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Research article

The singularity of two kinds of tricyclic graphs

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Abstract: Let G be a finite simple graph and let $A(G)$ be its adjacency matrix. Then G is *singular* if $A(G)$ is singular. Suppose $P_{b_1}, P_{b_2}, P_{b_3}$ are three paths with disjoint vertices, where $b_i \geq 2 (i = 1, 2, 3)$, and at most one of them is 2. Coalescing together one of the two end vertices of each of the three paths, and coalescing together the other end vertex of each of the three paths, the resulting graph is called the θ-graph, denoted by $\theta(b_1, b_2, b_3)$. Let $\alpha(a, b_1, b_2, b_3, s)$ be the graph obtained by merging one end of the path P_s with one vertex of a cycle C_a, and merging the other end of the path P_s with one vertex of $\theta(b_1, b_2, b_3)$ of degree 3. If $s = 1$, denote $\beta(a, b_1, b_2, b_3) = \alpha(a, b_1, b_2, b_3, 1)$. In this paper, we give the necessity and sufficiency condition for the singularity of $\alpha(a, b_1, b_2, b_3, s)$ and $\beta(a, b_1, b_2, b_3)$, and we also prove that the probability that any given $\alpha(a, b_1, b_2, b_3, s)$ is a singular graph is equal to $\frac{35}{64}$, the probability that any given $\beta(a, b_1, b_2, b_3)$ is a singular graph is equal to $\frac{9}{16}$. From our main results we can conclude that such a $\alpha(a, b_1, b_2, b_3, s)$ graph ($\beta(a, b_1, b_2, b_3)$ graph) is singular if $4|a$ or three $b_i (i = 1, 2, 3)$ are all odd numbers or exactly two of the three $b_i (i = 1, 2, 3)$ are odd numbers and the length of the cycle formed by the two odd paths in $\alpha(a, b_1, b_2, b_3, s)$ graph ($\beta(a, b_1, b_2, b_3)$ graph) is a multiple of 4. The theoretical probability of these graphs being singular is more than half.

Keywords: adjacency matrix; singular graphs; probability; nullity
Mathematics Subject Classification: 05C50

1. Introduction

Let G be a finite undirected simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of G is defined to be the $n \times n$ matrix $A(G) = (a_{ij})$,

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise,} \end{cases}$$

where $v_i \sim v_j$ represents the vertices v_i and v_j are adjacent. Obviously, $A(G)$ is a real symmetric matrix with diagonal elements all 0, and its eigenvalues are all real numbers. The eigenvalues of $A(G)$ are also the eigenvalues of G and form the spectrum of G. The number of non-zero eigenvalues and zero eigenvalues in the spectrum of G are called the *rank* and the nullity of G, respectively, and denoted by $r(G)$ and $\eta(G)$. Obviously $r(G) + \eta(G) = n$. Then G is singular if $A(G)$ is a singular matrix. In other words, G is singular if and only if 0 is an eigenvalue of G. Singular graphs play a significant role in graph theory, and there are many applications in physics and chemistry, see for instance the survey article [10].

In chemistry, if we represent the atoms by vertices and the bonds by edges in a molecular, we can get a molecular graph [19, 20]. The nullity of a molecular graph has many important applications in chemistry [5–8, 11]. For example, the nullity of a graph equal to 0 is a necessary condition for the stability of the chemical properties of the molecule it represents [7]. In 1957, Collatz and Sinogowitz [8] proposed the problem of characterizing all graphs with nullity greater than zero (i.e. the problem of characterizing all singular graphs), which is a very difficult problem. So far, only some special results are known (see [1, 3, 12, 13, 17, 21–24, 29]). This problem encourages people to study the structural characteristics of singular graphs, and many authors have studied the interaction between the nullity of a graph and its structure (see [4, 9, 14, 15, 18]). Recent studies have also shown that singular graphs are related to other fields of mathematics, such as representation theory of finite groups, combinatorial mathematics, algebraic geometry and so on (see [2, 16, 26–28, 30]).

Let K_n, P_n and C_n be the complete graph, path and cycle with n vertices, respectively. Suppose $P_{b_1}, P_{b_2}, P_{b_3}$ are three paths with disjoint vertices, where $b_i \geq 2(i = 1, 2, 3)$, and at most one of them is 2. Coalescing together one of the two end vertices of each of the three paths, and coalescing together the other end vertex of each of the three paths, the resulting graph is called the θ-graph, denoted by $\theta(b_1, b_2, b_3)$. Let $\alpha(a, b_1, b_2, b_3, s)$ (or α-graph for short) be the graph obtained by merging one end of the path P_s with one vertex of a circle C_a, and merging the other end of the path P_s with one vertex of $\theta(b_1, b_2, b_3)$ of degree 3, see Figure 1.

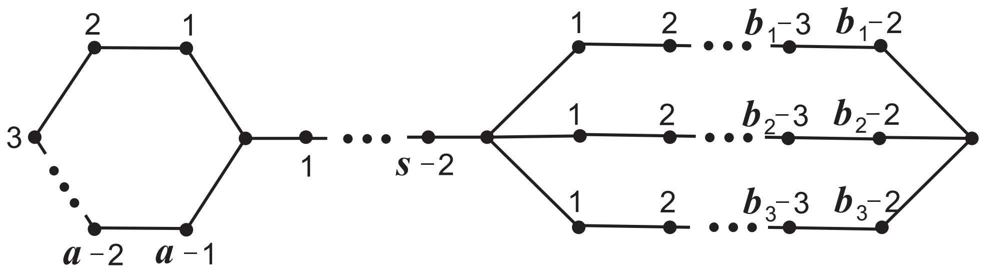

Figure 1. The graph $\alpha(a, b_1, b_2, b_3, s)$.

If $s = 1$, denote $\beta(a, b_1, b_2, b_3) = \alpha(a, b_1, b_2, b_3, 1)$ (or β-graph for short), see Figure 2. Let $G \cup H$ be the union of two graphs G and H. The vertex of degree 1 is called the *pendant vertex*, and the vertex adjacent to the pendant vertex is called the *quasi − pendant vertex*. For the notations and terms which are not defined above, please refer to [5].

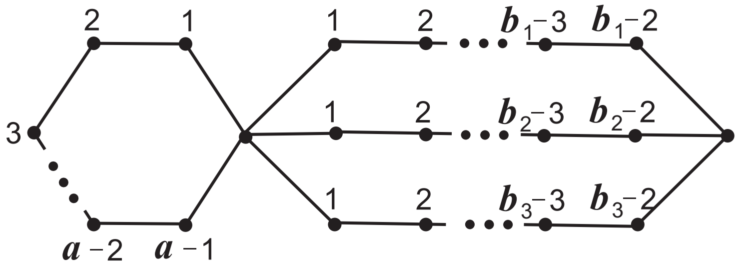

Figure 2. The graph $\beta(a, b_1, b_2, b_3)$.

We know that whether a graph is singular can be determined from two aspects: one is whether 0 is its eigenvalue, and the other is whether the determinant of its adjacency matrix is 0. Many previous research results mainly determine whether a graph is singular from the perspective of the 0 eigenvalue of graph, only few literatures describe singular graphs from the perspective of determinant, and also very limited literatures study the necessity and sufficiency of singularity of a class of graphs. In this paper, from the perspective of determinant, we give the necessity and sufficiency condition for the singularity of $\alpha(a, b_1, b_2, b_3, s)$ and $\beta(a, b_1, b_2, b_3)$, and we also prove that the probability that any given $\alpha(a, b_1, b_2, b_3, s)$ is a singular graph is equal to $\frac{35}{64}$, the probability that any given $\beta(a, b_1, b_2, b_3)$ is a singular graph is equal to $\frac{9}{16}$.

We will prove the following results in Section 3.

Theorem 1.1. *The graph $G = \alpha(a, b_1, b_2, b_3, s)$ is singular if and only if one of the following holds.*
(i) $4|a$.
(ii) *Three $b_i (i = 1, 2, 3)$ are all odd numbers.*
(iii) *Exactly two of the three $b_i (i = 1, 2, 3)$ are odd numbers. (1) The length of the cycle formed by the two odd paths in G is a multiple of 4; (2) a is even, s is odd.*
(iv) *Exactly one of three $b_i (i = 1, 2, 3)$ is odd number. a is even, s is even, and the length of the cycle formed by the two even paths in G is a multiple of 4.*
(v) *Three $b_i (i = 1, 2, 3)$ are all even numbers. a is even, s is odd.*

Corollary 1.2. *The graph $G = \beta(a, b_1, b_2, b_3)$ is singular if and only if one of the following holds.*
(i) $4|a$.
(ii) *Three $b_i (i = 1, 2, 3)$ are all odd numbers.*
(iii) *Exactly two of the three $b_i (i = 1, 2, 3)$ are odd numbers. (1) The length of the cycle formed by the two odd paths in G is a multiple of 4; (2) a is even.*
(iv) *Three $b_i (i = 1, 2, 3)$ are all even numbers. a is even.*

Theorem 1.3. *The probability that any given $\alpha(a, b_1, b_2, b_3, s)$ is a singular graph is equal to $\frac{35}{64}$, and the probability that any given $\beta(a, b_1, b_2, b_3)$ is a singular graph is equal to $\frac{9}{16}$.*

By Theorem 1.1 and Corollary 1.2, we know that the graph $\alpha(a, b_1, b_2, b_3, s)$ ($\beta(a, b_1, b_2, b_3)$) is singular if $4|a$ or three $b_i (i = 1, 2, 3)$ are all odd numbers or exactly two of the three $b_i (i = 1, 2, 3)$ are odd numbers and the length of the cycle formed by the two odd paths in G is a multiple of 4. The smallest singular graph with respect to the number of vertices in α-(β-) graphs is $\alpha(3, 2, 3, 3, 2)$ ($\beta(3, 2, 3, 3)$), it has 7 (6) vertices. By Theorem 1.3, we find that more than half of the α-(β-)graphs are singular, that is, the chemical properties of more than half of the chemical molecules with a molecular α-(β-) graphs are active and unstable by [7].

2. Some lemmas

Lemma 2.1. *[21] Suppose $G = G_1 \cup G_2 \cup \cdots \cup G_t$, where $G_i (i = 1, 2, ..., t)$ are connected components of G. Then $\eta(G) = \sum_{i=1}^{t} \eta(G_i)$. Equivalently, G is non-singular if and only if each $G_i (i = 1, 2, \ldots, t)$ is non-singular.*

Lemma 2.2. *[21] Let G be a graph with a pendant vertex v and a pendant edge vw, and let $H = G - v - w$ be the induced subgraph of G obtained by deleting the vertices v and w together with the edges incident to w. Then $\eta(G) = \eta(H)$. Equivalently, G non-singular if and only if H is non-singular.*

A subgraph H of G is an *elementary subgraph* (*Sachs subgraph*) if each component of H is either a copy of K_2 or a cycle of G. A *spanning elementary subgraph* (*spanning Sachs subgraph*) of G is an elementary subgraph which contains all vertices of G. A *matching* in G is a set of edges with no shared vertices and a *perfect matching* is a matching that saturates every vertex of G. Obviously, the number of vertices of a graph G with a perfect matching must be even.

Lemma 2.3. *[5] Let G be a graph of order n, $A(G)$ be the adjacency matrix of G, then*

$$\det(A(G)) = (-1)^n \sum_{H \in \mathcal{H}} (-1)^{p(H)} 2^{c(H)},$$

where $\mathcal{H}$ is the set of spanning Sachs subgraphs of G, $p(H)$ and $c(H)$ are the number of connected components and cycles in H, respectively.

3. Proofs of main results

3.1. Proof of Theorem 1.1

Note that $|V(G)| = a + b_1 + b_2 + b_3 + s - 6$. According to the parity of b_1, b_2, b_3 and the symmetry of graph $\alpha(a, b_1, b_2, b_3, s)$, we divided it into the following four cases to prove.

Case 1. Three $b_i (i = 1, 2, 3)$ are all odd numbers. In this case, G has no spanning Sachs subgraph, so G is singular.

Case 2. Exactly two of the three $b_i (i = 1, 2, 3)$ are odd numbers. Without losing generality, we assume that b_1 is an even number, b_2 and b_3 are two odd numbers. There are four subcases.

Subcase 2.1. Both a and s are odd numbers. In this subcase, G has no spanning Sachs subgraph containing two cycles, otherwise, the two cycles must be: $C_a \cup C_{b_i+b_j-2}$, $i, j = 1, 2, 3$, after deleting the above two cycles, the number of remaining vertices on the path P_s is odd, can not form a perfect matching. The spanning Sachs subgraph of G containing one cycle is $C_{b_2+b_3-2} \cup \frac{a+b_1+s-4}{2} P_2$ (In fact, there are four kinds of cycles: C_a, $C_{b_1+b_2-2}$, $C_{b_1+b_3-2}$ and $C_{b_2+b_3-2}$, but the first three kinds of the above cycles cannot form the spanning Sachs subgraph, because the subgraph obtained by deleting them does not contain perfect matching.), G has two perfect matchings. By Lemma 2.3,

$$\det(A(G)) = (-1)^n [(-1)^{\frac{a+b_1+s-4}{2}+1} \times 2 + (-1)^{\frac{a+b_1+b_2+b_3+s-6}{2}} \times 2].$$

So, G is singular if and only if

$$(-1)^{\frac{a+b_1+s-4}{2}+1} \times 2 + (-1)^{\frac{a+b_1+b_2+b_3+s-6}{2}} \times 2 = 0. \tag{1}$$

Simplify Eq (1), we get

$$(-1)^{\frac{b_2+b_3-2}{2}} - 1 = 0. \tag{2}$$

Equation (2) holds if and only if $4|b_2 + b_3 - 2$.

Subcase 2.2. a is odd, s is even. In this subcase, the spanning Sachs subgraphs of G containing two cycles is

$$C_a \cup C_{b_2+b_3-2} \cup \frac{b_1 + s - 4}{2}P_2,$$

and there are two spanning Sachs subgraphs of G containing one cycle, they have the form:

$$C_a \cup \frac{b_1 + b_2 + b_3 + s - 6}{2}P_2,$$

G has no perfect matching. By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{b_1+s-4}{2}+2} \times 2^2 + (-1)^{\frac{b_1+b_2+b_3+s-6}{2}+1} \times 2^2 = 0. \tag{3}$$

Equation (3) holds if and only if $4|b_2 + b_3 - 2$.

Subcase 2.3. a is even, s is odd. In this subcase, G has no spanning Sachs subgraph, so G is singular.

Subcase 2.4. Both a and s are even. In this subcase, the spanning Sachs subgraph of G containing two cycles is

$$C_a \cup C_{b_2+b_3-2} \cup \frac{b_1 + s - 4}{2}P_2,$$

and there are four spanning Sachs subgraphs of G containing one cycle, they have two of the following two forms:

$$C_a \cup \frac{b_1 + b_2 + b_3 + s - 6}{2}P_2, \;\; C_{b_2+b_3-2} \cup \frac{a + b_1 + s - 4}{2}P_2,$$

G has four perfect matchings. By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{b_1+s-4}{2}+2} \times 2^2 + (-1)^{\frac{b_1+b_2+b_3+s-6}{2}+1} \times 2^2 + (-1)^{\frac{a+b_1+s-4}{2}+1} \times 2^2 + (-1)^{\frac{a+b_1+b_2+b_3+s-6}{2}} \times 4 = 0. \tag{4}$$

Simplify Eq (4), we get

$$(-1)^{\frac{b_1}{2}} + (-1)^{\frac{b_1+b_2+b_3}{2}} - (-1)^{\frac{a+b_1}{2}} - (-1)^{\frac{a+b_1+b_2+b_3}{2}} = 0. \tag{5}$$

Equation (5) is equivalent to

$$[(-1)^{\frac{a}{2}} - 1][(-1)^{\frac{b_1}{2}} + (-1)^{\frac{b_1+b_2+b_3}{2}}] = 0. \tag{6}$$

Equation (6) holds if and only if $4|a$ or $4|b_2 + b_3 - 2$.

Case 3. Exactly one of the three $b_i (i = 1, 2, 3)$ is odd. Without losing generality, we assume that b_3 is the odd number, b_1 and b_2 are two even numbers. There are four subcases.

Subcase 3.1. Both a and s are odd numbers. In this subcase, G has no spanning Sachs subgraph containing two cycles. The spanning Sachs subgraphs of G containing one cycle are

$$C_a \cup \frac{b_1 + b_2 + b_3 + s - 6}{2}P_2, \;\; C_{b_1+b_3-2} \cup \frac{a + b_2 + s - 4}{2}P_2,$$

and

$$C_{b_2+b_3-2} \cup \frac{a+b_1+s-4}{2}P_2,$$

G has no perfect matching. By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{b_1+b_2+b_3+s-6}{2}+1} \times 2 + (-1)^{\frac{a+b_2+s-4}{2}+1} \times 2 + (-1)^{\frac{a+b_1+s-4}{2}+1} \times 2 = 0.$$

But this is impossible. So G is non-singular.

Subcase 3.2. a is odd, s is even. In this subcase, the spanning Sachs subgraphs of G containing two cycles are

$$C_a \cup C_{b_1+b_3-2} \cup \frac{b_2+s-4}{2}P_2$$

and

$$C_a \cup C_{b_2+b_3-2} \cup \frac{b_1+s-4}{2}P_2,$$

G has no spanning Sachs subgraph containing one cycle. G has one perfect matching. By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{b_2+s-4}{2}+2} \times 2^2 + (-1)^{\frac{b_1+s-4}{2}+2} \times 2^2 + (-1)^{\frac{a+b_1+b_2+b_3+s-6}{2}} = 0.$$

But this is impossible. So G is non-singular.

Subcase 3.3. a is even, s is odd. In this subcase, G has no spanning Sachs subgraph containing two cycles. The spanning Sachs subgraph of G containing one cycle is

$$C_a \cup \frac{b_1+b_2+b_3+s-6}{2}P_2,$$

G has two perfect matchings. By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{b_1+b_2+b_3+s-6}{2}+1} \times 2 + (-1)^{\frac{a+b_1+b_2+b_3+s-6}{2}} \times 2 = 0. \tag{7}$$

Simplify Eq (7), we get

$$(-1)^{\frac{a}{2}} - 1 = 0. \tag{8}$$

Equation (8) holds if and only if $4|a$.

Subcase 3.4. Both a and s are even numbers. In this subcase, the spanning Sachs subgraphs of G containing two cycles are

$$C_a \cup C_{b_1+b_3-2} \cup \frac{b_2+s-4}{2}P_2,$$

and

$$C_a \cup C_{b_2+b_3-2}\frac{b_1+s-4}{2}P_2.$$

There are four spanning Sachs subgraphs of G containing one cycle, they have two of the following two forms:

$$C_{b_1+b_3-2} \cup \frac{a+b_2+s-4}{2}P_2, \;\; C_{b_2+b_3-2} \cup \frac{a+b_1+s-4}{2}P_2,$$

G has no perfect matching. By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{b_2+s-4}{2}+2}\times 2^2+(-1)^{\frac{b_1+s-4}{2}+2}\times 2^2+(-1)^{\frac{a+b_2+s-4}{2}+1}\times 2^2+(-1)^{\frac{a+b_1+s-4}{2}+1}\times 2^2=0. \tag{9}$$

Simplify Eq (9), we get

$$[(-1)^{\frac{a}{2}}-1][(-1)^{\frac{b_1}{2}}+(-1)^{\frac{b_2}{2}}]=0. \tag{10}$$

Equation (10) holds if and only if $4|a$ or $4|b_1+b_2-2$.

Case 4. Three $b_i(i=1,2,3)$ are all even numbers. There are four subcases.

Subcase 4.1. Both a and s are odd numbers. In this subcase, G has no spanning Sachs subgraph containing two cycles. The spanning Sachs subgraphs of G containing one cycle are

$$C_{b_1+b_2-2}\cup\frac{a+b_3+s-4}{2}P_2,\ \ C_{b_1+b_3-2}\cup\frac{a+b_2+s-4}{2}P_2,$$

and

$$C_{b_2+b_3-2}\cup\frac{a+b_1+s-4}{2}P_2,$$

G has three perfect matchings. By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{a+b_3+s-4}{2}+1}\times 2+(-1)^{\frac{a+b_2+s-4}{2}+1}\times 2+(-1)^{\frac{a+b_1+s-4}{2}+1}\times 2+(-1)^{\frac{a+b_1+b_2+b_3+s-6}{2}}\times 3=0.$$

But this is impossible. So G is non-singular.

Subcase 4.2. a is odd, s is even. In this subcase, the spanning Sachs subgraphs of G containing two cycles are

$$C_a\cup C_{b_1+b_2-2}\cup\frac{b_3+s-4}{2}P_2,\ \ C_a\cup C_{b_1+b_3-2}\cup\frac{b_2+s-4}{2}P_2,$$

and

$$C_a\cup C_{b_2+b_3-2}\cup\frac{b_1+s-4}{2}P_2,$$

There are three spanning Sachs subgraphs of G containing one cycle, they all has the form:

$$C_a\cup\frac{b_1+b_2+b_3+s-6}{2}P_2,$$

G has no perfect matching. By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{b_3+s-4}{2}+2}\times 2^2+(-1)^{\frac{b_2+s-4}{2}+2}\times 2^2+(-1)^{\frac{b_1+s-4}{2}+2}\times 2^2+(-1)^{\frac{b_1+b_2+b_3+s-6}{2}+1}\times 2\times 3=0.$$

But this is impossible. So G is non-singular.

Subcase 4.3. a is even, s is odd. In this subcase, G has no spanning Sachs subgraph, so G is singular.

Subcase 4.4. Both a and s are even numbers. In this subcase, the spanning Sachs subgraphs of G containing two cycles are

$$C_a\cup C_{b_1+b_2-2}\cup\frac{b_3+s-4}{2}P_2,\ \ C_a\cup C_{b_1+b_3-2}\cup\frac{b_2+s-4}{2}P_2,$$

and

$$C_a\cup C_{b_2+b_3-2}\cup\frac{b_1+s-4}{2}P_2.$$

The spanning Sachs subgraphs containing one cycle includes: three $C_a \cup \frac{b_1+b_2+b_3+s-6}{2}P_2$, two $C_{b_1+b_2-2} \cup \frac{a+b_3+s-4}{2}P_2$, two $C_{b_1+b_3-2} \cup \frac{a+b_2+s-4}{2}P_2$ and two $C_{b_2+b_3-2} \cup \frac{a+b_1+s-4}{2}P_2$. G has six perfect matchings. By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{b_3+s-4}{2}+2} \times 2^2 + (-1)^{\frac{b_2+s-4}{2}+2} \times 2^2 + (-1)^{\frac{b_1+s-4}{2}+2} \times 2^2 + (-1)^{\frac{b_1+b_2+b_3+s-6}{2}+1} \times 2 \times 3$$

$$+(-1)^{\frac{a+b_3+s-4}{2}+1} \times 2 \times 2 + (-1)^{\frac{a+b_2+s-4}{2}+1} \times 2 \times 2 + (-1)^{\frac{a+b_1+s-4}{2}+1} \times 2 \times 2 + (-1)^{\frac{a+b_1+b_2+b_3+s-6}{2}} \times 6 = 0. \quad (11)$$

Multiply both sides of Eq (11) by

$$(-1)^{\frac{a+b_1+b_2+b_3+s-6}{2}} \times \frac{1}{2},$$

we get

$$[(-1)^{\frac{a+b_1+b_2-2}{2}} + (-1)^{\frac{a+b_1+b_3-2}{2}} + (-1)^{\frac{a+b_2+b_3-2}{2}}] \times 2 + [(-1)^3 \times (-1)^{\frac{b_1+b_2-2}{2}}$$

$$+(-1)^3 \times (-1)^{\frac{b_1+b_3-2}{2}} + (-1)^3 \times (-1)^{\frac{b_2+b_3-2}{2}}] \times 2 + [(-1)^{\frac{a}{2}+1} \times 3 + 3] = 0. \quad (12)$$

Simplify Eq (12), we get

$$[(-1)^{\frac{a}{2}} - 1][(-1)^{\frac{b_1+b_2-2}{2}} + (-1)^{\frac{b_1+b_3-2}{2}} + (-1)^{\frac{b_2+b_3-2}{2}}] \times 2 + [1 - (-1)^{\frac{a}{2}}] \times 3 = 0. \quad (13)$$

That is

$$[(-1)^{\frac{a}{2}} - 1][((-1)^{\frac{b_1+b_2-2}{2}} + (-1)^{\frac{b_1+b_3-2}{2}} + (-1)^{\frac{b_2+b_3-2}{2}}) \times 2 - 3] = 0. \quad (14)$$

Equation (14) holds if and only if

$$(-1)^{\frac{a}{2}} - 1 = 0.$$

So G is singular if and only if $4|a$.

To sum up, we have completed the proof of Theorem 1.1.

Take $s = 1$ in Theorem 1.1 and Corollary 1.2 is obviously.

Now, we prove Theorem 1.3.

3.2. *Proof of Theorem 1.3*

Let X be a random event, $p(X)$ represents the probability that event X will occur, $p(XY)$ represents the probability that events X and Y occur simultaneously, and $p(Y|X)$ represents the probability that event Y will occur under the condition that event X occurs.

Let U be the random event: the graph $\alpha(a, b_1, b_2, b_3, s)$ is a singular graph.

A indicates the random event: $4|a$.

B_1 indicates the random event: three $b_i(i = 1, 2, 3)$ are all odd numbers.

C_1 indicates the random event: exactly two of the three $b_i(i = 1, 2, 3)$ are odd numbers, and the length of the cycle formed by the two odd paths in G is a multiple of 4.

C_2 indicates the random event: exactly two of the three $b_i(i = 1, 2, 3)$ are odd numbers, a is even, and s is odd.

Suppose $B_2 = C_1 \cup C_2$.

B_3 indicates the random event: exactly one of three $b_i(i = 1, 2, 3)$ is odd number. a is even, s is even, and the length of the cycle formed by the two even paths in G is a multiple of 4.

B_4 indicates the random event: three $b_i(i = 1, 2, 3)$ are all even numbers. a is even, s is odd.

Then

$$p(A) = \frac{1}{4}, \quad p(B_1) = \binom{3}{3}(\frac{1}{2})^3 = \frac{1}{8},$$

$$p(B_2) = p(C_1 \cup C_2) = p(C_1) + p(C_2) - p(C_1C_2) = \binom{3}{2}(\frac{1}{2})^3(\frac{1}{2} + \frac{1}{4} - \frac{1}{8}) = \frac{15}{64},$$

$$p(B_3) = \binom{3}{1}(\frac{1}{2})^3 \times (\frac{1}{2})^3 = \frac{3}{64}, \quad p(B_4) = \binom{3}{0}(\frac{1}{2})^3 \times (\frac{1}{2})^2 = \frac{1}{32},$$
$$p(AB_1) = p(A)p(B_1) = \frac{1}{4} \times \frac{1}{8} = \frac{1}{32},$$

$$p(AB_2) = p(A)p(B_2|A) = p(A)[p(C_1|A) + p(C_2|A) - p(C_1C_2|A)] = \frac{1}{4}\binom{3}{2}(\frac{1}{2})^3(\frac{1}{2} + \frac{1}{2} - \frac{1}{4}) = \frac{9}{128},$$

$$p(AB_3) = p(A)p(B_3|A) = \frac{1}{4}\binom{3}{1}(\frac{1}{2})^3 \times (\frac{1}{2})^2 = \frac{3}{128},$$
$$p(AB_4) = p(A)p(B_4|A) = \frac{1}{4}\binom{3}{0}(\frac{1}{2})^3 \times \frac{1}{2} = \frac{1}{64}.$$

So

$$\begin{aligned}
p(U) &= p(A \cup B_1 \cup B_2 \cup B_3 \cup B_4) \\
&= p(A) + p(B_1) + p(B_2) + p(B_3) + p(B_4) - p(AB_1) - p(AB_2) - p(AB_3) - p(AB_4) \\
&= \frac{1}{4} + \frac{1}{8} + \frac{15}{64} + \frac{3}{64} + \frac{1}{32} - \frac{1}{32} - \frac{9}{128} - \frac{3}{128} - \frac{1}{64} = \frac{35}{64}.
\end{aligned}$$

Let V be the random event: the graph $\beta(a, b_1, b_2, b_3)$ is a singular graph.

A indicates the random event: $4|a$.

B_1 indicates the random event: three $b_i (i = 1, 2, 3)$ are all odd numbers.

C_1 indicates the random event: exactly two of the three $b_i (i = 1, 2, 3)$ are odd numbers, and the length of the cycle formed by the two odd paths in G is a multiple of 4.

C_2' indicates the random event: exactly two of the three $b_i (i = 1, 2, 3)$ are odd numbers, a is even.

Suppose $B_2' = C_1 \cup C_2'$.

B_4' represents a random event: three $b_i (i = 1, 2, 3)$ are all even numbers, a is even.

Then

$$p(A) = \frac{1}{4}, \quad p(B_1) = \frac{1}{8}, \quad p(AB_1) = \frac{1}{32},$$

$$p(B_2') = p(C_1 \cup C_2') = p(C_1) + p(C_2') - p(C_1C_2') = \binom{3}{2}(\frac{1}{2})^3(\frac{1}{2} + \frac{1}{2} - \frac{1}{4}) = \frac{9}{32},$$

$$p(B_4') = \binom{3}{0}(\frac{1}{2})^3 \times (\frac{1}{2}) = \frac{1}{16},$$

$$p(AB_2') = p(A)p(B_2'|A) = p(A)[p(C_1|A) + p(C_2'|A) - p(C_1C_2'|A)] = \frac{1}{4}\binom{3}{2}(\frac{1}{2})^3(\frac{1}{2} + 1 - \frac{1}{2}) = \frac{3}{32},$$

$$p(AB_4') = p(A)p(B_4'|A) = \frac{1}{4}\binom{3}{0}(\frac{1}{2})^3 = \frac{1}{32}.$$

So

$$\begin{aligned} p(V) &= p(A \cup B_1 \cup B_2' \cup B_4') \\ &= p(A) + p(B_1) + p(B_2') + p(B_4') - p(AB_1) - p(AB_2') - p(AB_4') \\ &= \frac{1}{4} + \frac{1}{8} + \frac{9}{32} + \frac{1}{16} - \frac{1}{32} - \frac{3}{32} - \frac{1}{32} = \frac{9}{16}. \end{aligned}$$

Hence we have finished the proof of Theorem 1.3.

4. Further discussion

We know that a connected graph G is a tree, a unicycle graph, a bicyclic graph and a tricyclic graph if and only if

$$|E(G)| = |V(G)| - 1,\ |V(G)|,\ |V(G)| + 1 \text{ and } |V(G)| + 2,$$

respectively. Merging one end of the path P_s with one vertex of C_a, and merging the other end of the path P_s with one vertex of C_b, the resulting graph is called the ∞-graph, denoted by $\infty(a, s, b)$, see Figure 3.

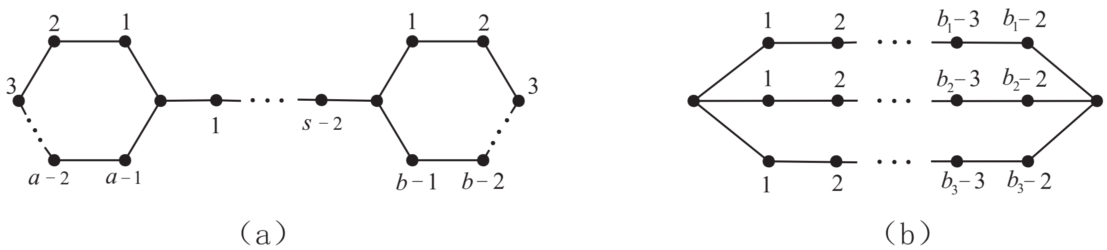

Figure 3. (a) The graph $\infty(a, s, b)$; (b) The graph $\theta(b_1, b_2, b_3)$.

Every unicyclic graph contains a cycle C_n as its induced subgraph. While the cycle C_n is singular if and only if $4|n$. Bicyclic graphs can be divided into two kinds, one is ∞-graph as its induced subgraph, the other is θ-graph as its induced subgraph.

Theorem 4.1. *The graph $G = \theta(b_1, b_2, b_3)$ is singular if and only if one of the following holds.*
(*i*) *Three $b_i(i = 1, 2, 3)$ are all odd numbers.*
(*ii*) *The parity of the three $b_i(i = 1, 2, 3)$ is not the same, and the length of the cycle formed by the two paths with the same parity in G is a multiple of 4.*

Proof. Note that $|V(G)| = b_1 + b_2 + b_3 - 4$. According to the parity of b_1, b_2, b_3 and the symmetry of graph $\theta(b_1, b_2, b_3)$, we divided it into the following four cases to prove.

Case 1. Three $b_i (i = 1, 2, 3)$ are all odd numbers. In this case, G has no spanning Sachs subgraph, so G is singular.

Case 2. Exactly two of the three $b_i (i = 1, 2, 3)$ are odd numbers. Without losing generality, we assume that b_1 is an even number, b_2 and b_3 are two odd numbers. The spanning Sachs subgraph of G containing one cycle is

$$C_{b_2+b_3-2} \cup \frac{b_1 - 2}{2} P_2,$$

G has two perfect matchings. By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{b_1-2}{2}+1} \times 2 + (-1)^{\frac{b_1+b_2+b_3-4}{2}} \times 2 = 0. \tag{15}$$

Simplify Eq (15), we get

$$(-1)^{\frac{b_2+b_3-2}{2}} - 1 = 0. \tag{16}$$

Equation (16) holds if and only if $4|b_2 + b_3 - 2$.

Case 3. Exactly one of the three $b_i (i = 1, 2, 3)$ is odd. Without losing generality, we assume that b_3 is the odd number, b_1 and b_2 are two even numbers. The spanning Sachs subgraph of G containing one cycle are

$$C_{b_1+b_3-2} \cup \frac{b_2 - 2}{2} P_2,$$

and

$$C_{b_2+b_3-2} \cup \frac{b_1 - 2}{2} P_2,$$

G has no perfect matching. By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{b_2-2}{2}+1} \times 2 + (-1)^{\frac{b_1-2}{2}+1} \times 2 = 0. \tag{17}$$

Equation (17) holds if and only if $4|b_1 + b_2 - 2$.

Case 4. Three $b_i (i = 1, 2, 3)$ are all even numbers. The spanning Sachs subgraph of G containing one cycle are

$$C_{b_1+b_2-2} \cup \frac{b_3 - 2}{2} P_2, \ \ C_{b_1+b_3-2} \cup \frac{b_2 - 2}{2} P_2,$$

and

$$C_{b_2+b_3-2} \cup \frac{b_1 - 2}{2} P_2,$$

G has three perfect matchings. By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{b_3-2}{2}+1} \times 2 + (-1)^{\frac{b_2-2}{2}+1} \times 2 + (-1)^{\frac{b_1-2}{2}+1} \times 2 + (-1)^{\frac{b_1+b_2+b_3-4}{2}} \times 3 = 0.$$

But this is impossible. So G is non-singular.

Hence we have finished the proof of Theorem 4.1.

Theorem 4.2. *The graph $G = \infty(a, s, b)$ is singular if and only if one of the following holds.*
(i) *The length of at least one of the two cycles is a multiple of 4, i.e. $4|a$ or $4|b$.*
(ii) *s is odd. (1) The length of two cycles is even; (2) The length of two cycles is odd, but module 4 is not congruence.*

Proof. Note that $|V(G)| = a+b+s-2$. According to the parity of a, b, s and the symmetry of graph $\infty(a, s, b)$, we divided it into the following two cases to prove.

Case 1. s is an even number.

Subcase 1.1. Both a and b are even numbers. Where the spanning Sachs subgraph of G with two cycles is $C_a \cup C_b \cup \frac{s-2}{2}P_2$. The spanning Sachs subgraphs containing one cycle includes two $C_a \cup \frac{b+s-2}{2}P_2$, two $C_b \cup \frac{a+s-2}{2}P_2$; G has 4 perfect matchings.

By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{s-2}{2}+2} \times 2^2 + (-1)^{\frac{b+s-2}{2}+1} \times 2 \times 2 + (-1)^{\frac{a+s-2}{2}+1} \times 2 \times 2 + (-1)^{\frac{a+b+s-2}{2}} \times 4 = 0,$$

multiply both sides of above equation by $(-1)^{\frac{a+b+s-2}{2}} \times \frac{1}{4}$, we get

$$(-1)^{\frac{a+b}{2}+2} + (-1)^{\frac{a}{2}+1} + (-1)^{\frac{b}{2}+1} + 1 = 0,$$

that is

$$(-1)^{\frac{a+b}{2}} - (-1)^{\frac{a}{2}} - (-1)^{\frac{b}{2}} + 1 = 0,$$

the above equation holds if and only if

$$[(-1)^{\frac{a}{2}} - 1][(-1)^{\frac{b}{2}} - 1] = 0,$$

if and only if $4|a$ or $4|b$.

Subcase 1.2. Only one of a and b is even. Without losing generality, we assume that a is an even number, b is an odd number. Where the spanning Sachs subgraph of G with two cycles is: $C_a \cup C_b \cup \frac{s-2}{2}P_2$. The spanning Sachs subgraph of G with one cycle includes two $C_b \cup \frac{a+s-2}{2}P_2$. G has no perfect matching.

By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{s-2}{2}+2} \times 2^2 + (-1)^{\frac{a+s-2}{2}+1} \times 2 \times 2 = 0,$$

multiply both sides of above equation by $(-1)^{\frac{a+s-2}{2}} \times \frac{1}{4}$, we get

$$(-1)^{\frac{a}{2}} - 1 = 0,$$

if and only if $4|a$.

Subcase 1.3. Both a and b are odd numbers. Where the spanning Sachs subgraph of G containing two cycles is $C_a \cup C_b \cup \frac{s-2}{2}P_2$. G has no spanning Sachs subgraph containing one cycle, and G has one perfect matching.

By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{s-2}{2}+2} \times 2^2 + (-1)^{\frac{a+b+s-2}{2}} = 0.$$

But this is impossible. So G is non-singular.

Case 2. s is an odd number.

Subcase 2.1. Both a and b are even numbers. In this subcase, G has no spanning Sachs subgraph, so G is singular.

Subcase 2.2. Only one of a and b is even. Without losing generality, we assume that a is an even number, b is an odd number. There is no the spanning Sachs subgraph of G with two cycles. The spanning Sachs subgraph of G with one cycle is $C_a \cup \frac{b+s-2}{2}P_2$. G has 2 perfect matchings.

By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{b+s-2}{2}+1} \times 2 + (-1)^{\frac{a+b+s-2}{2}} \times 2 = 0,$$

multiply both sides of above equation by $(-1)^{\frac{a+b+s-2}{2}} \times \frac{1}{2}$, we get

$$(-1)^{\frac{a}{2}} - 1 = 0,$$

if and only if $4|a$.

Subcase 2.3. Both a and b are odd numbers. Where there is no spanning Sachs subgraph of G with two cycles. The spanning Sachs subgraph of G with one cycle are: $C_a \cup \frac{b+s-2}{2}P_2$, $C_b \cup \frac{a+s-2}{2}P_2$. G has no perfect matching.

By Lemma 2.3, G is singular if and only if

$$(-1)^{\frac{b+s-2}{2}+1} \times 2 + (-1)^{\frac{a+s-2}{2}+1} \times 2 = 0,$$

multiply both sides of above equation by $(-1)^{\frac{a+s-2}{2}} \times \frac{1}{2}$, we get

$$(-1)^{\frac{a+b}{2}} - 1 = 0,$$

if and only if a and b module 4 is not congruence.

Hence we have finished the proof of Theorem 4.2.

By Theorems 4.1 and 4.2, it is easy to calculate that the probability that $\theta(b_1, b_2, b_3)$ is a singular graph is equal to $\frac{1}{2}$, the probability that $\infty(a, s, b)$ is a singular graph is equal to $\frac{17}{32}$.

Similarly to the bicyclic graphs, the tricyclic graphs can be divided into 15 kinds according to the induced subgraphs it contains. Denote the set of tricyclic graphs which contains the α-graph and the β-graph as its induced subgraphs by $\mathcal{T}(\alpha)$ and $\mathcal{T}(\beta)$, respectively. Then using Lemmas 2.1 and 2.2, Theorems 1.1, 4.1 and 4.2, and Corollary 1.2, we can determine the singular graphs in the $\mathcal{T}(\alpha)$ and $\mathcal{T}(\beta)$.

For random graphs, some scholars have studied the variables on random graphs, for instance, Shang [25] obtained the lower and upper bounds of distance Estrada index of random bipartite graph. However, although the probability of singular graphs is mentioned, we do not consider the singularity of random graphs in this paper, so readers can study the singularity of random graphs.

5. Conclusions

In this paper, we first define two kinds of tricyclic graphs, the α-graph and β-graph, then give the necessity and sufficiency condition for the singularity of α-graph and β-graph, and obtain the probability that any given two kinds of graphs are singular graphs. Our results enriches the study of singularity of graphs.

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Conflict of interest

There are no conflicts interest for this paper.

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