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Research article

On the construction of stable periodic solutions for the dynamical motion of AC machines

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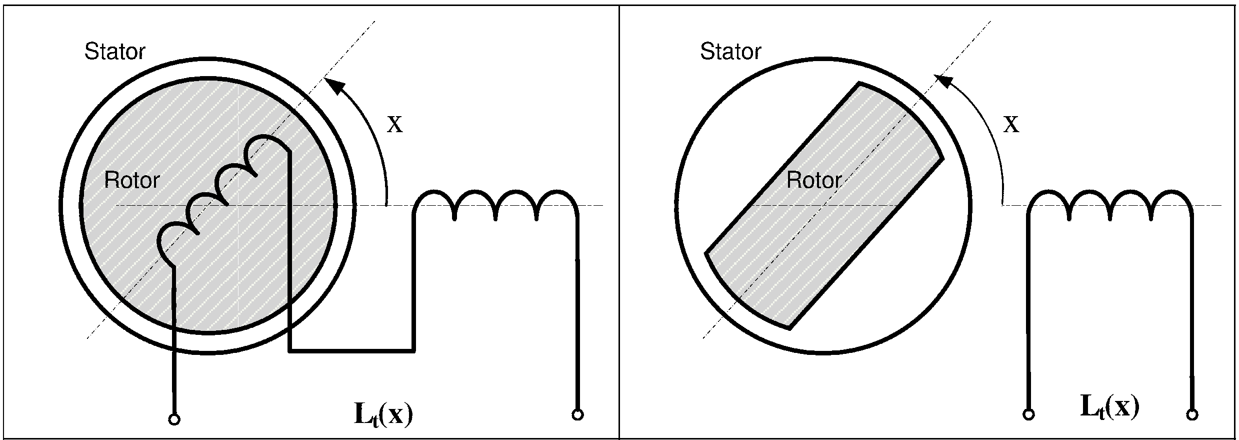
Abstract: This article discusses the stability of periodic responses for the dynamical motion of AC machines from the perspective of Lyapunov function approach. The dynamical motion of AC machines is prototypically modeled as an equivalent linear RLC series circuit with time-variant inductance represented by a linear differential equation with periodic coefficients. Based on the deduced stability conditions, some special identities among the equivalent circuit parameters to ensure the stability of responses and their periodic structures are concluded. Through these conditions, the periodic structure of responses is obtained by using the method of strained parameters. Through a comparison with the experimental results from the specialized practical literatures, a strong agreement with the obtained analytical results is achieved. In addition, from a practical point of views, some future points within the discussion are raised to improve the mathematical modeling of AC machines to obtain a better model and simulation.

Keywords: linear differential equations; stability theory; periodic solutions; perturbation techniques
Mathematics Subject Classification: 34A30, 34C25, 34D20, 70K60

1. Introduction

Nowadays, AC machines are considered an important type of electric power supply systems for every industrial establishment, since they convert mechanical energy into electrical energy (generators) or vice versa (motors). Most of the electrical energy is obtained from mechanical sources (e.g., waterfalls and wind) or by using a mechanical station as an intermediate stage such as steam and gas power stations. In such systems, the generation process is known as electromechanical energy conversion. Normally, a magnetic field is used as a conversion medium. That field can be obtained from a separate source or through some type of self-excitation which uses a part of the generated power for field production.

The basic operation of many types of electrical generators relies on periodic inductance time variation. This variation is possible, as shown in Figure 1, by using two methods. The first method depends on the variation of magnetic coupling between two series-connected coils, where one is fixed while the other is rotating. The second method is obtained if the magnetic reluctance to a coil MMF varies on rotation due to the magnetic saliency of the rotor member. For self-excitation, the varying inductance (regardless the source of variation) is connected to a capacitor with a suitable amount of capacitance that depends on the circuit parameters and mechanical speed. Consequently, the whole system can be described by using (non)linear differential equations. To ensure successful generation, the system has to have some residual magnetism in its iron parts and/or a capacitor charge. These amounts represent the necessary initial conditions required for a non-trivial solution. So, this type of generator represents a simple and reliable type of energy converter which is expected to be suitable for use with renewable energy sources and energy harvesting applications.



a) Mutual inductance variation | b) Magnetic reluctance variation

Figure 1. Basic construction of AC machines.

The dynamic analysis of such power machines is as necessary as the need to obtain particular sustained periodic stable output modes that have to be achieved. So, finding an exact mathematical model by considering real variations is needed to investigate the dynamic stability of the machines and their periodic responses under transient events such as those induced by the saturation effect, slow change in the resistances and the existence of relative movement between the stator and rotor. In most cases, AC machines in general might be considered as a dynamic RLC circuit which is represented by non-autonomous (non)linear second order differential equations, cf. [1, 2]. For instance, the existence of relative movement between the stator and the rotor in AC machines considers inductances as variable coefficients in the equivalent circuit structure, cf. [2, 3].

The stability analysis of a dynamical system is considered to be the key step that facilitates an understanding of how changes can affect such a system. Indeed, the design parameters and the design of stabilizing feedback controllers might cause a bounded or unbounded response. Recently, the appearance of the three common mathematical phenomena in nature, i.e, periodic coefficients, fractional order derivatives and time delays that are included in modern models of real processes makes the study of stability and control much more attractive, cf. [4–10]. Specifically, systems with periodic coefficients are often relevant to models of engineering and physical problems which have been studied carefully, as described by Mathieu in [11] for free oscillations of an elliptic membrane; also see [12–18].

It is generally agreed upon that, such systems with periodic coefficients are still a hot point in mathematical research. However, an analytical solution of the general Mathieu equation is still an attractive study for its applications. In particular, the study of parametric resonances (which are subharmonic resonances that exemplify the response of a system to a kind of periodic force) and transition curves require relatively high accuracy to ensure that the practical use does not depend on numerical calculations. In general, somehow, certain approaches are designed in order to introduce semi-analytical approximations of periodic solutions for such systems, cf. [19–22].

Due to the electrical construction of AC machines, they might be expressed by a circuit structure. Additionally, the difference between AC machines and general static machines is the existence of relative movement between the stator and the rotor of the former. Hence, some loop inductances of AC machines depend upon the rotor position, i.e., these inductances are considered to be variable coefficients, cf. [23, 24].

Consequently, regardless of the power measurements and control accessories, the complete governing equation of the natural response (y) in such AC machines under a periodic variation of inductances reads as follows:

$$(1 + h\cos 2x)y'' + \frac{Q}{\alpha}y' + \frac{1}{\alpha^2}y = 0, y(0) = c_1, y'(0) = c_2, \tag{1.1}$$

$$\omega_0 = \frac{1}{\sqrt{L_0 C}} > 0, \alpha = \frac{\omega}{\omega_0} > 0, Q = \frac{R}{\omega_0 L_0} \geq 0,$$

where h is a constant representing a system perturbation on the inductance due to the relative movement between the stator and rotor of the AC machine, L_0 is the inductance when $h \to 0$, C is the capacitance, R is the resistance, ω is the operating frequency due to the existence of h, and $x = \omega t$ (t refers to the time). However, if it is considered that, the system is undamped ($Q = 0$), then the solution can be easily obtained with the aid of Jacobian elliptic functions, cf. [25–27].

The contribution of this work is that: it is an investigation of the dynamic stability of linear dynamic models of AC machines via the Lyapunov second approach which is needed to construct approximate forms of stable periodic responses. Based on the equivalent circuit parameters, it is essential to obtain the intervals of such parameters to get a stable periodic output. Moreover, the prediction of resonances defined on the stable domains is considered. Consequently, the stability charts have been drawn to illustrate the stable and unstable regions separated by the transition curves via the method of strained parameters based on the deduced stability conditions. In addition to, this work is introduced to complete the work of the mathematical development for certain interesting engineering problems through the use of the harmonic balance and variational iteration methods in [23, 24].

This paper is organized as follows. In Section 2, we introduce the model of AC machines and the derivations of the stability conditions through the use of the Lyapunov function approach. In Section 3, the existence of a periodic solution is proved by using the method of strained parameters based on the stability conditions. In Section 4, transition curves are obtained with the aid of relations between the $\alpha - h$ parameters. In Section 5, we predict the stability of periodic motion by using the change of energy due to the perturbation and dissipation of the governing system. In the last section, the conclusion, discussion and future outlook are given.

2. Stability analysis

This section is devoted to obtaining the relationships among the model parameters in order to establish necessary and sufficient conditions for the asymptotic stability of the natural response. However, it would be better to apply the second approach of Lyapunov by constructing a suitable Lyapunov function for the required analysis. However, optimal construction of the Lyapunov function is dependent on the system parameters for the derivation of consistent conditions; this is required to obtain an asymptotically stable response and capture the ranges of parameters used for the construction of the periodic solutions, cf. [28–31]. To generalize the stability results for the case of time varying inductance, the governing equation is transformed to the following general form:

$$y'' + f(x)y' + g(x)y = 0, \tag{2.1}$$

where

$$f(x) = \frac{Q}{\alpha(1 + h\cos 2x)}, \; g(x) = \frac{1}{\alpha^2(1 + h\cos 2x)}. \tag{2.2}$$

In the proofs of the stated theorems, we propose different constructions for the Lyapunov function (V), which is required to approach a reliable accuracy of the real energy functional needed to obtain more specified conditions in the studied system.

Theorem 1. *The zero solution of the governing equation is uniformly asymptotically stable if the following conditions hold*

$$|h| < 1 and h < \frac{Q}{\alpha}. \tag{2.3}$$

Proof. <u>1^{st} **construction of** V**:**</u> Let the Lyapunov function with the aid of [32] be the following

$$V = V(x, y, y') = y^2 + cyy' + \frac{y'^2}{g(x)}, \tag{2.4}$$

where

$$c = min\{\frac{1}{\sqrt{\overline{g}}}, \frac{2\underline{g}\underline{p}}{\overline{g}(2\underline{g} + \overline{f}^2)}\} > 0, \quad \underline{g}, \overline{g}, \overline{f}, \underline{p} > 0 \tag{2.5}$$

and

$$p(x) = \frac{1}{2}\frac{g'(x)}{g(x)} + f(x). \tag{2.6}$$

Now, we have to prove that V is positive definite and decreasing, and that the first derivative (V') is negative definite until the zero solution of the governing equation becomes uniformly asymptotically stable. Thus, we will write Eq (2.4) as

$$V = \frac{1}{2}y^2 + \frac{1}{2}(y + cy')^2 + (\frac{1}{g(x)} - \frac{c^2}{2})y^2. \tag{2.7}$$

With the aid of Eq (2.5), and by using the following relation

$$\underline{g} \leq g(x) \leq \overline{g}, \tag{2.8}$$

Eq (2.7) is converted to the following inequality

$$V \geq \frac{1}{2}y^2 + (\frac{1}{\overline{g}} - \frac{c^2}{2})y'^2 \geq \frac{1}{2}(y^2 + \frac{y'^2}{\overline{g}}) > 0. \tag{2.9}$$

We also can write Eq (2.4) as follows

$$V = 2y^2 - (y - \frac{cy'}{2})^2 + (\frac{1}{g(x)} + \frac{c^2}{4})y'^2. \tag{2.10}$$

With the aid of Eq (2.8), Eq (2.10) is converted to the following inequality

$$V \leq 2y^2 + (\frac{1}{\underline{g}} + \frac{c^2}{4})y'^2. \tag{2.11}$$

From Eqs (2.7) and (2.10), V is positive definite and decreasing.

Now, we will calculate the first derivative of V which is given by

$$V' = (2y + cy')y' + (cy + \frac{2y}{g(x)}) - (g(x)y - f(x)y) - \frac{g'(x)y'^2}{g^2(x)}. \tag{2.12}$$

Eq (2.12) is converted to

$$V' = -cg(x)y^2 - cf(x)yy' - (\frac{2p(x)}{g(x)} - c)y'^2, \tag{2.13}$$

and with the aid of the following relations

$$|f(x)| \leq \overline{f}, \tag{2.14}$$

and

$$p(x) \geq \underline{p}, \tag{2.15}$$

Eq (2.13) is converted to the following inequality:

$$V' \leq -c[\underline{g}y^2 - \overline{f}\,|y|\,|y'| + (\frac{2\underline{p}}{c\overline{g}} - 1)y^2]. \tag{2.16}$$

From Eq (2.5), and by simplifying Eq (2.16), we get

$$V' \leq -c[\frac{\underline{g}y^2}{2} + \frac{\underline{g}}{2}(|y| - \frac{\overline{f}\,|y'|}{\underline{g}})^2 + (\frac{2\underline{p}}{c\overline{g}} - \frac{\overline{f}^2}{2\underline{g}} - 1)y'^2]. \tag{2.17}$$

Hence, with further simplification, we obtain

$$V' \leq -c[\frac{\underline{g}y^2}{2} + (\frac{2\underline{p}}{c\overline{g}} - \frac{\overline{f}^2}{2\underline{g}} - 1)y'^2] \leq -c[\frac{1}{2}\underline{g}y^2 + \frac{\underline{p}}{c\overline{g}}y'^2]. \tag{2.18}$$

Thus, from Eq (2.18), V' is negative definite.

Finally, the zero solution of the governing equation is uniformly asymptotically stable if the following conditions are fulfilled:

i. $|f(x)| \leq \overline{f}$, ii. $\underline{g} \leq g(x) \leq \overline{g}$, iii. $p(x) \geq \underline{p}$.

Now we will apply the previous conditions to our practical system.

Condition (i) $|f(x)| \leq \overline{f} \leq \frac{Q}{\alpha}$ is satisfied.

Condition (ii) $0 < \frac{1}{\alpha^2(1-h)} \leq g(x) \leq \frac{1}{\alpha^2(1+h)}$ is fulfilled if

$$|h| < 1. \tag{2.19}$$

Condition (iii) $p(x) = \frac{h\sin 2x + \frac{Q}{\alpha}}{1 + h\cos 2x} \geq \underline{p} > 0$ is fulfilled if

$$h < \left|\frac{Q}{\alpha}\right|. \tag{2.20}$$

2^{nd} construction of V: In this construction, the two functions $f(x)$ and $g(x)$ are bounded as follows:

$$|f(x)| < M_1, \quad |g(x)| < M_2, \quad |g'(x)| < M_3. \tag{2.21}$$

Now, with the aid of [33], let the Lyapunov function for this system be

$$V = \frac{1}{2}[y^2 + 2B\frac{yy'}{\sqrt{g(x)}} + \frac{y'^2}{g(x)}]. \tag{2.22}$$

The first derivative of V is given by

$$V' = \frac{1}{\sqrt{g(x)}}((\frac{-p(x)}{\sqrt{g(x)}} + B)y'^2 - Bp(x)yy' - Bg(x)y^2), \tag{2.23}$$

where

$$0 < B < min[1, \frac{\alpha_2}{2\sqrt{M_2}}, \frac{8\alpha_1^3\alpha_2}{(M_3 + 2\alpha_1 M_1)^2\sqrt{M_2}}], \quad \alpha_1, \alpha_2 > 0. \tag{2.24}$$

The zero solution of the governing equation is uniformly asymptotically stable if the following conditions are satisfied:

(i) $g(x) > \alpha_1 > 0$,
(ii) $p(x) > \alpha_2 > 0$.

Now we will apply the previous conditions to our practical system.

Condition (i) is satisfied if

$$|h| < 1. \tag{2.25}$$

Condition (ii) is satisfied if

$$h < \frac{Q}{\alpha}. \tag{2.26}$$

From [33], if the two conditions are fulfilled, then there are exist positive numbers B_1 and B_2 such that for $x > x_0 \geq 0$, the following inequalities hold

$$|y(x)| < B_1 e^{-B_2(x-x_0)} \quad , \quad |y'(x)| < B_1 e^{-B_2(x-x_0)}. \tag{2.27}$$

<u>3^{rd} **construction of** V**:**</u> In this construction, we follow [34] and let the Lyapunov function be

$$V = \frac{1}{2}y^2 + \frac{1}{2g(x)}y'^2, \tag{2.28}$$

with the first derivative given by

$$V' = -\frac{1}{2g(x)^2}(2g(x)f(x) + g'(x))y'^2. \tag{2.29}$$

The zero solution of the governing system is uniformly asymptotically stable if the following conditions are satisfied

(i) $\exists\, \underline{g}, \overline{g} > 0 \quad \forall\, x \geq 0 : 0 < \underline{g} \leq g(x) \leq \overline{g}$,
(ii) $\exists\, \overline{f} > 0 \quad \forall\, x \geq 0 : |f(x)| \leq \overline{f}$,
(iii) $\exists\, \underline{a} > 0 \quad \forall\, x \geq 0 : 0 < \underline{a} \leq g'(x) + 2g(x)f(x)$.

Condition (i) is satisfied if

$$|h| < 1, \tag{2.30}$$

and Condition (ii) is consequently satisfied.

Also, from Condition (iii), $g'(x) + 2g(x)f(x) = \dfrac{2}{\alpha^2(1 + h\cos 2x)^2}(h\sin 2x + \dfrac{Q}{\alpha}) > \underline{a} > 0$, is satisfied if

$$h < \frac{Q}{\alpha}. \tag{2.31}$$

Then, the two conditions hold.

<u>4^{th} **construction of** V**:**</u> In this construction, following [34], the Lyapunov function is

$$V = (y + Py')^2 + 2(q(x) - P^2)y'^2, \tag{2.32}$$

and the first derivative reads as follows:

$$V' = -2Pg(x)V(x) - d(x)y'^2, \tag{2.33}$$

where

$$P = min(\frac{1}{2\overline{f}}, \frac{E}{M + 4 + 4P^2\overline{g} + \overline{f}E}),$$
$$q(x) = \frac{2P^2 g(x) - Pf(x) + 1}{g(x)} \tag{2.34}$$
$$d(x) = -2P - q'(x) - 2Pg(x)q(x) + 2f(x) + 1,$$
$$E, M, \underline{g}, \overline{g}, \overline{f} > 0.$$

The zero solution of the governing equation is uniformly asymptotically stable if the following conditions are fulfilled:

(i) $\int_0^x g(s)ds \to \infty$ as $x \to \infty$,
(ii) $\forall\, x \geq 0 \quad 0 < \underline{g} \leq g(x) \leq \overline{g}$,
(iii) $\forall\, x \geq 0 \quad |f(x)| \leq \overline{f}$ and $|f'(x)| < M\,|g(x)|$,
(iv) $\forall\, x \geq 0 \quad 0 < Eg(x)^2 \leq g'(x) + 2g(x)f(x)$.

From Condition (i), we have

$$\frac{1}{\alpha^2}\int_0^x \frac{ds}{1 + h\cos 2s} = \frac{1}{\alpha^2\sqrt{1-h^2}}\tan^{-1}(\frac{\sqrt{1-h^2}\tan x}{1+h}), \quad h^2 < 1,$$

and the integration $\to \infty$ as $x \to \infty$ if $h \ll 1$; then, Condition (i) is satisfied.

Condition (ii) is satisfied if

$$|h| < 1. \tag{2.35}$$

From Condition (iii)

$$|f(x)| \leq \overline{f} \leq \frac{Q}{\alpha} \quad \text{and} \quad |f'(x)| < M\,|g(x)|$$

are satisfied if $Q < \frac{M}{\alpha}\frac{1-h}{h}$ and this is fulfilled if

$$|h| < 1. \tag{2.36}$$

Condition (iv) is satisfied if

$$h < \frac{Q}{\alpha}. \tag{2.37}$$

Hence, the conclusion holds and the proof of the theorem is complete. □

Theorem 2. *The zero solution of the governing equation is hyperbolic and asymptotically stable if the following condition is fulfilled*

$$\alpha > \frac{1}{\sqrt[5]{1-h^2}}\sqrt{1 - h^2(1 + \frac{Q^2}{4})}, \tag{2.38}$$

yielding

$$|h| < 1, \tag{2.39}$$
$$Q \leq \frac{2}{h}\sqrt{1-h^2}. \tag{2.40}$$

Proof. Let us transform the governing equation to the equivalent Hill's equation, cf. [35, 36], by imposing the conditions that $f(x)$ and $g(x)$ are continuously differentiable with

$$y(x) = z(x)e^{-\frac{1}{2}\int f(x)dx}. \tag{2.41}$$

Then, the governing equation is transformed to

$$z''(x) + \beta(x)z(x) = 0, \tag{2.42}$$

where

$$\beta(x) = g(x) - \frac{1}{2}f'(x) - \frac{1}{4}f^2(x). \tag{2.43}$$

The function $\beta(x)$ is continuous and π-periodic

$$\beta(x) = \frac{1}{\alpha^2(1 + h\cos 2x)} - \frac{Qh\sin 2x}{\alpha(1 + h\cos 2x)^2} - \frac{Q^2}{4\alpha^2(1 + h\cos 2x)^2}. \tag{2.44}$$

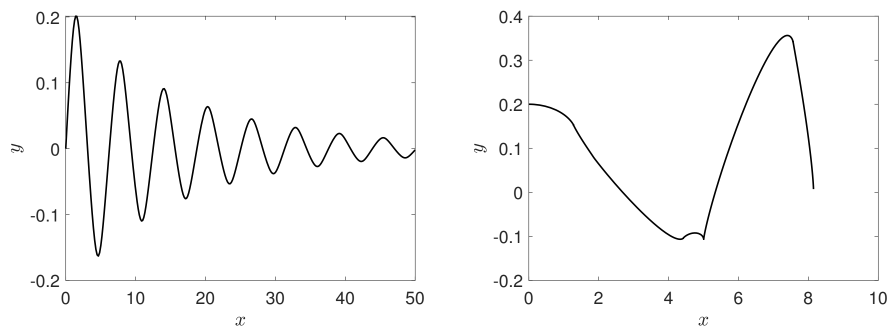
If the following conditions are fulfilled: i. $\int_0^\pi f(x)dx \geq 0$, ii. $\int_0^\pi \beta(x)dx \geq 0$, iii. $\int_0^\pi \beta(x)dx \leq \frac{2}{\pi}$, then solutions are bounded and the characteristic exponents of Eq (2.42) do not have a positive real part; then, the zero solution is hyperbolic and asymptotically stable, cf. [28].

Now, let us apply these conditions to the governing system. First, by using contour integration, we calculate the following definite integrals:

$$\int_0^\pi f(x)dx = \frac{\pi Q}{\alpha\sqrt{1-h^2}},$$
$$\int_0^\pi \beta(x)dx = \frac{\pi}{\alpha^2\sqrt{1-h^2}}[1 - \frac{h^2Q^2}{4(1-h^2)}];$$

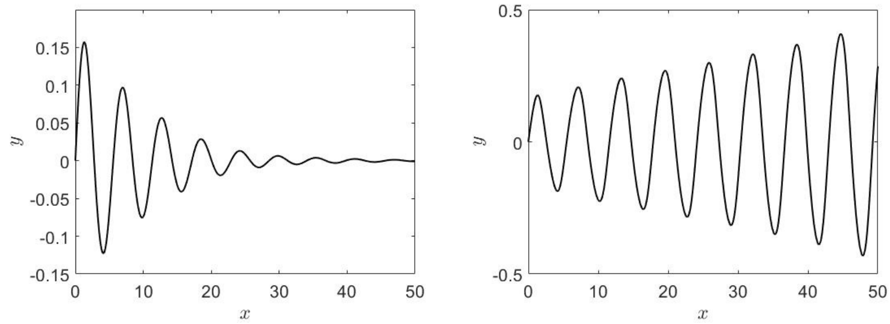
then, the conclusion holds. □

These results are verified numerically by constructing the figures shown in Figures 2–6 using the 4^{th}-order Runge-Kutta method. The stability of the solutions is clearly shown under the conditions derived from Theorems 1 and 2 as well as the instability.



(a) Stable state for $h < 1$ ($h = 0.1$, $Q = 0.2$ and $\alpha = 0.9$). **(b)** Unstable state for $h > 1$ ($h = 1.2$, $Q = 0.2$ and $\alpha = 0.9$).

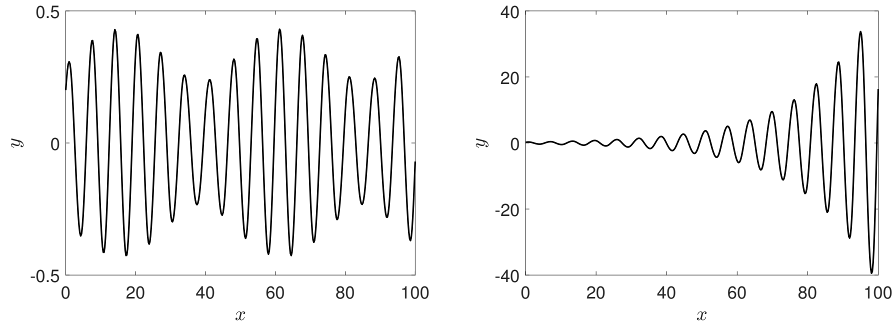
Figure 2. Stable and unstable states according to the conditions of Theorem 1.



(a) Stable state for $h < \dfrac{Q}{\alpha}$ ($h = 0.4$, $Q = 0.5$ and $\alpha = 1$).

(b) Unstable state for $h > \dfrac{Q}{\alpha}$ ($h = 0.5$, $Q = 0.2$ and $\alpha = 1$).

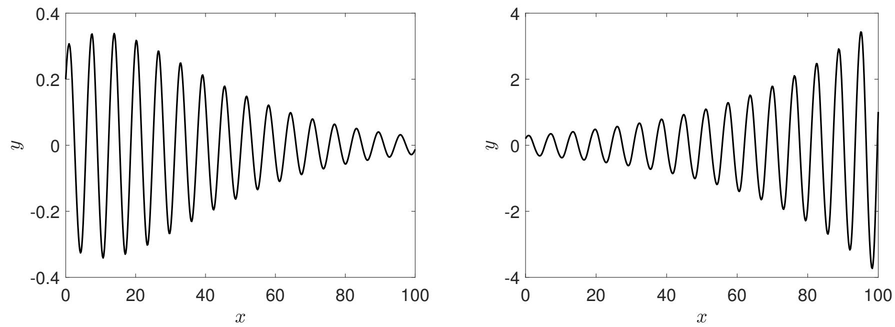
Figure 3. Stable and unstable states according to the conditions of Theorem 1.



(a) Stable solution at $h = 0.2$, $Q = 0$ and $\alpha = 1.1$.

(b) Unstable solution at $h = 0.2$, $Q = 0$ and $\alpha = 1$.

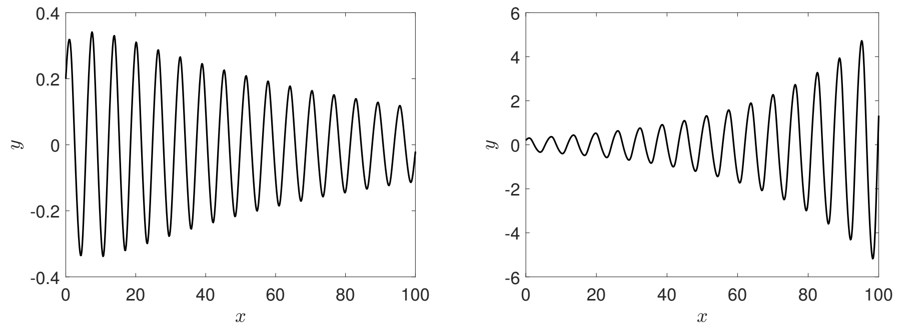
Figure 4. Stable and unstable states according to the conditions of Theorem 2.



(a) Stable solution at $h = 0.3$, $Q = 0.1$ and $\alpha = 1.1$.

(b) Unstable solution at $h = 0.3$, $Q = 0.1$ and $\alpha = 1.02$.

Figure 5. Stable and unstable states according to the conditions of Theorem 2.



(a) Stable solution at $h = 0.5$, $Q = 0.2$ and $\alpha = 1.17$.
(b) Unstable solution at $h = 0.5$, $Q = 0.2$ and $\alpha = 1.06$.

Figure 6. Stable and unstable states according to the conditions of Theorem 2.

3. Construction of periodic solutions

Based on the stability results presented in Section 2, in this section, we derive an approximate form of periodic solutions of the governing equation in accordance with some specific regions of $|h| < 1$. Hence, h can be a small perturbing parameter in the applied perturbation technique. However, several methods can be applied based on h such as the Linstedt-Poincare method, multiple scale method and averaging method, cf. [37–39]. An efficient method is needed to capture the behavior of the periodic solutions for specific values of h that yield somewhat accurate relations for the transition curves. In [40, 41], it is explicated that the straightforward expansion methods mostly fails to obtain a form of periodic solution and consequently the prediction of the transition curves for linear problems with periodic coefficients such as the classical Mathieu equation. Thus, here, we introduce the method of strained parameters to deduce the required results for the construction of periodic solutions and hence obtain a more accurate description of transition curves along them, cf. [42, 43].

The necessary condition for asymptotic stability is covered by $|h| < 1$ which allows us to construct a convergent stable periodic solution as a function of $y = y(x, h)$. We have eliminated the terms that produce secular parts which do not lead to convergent solutions with non-resonant parts.

In order to facilitate the perturbation methods, we scale the damping coefficient Q to be hQ_0; then, Eq (1.1) reads as follows:

$$y'' + \delta^2 y = -h\cos(2x)y'' - hQ_0\delta y', \tag{3.1}$$

where

$$\delta = \frac{1}{\alpha} = \frac{\omega_0}{\omega}. \tag{3.2}$$

By using the method of strained parameters, δ is needed to expand around $\delta_0 = n^2$, where $n \in \mathbb{Z}^+$ in powers of h, with the expansions of the solution $y(x; h)$ and the coefficient Q_0. Thus, we seek a uniform expansion in the following form:

$$y(x, h) = y_0 + hy_1 + h^2 y_2 + h^3 y_3 + O(h^4), \tag{3.3}$$

$$\delta = n^2 + h\delta_1 + h^2\delta_2 + h^3\delta_3 + O(h^4), \tag{3.4}$$

$$\delta^2 = n^4 + h(2n^2\delta_1) + h^2(\delta_1^2 + 2n^2\delta_2) + h^3(2\delta_1\delta_2 + 2n^2\delta_3) + O(h^4). \tag{3.5}$$

By substitution and upon equating the coefficients of each power of h, then we have

$$y_0'' + n^4 y_0 = 0, \tag{3.6}$$

$$y_1'' + n^4 y_1 = -(2n^2\delta_1 y_0 + \cos(2x) y_0'' + Q_0 n^2 y_0'), \tag{3.7}$$

$$y_2'' + n^4 y_2 = -((\delta_1^2 + 2n^2\delta_2) y_0 + 2n^2\delta_1 y_1 + \cos(2x) y_1'' + Q_0\delta_1 y_0' + Q_0 n^2 y_1'), \tag{3.8}$$

$$y_3'' + n^4 y_3 = -((2\delta_1\delta_2 + 2n^2\delta_3) y_0 + (\delta_1^2 + 2n^2\delta_2) y_1 + \cos(2x) y_2'' + Q_0 n^2 y_2' + Q_0\delta_1 y_1' + Q_0\delta_2 y_0'). \tag{3.9}$$

Hence, we seek the periodic solutions at different values of n. First, when $n = 0$, the periodic solution is zero, thus this case leads to the trivial solution. Typically, the period of the solution is based on the oddness or evenness of n, within 2π or π respectively. In what follows, it is considered only for some different cases for n. Consequently, the transition curves that correspond to the positive and negative δ values are derived.

Case of $n = 1$:

The periodic solution with period 2π reads as follows:

$$\begin{aligned} y(x,h) =& a\cos x + b\sin x - \frac{1}{16}h[a\cos 3x + b\sin 3x] \\ &+ h^2[-\frac{1}{8}(A_1\cos 3x + A_2\sin 3x) + \frac{3}{256}(a\cos 5x + b\sin 5x)] \\ &+ h^3[\frac{1}{8}(A_3\cos 3x + A_4\sin 3x) + \frac{1}{24}(A_5\cos 5x + A_6\sin 3x) \\ &+ \frac{25}{8192}(a\cos 7x + b\sin 7x)] + O(h^4), \end{aligned} \tag{3.10}$$

where the coefficients are

$$\begin{aligned} a &= -\frac{Q_0}{2\delta_1 - \frac{1}{2}}b \quad \text{for} \quad \delta^+, \qquad b = \frac{Q_0}{2\delta_1 + \frac{1}{2}}a \quad \text{for} \quad \delta^-, \\ A_1 &= \frac{1}{8}\delta_1 a + \frac{3}{16}Q_0 b, \qquad A_2 = \frac{1}{8}\delta_1 b - \frac{3}{16}Q_0 a, \\ A_3 &= -\frac{1}{16}((\delta_1^2 + 2\delta_2)a - 6Q_0A_2 + 3Q_0\delta_1 b), \\ A_4 &= -\frac{1}{16}((\delta_1^2 + 2\delta_2)b + 6Q_0A_1 - 3Q_0\delta_1 a), \\ A_5 &= -\frac{1}{16}(9A_1 - \frac{15}{16}Q_0 b), \qquad A_6 = -\frac{1}{16}(9A_2 + \frac{15}{16}Q_0 a), \\ \delta_1^+ &= \frac{1}{4} - \frac{1}{2}Q_0, \qquad \delta_1^- = -\frac{1}{4} + \frac{1}{2}Q_0, \\ \delta_2^+ &= \frac{1}{2}(\delta_1^2 - \frac{1}{2}\delta_1 - \frac{9}{32}), \qquad \delta_2^- = \frac{1}{2}(\delta_1^2 + \frac{1}{2}\delta_1 - \frac{9}{32}), \\ \delta_3^+ &= -\frac{1}{2}\delta_2 + \frac{9}{64}\delta_1 - \frac{27}{512}, \qquad \delta_3^- = \frac{1}{2}\delta_2 + \frac{9}{64}\delta_1 + \frac{27}{512}. \end{aligned}$$

a and b are arbitrary constants to be determined from the initial conditions.

The corresponding positive and negative values of δ read as follows:

$$\begin{aligned}\delta^{+} =&1 + h(\frac{1}{4} - \frac{1}{2}Q_0) + \frac{1}{2}h^2(\delta_1^2 - \frac{1}{2}\delta_1 - \frac{9}{32})\\ &+ h^3(-\frac{1}{2}\delta_2 + \frac{9}{64}\delta_1 - \frac{27}{512}) + O(h^4),\end{aligned} \tag{3.11}$$

$$\begin{aligned}\delta^{-} =&1 + h(\frac{1}{2}Q_0 - \frac{1}{4}) + \frac{1}{2}h^2(\delta_1^2 + \frac{1}{2}\delta_1 - \frac{9}{32})\\ &+ h^3(\frac{1}{2}\delta_2 + \frac{9}{64}\delta_1 + \frac{27}{512}) + O(h^4)\end{aligned} \tag{3.12}$$

Special Case of $Q_0 = 0$:

The values of δ read as follows:

$$\delta^{+} = 1 + \frac{1}{4}h - \frac{11}{64}h^2 + \frac{35}{1024}h^3 - \frac{479}{8192}h^4 + O(h^5), \tag{3.13}$$

$$\delta^{-} = 1 - \frac{1}{4}h - \frac{11}{64}h^2 - \frac{35}{1024}h^3 - \frac{479}{8192}h^4 + O(h^5). \tag{3.14}$$

Case of $n = \sqrt{2}$:

The periodic solution with period π reads as follows:

$$\begin{aligned}y(x,h) =&a\cos 2x + b\sin 2x + h[\frac{a}{2} - \frac{1}{6}(a\cos 4x + b\sin 4x)]\\ &- h^2[\frac{\delta_1}{2}a + \frac{1}{12}(A_1\cos 4x + A_2\sin 4x)\\ &- \frac{1}{24}(a\cos 6x + b\sin 6x)] - h^3[\frac{a}{4}(\frac{1}{2}\delta_1^2 + 2\delta_2)\\ &+ \frac{1}{12}(A_3\cos 4x + A_4\sin 4x) + \frac{1}{32}(A_5\cos 6x + A_6\sin 6x)\\ &+ \frac{1}{80}(a\cos 8x + b\sin 8x)] + O(h^4),\end{aligned} \tag{3.15}$$

where the coefficients are

$$\begin{aligned}a &= -\frac{Q_0}{\delta_1}b \quad \text{for} \quad \delta^{+}, \qquad b = \frac{Q_0}{\delta_1}a \quad \text{for} \quad \delta^{-},\\ A_1 &=\frac{2}{3}\delta_1 a + 4\frac{Q_0}{3}b, \qquad A_2 = \frac{2}{3}\delta_1 b - 4\frac{Q_0}{3}a,\\ A_3 &=(\frac{\delta_1^2 + 4\delta_2}{6} + \frac{3}{4})a + (\frac{2}{3}Q_0\delta_1)b + \frac{2}{3}Q_0A_2,\\ A_4 &=(\frac{\delta_1^2 + 4\delta 2}{6} + \frac{3}{4})b - (\frac{2}{3}Q_0\delta_1)a - \frac{2}{3}Q_0A_1,\\ A_5 &=\frac{2}{3}A_1 - \frac{1}{2}Q_0b, \qquad A_6 = \frac{2}{3}A_2 - \frac{1}{2}Q_0a,\end{aligned}$$

$$\delta_1^+ = Q_0, \qquad \delta_1^- = -Q_0,$$
$$\delta_2^+ = \frac{1}{4}(\delta_1^2 - \frac{4}{3}), \qquad \delta_2^- = \frac{1}{4}(\delta_1^2 - \frac{4}{3}),$$
$$\delta_3^+ = -\frac{1}{4}\delta_1(\delta_2 - \frac{4}{9}), \qquad \delta_3^- = -\frac{1}{4}\delta_1(\delta_2 - \frac{4}{9}).$$

The corresponding values of δ read as follows:

$$\delta^+ = 2 + h(Q_0) + h^2[\frac{1}{4}(\delta_1^2 - \frac{4}{3})] + h^3(-\frac{1}{4}\delta_1(\delta_2 - \frac{4}{9})) + O(h)^4, \tag{3.16}$$
$$\delta^- = 2 + h(-Q_0) + h^2([\frac{1}{4}(\delta_1^2 - \frac{4}{3})] + h^3(-\frac{1}{4}\delta_1(\delta_2 - \frac{4}{9})) + O(h)^4. \tag{3.17}$$

Case of $n = \sqrt{3}$

The periodic solution with period 2π reads as follows:

$$\begin{aligned} y(x,h) = {} & a\cos 3x + b\sin 3x + h\frac{9}{16}[a\cos x + b\sin x \\ & - \frac{1}{2}(a\cos 5x + b\sin 5x)] + h^2[-\frac{1}{64}((A_1\cos x + A_2\sin x) \\ & + \frac{1}{2}(A_3\cos 5x + A_4\sin 5x) - \frac{15}{4}(a\cos 7x + b\sin 7x))] \\ & + h^3[y_3 - \frac{1}{128}(A_5\cos x + A_6\sin x) + \frac{1}{16}(A_7\cos 5x + A_8\sin 5x) \\ & + \frac{1}{2048}(A_9\cos 7x + A_{10}\sin 7x) - \frac{245}{6144}(a\cos 9x + b\sin 9x)] + O(h^4), \end{aligned} \tag{3.18}$$

where the coefficients are

$$a = -\frac{3}{2}\frac{Q_0}{\delta_1}b \quad \text{for} \quad \delta^+, \qquad b = -\frac{3}{2}\frac{Q_0}{\delta_1}a \quad \text{for} \quad \delta^-,$$
$$A_1 = (27\delta_1 - \frac{9}{4})a + \frac{Q_0}{6}b,, \qquad A_2 = (27\delta_1 + \frac{9}{4})b - \frac{Q_0}{6}a,$$
$$A_3 = 9\delta_1 a + \frac{45}{4}Q_0 b,, \qquad A_4 = 9\delta_1 b - \frac{45}{4}Q_0 a,$$
$$A_5 = 9(\delta_1^2 + 6\delta_2)a + 9Q_0\delta_1 b + \frac{1}{4}(3Q_0A_2 - \frac{1}{2}A_1),$$
$$A_6 = 9(\delta_1^2 + 6\delta_2)b - 9Q_0\delta_1 a - \frac{1}{4}(3Q_0A_1 - \frac{1}{2}A_2),$$
$$A_7 = \frac{9}{32}[(\delta_1^2 + 6\delta_2 - \frac{735}{256})a + \frac{5}{2}b + \frac{5}{128}A_4],$$
$$A_8 = \frac{9}{32}[(\delta_1^2 + 6\delta_2 - \frac{735}{256})b - \frac{5}{2}a - \frac{5}{128}A_3],$$
$$A_9 = 5A_3 - 21b, \qquad A_{10} = 5A_4 + 21a,$$
$$\delta_1^+ = \frac{3}{2}Q_0, \qquad \delta_1^- = -\frac{3}{2}Q_0$$

$$\delta_2^+ = \frac{1}{6}(\delta_1^2 - \frac{207}{64}), \qquad \delta_2^- = \frac{1}{6}(\delta_1^2 - \frac{207}{64}),$$
$$\delta_3^+ = -\frac{1}{54}(\frac{1615}{9}\delta_1 - 9), \qquad \delta_3^- = -\frac{1}{54}(\frac{1615}{9}\delta_1 + 9).$$

The corresponding positive and negative values of δ read as follows:

$$\delta^+ = 3 + h(\frac{3}{2}Q_0) + \frac{1}{6}h^2[\delta_1^2 - \frac{207}{64}] - \frac{1}{54}h^3[\frac{1615}{9}\delta_1 - 9] + O(h)^4, \tag{3.19}$$
$$\delta^- = 3 + h(-\frac{3}{2}Q_0) + \frac{1}{6}h^2[\delta_1^2 - \frac{207}{64}] - \frac{1}{54}h^3[\frac{1615}{9}\delta_1 + 9] + O(h)^4. \tag{3.20}$$

<u>Case of $n = 2$</u>

The periodic solution with period π reads as follows:

$$\begin{aligned} y(x,h) =& a\cos 4x + b\sin 4x + h[\frac{2}{3}(a\cos 2x + b\sin 2x) \\ &- \frac{2}{5}(a\cos 6x + b\sin 6x)] + h^2[\frac{1}{12}a - \frac{4}{9}(A_1\cos 2x + A_2\sin 2x) \\ &+ \frac{4}{25}(A_2\cos 6x + A_3\sin 6x) + \frac{3}{20}(a\cos 8x + b\sin 8x)] \\ &+ h^3[-\frac{1}{18}(A_5\cos 2x + A_6\sin 2x) - \frac{1}{50}(A_7\cos 6x + A_8\sin 6x) \\ &+ \frac{1}{10}(\frac{3}{5}A_3 + b)\cos 8x + \frac{1}{10}(\frac{3}{5}A_4 - a)\sin 8x - \frac{1}{18}A_1] + O(h^4), \end{aligned} \tag{3.21}$$

where the coefficients are

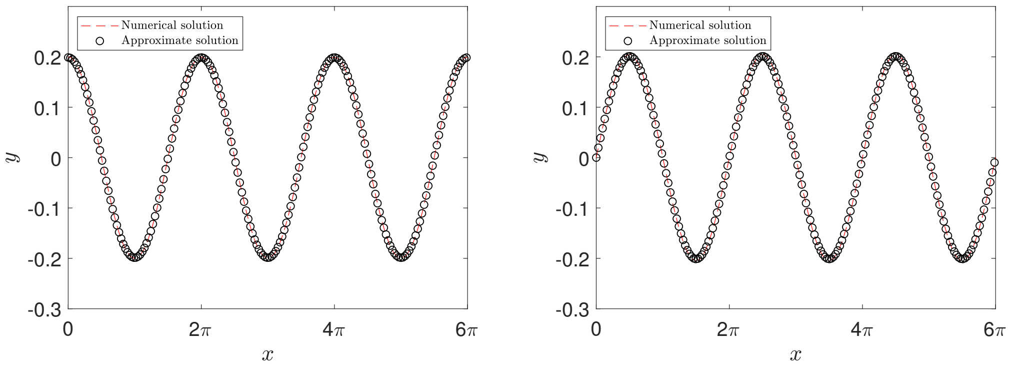
$$a = -\frac{2Q_0}{\delta_1}b \quad \text{for} \quad \delta^+, \quad b = \frac{2Q_0}{\delta_1}a \quad \text{for} \quad \delta^-,$$
$$A_1 = \delta_1 a + Q_0 b, \qquad A_2 = \delta_1 b - Q_0 a,$$
$$A_3 = \delta_1 a + 3Q_0 b, \qquad A_4 = \delta_1 b - 3Q_0,$$
$$A_5 = (\delta_1^2 + 4\delta_2)a + (\frac{8}{3}Q_0\delta_1)b - \frac{32}{9}Q_0 A_2,$$
$$A_6 = (\delta_1^2 + 4\delta_2)a - (\frac{8}{3}Q_0\delta_1)b - \frac{32}{9}Q_0 A_1,$$
$$A_7 = (\delta_1^2 + 4\delta_2)a + (6Q_0\delta_1)b + \frac{48}{5}Q_0 A_4,$$
$$A_8 = (\delta_1^2 + 4\delta_2)b - (6Q_0\delta_1)a - \frac{48}{5}Q_0 A_4,$$
$$\delta_1^+ = 2Q_0, \qquad \delta_1^- = -2Q_0,$$
$$\delta_2^+ = \frac{1}{8}(\delta_1^2 + \frac{88}{15}), \qquad \delta_2^- = \frac{1}{8}(\delta_1^2 + \frac{88}{15}),$$
$$\delta_3^+ = \frac{14}{225}\delta_1, \qquad \delta_3^- = \frac{14}{225}\delta_1.$$

The corresponding positive and negative values of δ read as follows:

$$\delta^+ = 4 + h(2Q_0) + \frac{1}{8}h^2(\delta_1^2 + \frac{88}{15}) + h^3(\frac{14}{225}\delta_1) + O(h^4), \tag{3.22}$$

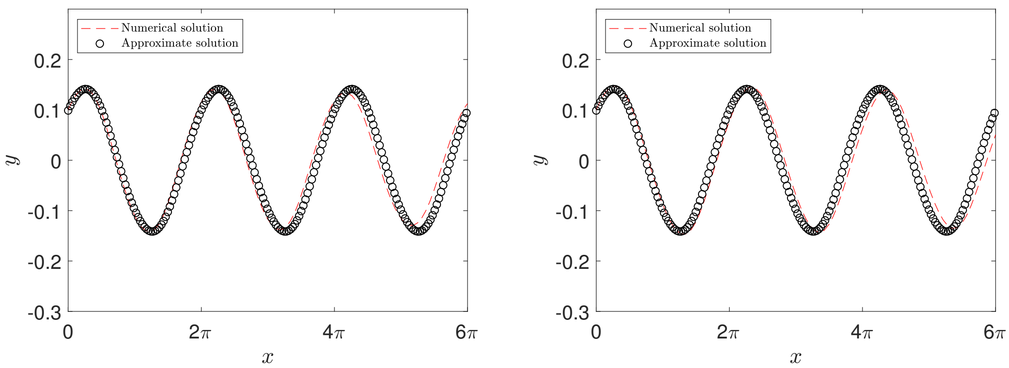
$$\delta^{-} = 4 + h(-2Q_0) + \frac{1}{8}h^2(\delta_1^2 + \frac{88}{15}) + h^3(\frac{14}{225}\delta_1) + O(h^4). \tag{3.23}$$

The numerical verification of such results is shown in Figures 7 and 8. The first shows the stable periodic solution with period 2π at $Q_0 = 0$; the numerical solution is so closed to the approximate solution. Also, the second shows that if Q_0 is increased by increasing the resistance of the system then it leads to changes in the positions of the periodic solution.



(a) α=0.977248 **(b)** α=1.027494

Figure 7. Periodic solutions for $Q_0 = 0$ and $h = 0.1$.



(a) α=0.967436 **(b)** α=1.049660

Figure 8. Periodic solutions for $Q_0 = 0.1$ and $h = 0.2$.

4. Prediction of transition curves

In this section, we are concerned with the existence of transition curves for the governing equation of the studied problem. This means that, there exists critical boundary curves which divide the space (δ, h) into regions where the numbers of unstable characteristic exponents might be constant according to the derived relations from the method of strained parameters. Therefore, the changes of these numbers along the boundary curves by the relationships between δ and h create regions separated by such transition curves exhibiting the stability and instability regions for the studied system.

The plot shown in Figure 9 illustrates the transition curves for each value of the damping coefficient (Q) as a barrier between the stability and instability regions for different values of δ. So, it explains the transition from stability to instability around the integer values of ($\delta = \frac{\omega_0}{\omega}$). Practically, in accordance with the presented theoretical analysis of stability, the region of the parameter h lies on (-1,1) for stable motion coinciding with the machine operation points. So, as shown in Figures 10 and 11, around ($\delta = \frac{\omega_0}{\omega} = 1$), a set of transition curves separate the domain for $0 < h < 1$ to stability and instability regions depending on the damping coefficient (Q). As can be clearly noticed from the depicted figures, when the damping coefficient (Q) is increased, the region of stability increases until the instability region completely disappears.

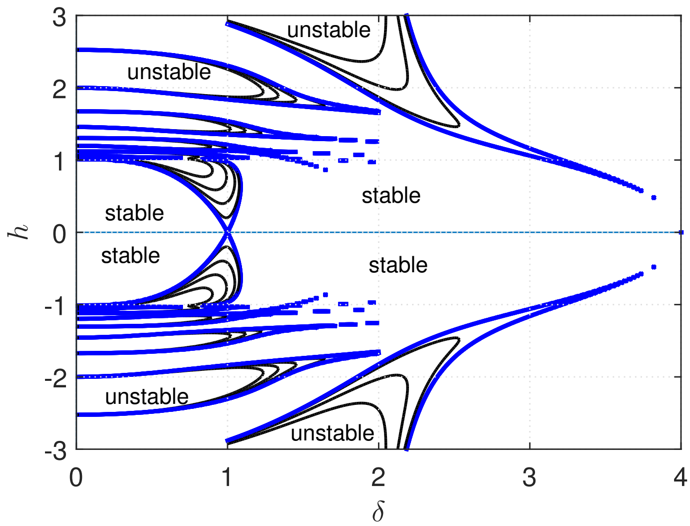


Figure 9. Transition curves for different values of Q and n.

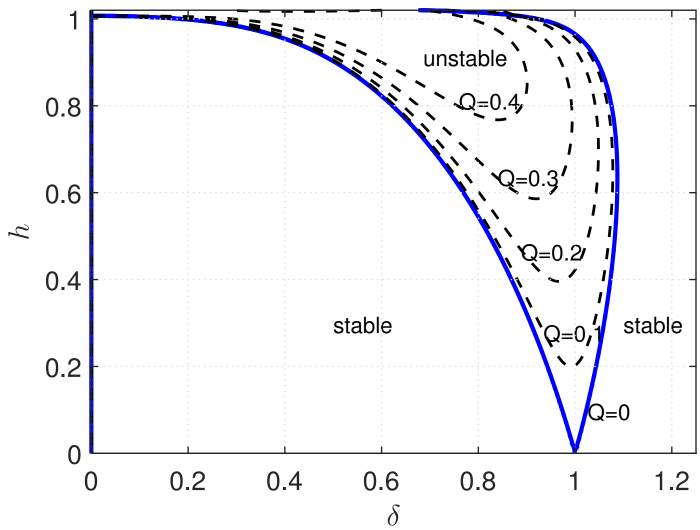


Figure 10. Transition curves for different values of Q and $n = 1$ in the $\delta - h$ plane.

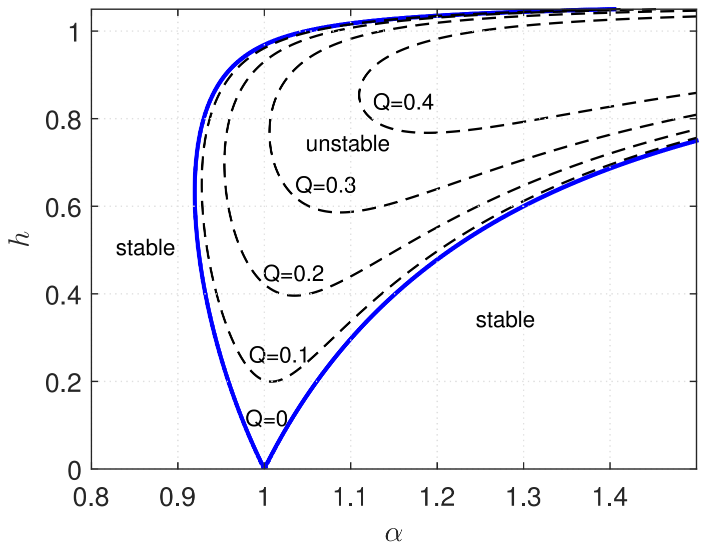


Figure 11. Transition curves for different values of Q and $n = 1$.

5. Stability of periodic solutions

The stability of periodic solutions can be readily predicted from Figure 9 based on the value of h on the interval $(-1, 1)$. Additionally, it is mostly emitted from the integer number for the ratio $\alpha = \frac{\omega}{\omega_0} = n$, $n \in \mathbb{Z}^+$; otherwise, the periodic solution is unstable; this is obviously shown in Figures 12–13 when $Q = 0$. Typically, this can be proved by calculating the average of the energy change $(\Delta\mathbb{H}_{av}(x))$ over the period $[0, 2\pi\alpha]$ for the basic generating solution $y = a \sin \frac{1}{\alpha} x$ as follows

$$\Delta\mathbb{H}_{av}(\alpha) = -\frac{h}{2\pi} \int_0^{2\pi\alpha} y'(\cos 2xy'' + \frac{Q_0}{\alpha} y')dx. \tag{5.1}$$

Then, the stability of periodic solutions is satisfied if the following condition holds

$$\frac{\partial}{\partial a}(\Delta\mathbb{H}_{av}(\alpha)) = \Delta\mathbb{H}'_{av}(\alpha) < 0. \tag{5.2}$$

Indeed, if this fraction is odd the curves show the periodic solutions with period 2π and if it is even the curves show the periodic solutions with period π.

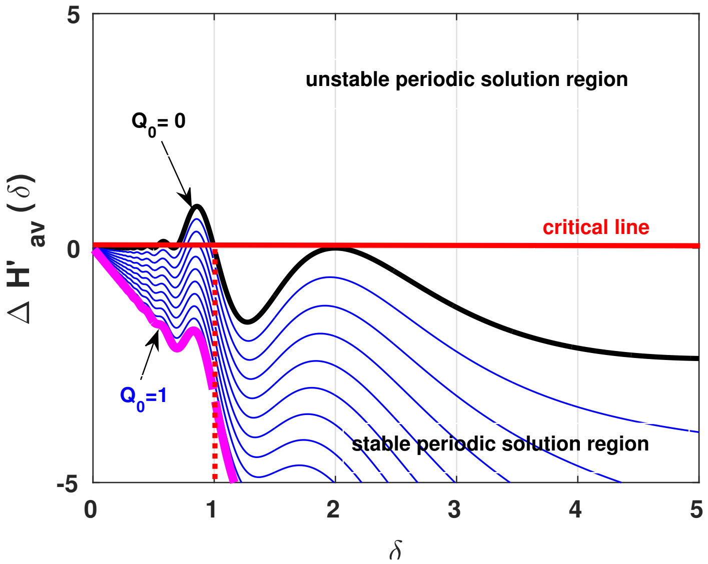


Figure 12. Stability domains of periodic solutions based on δ values.

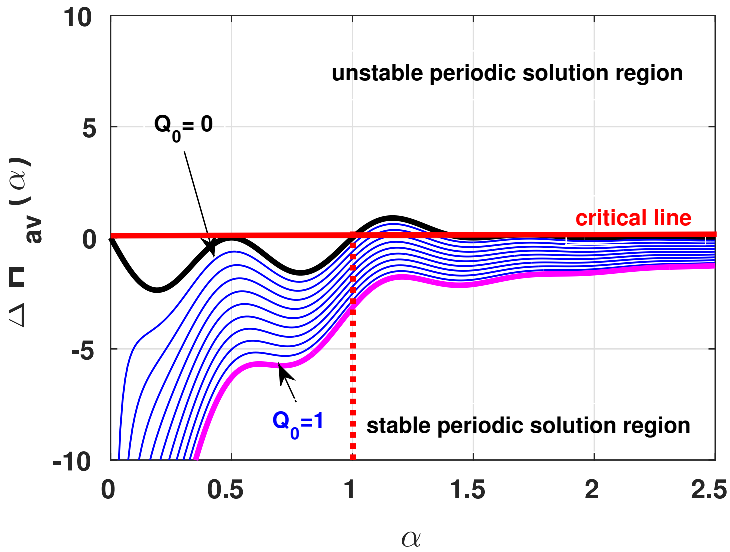
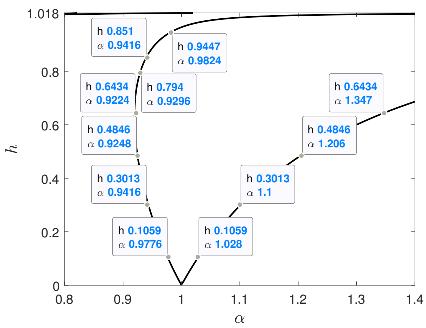
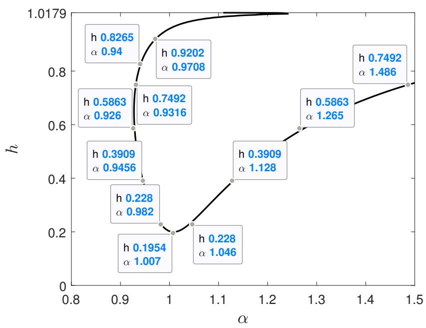


Figure 13. Stability domains of periodic solutions based on α values.

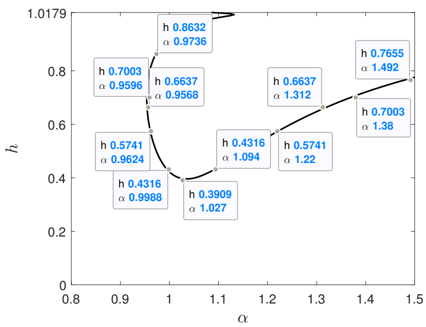
Normally, increasing damping coefficient (Q)(resulted by increasing the dissipated energy in the system) does not affect the stability of the periodic solutions, but it increases the stable regions by default. Finally, we attempted the numerical verification for the existence of the periodic solution in certain (α, h) curves. Thus, the periodic solutions we are checked at many points along the transition curves for different values of Q_0=0, 0.1, 0.2 and 0.3 emitted from $\alpha = 1$ with period 2π. First, the values of h and α at some points were set as shown in Figure 14. Then, we found the numerical solutions with sustained periods which are shown in Figures 15–18. These figures show the existence of periodic solutions for different values of h in $(-1, 1)$ and the change in their configurations when h and α are consequently changed.



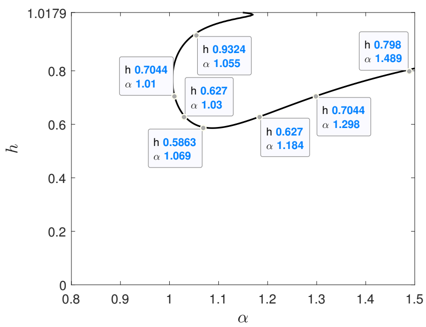
(a) Periodic solution at $Q = 0$.



(b) Periodic solution at $Q = 0.1$.

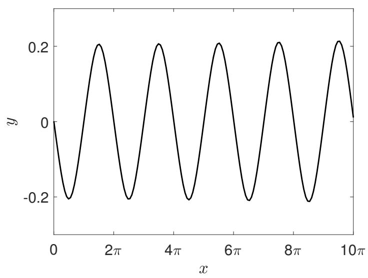


(c) Periodic solution at $Q = 0.2$.

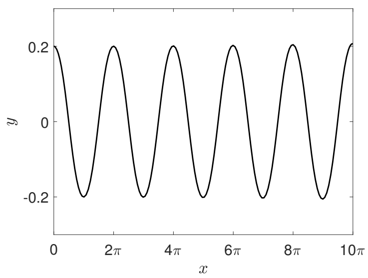


(d) Periodic solution at $Q = 0.3$.

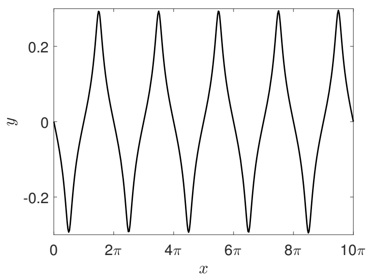
Figure 14. Values of h and α in the periodic solutions given Q=0, 0.1, 0.2 and 0.3.



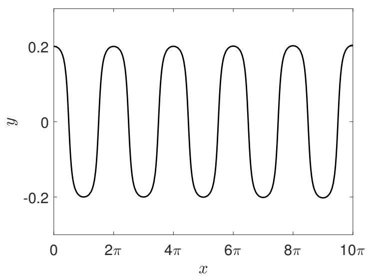
(a) $h = 0.1059, \quad \alpha = 0.9776.$



(b) $h = 0.1059 \quad \alpha = 1.028.$

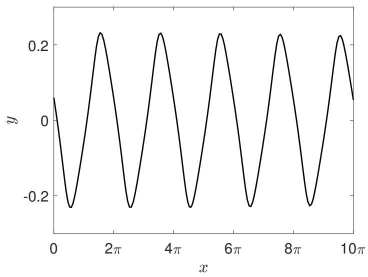


(c) $h = 0.851, \quad \alpha = 0.9396.$

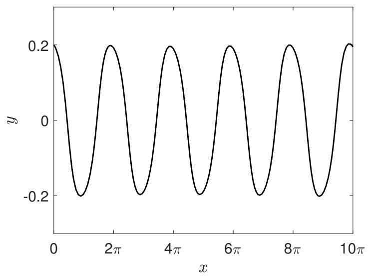


(d) $h = 0.851, \quad \alpha = 1.758.$

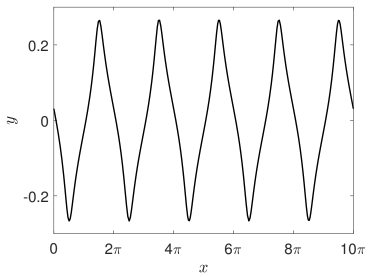
Figure 15. Periodic solutions given $Q = 0$.



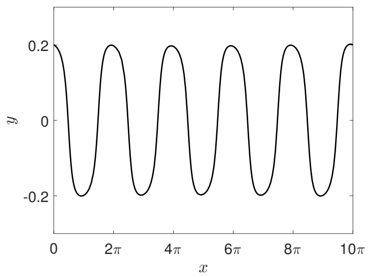
(a) $h = 0.3909, \alpha = 0.945.$



(b) $h = 0.3909, \quad \alpha = 1.128.$

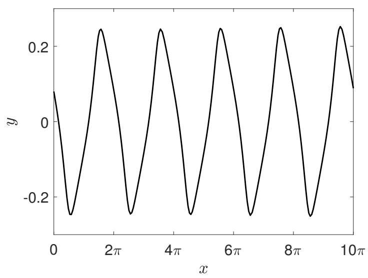


(c) $h = 0.7492, \quad \alpha = 0.9312.$

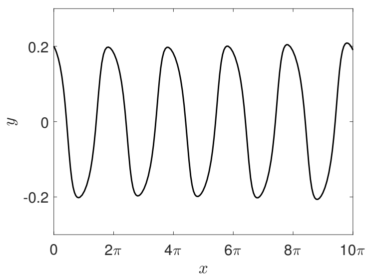


(d) $h = 0.7492, \quad \alpha = 1.502.$

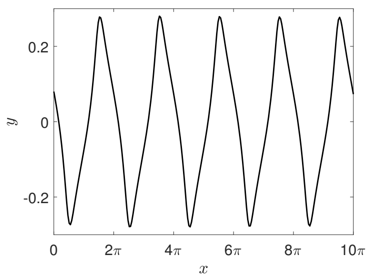
Figure 16. Periodic solutions given $Q = 0.1$.



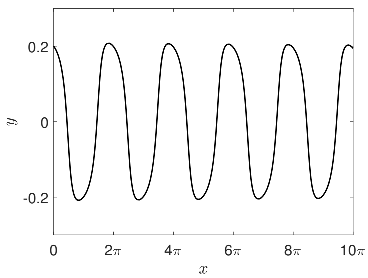
(a) $h = 0.5741, \quad \alpha = 0.9624.$



(b) $h = 0.5741, \quad \alpha = 1.191.$

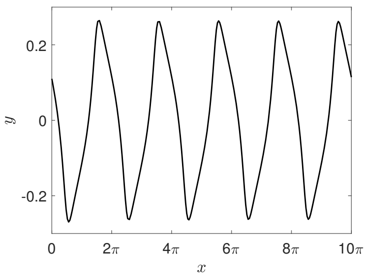


(c) $h = 0.7003, \quad \alpha = 0.9535.$

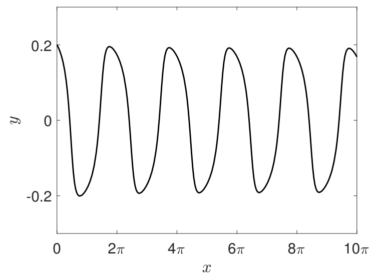


(d) $h = 0.7003, \quad \alpha = 1.421.$

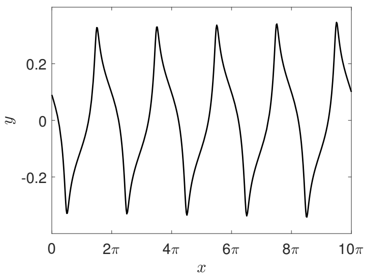
Figure 17. Periodic solutions given $Q = 0.2$.



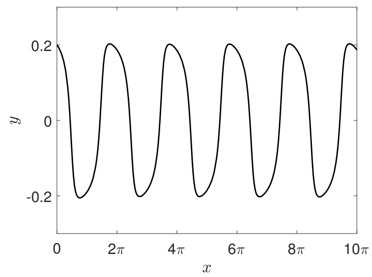
(a) $h = 0.7044, \quad \alpha = 1.01.$



(b) $h = 0.7044 \quad \alpha = 1.298.$



(c) $h = 0.9324, \quad \alpha = 1.05.$



(d) $h = 0.798, \quad \alpha = 1.473.$

Figure 18. Periodic solutions given $Q = 0.3$.

6. Conclusions

This work was performed to explain and clarify the proper selections of parameters for the dynamical motion of AC machines by using the stability analysis of the presented model. The used model is considered to be a linear prototype model to obtain sustained periodic responses. The qualitative study used to predict the stability regions, and it was carried out by using the Lyapunov second approach. The concluded results are reasonably consistent with the experimental results. In addition, the results were compared with the numerical solutions to realize a satisfactory agreement with the theoretical ones, specifically in cases of small perturbations (h). The method of strained parameters was used to obtain reasonably accurate forms of periodic solutions inside the required stable regions. In particular, it showed that the operating regions of machines lies in $|h| < 1$; it also revealed the integer numbers of the frequency ratio(α) required to obtain sustained stable periodic modes of output. These results were checked experimentally for some configurations of AC induction machines, and it was found that the two results matched relatively well.

Regarding the improvement of the study, this prototype will be enhanced by nonlinear terms due to saturation and hysteresis of induction machines. Additionally, some other natural phenomena might be taken into the account such as fractionalization, synchronization and delay effects, according to the nature of the problem; this will help to obtain the best scenario of desirable results. Indeed, the solution of the linear system might give conditions under which the sustained oscillations might occur. Typically, this type of output represents the most usual forms of voltage and current in AC machines. However, it is not possible to attain steady-state responses in the absence of some forms of controlling actions, i.e., either via external means or due to some parameter variations. Accordingly, linear analysis is not sufficient to allow for the study of voltage/current build-up behavior which is essential in self-excited generators. Often, the most important phenomenon associated with this type of problem is the magnetic saturation, which results in a decrease in inductance with an increase in magneto-motive force (as a result of current) or magnetic flux (as a result of voltage per frequency). So, the magnetic saturation phenomenon could be considered to have the first impact to generate nonlinearities in the mathematical modeling of AC machines.

However, due to the presence of such phenomenon within this engineering issue that cannot be ignored in the study, development and improvement of the mathematical model have become necessary to enhance the results to a level of perfect consistency with the corresponding practical ones.

Data availability

The data used to support the findings of this study are available from the authors upon request.

Conflicts of interest

The authors declare that they have no conflicts of interest.

Authors contributions

All authors contributed significantly in this paper. All authors read and approved the final manuscript.

References

1. J. D. Gao, X. H. Wang, L. Z. Zhang, *AC Machine Systems: Mathematical Model and Parameters, Analysis, and System Performance*, Berlin, Heidelberg: Springer, 2009. https://doi.org/10.1007/978-3-642-01153-5
2. A. S. Mostafa, A. L. Mohamadein, E. M. Rashad, Application of Floquet's theory to the analysis of series-connected wound-rotor self-excited synchronous generator, *IEEE Trans. Energy Convers.*, **8** (1993), 369–376. http://doi.org/10.1109/60.257047
3. A. S. Mostafa, A. L. Mohamadein, E. M. Rashad, Analysis of series-connected wound-rotor self-excited induction generator, *IEE Proc. B Electr. Power Appl.*, **140** (1993), 329–336. http://doi.org/10.1049/ip-b.1993.0041
4. M. El-Borhamy, Chaos transition of the generalized fractional duffing oscillator with a generalized time delayed position feedback, *Nonlinear Dyn.*, **111** (2020), 2471–2487. https://doi.org/10.1007/s11071-020-05840-y
5. M. El-Borhamy, Z. El-Sheikh, M. E. Ali, Modeling and dynamic analysis for a motion of mounted-based axisymmetric rigid body under self-excited vibrations in an attractive Newtonian field, *Math. Probl. Eng.*, 2022, 4329906. https://doi.org/10.1155/2022/4329906
6. M. El-Borhamy, T. Medhat, M. E. Ali, Chaos prediction in fractional delayed energy-based models of capital accumulation, *Complexity*, 2021, 8751963. https://doi.org/10.1155/2020/8751963

7. A. Y. Leung, Z. Guo, H. X. Yang, Fractional derivative and time delay damper characteristics in Duffing-van der Pol oscillators, *Commun. Nonlinear Sci. Numer. Simul.*, **18** (2013), 2900–2915. https://doi.org/10.1016/j.cnsns.2013.02.013

8. A. Y. Leung, H. X. Yang, P. Zhu, Periodic bifurcation of Duffing-van der Pol oscillators having fractional derivatives and time delay, *Commun. Nonlinear Sci. Numer. Simul.*, **19** (2014), 1142–1155. https://doi.org/10.1016/j.cnsns.2013.08.020

9. Y. Yu, Z. Zhang, Q. Bi, Multistability and fast-slow analysis for van der Pol-Duffing oscillator with varying exponential delay feedback factor, *Appl. Math. Model.*, **57** (2018), 448–458. https://doi.org/10.1016/j.apm.2018.01.010

10. S. Wen, Y. Shen, S. Yang, J. Wang, Dynamical response of Mathieu–Duffing oscillator with fractional-order delayed feedback, *Chaos Soliton Fract.*, **94** (2017), 54–62. https://doi.org/10.1016/j.chaos.2016.11.008

11. É. Mathieu, Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique, *J. Math. Pures Appl.*, **13** (1868), 137–203.

12. D. Frenkel, R. Portugal, Algebraic methods to compute Mathieu functions, *J. Phys.*, **34** (2001), 3541–3551. http://doi.org/10.1088/0305-4470/34/17/302

13. M. Gadella, H. Giacomini, L. P. Lara, Periodic analytic approximate solutions for the Mathieu equation, *Appl. Math. Comput.*, **271** (2015), 436–445. http://doi.org/10.1016/j.amc.2015.09.018

14. I. Kovacic, R. Rand, S. M. Sah, Mathieu's equation and its generalizations: Overview of stability charts and their features, *Appl. Mech. Rev.*, **70** (2018), 020802. http://doi.org/10.1115/1.4039144

15. J. A. Richards, *Analysis of Periodically Time-Varying Systems*, Berlin, Heidelberg: Springer, 1983. https://doi.org/10.1007/978-3-642-81873-8

16. S. A. Wilkinson, N. Vogt, D. S. Golubev, J. H. Cole, Approximate solutions to Mathieu's equation, *Physica E Low Dimens.*, **100** (2018), 24–30. http://doi.org/10.1016/j.physe.2018.02.019

17. D. Younesian, E. Esmailzadeh, R. Sedaghati, Existence of periodic solutions for the generalized form of Mathieu equation, *Nonlinear Dyn.*, **39** (2005), 335–348. http://doi.org/10.1007/s11071-005-4338-y

18. D. Younesian, E. Esmailzadeh, R. Sedaghati, Asymptotic solutions and stability analysis for generalized non-homogeneous Mathieu equation, *Commun. Nonlinear Sci. Numer. Simul.*, **12** (2007), 58–71. http://doi.org/10.1016/j.cnsns.2006.01.005

19. W. S. Loud, Stability regions for Hill's equation, *J. Differ. Equ.*, **19** (1975), 226–241. https://doi.org/10.1016/0022-0396(75)90003-0

20. A. Parra-Hinojosa, J. C. Gutierrez-Vega, Fractional Ince equation with a Riemann-Liouville fractional derivative, *Appl. Math. Comput.*, **219** (2013), 10695–10705. https://doi.org/10.1016/j.amc.2013.04.044

21. F. J. Poulin, G. R. Flierl, J. Pedlosky, Parametric instability in oscillatory shear flows, *J. Fluid Mechanics*, **481** (2003), 329–353. https://doi.org/10.1017/S0022112003004051

22. R. H. Rand, S. M. Sah, M. K. Suchorsky, Fractional Mathieu equation, *Commun. Nonlinear Sci. Numer. Simul.*, **15** (2010), 3254–3262. https://doi.org/10.1016/j.cnsns.2009.12.009

23. M. El-Borhamy, E. M. Rashad, I. Sobhy, Floquet analysis of linear dynamic RLC circuits, *Open Phys.*, **18** (2020), 264–277. http://doi.org/10.1515/phys-2020-0136

24. M. El-Borhamy, E. M. Rashad, I. Sobhy, M. El-sayed, Modeling and semi-analytic stability analysis for dynamics of AC machines, *Mathematics*, **9** (2021), 644. https://doi.org/10.3390/math9060644

25. M. Batista, Elfun18-A collection of MATLAB functions for the computation of elliptic integrals and Jacobian elliptic functions of real arguments, *SoftwareX*, **10** (2019), 100245. https://doi.org/10.1016/j.softx.2019.100245

26. M. El-Borhamy, On the existence of new integrable cases for Euler-Poisson equations in Newtonian fields, *Alex. Eng. J.*, **58** (2019), 733–744. https://doi.org/10.1016/j.aej.2019.06.004

27. D. Zwillinger, *Handbook of Integration*, Boca Raton: CRC Press, 1992.

28. C. Chicone, *Ordinary Differential Equations with Applications*, New York: Springer, 2006. https://doi.org/10.1007/b97645

29. A. A. Martynyuk, *Stability by Liapunov's Matrix Function Method with Applications*, Boca Raton: CRC Press, 1998.

30. D. R. Merkin, *Introduction to the Theory of Stability*, New York: Springer, 1997.

31. S. K. Nikravesh, *Nonlinear Systems Stability Analysis: Lyapunov-Based Approach*, Boca Raton: CRC Press, 2013.

32. M. Onitsuka, Uniform asymptotic stability for damped linear oscillators with variable parameters, *Appl. Math. Comput.*, **218** (2011), 1436–1442. https://doi.org/10.1016/j.amc.2011.06.025

33. A. O. Ignatyev, Stability of a linear oscillator with variable parameters, *Electron. J. Differ. Eq.*, **17** (1997), 1–6.

34. L. Duc, A. Ilchmann, S. Siegmund, P. Taraba, On stability of linear time-varying second-order differential equations, *Q. Appl. Math.*, **64** (2006), 137–151.

35. M. Grau, D. Peralta-Salas, A note on linear differential equations with periodic coefficients, *Nonlinear Anal. Theory Methods Appl.*, **71** (2009), 3197–3202. https://doi.org/10.1016/j.na.2009.01.199

36. J. H. Hale, *Ordinary Differential Equations*, Dover Publications, 2009.

37. C. S. Liu, Y. W. Chen, A simplified Lindstedt-Poincare method for saving computational cost to determine higher order nonlinear free vibrations, *Mathematics*, **9** (2021), 3070. https://doi.org/10.3390/math9233070

38. A. H. Nayfeh, *Introduction to Perturbation Techniques*, Wiley, 2011.

39. B. K. Shivamoggi, *Perturbation Methods for Differential Equations*, Boston: Birkhäuser, 2012.

40. M. El-Borhamy, N. Mosalam, On the existence of periodic solution and the transition to chaos of Rayleigh-Duffing equation with application of gyro dynamic, *Appl. Math. Nonl. Sci.*, **5** (2020), 43–58. http://doi.org/10.2478/amns.2020.1.00010

41. E. J. Hinch, *Perturbation Methods*, Cambridge University Press, 1991. https://doi.org/10.1017/CBO9781139172189

42. N. Karjanto, On the method of strained parameters for a KdV type of equation with exact dispersion property, *IMA J. Appl. Math.*, **80** (2015), 893–905. https://doi.org/10.1093/imamat/hxu020

43. D. P. Mason, On the method of strained parameters and the method of averaging, *Q. Appl. Math.*, **42** (1984), 77–85.