aligned} I_{21}(\varepsilon) \ll& \varepsilon^{2b+1} \sum_{n \leq b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}\Delta_n^{1/2}}{n} \\ =& \frac{\sum_{n\leq b(M,\varepsilon)} \left((\ln n)^{b-1/2}\Delta_n^{1/2}/n\right)}{\varepsilon^{-2b-1}} \\ \ll& \frac{\sum_{n\leq b(M,\varepsilon)} \left((\ln n)^{b-1/2}\Delta_n^{1/2}/n\right)}{\sum_{n\leq b(M,\varepsilon)} ((\ln n)^{b-1/2}/n)} \longrightarrow 0, \varepsilon \longrightarrow 0. \end{aligned} \tag{4.6}$$

That is, (4.4) is established.

Next, we prove (4.5). For $0 < \mu < 1$, let $\varphi(x) \in C_{l,Lip}(\mathbb{R})$ be an even and nondecreasing function on $x \geq 0$ such that $0 \leq \varphi(x) \leq 1$ for all x and $\varphi(x) = 0$ if $|x| \leq \mu$, $\varphi(x) = 1$ if $|x| \geq 1$. Hence,

$$I\left(|x| \geq 1\right) \leq \varphi\left(x\right) \leq I\left(|x| \geq \mu\right). \tag{4.7}$$

Therefore, by (2.2), (4.7) and the identical distribution of X, X_i, for any $x > 0$,

$$\mathbb{V}\left(|X_i| \geq x\right) \leq \hat{\mathbb{E}}\left(\varphi\left(\frac{X_i}{x}\right)\right) = \hat{\mathbb{E}}\left(\varphi\left(\frac{X}{x}\right)\right) \leq \mathbb{V}\left(|X| \geq \mu x\right). \tag{4.8}$$

By (4.8) and taking $x = \varepsilon\sqrt{n \ln n}$ and $y = \varepsilon\sqrt{n \ln n}/(2+b)$ in (2.3) of Lemma 2.1, for $n > b(M,\varepsilon) > \exp(6/\varepsilon^2)$, we get

$$\begin{aligned} &\mathbb{V}\left(\max_{k\leq n} S_k \geq \varepsilon\sqrt{n\ln n}\right) \\ \leq& \sum_{i=1}^{n} \mathbb{V}\left(|X_i| \geq \frac{\varepsilon\sqrt{n\ln n}}{2+b}\right) + \exp\left\{(2+b) - (2+b)\ln\left(1 + \frac{\varepsilon^2 \ln n}{2+b}\right)\right\} \\ \ll& n\mathbb{V}\left(|X| \geq c\varepsilon\sqrt{n\ln n}\right) + \frac{1}{\varepsilon^{2(2+b)}(\ln n)^{(2+b)}}. \end{aligned}$$

Since $\{-X, -X_i\}$ also satisfies the conditions of Theorem 3.1, we replace the $\{X, X_i\}$ with the $\{-X, -X_i\}$ in the upper form:

$$\mathbb{V}\left(\max_{k\leq n}(-S_k) \geq \varepsilon\sqrt{n\ln n}\right) \ll n\mathbb{V}\left(|X| \geq c\varepsilon\sqrt{n\ln n}\right) + \frac{1}{\varepsilon^{2(2+b)}(\ln n)^{(2+b)}}.$$

Therefore,

$$\begin{aligned} \mathbb{V}\left(\max_{k\leq n}|S_k| \geq \varepsilon\sqrt{n\ln n}\right) \leq& \mathbb{V}\left(\max_{k\leq n} S_k \geq \varepsilon\sqrt{n\ln n}\right) + \mathbb{V}\left(\max_{k\leq n}(-S_k) \geq \varepsilon\sqrt{n\ln n}\right) \\ \ll& n\mathbb{V}\left(|X| \geq c\varepsilon\sqrt{n\ln n}\right) + \frac{1}{\varepsilon^{2(2+b)}(\ln n)^{(2+b)}}. \end{aligned}$$

More generally, for any $x > 0$ and $n > \exp(M/\varepsilon^2)$, we have

$$\begin{aligned} \mathbb{V}\left(\max_{k\leq n}|S_k| \geq (x+\varepsilon)\sqrt{n\ln n}\right) \ll& n\mathbb{V}\left(|X| \geq c(x+\varepsilon)\sqrt{n\ln n}\right) + \frac{1}{(x+\varepsilon)^{2(2+b)}(\ln n)^{(2+b)}} \\ :=& II_1(\varepsilon) + II_2(\varepsilon). \end{aligned} \tag{4.9}$$

Combining with (4.9), we get

$$
\begin{aligned}
I_{22}(\varepsilon) =& \varepsilon^{2b+1} \sum_{n>b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}}{n^{3/2}} \int_0^\infty \mathbb{V}(|S_n| \geq x + \varepsilon\sqrt{n \ln n}) \mathrm{d}x \\
\leq& \varepsilon^{2b+1} \sum_{n>b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}}{n^{3/2}} \int_0^\infty \mathbb{V}(\max_{k\leq n} |S_k| \geq x + \varepsilon\sqrt{n \ln n}) \mathrm{d}x \\
=& \varepsilon^{2b+1} \sum_{n>b(M,\varepsilon)} \frac{(\ln n)^{b}}{n} \int_0^\infty \mathbb{V}(\max_{k\leq n} |S_k| \geq (x + \varepsilon)\sqrt{n \ln n}) \mathrm{d}x \\
\ll& \varepsilon^{2b+1} \sum_{n>b(M,\varepsilon)} \frac{(\ln n)^{b}}{n} \int_0^\infty (II_1(\varepsilon) + II_2(\varepsilon))\, \mathrm{d}x \\
=& \varepsilon^{2b+1} \sum_{n>b(M,\varepsilon)} \frac{(\ln n)^{b}}{n} \int_0^\infty n\mathbb{V}\left(|X| \geq c(x + \varepsilon)\sqrt{n \ln n}\right) \mathrm{d}x \\
&+ \varepsilon^{2b+1} \sum_{n>b(M,\varepsilon)} \frac{(\ln n)^{b}}{n} \int_0^\infty \frac{1}{(x+\varepsilon)^{2(2+b)}(\ln n)^{(2+b)}} \mathrm{d}x \\
:=& II_{11}(\varepsilon) + II_{22}(\varepsilon).
\end{aligned} \tag{4.10}
$$

By the Markov's inequality, and $\hat{\mathbb{E}}\left(|X|^{2+\delta}\right) < \infty$, we assume that $\varepsilon^2 < 1/16$, then

$$
\begin{aligned}
II_{11}(\varepsilon) &\ll \varepsilon^{2b+1} \sum_{n>b(M,\varepsilon)} \frac{(\ln n)^{b}}{n} \int_0^\infty \frac{n\hat{\mathbb{E}}\,|X|^{2+\delta}}{(x+\varepsilon)^{2+\delta}(n \ln n)^{1+\delta/2}} \mathrm{d}x \\
&\ll \varepsilon^{2b+1} \sum_{n>b(M,\varepsilon)} \frac{(\ln n)^{b-1-\delta/2}}{n^{1+\delta/2}} \int_0^\infty \frac{1}{(x+\varepsilon)^{2+\delta}} \mathrm{d}x \\
&\ll \varepsilon^{2b+1} \sum_{n>b(M,\varepsilon)} \frac{(\ln n)^{b-1-\delta/2}}{n^{1+\delta/2}} \ \varepsilon^{-1-\delta} \\
&\ll \varepsilon^{2b-\delta} (\ln (b(M,\varepsilon)))^{b-1-\delta/2} (b(M,\varepsilon))^{-1-\delta/2+1} \\
&= \varepsilon^{2b-\delta} (M\varepsilon^{-2})^{b-1-\delta/2} (\mathrm{e}^{M\varepsilon^{-2}})^{-\delta/2} \\
&= \varepsilon^{2} M^{b-1-\delta/2} \frac{1}{\mathrm{e}^{M\delta/2\varepsilon^2}} \\
&\ll M^{b-1-\delta/2} \frac{1}{\mathrm{e}^{8M\delta}} \\
&\longrightarrow 0, M \longrightarrow \infty,
\end{aligned} \tag{4.11}
$$

uniformly for $0 < \varepsilon < 1/4$.

Since,

$$
\begin{aligned}
II_{22}(\varepsilon) &= \varepsilon^{2b+1} \sum_{n>b(M,\varepsilon)} \frac{1}{n\,(\ln n)^2} \int_0^\infty \frac{1}{(x+\varepsilon)^{2(2+b)}} \mathrm{d}x \\
&\ll \varepsilon^{2b+1} \sum_{n>b(M,\varepsilon)} \frac{1}{n\,(\ln n)^2} \ \varepsilon^{-3-2b}
\end{aligned}
$$

$$\begin{aligned}
&\sim \varepsilon^{-2}\int_{b(M,\varepsilon)}^{\infty}\frac{1}{x(\ln x)^2}\mathrm{d}x\\
&\ll \varepsilon^{-2}\left(\ln\left(b(M,\varepsilon)\right)\right)^{-1}\\
&= \varepsilon^{-2}\left(M\varepsilon^{-2}\right)^{-1}\\
&= M^{-1}\longrightarrow 0, M\longrightarrow\infty.
\end{aligned}\tag{4.12}$$

Therefore, combining with (4.10)–(4.12), we obtain

$$\lim_{M\longrightarrow\infty} I_{22}(\varepsilon)=0,\tag{4.13}$$

uniformly for $0<\varepsilon<1/4$.

Finally, it follows from Lemma 4 in Wu [14] that $\hat{\mathbb{E}}|\xi|^p<\infty$, by Markov's inequality, for $p>2b+2$, we have

$$\begin{aligned}
I_{23}(\varepsilon) =&\varepsilon^{2b+1}\sum_{n>b(M,\varepsilon)}\frac{(\ln n)^{b-1/2}}{n}\int_0^{\infty}\mathbb{V}(|\xi|\geq x+\varepsilon\sqrt{\ln n})\mathrm{d}x\\
=&\varepsilon^{2b+1}\sum_{n>b(M,\varepsilon)}\frac{(\ln n)^{b}}{n}\int_0^{\infty}\mathbb{V}(|\xi|\geq (x+\varepsilon)\sqrt{\ln n})\mathrm{d}x\\
\leq&\varepsilon^{2b+1}\sum_{n>b(M,\varepsilon)}\frac{(\ln n)^{b}}{n}\int_0^{\infty}\frac{\hat{\mathbb{E}}|\xi|^p}{(x+\varepsilon)^p(\ln n)^{p/2}}\mathrm{d}x\\
\ll&\varepsilon^{2b+1}\sum_{n>b(M,\varepsilon)}\frac{(\ln n)^{b-p/2}}{n}\int_0^{\infty}\frac{1}{(x+\varepsilon)^p}\mathrm{d}x\\
\ll&\varepsilon^{2b+1}\sum_{n>b(M,\varepsilon)}\frac{(\ln n)^{b-p/2}}{n}\varepsilon^{-p+1}\\
\sim&\varepsilon^{2b+2-p}\int_{b(M,\varepsilon)}^{\infty}\frac{(\ln x)^{b-p/2}}{x}\mathrm{d}x\\
\ll&\varepsilon^{2b+2-p}\left(\ln\left(b(M,\varepsilon)\right)\right)^{b+1-p/2}\\
=&\varepsilon^{2b+2-p}\left(M\varepsilon^{-2}\right)^{b+1-p/2}=M^{b+1-p/2}\longrightarrow 0, M\rightarrow\infty,
\end{aligned}$$

uniformly for $0<\varepsilon<1/4$.

From this, combining with (4.6) and (4.13), (4.2) is established. This completes the proof of Theorem 3.1.

4.2. Proof of Theorem 3.2

Note that

$$\begin{aligned}
&\varepsilon^{2b+1}\sum_{n=2}^{\infty}\frac{(\ln n)^{b-1/2}}{n^{3/2}}C_{\mathbb{V}}\left(M_n I(M_n\geq\varepsilon\sqrt{n\ln n})\right)\\
=&\varepsilon^{2b+1}\sum_{n=2}^{\infty}\frac{(\ln n)^{b-1/2}}{n}C_{\mathbb{V}}\left(\max_{0\leq t\leq 1}|W(t)|\,I(\max_{0\leq t\leq 1}|W(t)|\geq\varepsilon\sqrt{\ln n})\right)
\end{aligned}$$

$$+\varepsilon^{2b+1}\sum_{n=2}^{\infty}\frac{(\ln n)^{b-1/2}}{n}\Big\{n^{-1/2}C_{\mathbb{V}}\left(M_nI(M_n\geq\varepsilon\sqrt{n\ln n})\right)$$
$$-C_{\mathbb{V}}(\max_{0\le t\le 1}|W(t)|\,I(\max_{0\le t\le 1}|W(t)|\geq\varepsilon\sqrt{\ln n}))\Big\}:=H_1(\varepsilon)+H_2(\varepsilon).$$

Hence, in order to establish (3.2), it suffices to prove that

$$\lim_{\varepsilon\to 0}H_1(\varepsilon)=\frac{2E\,|N|^{2(b+1)}}{(b+1)(2b+1)}\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)^{2(b+1)}},\tag{4.14}$$

and

$$\lim_{\varepsilon\to 0}H_2(\varepsilon)=0.\tag{4.15}$$

We first prove (4.14). Combining with (2.9) in Lemma 2.4, for any $b>-1/2$, we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}H_1(\varepsilon)=&\lim_{\varepsilon\to 0}\varepsilon^{2b+1}\sum_{n=2}^{\infty}\frac{(\ln n)^{b-1/2}}{n}\int_{\varepsilon\sqrt{\ln n}}^{\infty}\mathbb{V}(\max_{0\le t\le 1}|W(t)|\geq x)\mathrm{d}x\\
=&\lim_{\varepsilon\to 0}\varepsilon^{2b+1}\int_{2}^{\infty}\frac{(\ln y)^{b-1/2}}{y}\mathrm{d}y\int_{\varepsilon\sqrt{\ln y}}^{\infty}\mathbb{V}(\max_{0\le t\le 1}|W(t)|\geq x)\mathrm{d}x\\
=&4\lim_{\varepsilon\to 0}\int_{\varepsilon\sqrt{\ln 2}}^{\infty}u^{2b}\mathrm{d}u\int_{u}^{\infty}G(x)\mathrm{d}x \qquad \left(u=\varepsilon\sqrt{\ln y}\right)\\
=&4\sum_{i=0}^{\infty}(-1)^i\lim_{\varepsilon\to 0}\int_{\varepsilon\sqrt{\ln 2}}^{\infty}u^{2b}\mathrm{d}u\int_{u}^{\infty}P(|N|\geq(2i+1)x)\mathrm{d}x\\
=&4\sum_{i=0}^{\infty}(-1)^i\lim_{\varepsilon\to 0}\int_{\varepsilon\sqrt{\ln 2}}^{\infty}P(|N|\geq(2i+1)x)\mathrm{d}x\int_{\varepsilon\sqrt{\ln 2}}^{x}u^{2b}\mathrm{d}u\\
=&\frac{4}{(2b+1)}\sum_{i=0}^{\infty}(-1)^i\lim_{\varepsilon\to 0}\int_{\varepsilon\sqrt{\ln 2}}^{\infty}P(|N|\geq(2i+1)x)\left(x^{2b+1}-(\varepsilon\sqrt{\ln 2})^{2b+1}\right)\mathrm{d}x\\
=&\frac{4}{(2b+1)}\sum_{i=0}^{\infty}(-1)^i\int_{0}^{\infty}x^{2b+1}P\left(|N|\geq(2i+1)x\right)\mathrm{d}x\\
&-\frac{4}{(2b+1)}\sum_{i=0}^{\infty}(-1)^i\lim_{\varepsilon\to 0}(\varepsilon\sqrt{\ln 2})^{2b+1}\int_{\varepsilon\sqrt{\ln 2}}^{\infty}P\left(|N|\geq(2i+1)x\right)\mathrm{d}x\\
=&\frac{2E\,|N|^{2(b+1)}}{(b+1)(2b+1)}\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)^{2(b+1)}}.
\end{aligned}$$

Next, we prove (4.15). Note that

$$\begin{aligned}
|H_2(\varepsilon)|\leq&\varepsilon^{2b+1}\sum_{n\leq b(M,\varepsilon)}\frac{(\ln n)^{b-1/2}}{n}\Big|n^{-1/2}C_{\mathbb{V}}\left(M_nI(M_n\geq\varepsilon\sqrt{n\ln n})\right)\\
&-C_{\mathbb{V}}\left(\max_{0\le t\le 1}|W(t)|\,I(\max_{0\le t\le 1}|W(t)|\geq\varepsilon\sqrt{\ln n})\right)\Big|
\end{aligned}$$

$$+\varepsilon^{2b+1}\sum_{n>b(M,\varepsilon)}\frac{(\ln n)^{b-1/2}}{n^{3/2}}C_{\mathbb{V}}\left(M_nI(M_n\geq\varepsilon\sqrt{n\ln n})\right)$$
$$+\varepsilon^{2b+1}\sum_{n>b(M,\varepsilon)}\frac{(\ln n)^{b-1/2}}{n}C_{\mathbb{V}}\left(\max_{0\leq t\leq 1}|W(t)|\,I(\max_{0\leq t\leq 1}|W(t)|\geq\varepsilon\sqrt{\ln n})\right)$$
$$:=H_{21}(\varepsilon)+H_{22}(\varepsilon)+H_{23}(\varepsilon).$$

Hence, in order to establish (4.15), it suffices to prove that

$$\lim_{\varepsilon\to 0}H_{21}(\varepsilon)=0, \tag{4.16}$$

and

$$\lim_{M\to\infty}H_{22}(\varepsilon)=0, \qquad \lim_{M\to\infty}H_{23}(\varepsilon)=0, \tag{4.17}$$

uniformly for $0<\varepsilon<1/4$.

Now, we prove (4.16). Note that

$$\begin{aligned}H_{21}(\varepsilon)=&\varepsilon^{2b+1}\sum_{n\leq b(M,\varepsilon)}\frac{(\ln n)^{b-1/2}}{n}\Big|n^{-\frac{1}{2}}\int_0^\infty\mathbb{V}(M_n\geq x+\varepsilon\sqrt{n\ln n})\mathrm{d}x\\&-\int_0^\infty\mathbb{V}(\max_{0\leq t\leq 1}|W(t)|\geq x+\varepsilon\sqrt{\ln n})\mathrm{d}x\Big|\\\leq&\varepsilon^{2b+1}\sum_{n\leq b(M,\varepsilon)}\frac{(\ln n)^{b}}{n}\int_0^\infty\Big|\mathbb{V}(M_n\geq(x+\varepsilon)\sqrt{n\ln n})-\mathbb{V}(\max_{0\leq t\leq 1}|W(t)|\geq(x+\varepsilon)\sqrt{\ln n})\Big|\mathrm{d}x\\=&\varepsilon^{2b+1}\sum_{n\leq b(M,\varepsilon)}\frac{(\ln n)^{b}}{n}\int_0^{\theta_n}\Big|\mathbb{V}(M_n\geq(x+\varepsilon)\sqrt{n\ln n})-\mathbb{V}(\max_{0\leq t\leq 1}|W(t)|\geq(x+\varepsilon)\sqrt{\ln n})\Big|\mathrm{d}x\\&+\varepsilon^{2b+1}\sum_{n\leq b(M,\varepsilon)}\frac{(\ln n)^{b}}{n}\int_{\theta_n}^\infty\Big|\mathbb{V}(M_n\geq(x+\varepsilon)\sqrt{n\ln n})-\mathbb{V}(\max_{0\leq t\leq 1}|W(t)|\geq(x+\varepsilon)\sqrt{\ln n})\Big|\mathrm{d}x\\:=&J_1(\varepsilon)+J_2(\varepsilon).\end{aligned}$$

Let $\theta_n=(\ln n)^{-1/2}l_n^{-1/2}$ and $l_n=\sup_{x\geq 0}\left|\mathbb{V}(M_n\geq x\sqrt{n})-\mathbb{V}(\max_{0\leq t\leq 1}|W(t)|\geq x)\right|$. Similarly to the proof of I_{21}, it follows from (2.9) of Lemma 2.4 that $l_n\longrightarrow 0$ as $n\longrightarrow\infty$, then

$$\begin{aligned}J_1(\varepsilon)&=\varepsilon^{2b+1}\sum_{n\leq b(M,\varepsilon)}\frac{(\ln n)^{b}}{n}\int_0^{\theta_n}l_n\,\mathrm{d}x\\&\leq\varepsilon^{2b+1}\sum_{n\leq b(M,\varepsilon)}\frac{(\ln n)^{b-1/2}}{n}l_n^{1/2}.\end{aligned}\tag{4.18}$$

Since,

$$\begin{aligned}J_2(\varepsilon)\leq&\varepsilon^{2b+1}\sum_{n\leq b(M,\varepsilon)}\frac{(\ln n)^{b}}{n}\int_{\theta_n}^\infty\mathbb{V}\left(M_n\geq(x+\varepsilon)\sqrt{n\ln n}\right)\mathrm{d}x\\&+\varepsilon^{2b+1}\sum_{n\leq b(M,\varepsilon)}\frac{(\ln n)^{b}}{n}\int_{\theta_n}^\infty\mathbb{V}\left(\max_{0\leq t\leq 1}|W(t)|\geq(x+\varepsilon)\sqrt{\ln n}\right)\mathrm{d}x\\:=&J_{21}(\varepsilon)+J_{22}(\varepsilon).\end{aligned}\tag{4.19}$$

For $J_{21}(\varepsilon)$, is similar considerations to (4.9)–(4.13), we obtain

$$\begin{aligned}
J_{21}(\varepsilon) =&\varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{(\ln n)^b}{n} \int_{\theta_n}^{\infty} \mathbb{V}\left(\max_{k\le n} |S_k| \ge (x+\varepsilon)\sqrt{n\ln n}\right) \mathrm{d}x \\
\le&\varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{(\ln n)^b}{n} \int_{\theta_n}^{\infty} n\mathbb{V}\left(|X| \ge c(x+\varepsilon)\sqrt{n\ln n}\right) \mathrm{d}x \\
&+ \varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{(\ln n)^b}{n} \int_{\theta_n}^{\infty} \frac{1}{(x+\varepsilon)^{2(2+b)}(\ln n)^{(2+b)}} \mathrm{d}x \\
\ll&\varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{(\ln n)^{b-1-\delta/2}}{n^{1+\delta/2}} \int_{\theta_n}^{\infty} \frac{1}{(x+\varepsilon)^{2+\delta}} \mathrm{d}x \\
&+ \varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{1}{n(\ln n)^2} \int_{\theta_n}^{\infty} \frac{1}{(x+\varepsilon)^{2(2+b)}} \mathrm{d}x \\
\ll&\varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{(\ln n)^{b-1-\delta/2}}{n^{1+\delta/2}} \left((\ln n)^{-1/2} l_n^{-1/2}\right)^{-1-\delta} \\
&+ \varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{1}{n(\ln n)^2} \left((\ln n)^{-1/2} l_n^{-1/2}\right)^{-3-2b} \\
\ll&\varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}}{n^{1+\delta/2}} l_n^{1/2+\delta/2} + \varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}}{n} l_n^{3/2+b} \\
\ll&\varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}}{n} l_n^{1/2}.
\end{aligned} \tag{4.20}$$

Combining with (2.9) in Lemma 2.4, and Markov's inequality, we get

$$\begin{aligned}
J_{22}(\varepsilon) =&2\varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{(\ln n)^b}{n} \int_{\theta_n}^{\infty} G\left((x+\varepsilon)\sqrt{\ln n}\right) \mathrm{d}x \\
\le&2\varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{(\ln n)^b}{n} \left| \sum_{i=0}^{\infty} (-1)^i \int_{\theta_n}^{\infty} P\left(|N| \ge (2i+1)(x+\varepsilon)\sqrt{\ln n}\right) \mathrm{d}x \right| \\
\le&2\varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{(\ln n)^b}{n} \left| \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2} \int_{\theta_n}^{\infty} \frac{E|N|^2}{(x+\varepsilon)^2 \ln n} \mathrm{d}x \right| \\
\ll&\varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{(\ln n)^{b-1}}{n} \int_{\theta_n}^{\infty} \frac{1}{(x+\varepsilon)^2} \mathrm{d}x \\
\ll&\varepsilon^{2b+1} \sum_{n\le b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}}{n} l_n^{1/2}.
\end{aligned} \tag{4.21}$$

For the results of (4.18)–(4.21), are similar considerations to (4.6). Combining with Toeplitz's lemma, we get that

$$
\begin{aligned}
H_{21}(\varepsilon) &\ll J_1(\varepsilon) + J_{21}(\varepsilon) + J_{22}(\varepsilon) \\
&\ll \varepsilon^{2b+1} \sum_{n \le b(M,\varepsilon)} \frac{(\ln n)^{b-1/2} l_n^{1/2}}{n} \\
&\ll \frac{\sum_{n \le b(M,\varepsilon)} \left((\ln n)^{b-1/2} l_n^{1/2}/n \right)}{\sum_{n \le b(M,\varepsilon)} \left((\ln n)^{b-1/2}/n \right)} \longrightarrow 0, \varepsilon \longrightarrow 0.
\end{aligned}
\tag{4.22}
$$

That is, (4.16) is established.

Next, we prove (4.17). Since the proof for $H_{22}(\varepsilon)$ is the same as the proof for $I_{22}(\varepsilon)$, by (4.9)–(4.13), we get

$$
\lim_{M \to \infty} H_{22}(\varepsilon) = 0, \tag{4.23}
$$

uniformly for $0 < \varepsilon < 1/4$.

Finally, combining with (2.9) in Lemma 2.4, Markov's inequality and $E\,|N|^\beta < \infty$, for $\beta > 2b + 2$, we obtain

$$
\begin{aligned}
H_{23} =& \varepsilon^{2b+1} \sum_{n > b(M,\varepsilon)} \frac{(\ln n)^{b-1/2}}{n} \int_0^\infty \mathbb{V}\left(\max_{0 \le t \le 1} |W(t)| \ge x + \varepsilon \sqrt{\ln n} \right) \mathrm{d}x \\
=& 2\varepsilon^{2b+1} \sum_{n > b(M,\varepsilon)} \frac{(\ln n)^b}{n} \int_0^\infty G\left((x + \varepsilon) \sqrt{\ln n} \right) \mathrm{d}x \\
\ll& \varepsilon^{2b+1} \left| \sum_{i=0}^\infty (-1)^i \sum_{n > b(M,\varepsilon)} \frac{(\ln n)^b}{n} \int_0^\infty P\left(|N| \ge (2i+1)(x+\varepsilon) \sqrt{\ln n} \right) \mathrm{d}x \right| \\
\le& \varepsilon^{2b+1} \left| \sum_{i=0}^\infty (-1)^i \sum_{n > b(M,\varepsilon)} \frac{(\ln n)^b}{n} \int_0^\infty \frac{E\,|N|^\beta}{(2i+1)^\beta (x+\varepsilon)^\beta (\ln n)^{\beta/2}} \mathrm{d}x \right| \\
\ll& \varepsilon^{2b+1} \left| \sum_{i=0}^\infty \frac{(-1)^i}{(2i+1)^\beta} \sum_{n > b(M,\varepsilon)} \frac{(\ln n)^{b-\beta/2}}{n} \int_0^\infty \frac{1}{(x+\varepsilon)^\beta} \mathrm{d}x \right| \\
\ll& \varepsilon^{2b+2-\beta} \int_{b(M,\varepsilon)}^\infty \frac{(\ln y)^{b-\beta/2}}{y} \mathrm{d}y \\
\le& \varepsilon^{2b+2-\beta} (\ln (b(M,\varepsilon)))^{b-\beta/2+1} \\
=& \varepsilon^{2b+2-\beta} (M\varepsilon^{-2})^{b-\beta/2+1} \\
=& M^{b+1-\beta/2} \longrightarrow 0, M \longrightarrow \infty,
\end{aligned}
$$

uniformly for $0 < \varepsilon < 1/4$.

From this, combining with (4.22) and (4.23), (4.15) is established. This completes the proof of Theorem 3.2.

5. Conclusions

The aim of this study is to research the precise asymptotics of independent identically distributed random variables for complete integral convergence under the sub-linear expectation space. Compared with the traditional probability space, the expectation and capacities of sub-linear expectation space are no longer additive. Moreover, many tools and methods applied to probability space no longer apply to sub-linear expectation space. Therefore, the methods and tools for studying precise asymptotics in this paper are different from those for researching precise asymptotics in probability space. In this paper, our research mainly refers to central limit theorem in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ by Zhang [31], which provides a powerful tool for our proof process.

We use central limit theorem in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ to prove the precise asymptotics of a sequence of independent identically distributed random variables under the sub-linear expectation space. The results of this paper extend the precise asymptotics of complete integral convergence of independent identically distributed random variables in probability space to sub-linear expectation space. In the future research, we will further study the precise asymptotics of a wider range of random variables and explore more precise asymptotics theorems with practical significance.

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Conflict of interest

In this article, all authors disclaim any conflicts of interest.

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