Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 8811–8828.
DOI:10.3934/math.2023442
Received: 18 November 2022
Revised: 25 January 2023
Accepted: 29 January 2023
Published: 08 February 2023

Research article

Classical Darboux transformation and exact soliton solutions of a two-component complex short pulse equation

Qiulan Zhao*, **Muhammad Arham Amin and Xinyue Li**

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, China

* **Correspondence:** Email: qlzhao@sdust.edu.cn.

Abstract: This paper investigates soliton solutions to a two-component complex short pulse (c-SP) equation. Based on the known Lax pair representation of this equation, we verify the integrability of a two-component c-SP equation and find an equivalent convenient Lax pair through hodograph transformation. The classical Darboux transformation (DT) is utilized to construct multi-soliton solutions for the two-component c-SP equation as an ordinary determinant. Furthermore, the details of one-soliton and two-soliton solutions are presented and generalized for N-fold soliton solutions. We also derive exact soliton solutions in explicit form using suitable reduction constraints from various "seed" solutions and explore them via graphs.

Keywords: integrability; hodograph transformation; Darboux transformation; exact soliton solutions
Mathematics Subject Classification: 35Q35, 37K10, 37K40

1. Introduction

In fiber-optic transmission systems, solitons are ideal information carriers due to their inherent stability over long distances. The nonlinear and linear effects of fiber optics produce optical solitons that are unique to fiber optics [1]. Firstly, optical solitons were originated by Garmire in 1964 when he observed self-focusing optical rays on super-lattice waveguides, and the first optical soliton was predicted in 1973 by Hasegawa and Tappert [2]. In nonlinear optics, several fundamental physical models can be used to describe the dynamic characteristics of the systems. The Nonlinear Schrödinger (NLS) equation is an important model for understanding the propagation of wide-range pulses or waves with slowly varying envelopes [3, 4]. It is used to describe the transmission characteristics of optical solitons in fiber-optic systems due to its ability to accurately represent the associated dynamics. In strongly nonlocal nonlinear media, the NLS equation is augmented with nonlocal terms which allow researchers to study the propagation dynamics of various transmission characteristics, such as tripole breathers [5]. These tripole breathers are special waves that travel in nonlocal nonlinear media without dispersion or distortion even over long distances, making them ideal candidates for transmission in fiber optics [6]. Furthermore, the NLS equation can be used to study the stability, nonlocality, and wave interaction aspects of nonlinear media [7]. However, the accuracy of the NLS equation decreases at wavelengths in the femtosecond range when ultra-short pulses are used [8, 9]. Therefore, in order to accurately represent ultra-short processes using the NLS equation, it is necessary to modify existing model to reflect the details of ultra-short processes. Schäfer-Wayne developed a new modified model, an alternative to the NLS equation to approximate the evolution of very short optical pulses, known as the short pulse (SP) equation [10]. The proposed SP equation plays a vital role in the study of nonlinear optical fibers [11]. In numerical analysis, it was found that the SP equation has a better approximation to Maxwell's equation as the pulse length shorten than NLS equation [12].

The ideal model of the SP equation has the following form

$$\partial_x\partial_t p = p + \frac{1}{6}\partial_x^2(p^3), \tag{1.1}$$

where the dynamical real-valued function $p = p(x, t)$ represents the electric pulse's magnitude and the subscripts indicate space-time partial derivatives. In addition to the nonlinear optical context, the SP equation is also extracted from the integrable differential equation with respect to pseudo-spherical surfaces [13]. In their studies, S. Sakovich and A. Sakovich examined the integrability of the SP equation from different perspectives and demonstrated that the SP equation is an integrable nonlinear equation [14]. The SP Eq (1.1) has integrable properties like the Lax pair (corresponding to linear combination), representation of zero-curvature, infinitely many conservation laws, and so on. An implicit transformation is applied to transform the SP Eq (1.1) into the Sine-Gordon (SG) equation, which is called the hodograph transformation [15]. The connection generated by the hodograph transformation between the SP equation and the SG equation was utilized to obtain exact soliton solutions of the SP equation [16]. Further, the N-soliton solutions involving multi-loop and multi-breather solutions were obtained using the well-known Hirota bi-linear method [17]. In addition, the SP equation is the basic version of a much more general equation, the Wadati-Konno-Ichikawa (WKI) equation, which is defined on multiple dimensions and holds a much greater complexity when compared to the SP equation [18]. This equation can be further generalized into the complex short pulse (c-SP) equation which is used to describe nonlinear phenomena like solitary waves, rogue waves, and other nonlinear structures [19].

In the present paper [20], the following c-SP equation is proposed and investigated

$$\partial_x\partial_t p = \tfrac{1}{2}\partial_x(|p|^2\partial_x p) + p, \tag{1.2}$$

and its generalized two-component equations

$$\partial_x\partial_t p = \tfrac{1}{2}\partial_x((|p|^2 + |q|^2)\partial_x p) + p, \tag{1.3}$$

$$\partial_x\partial_t q = \tfrac{1}{2}\partial_x((|p|^2 + |q|^2)\partial_x q) + q. \tag{1.4}$$

As shown in current studies, the c-SP equation and its generalized two-component equations are both integrable, as evidenced by the presence of the Lax pair and infinitely many conservation laws.

Further, the integrability of the c-SP equation helps us to analyze interesting dynamical characteristics in terms of multi-soliton, dark-bright soliton, loop soliton, and loop-breather soliton solutions [21]. The c-SP equation works better for manipulating multiple wave interactions than the SP equation due to pulse waves' complex representation. Specifically, the c-SP equation can better approximate the dispersion, and nonlinearity. These effects are important to consider when modeling the behavior of optical fibers and predicting the nonlinear dynamics of the system. The c-SP equation also incorporates other factors such as higher order dispersion, which is not considered in the simple SP equation [22]. In the same way as other nonlinear physical models, the c-SP equation and its generalized two-component equations also have N-soliton solutions that can be used to investigate dynamical features graphically. We present DT technique to derive N-soliton solutions for the c-SP equation and its two-component equations because the DT technique is widely used to construct multi-soliton solutions of several fundamental physical models such as the generalized Korteweg-de Vries (KdV) equation, NLS equation, and SP equation, etc [23]. This is a valuable technique that is not only effective for providing soliton solutions but also for determining exact soliton solutions in explicit form [24]. The DT originated by Jean Gaston Darboux in 1882 through the representation of the Darboux theorem. Moreover, the book describing DT's relationship with the theory of solitons belongs to Matveev and Salle [25].

In this paper, the outline is classified as follows. In Section 2, we present a Lax pair representation of the two-component c-SP equation, as well as a Lax integrability analysis of that equation. We also introduce the hodograph transformation and deform the two-component c-SP equation into equivalent nonlinear differential equations. This transformation provides a convenient Lax pair that will be used to construct DT. The DT is then defined in Section 3, to compute generalized N-fold soliton solutions for the two-component c-SP equation in the ratio of the ordinary determinant. The one-fold DT is expressed in details, while the two-fold and generalized N-fold DT are presented explicitly. In Section 4, we use derived soliton solutions to find exact soliton solutions in explicit form by choosing suitable seed solutions. Furthermore, the dynamics of exact soliton solutions are also analyzed via graphs using different parameters. In the final section, we summarize our conclusions and make a few remarks.

2. Lax pair and integrability

The Lax pair for an integrable two-component SP equation is derived explicitly from a generalized matrix of the SP equation using a zero-curvature representation [26]. In the present section, we construct the Lax pair for a two-component c-SP equation with another generalized matrix of the c-SP equation and show its integrability.

The linear matrix-valued system (Lax pair) associated with Eq (1.2) is

$$\partial_x \Psi = U\Psi, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda + \begin{pmatrix} 0 & p_x \\ p_x^* & 0 \end{pmatrix} \lambda, \tag{2.1}$$

$$\partial_t \Psi = V\Psi, \quad V = \tfrac{1}{2}|p|^2 \begin{pmatrix} 1 & p_x \\ p_x^* & -1 \end{pmatrix} \lambda + \tfrac{1}{2} \begin{pmatrix} 0 & -p \\ p^* & 0 \end{pmatrix} + \tfrac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^{-1}. \tag{2.2}$$

The zero-curvature representation $\partial_t U - \partial_x V + [U, V] = 0$ is the condition for compatibility of linear matrix-valued system that gives Eq (1.2). Consider the following Lax pair for associated Eqs (1.3) and (1.4)

$$\partial_x \Psi = S\Psi, \quad S = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}\lambda + \begin{pmatrix} 0 & P_x \\ Q_x & 0 \end{pmatrix}\lambda, \tag{2.3}$$

$$\partial_t \Psi = T\Psi, \quad T = \tfrac{1}{2}\begin{pmatrix} PQ & PQP_x \\ QPQ_x & -QP \end{pmatrix}\lambda + \tfrac{1}{2}\begin{pmatrix} 0 & -P \\ Q & 0 \end{pmatrix} + \tfrac{1}{4}\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}\lambda^{-1}, \tag{2.4}$$

where $I_{2\times 2}$ is the identity matrix and P, Q are defined as second-order 2-by-2 matrices

$$P = \begin{pmatrix} p & q \\ -q^* & p^* \end{pmatrix}, \quad Q = \begin{pmatrix} p^* & -q \\ q^* & p \end{pmatrix}.$$

Note that $Q = P^\dagger$, thus $PQ = QP = (|p|^2 + |q|^2)I_2$.

The compatibility condition $\partial_x\partial_t\Psi = \partial_t\partial_x\Psi$ gives zero-curvature equation $S_t - T_x + ST - TS = 0$, which provides a generalized two-component Eqs (1.3) and (1.4) of the c-SP equation, and shows the complete integrability of these coupled equations. In spite of the inconvenient nature of the Lax pair Eqs (2.3) and (2.4) for solving a c-SP equation, the hodograph transformation is commonly used as a special coordinate transformation for solving some nonlinear multi-valued function equations [27,28]. Several novel nonlinear evolution equations can also be generated by it, such as the modified KdV equation, dispersion-less Toda equation, and SG equation [29].

The conservation law associated with the c-SP equation has the following form

$$\partial_t \omega = \frac{1}{2}\partial_x(pp^*\omega). \tag{2.5}$$

The relation between the old variable p and the new variable ω is given by

$$\omega = \sqrt{1 + \partial_x p \partial_x p^*}. \tag{2.6}$$

The induced hodograph transformation replaces the independent variables (x, t) into new ones (y, z) by the following simultaneous equations

$$dy = \omega dx + \tfrac{1}{2}pp^*\omega dt, \quad dz = dt. \tag{2.7}$$

Now, the c-SP equation has transformed into an equivalent pair of equations with new variables

$$\partial_y\partial_z x = -\frac{1}{2}\partial_y(pp^*), \tag{2.8}$$

$$\partial_y\partial_z p = p\partial_y x. \tag{2.9}$$

We need to transform the Eqs (2.8) and (2.9) into matrix generalization, so

$$\mathcal{X}_{yz} = -\frac{1}{2}(P_y Q + PQ_y), \tag{2.10}$$

$$P_{yz} = \frac{1}{2}(\mathcal{X}_y P + P\mathcal{X}_y), \tag{2.11}$$

where, the functions χ, Q, and P are 2-by-2 second-order matrices. The Lax pair (2.3) and (2.4) has been transformed into a new compatible Lax pair using integrable Eqs (2.10) and (2.11) as

$$\varphi_y = L(y, z; \lambda)\varphi, \tag{2.12}$$

$$\varphi_z = M(y, z; \lambda)\varphi. \tag{2.13}$$

Here, $L(y, z; \lambda)$ and $M(y, z; \lambda)$ are 2-by-2 second-order matrix-valued functions which are equivalent to

$$L = \lambda\partial_y Z = \frac{\partial}{\partial y}\begin{pmatrix} \chi & P \\ Q & -\chi \end{pmatrix}\lambda, \tag{2.14}$$

$$M = N + \frac{1}{\lambda}R = \frac{1}{2}\begin{pmatrix} 0 & -P \\ Q & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}\lambda^{-1}, \tag{2.15}$$

and the matrices Z, N, and R are given by

$$Z = \begin{pmatrix} \chi & P \\ Q & -\chi \end{pmatrix}, \quad N = \frac{1}{2}\begin{pmatrix} 0 & -P \\ Q & 0 \end{pmatrix}, \quad R = \frac{1}{4}\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

It follows [30] that the new Lax pair has compatibility constraint $\varphi_{yz} = \varphi_{zy}$, which leads to zero-curvature representation $M_y(y, z, \lambda) - L_z(y, z, \lambda) + [M(y, z, \lambda), L(y, z, \lambda)] = 0$, for the Lax pair Eqs (2.12) and (2.13) and yield is equivalent to the Eqs (2.10) and (2.11). It is found that the first term of the second-order matrix T in Eq (2.4) does not appear in Eq (2.15) when comparing the time spectrum of the Lax pair before and after the hodograph transformation. This reduces our computational work in the next section as well. Now, we are in a position to apply the well-known DT technique based on the new Lax pair Eqs (2.12) and (2.13) for solving the two-component c-SP equation to get multi-soliton solutions.

3. Darboux transformation

The DT is the most powerful and effective tool to generate soliton solutions to an integrable system of equations. In this section, we derive scalar soliton solutions to the two-component c-SP equation for the N-fold DT by choosing a suitable Darboux matrix [31]. Firstly, we define matrix-valued solution $\varphi^{[1]}$ of co-variant Lax pair Eqs (2.12) and (2.13) for one-fold DT as

$$\varphi^{[1]} = H^{[1]}\varphi, \tag{3.1}$$

where Darboux matrix $H^{[1]}$ is equal to

$$H^{[1]} = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}\lambda^{-1} + \begin{pmatrix} A_1^{[1]} & B_1^{[1]} \\ C_1^{[1]} & -D_1^{[1]} \end{pmatrix}. \tag{3.2}$$

Here, $A_1^{[1]}$, $B_1^{[1]}$, $C_1^{[1]}$, and $D_1^{[1]}$ are matrix-valued functions associated with variables y and z. The equivalent form of the Lax pair Eqs (2.12) and (2.13) is given by

$$\varphi_y^{[1]} = L^{[1]}\varphi^{[1]}, \tag{3.3}$$

$$\varphi_z^{[1]} = M^{[1]}\varphi^{[1]}. \tag{3.4}$$

At this point, the second-order functions $L^{[1]}$ and $M^{[1]}$ are equal to

$$L^{[1]} = \lambda \begin{pmatrix} \mathcal{X}_y^{[1]} & P_y^{[1]} \\ Q_y^{[1]} & -\mathcal{X}_y^{[1]} \end{pmatrix}, \quad M^{[1]} = \begin{pmatrix} \frac{1}{4\lambda} I_2 & -\frac{1}{2} P^{[1]} \\ \frac{1}{2} Q^{[1]} & -\frac{1}{4\lambda} I_2 \end{pmatrix}. \tag{3.5}$$

The compatibility condition $\varphi_{yz}^{[1]} = \varphi_{zy}^{[1]}$ gives the governing equations

$$H_y^{[1]} + H^{[1]} L = L^{[1]} H^{[1]}, \quad H_z^{[1]} + H^{[1]} M = M^{[1]} H^{[1]}. \tag{3.6}$$

We use pair of Eq (3.6) to find out the relationship between the newly proposed solution, the seed solution, and elements of the Darboux matrix $H^{[1]}$. Substituting the Darboux matrix of $H^{[1]}$ and other corresponding values into pair of Eq (3.6) , then by comparing the coefficients of λ, we get the pair of equations

$$B_1^{[1]} = P^{[1]} - P, \tag{3.7}$$

$$C_1^{[1]} = Q^{[1]} - Q, \tag{3.8}$$

and

$$A_1^{[1]} = \mathcal{X}^{[1]} - \mathcal{X}, \tag{3.9}$$

$$D_1^{[1]} = \mathcal{X}^{[1]} - \mathcal{X}. \tag{3.10}$$

Since $Q = P^\dagger$ (the sign † named dagger of P), we obtain $C_1^{[1]} = B_1^{\dagger[1]}$ and $A_1^{[1]} = D_1^{[1]}$ by equating Eqs (3.7) and (3.8) as well as Eqs (3.9) and (3.10), then the Darboux matrix $H^{[1]}$ becomes the form

$$H^{[1]}(\lambda) = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} A_1^{[1]} & B_1^{[1]} \\ B_1^{\dagger[1]} & -A_1^{[1]} \end{pmatrix}, \tag{3.11}$$

where $A_1^{[1]}$ and $B_1^{[1]}$ are unknown matrix-valued functions as

$$A_1^{[1]} = \begin{pmatrix} (a_1^{[1]})_{11} & (a_1^{[1]})_{12} \\ (a_1^{[1]})_{21} & (a_1^{[1]})_{22} \end{pmatrix}, \quad B_1^{[1]} = \begin{pmatrix} (b_1^{[1]})_{11} & (b_1^{[1]})_{12} \\ (b_1^{[1]})_{21} & (b_1^{[1]})_{22} \end{pmatrix},$$

and the generated one-fold soliton solutions are

$$\mathcal{X}^{[1]}(y,z) = \mathcal{X}(y,z) + A_1^{[1]}, \quad P^{[1]}(y,z) = P(y,z) + B_1^{[1]}. \tag{3.12}$$

The next step is to compute the matrix-valued components of unknown matrix-valued functions $A_1^{[1]}$ and $B_1^{[1]}$ that will be interlinked with seed solutions. We use the following constraint $H^{[1]}(\lambda)\varphi|_{\lambda=\lambda_i} = 0$, with associated eigen-valued function $\varphi = \varphi_i = \begin{pmatrix} g_{11}^{(i)} & g_{12}^{(i)} & g_{13}^{(i)} & g_{14}^{(i)} \end{pmatrix}^T$, where i varies from 0 to 4. The simultaneous governing equations are derived from this constraint and solved through the well-known Cramer's rule. Then we get

$$(a_1^{[1]})_{11} = \frac{(\Lambda_1^{[1]})_{11}}{\Theta_1} = \frac{\det \begin{pmatrix} -\lambda_1^{-1} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ -\lambda_2^{-1} g_{11}^{(2)} & g_{12}^{(2)} & g_{13}^{(2)} & g_{14}^{(2)} \\ -\lambda_3^{-1} g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ -\lambda_4^{-1} g_{11}^{(4)} & g_{12}^{(4)} & g_{13}^{(4)} & g_{14}^{(4)} \end{pmatrix}}{\det \begin{pmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ g_{11}^{(2)} & g_{12}^{(2)} & g_{13}^{(2)} & g_{14}^{(2)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ g_{11}^{(4)} & g_{12}^{(4)} & g_{13}^{(4)} & g_{14}^{(4)} \end{pmatrix}}, \tag{3.13}$$

$$(a_1^{[1]})_{12} = \frac{(\Lambda_1^{[1]})_{12}}{\Theta_1} = \frac{det\begin{pmatrix} g_{11}^{(1)} & -\lambda_1^{-1}g_{11}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ g_{11}^{(2)} & -\lambda_2^{-1}g_{11}^{(2)} & g_{13}^{(2)} & g_{14}^{(2)} \\ g_{11}^{(3)} & -\lambda_3^{-1}g_{11}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ g_{11}^{(4)} & -\lambda_4^{-1}g_{11}^{(4)} & g_{13}^{(4)} & g_{14}^{(4)} \end{pmatrix}}{\det\begin{pmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ g_{11}^{(2)} & g_{12}^{(2)} & g_{13}^{(2)} & g_{14}^{(2)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ g_{11}^{(4)} & g_{12}^{(4)} & g_{13}^{(4)} & g_{14}^{(4)} \end{pmatrix}}, \tag{3.14}$$

$$(a_1^{[1]})_{21} = \frac{(\Lambda_1^{[1]})_{21}}{\Theta_1} = \frac{det\begin{pmatrix} -\lambda_1^{-1}g_{12}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ -\lambda_2^{-1}g_{12}^{(2)} & g_{12}^{(2)} & g_{13}^{(2)} & g_{14}^{(2)} \\ -\lambda_3^{-1}g_{12}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ -\lambda_4^{-1}g_{12}^{(4)} & g_{12}^{(4)} & g_{13}^{(4)} & g_{14}^{(4)} \end{pmatrix}}{\det\begin{pmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ g_{11}^{(2)} & g_{12}^{(2)} & g_{13}^{(2)} & g_{14}^{(2)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ g_{11}^{(4)} & g_{12}^{(4)} & g_{13}^{(4)} & g_{14}^{(4)} \end{pmatrix}}, \tag{3.15}$$

$$(a_1^{[1]})_{22} = \frac{(\Lambda_1^{[1]})_{22}}{\Theta_1} = \frac{\det\begin{pmatrix} g_{11}^{(1)} & -\lambda_1^{-1}g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ g_{11}^{(2)} & -\lambda_2^{-1}g_{12}^{(2)} & g_{13}^{(2)} & g_{14}^{(2)} \\ g_{11}^{(3)} & -\lambda_3^{-1}g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ g_{11}^{(4)} & -\lambda_4^{-1}g_{12}^{(4)} & g_{13}^{(4)} & g_{14}^{(4)} \end{pmatrix}}{\det\begin{pmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ g_{11}^{(2)} & g_{12}^{(2)} & g_{13}^{(2)} & g_{14}^{(2)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ g_{11}^{(4)} & g_{12}^{(4)} & g_{13}^{(4)} & g_{14}^{(4)} \end{pmatrix}}, \tag{3.16}$$

also

$$(b_1^{[1]})_{11} = \frac{(\Delta_1^{[1]})_{11}}{\Theta_1} = \frac{\det\begin{pmatrix} g_{11}^{(1)} & g_{12}^{(1)} & -\lambda_1^{-1}g_{11}^{(1)} & g_{14}^{(1)} \\ g_{11}^{(2)} & g_{12}^{(2)} & -\lambda_2^{-1}g_{11}^{(2)} & g_{14}^{(2)} \\ g_{11}^{(3)} & g_{12}^{(3)} & -\lambda_3^{-1}g_{11}^{(3)} & g_{14}^{(3)} \\ g_{11}^{(4)} & g_{12}^{(4)} & -\lambda_4^{-1}g_{11}^{(4)} & g_{14}^{(4)} \end{pmatrix}}{\det\begin{pmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ g_{11}^{(2)} & g_{12}^{(2)} & g_{13}^{(2)} & g_{14}^{(2)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ g_{11}^{(4)} & g_{12}^{(4)} & g_{13}^{(4)} & g_{14}^{(4)} \end{pmatrix}}, \tag{3.17}$$

$$(b_1^{[1]})_{12} = \frac{(\Delta_1^{[1]})_{12}}{\Theta_1} = \frac{\det\begin{pmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & -\lambda_1^{-1} g_{11}^{(1)} \\ g_{11}^{(2)} & g_{12}^{(2)} & g_{13}^{(2)} & -\lambda_2^{-1} g_{11}^{(2)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & -\lambda_3^{-1} g_{11}^{(3)} \\ g_{11}^{(4)} & g_{12}^{(4)} & g_{13}^{(4)} & -\lambda_4^{-1} g_{11}^{(4)} \end{pmatrix}}{\det\begin{pmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ g_{11}^{(2)} & g_{12}^{(2)} & g_{13}^{(2)} & g_{14}^{(2)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ g_{11}^{(4)} & g_{12}^{(4)} & g_{13}^{(4)} & g_{14}^{(4)} \end{pmatrix}}, \tag{3.18}$$

$$(b_1^{[1]})_{21} = \frac{(\Delta_1^{[1]})_{21}}{\Theta_1} = \frac{\det\begin{pmatrix} g_{11}^{(1)} & g_{12}^{(1)} & -\lambda_1^{-1} g_{12}^{(1)} & g_{14}^{(1)} \\ g_{11}^{(2)} & g_{12}^{(2)} & -\lambda_2^{-1} g_{12}^{(2)} & g_{14}^{(2)} \\ g_{11}^{(3)} & g_{12}^{(3)} & -\lambda_3^{-1} g_{12}^{(3)} & g_{14}^{(3)} \\ g_{11}^{(4)} & g_{12}^{(4)} & -\lambda_4^{-1} g_{12}^{(4)} & g_{14}^{(4)} \end{pmatrix}}{\det\begin{pmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ g_{11}^{(2)} & g_{12}^{(2)} & g_{13}^{(2)} & g_{14}^{(2)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ g_{11}^{(4)} & g_{12}^{(4)} & g_{13}^{(4)} & g_{14}^{(4)} \end{pmatrix}}, \tag{3.19}$$

$$(b_1^{[1]})_{22} = \frac{(\Delta_1^{[1]})_{22}}{\Theta_1} = \frac{\det\begin{pmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & -\lambda_1^{-1} g_{12}^{(1)} \\ g_{11}^{(2)} & g_{12}^{(2)} & g_{13}^{(2)} & -\lambda_2^{-1} g_{12}^{(2)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & -\lambda_3^{-1} g_{12}^{(3)} \\ g_{11}^{(4)} & g_{12}^{(4)} & g_{13}^{(4)} & -\lambda_4^{-1} g_{12}^{(4)} \end{pmatrix}}{\det\begin{pmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ g_{11}^{(2)} & g_{12}^{(2)} & g_{13}^{(2)} & g_{14}^{(2)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ g_{11}^{(4)} & g_{12}^{(4)} & g_{13}^{(4)} & g_{14}^{(4)} \end{pmatrix}}. \tag{3.20}$$

Using the set of Eqs (3.13)–(3.20) into pair of Eq (3.12), the one-fold DT on the scalar solutions of the two-component c-SP equation is given as

$$x^{[1]} = x + \frac{1}{2}\{\frac{(\Lambda_1^{[1]})_{11}}{\Theta_1} + \frac{(\Lambda_1^{[1]})_{22}}{\Theta_1}\}, \quad p^{[1]} = p + \frac{(\Delta_1^{[1]})_{11}}{\Theta_1}, \quad q^{[1]} = q + \frac{(\Delta_1^{[1]})_{12}}{\Theta_1}. \tag{3.21}$$

In the second succession followed by the one-fold DT, the Darboux matrix $H^{[2]}$ for the two-component c-SP equation is in the form

$$H^{[2]}(\lambda) = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}\lambda^{-2} + \begin{pmatrix} A_2^{[1]} & B_2^{[1]} \\ -B_2^{\dagger[1]} & A_2^{[1]} \end{pmatrix}\lambda^{-1} + \begin{pmatrix} A_2^{[2]} & B_2^{[2]} \\ B_2^{\dagger[2]} & -A_2^{[2]} \end{pmatrix}, \tag{3.22}$$

where we take

$$A_2^{[1]} = \begin{pmatrix} (a_2^{[1]})_{11} & (a_2^{[1]})_{12} \\ (a_2^{[1]})_{21} & (a_2^{[1]})_{22} \end{pmatrix}, \quad A_2^{[2]} = \begin{pmatrix} (a_2^{[2]})_{11} & (a_2^{[2]})_{12} \\ (a_2^{[2]})_{21} & (a_2^{[2]})_{22} \end{pmatrix},$$

$$B_2^{[1]} = \begin{pmatrix} (b_2^{[1]})_{11} & (b_2^{[1]})_{12} \\ (b_2^{[1]})_{21} & (b_2^{[1]})_{22} \end{pmatrix}, \quad B_2^{[2]} = \begin{pmatrix} (b_2^{[2]})_{11} & (b_2^{[2]})_{12} \\ (b_2^{[2]})_{21} & (b_2^{[2]})_{22} \end{pmatrix},$$

and the generated corresponding novel solutions for two-fold DT are

$$\chi^{[2]}(y,z) = \chi(y,z) + A_2^{[1]}, \quad P^{[2]}(y,z) = P(y,z) + B_2^{[1]}. \tag{3.23}$$

Similarly, utilize the Cramer's rule to solve the following constraint $H^{[2]}(\lambda)\varphi|_{\lambda=\lambda_i} = 0$ for two-fold DT, where i varies from 1 to 8. The linear governing system of equations gives

$$(a_2^{[1]})_{11} = \frac{(\Lambda_2^{[1]})_{11}}{\Theta_2}, \quad (a_2^{[1]})_{12} = \frac{(\Lambda_2^{[1]})_{12}}{\Theta_2}, \tag{3.24}$$

$$(a_2^{[1]})_{21} = \frac{(\Lambda_2^{[1]})_{21}}{\Theta_2}, \quad (a_2^{[1]})_{22} = \frac{(\Lambda_2^{[1]})_{22}}{\Theta_2}, \tag{3.25}$$

$$(b_2^{[1]})_{11} = \frac{(\Delta_2^{[1]})_{11}}{\Theta_2}, \quad (b_2^{[1]})_{12} = \frac{(\Delta_2^{[1]})_{12}}{\Theta_2}, \tag{3.26}$$

$$(b_2^{[1]})_{21} = \frac{(\Delta_2^{[1]})_{21}}{\Theta_2}, \quad (b_2^{[1]})_{22} = \frac{(\Delta_2^{[1]})_{22}}{\Theta_2}, \tag{3.27}$$

where

$$\begin{aligned}
\Theta_2 &= \det\begin{pmatrix} \lambda_i^{-1} g_{11}^{(i)} & g_{11}^{(i)} & \lambda_i^{-1} g_{12}^{(i)} & g_{12}^{(i)} & \lambda_i^{-1} g_{13}^{(i)} & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix},\\
(\Lambda_2^{[1]})_{11} &= \det\begin{pmatrix} -\lambda_i^{-2} g_{11}^{(i)} & g_{11}^{(i)} & \lambda_i^{-1} g_{12}^{(i)} & g_{12}^{(i)} & \lambda_i^{-1} g_{13}^{(i)} & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix},\\
(\Lambda_2^{[1]})_{12} &= \det\begin{pmatrix} \lambda_i^{-1} g_{11}^{(i)} & g_{11}^{(i)} & -\lambda_i^{-2} g_{11}^{(i)} & g_{12}^{(i)} & \lambda_i^{-1} g_{13}^{(i)} & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix},\\
(\Lambda_2^{[1]})_{21} &= \det\begin{pmatrix} -\lambda_i^{-2} g_{12}^{(i)} & g_{11}^{(i)} & \lambda_i^{-1} g_{12}^{(i)} & g_{12}^{(i)} & \lambda_i^{-1} g_{13}^{(i)} & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix},\\
(\Lambda_2^{[1]})_{22} &= \det\begin{pmatrix} \lambda_i^{-1} g_{11}^{(i)} & g_{11}^{(i)} & -\lambda_i^{-2} g_{12}^{(i)} & g_{12}^{(i)} & \lambda_i^{-1} g_{13}^{(i)} & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
(\Delta_2^{[1]})_{11} &= \det\begin{pmatrix} \lambda_i^{-1} g_{11}^{(i)} & g_{11}^{(i)} & \lambda_i^{-1} g_{12}^{(i)} & g_{12}^{(i)} & -\lambda_i^{-2} g_{11}^{(i)} & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix},\\
(\Delta_2^{[1]})_{12} &= \det\begin{pmatrix} \lambda_i^{-1} g_{11}^{(i)} & g_{11}^{(i)} & \lambda_i^{-1} g_{12}^{(i)} & g_{12}^{(i)} & \lambda_i^{-1} g_{13}^{(i)} & g_{13}^{(i)} & -\lambda_i^{-2} g_{11}^{(i)} & g_{14}^{(i)} \end{pmatrix},\\
(\Delta_2^{[1]})_{21} &= \det\begin{pmatrix} \lambda_i^{-1} g_{11}^{(i)} & g_{11}^{(i)} & \lambda_i^{-1} g_{12}^{(i)} & g_{12}^{(i)} & -\lambda_i^{-2} g_{12}^{(i)} & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix},\\
(\Delta_N^{[1]})_{22} &= \det\begin{pmatrix} \lambda_i^{-1} g_{11}^{(i)} & g_{11}^{(i)} & \lambda_i^{-1} g_{12}^{(i)} & g_{12}^{(i)} & \lambda_i^{-1} g_{13}^{(i)} & g_{13}^{(i)} & -\lambda_i^{-2} g_{12}^{(i)} & g_{14}^{(i)} \end{pmatrix}.
\end{aligned}$$

Inserting the set of Eqs (3.24)–(3.27) into pair of Eq (3.23), it provides the scalar solutions of the two-component c-SP equation for two-fold DT

$$x^{[2]} = x + \frac{1}{2}\{\frac{(\Lambda_2^{[1]})_{11}}{\Theta_2} + \frac{(\Lambda_2^{[1]})_{22}}{\Theta_2}\}, \quad p^{[2]} = p + \frac{(\Delta_2^{[1]})_{11}}{\Theta_2}, \quad q^{[2]} = q + \frac{(\Delta_2^{[1]})_{12}}{\Theta_2}. \tag{3.28}$$

In this generalized step, we derive the N-fold DT of the two-component c-SP equation by following one-fold and two-fold iterations. Consider the following generalized Darboux matrix

$$H^{[N]}(\lambda) = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} \lambda^{-N} + \begin{pmatrix} \sum_{i=1}^{N} \lambda^{i-N} A_N^{[i]} & \sum_{i=1}^{N} \lambda^{i-N} B_N^{[i]} \\ \sum_{i=1}^{N} (-1)^{N-i} \lambda^{i-N} B_N^{\dagger[i]} & -\sum_{i=1}^{N} (-1)^{N-i} \lambda^{i-N} A_N^{[i]} \end{pmatrix}, \tag{3.29}$$

the unknown-valued functions are

$$A_N^{[i]} = \begin{pmatrix} (a_N^{[i]})_{11} & (a_N^{[i]})_{12} \\ (a_N^{[i]})_{21} & (a_N^{[i]})_{22} \end{pmatrix}, \qquad B_N^{[i]} = \begin{pmatrix} (b_N^{[i]})_{11} & (b_N^{[i]})_{12} \\ (b_N^{[i]})_{21} & (b_N^{[i]})_{22} \end{pmatrix}.$$

Similarly, under the action of the DT technique, the generalized N-fold soliton solutions are

$$\chi^{[N]}(y,z) = \chi(y,z) + A_N^{[1]}, \quad P^{[N]}(y,z) = P(y,z) + B_N^{[1]}. \tag{3.30}$$

The simultaneous algebraic equations $H^{[N]}(\lambda)\varphi|_{\lambda=\lambda_i} = 0$, $i = 1,2,3,4,....,N$ are solved through Cramer's rule for N-fold DT. We get the following yields

$$(a_N^{[1]})_{11} = \frac{(\Lambda_N^{[1]})_{11}}{\Theta_N}, \quad (a_N^{[1]})_{12} = \frac{(\Lambda_N^{[1]})_{12}}{\Theta_N}, \tag{3.31}$$

$$(a_N^{[1]})_{21} = \frac{(\Lambda_N^{[1]})_{21}}{\Theta_N}, \quad (a_N^{[1]})_{22} = \frac{(\Lambda_N^{[1]})_{22}}{\Theta_N}, \tag{3.32}$$

$$(b_N^{[1]})_{11} = \frac{(\Delta_N^{[1]})_{11}}{\Theta_N}, \quad (b_N^{[1]})_{12} = \frac{(\Delta_N^{[1]})_{12}}{\Theta_N}, \tag{3.33}$$

$$(b_N^{[1]})_{21} = \frac{(\Delta_N^{[1]})_{21}}{\Theta_N}, \quad (b_N^{[1]})_{22} = \frac{(\Delta_N^{[1]})_{22}}{\Theta_N}, \tag{3.34}$$

with

$$\begin{aligned}
\Theta_N &= \det\begin{pmatrix} \lambda_1^{1-N} g_{11}^{(i)} & \lambda_i^{1-N} g_{12}^{(i)} & \lambda_i^{1-N} g_{13}^{(i)} & \dots & g_{12}^{(i)} & \lambda_1^{-1} g_{13}^{(i)} & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix}, \\
(\Lambda_N^{[1]})_{11} &= \det\begin{pmatrix} -\lambda_i^{-N} g_{11}^{(i)} & \lambda_i^{1-N} g_{12}^{(i)} & \lambda_i^{1-N} g_{13}^{(i)} & \dots & g_{12}^{(i)} & \lambda_i^{-1} g_{13}^{(i)} & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix}, \\
(\Lambda_N^{[1]})_{12} &= \det\begin{pmatrix} \lambda_i^{1-N} g_{11}^{(i)} & -\lambda_i^{-N} g_{11}^{(i)} & \lambda_i^{1-N} g_{13}^{(i)} & \dots & g_{12}^{(i)} & \lambda_i^{-1} g_{13}^{(i)} & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix}, \\
(\Lambda_N^{[1]})_{21} &= \det\begin{pmatrix} -\lambda_i^{-N} g_{12}^{(i)} & \lambda_i^{1-N} g_{12}^{(i)} & \lambda_i^{1-N} g_{13}^{(i)} & \dots & g_{12}^{(i)} & \lambda_i^{-1} g_{13}^{(i)} & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix}, \\
(\Lambda_N^{[1]})_{22} &= \det\begin{pmatrix} \lambda_i^{1-N} g_{11}^{(i)} & -\lambda_i^{-N} g_{12}^{(i)} & \lambda_i^{1-N} g_{13}^{(i)} & \dots & g_{12}^{(i)} & \lambda_i^{-1} g_{13}^{(i)} & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
(\Delta_N^{[1]})_{11} &= \det\begin{pmatrix} \lambda_i^{1-N} g_{11}^{(i)} & \lambda_i^{1-N} g_{12}^{(i)} & -\lambda_i^{-N} g_{11}^{(i)} & \dots & g_{12}^{(i)} & \lambda_i g_{13}^{(i)} & g_{13}^{(i)} & \lambda_i g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix}, \\
(\Delta_N^{[1]})_{12} &= \det\begin{pmatrix} \lambda_i^{1-N} g_{11}^{(i)} & \lambda_i^{1-N} g_{12}^{(i)} & \lambda_i^{1-N} g_{13}^{(i)} & -\lambda_i^{-N} g_{11}^{(i)} & \dots & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix},
\end{aligned}$$

$$(\Delta_N^{[1]})_{21} = \det\begin{pmatrix} \lambda_i^{1-N} g_{11}^{(i)} & \lambda_i^{1-N} g_{12}^{(i)} & -\lambda_i^{-N} g_{12}^{(i)} & \dots & g_{12}^{(i)} & \lambda_i^{-1} g_{13}^{(i)} & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix},$$

$$(\Delta_N^{[1]})_{22} = \det\begin{pmatrix} \lambda_i^{1-N} g_{11}^{(i)} & \lambda_i^{1-N} g_{12}^{(i)} & \lambda_i^{1-N} g_{13}^{(i)} & -\lambda_i^{-N} g_{12}^{(i)} & \dots & g_{13}^{(i)} & \lambda_i^{-1} g_{14}^{(i)} & g_{14}^{(i)} \end{pmatrix}.$$

The scalar solutions of the c-SP equation for generalized N-fold DT are obtained using bunch of Eqs (3.31)–(3.34) into a pair of Eq (3.30). The results are given as

$$x^{[N]} = x + \frac{1}{2}\{\frac{(\Lambda_N^{[1]})_{11}}{\Theta_N} + \frac{(\Lambda_N^{[1]})_{22}}{\Theta_N}\}, \quad p^{[N]} = p + \frac{(\Delta_N^{[1]})_{11}}{\Theta_N}, \quad q^{[N]} = q + \frac{(\Delta_N^{[1]})_{12}}{\Theta_N}. \tag{3.35}$$

In the next section, we will use the above-determined scalar solutions for N-fold DT to find exact soliton solutions in explicit form.

4. Exact soliton solutions

In this section, the exact soliton solutions of the two-component c-SP equation are computed for multi-soliton solutions by taking two different seed solutions and analyzing them via graphs [32]. Note that we can only choose suitable seed solutions that satisfy the transformed linear system of Eqs (2.12) and (2.13).

Case I. In the first case, let us take a seed solution $\chi_y = I_{2\times 2}$ and $P = O_{2\times 2}$ that is true for the hodograph transformation of the c-SP equation. The resulting Lax pair Eqs (2.12) and (2.13) becomes

$$\varphi_y = \begin{pmatrix} \lambda I_2 & O_2 \\ O_2 & -\lambda I_2 \end{pmatrix}\varphi, \quad \varphi_z = \begin{pmatrix} \frac{1}{4\lambda} I_2 & O_2 \\ O_2 & -\frac{1}{4\lambda} I_2 \end{pmatrix}\varphi, \tag{4.1}$$

where $\varphi = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \end{pmatrix}^T$, I_2, and O_2 simultaneously represent the eigenfunction, 2-by-2 identity matrix, and second-order null matrix. The integral solution to Eq (4.1) gives

$$g_{11} = g_{12} = c_1 \exp(\lambda y + 0.25\lambda^{-1} z), \tag{4.2}$$

$$g_{13} = g_{14} = c_3 \exp(-\lambda y - 0.25\lambda^{-1} z). \tag{4.3}$$

Here, c_1 and c_3 are constants. At $\lambda = \lambda_i$, the particular solutions are defined as follows

$$g_{11}^{(i)} = g_{12}^{(i)} = c_1 \exp(\lambda_i y + 0.25\lambda_i^{-1} z), \tag{4.4}$$

$$g_{13}^{(i)} = g_{14}^{(i)} = c_3 \exp(-\lambda_i y - 0.25\lambda_i^{-1} z). \tag{4.5}$$

The particular solution is in complex conjugate form

$$g_{11}^{*(i)} = g_{12}^{*(i)} = c_1 \exp(\lambda_i^* y + 0.25\lambda_i^{-1*} z), \tag{4.6}$$

$$g_{13}^{*(i)} = g_{14}^{*(i)} = c_3 \exp(-\lambda_i^* y - 0.25\lambda_i^{-1*} z). \tag{4.7}$$

The following reduction requirements for a one-fold soliton solution are satisfied for the two-component c-SP equation

$$g_{11}^{(2)} = -g_{12}^{*(1)}, \quad g_{12}^{(2)} = g_{11}^{*(1)}, \quad g_{13}^{(2)} = -g_{14}^{*(1)}, \quad g_{14}^{(2)} = g_{13}^{*(1)}, \quad \lambda_2 = -\lambda_1^*, \tag{4.8}$$

$$g_{11}^{(4)} = -g_{12}^{*(3)}, \quad g_{12}^{(4)} = g_{11}^{*(3)}, \quad g_{13}^{(4)} = -g_{14}^{*(3)}, \quad g_{14}^{(4)} = g_{13}^{*(3)}, \quad \lambda_4 = -\lambda_3^*. \tag{4.9}$$

Put N=1 in a pair of Eq (3.35) and using reduction constraints from Eqs (4.8) and (4.9), the components of one-fold soliton solutions give exact soliton solutions in the following form

$$x^{[1]} = x + \frac{1}{2}\left(\frac{\begin{vmatrix} -\lambda_1^{-1} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ -\lambda_1^{-1*} g_{12}^{*(1)} & g_{11}^{*(1)} & -g_{14}^{*(1)} & g_{13}^{*(1)} \\ -\lambda_3^{-1} g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ -\lambda_3^{-1*} g_{12}^{*(3)} & g_{11}^{*(3)} & -g_{14}^{*(3)} & g_{13}^{*(3)} \end{vmatrix}}{\begin{vmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ -g_{12}^{*(1)} & g_{11}^{*(1)} & -g_{14}^{*(1)} & g_{13}^{*(1)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ -g_{12}^{*(3)} & g_{11}^{*(3)} & -g_{14}^{*(3)} & g_{13}^{*(3)} \end{vmatrix}} + \frac{\begin{vmatrix} g_{11}^{(1)} & -\lambda_1^{-1} g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ -g_{12}^{*(1)} & \lambda_1^{-1*} g_{11}^{*(1)} & -g_{14}^{*(1)} & g_{13}^{*(1)} \\ g_{11}^{(3)} & -\lambda_3^{-1} g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ -g_{12}^{*(3)} & \lambda_3^{-1*} g_{11}^{*(3)} & -g_{14}^{*(3)} & g_{13}^{*(3)} \end{vmatrix}}{\begin{vmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ -g_{12}^{*(1)} & g_{11}^{*(1)} & -g_{14}^{*(1)} & g_{13}^{*(1)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ -g_{12}^{*(3)} & g_{11}^{*(3)} & -g_{14}^{*(3)} & g_{13}^{*(3)} \end{vmatrix}} \right), \tag{4.10}$$

$$p^{[1]} = \frac{\begin{vmatrix} g_{11}^{(1)} & g_{12}^{(1)} & -\lambda_1^{-1} g_{11}^{(1)} & g_{14}^{(1)} \\ -g_{12}^{*(1)} & g_{11}^{*(1)} & -\lambda_1^{-1*} g_{12}^{*(1)} & g_{13}^{*(1)} \\ g_{11}^{(3)} & g_{12}^{(3)} & -\lambda_3^{-1} g_{11}^{(3)} & g_{14}^{(3)} \\ -g_{12}^{*(3)} & g_{11}^{*(3)} & -\lambda_3^{-1*} g_{12}^{*(3)} & g_{13}^{*(3)} \end{vmatrix}}{\begin{vmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ -g_{12}^{*(1)} & g_{11}^{*(1)} & -g_{14}^{*(1)} & g_{13}^{*(1)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ -g_{12}^{*(3)} & g_{11}^{*(3)} & -g_{14}^{*(3)} & g_{13}^{*(3)} \end{vmatrix}}, \quad q^{[1]} = \frac{\begin{vmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & -\lambda_1^{-1} g_{11}^{(1)} \\ -g_{12}^{*(1)} & g_{11}^{*(1)} & -g_{14}^{*(1)} & -\lambda_1^{-1*} g_{12}^{*(1)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & -\lambda_3^{-1} g_{11}^{(3)} \\ -g_{12}^{*(3)} & g_{11}^{*(3)} & -g_{14}^{*(3)} & -\lambda_3^{-1*} g_{12}^{*(3)} \end{vmatrix}}{\begin{vmatrix} g_{11}^{(1)} & g_{12}^{(1)} & g_{13}^{(1)} & g_{14}^{(1)} \\ -g_{12}^{*(1)} & g_{11}^{*(1)} & -g_{14}^{*(1)} & g_{13}^{*(1)} \\ g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} & g_{14}^{(3)} \\ -g_{12}^{*(3)} & g_{11}^{*(3)} & -g_{14}^{*(3)} & g_{13}^{*(3)} \end{vmatrix}}. \tag{4.11}$$

Using particular column solutions and their complex conjugates from Eqs (4.4)–(4.7) into Eqs (4.10) and (4.11), we get the exact soliton solutions in the explicit form shown in Figure 1. There are also dynamic characteristics associated with the computed single-soliton solutions such as dark solitons and dark-bright solitons with single peaks.

$$x^{[1]} = x + \frac{\kappa_1(\lambda_3^*|\lambda_1|^2 - \lambda_1|\lambda_3|^2) + \kappa_2(\lambda_3|\lambda_1|^2 - \lambda_3^*|\lambda_1|^2) + \lambda_1|\lambda_3|^2\kappa_3 - \lambda_3|\lambda_1|^2\kappa_4}{2|\lambda_1|^2|\lambda_3|^2\Gamma_1\Gamma_2}, \tag{4.12}$$

$$\begin{aligned} p^{[1]} =& \frac{c_1\left(\kappa_5(\lambda_1^*|\lambda_3|^2 - \lambda_3^*|\lambda_1|^2) + \kappa_6(\lambda_3^*|\lambda_1|^2 - \lambda_1^*|\lambda_3|^2)\right)}{2c_3|\lambda_1|^2|\lambda_3|^2\Gamma_1\Gamma_2} \\ &+ \frac{c_1\left(\kappa_7(\lambda_1|\lambda_3|^2 - \lambda_3|\lambda_1|^2) + \kappa_8(\lambda_3|\lambda_1|^2 - \lambda_1|\lambda_3|^2)\right)}{2c_3|\lambda_1|^2|\lambda_3|^2\Gamma_1\Gamma_2}, \end{aligned} \tag{4.13}$$

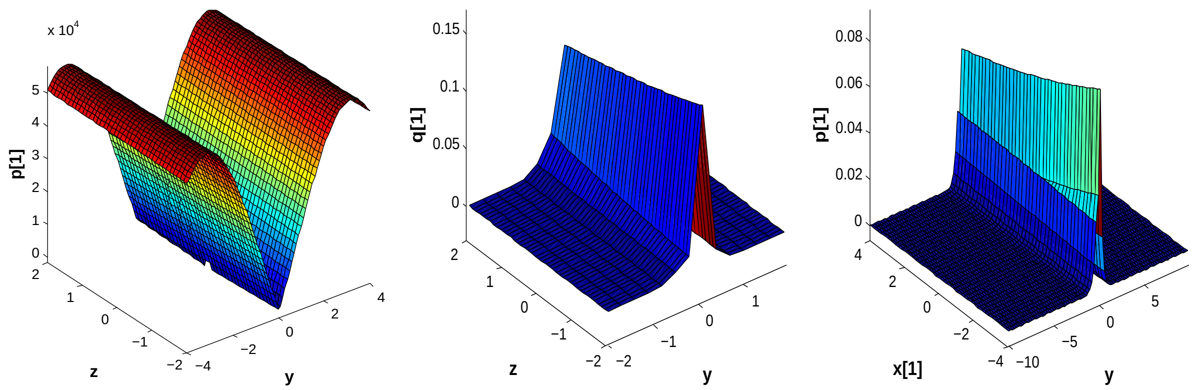
$$q^{[1]} = \frac{c_1\left(\kappa_5(\lambda_1^*|\lambda_3|^2 - \lambda_3^*|\lambda_1|^2) + \kappa_6(\lambda_3^*|\lambda_1|^2 - \lambda_1^*|\lambda_3|^2)\right)}{2c_3|\lambda_1|^2|\lambda_3|^2\Gamma_1\Gamma_2} + \frac{c_1\left(\kappa_7(\lambda_3|\lambda_1|^2 - \lambda_1|\lambda_3|^2) + \kappa_8(\lambda_1|\lambda_3|^2 - \lambda_3|\lambda_1|^2)\right)}{2c_3|\lambda_1|^2|\lambda_3|^2\Gamma_1\Gamma_2}, \tag{4.14}$$

where

$$\begin{aligned}\Gamma_1 &= \exp(2\lambda_1 y + 0.5\lambda_1^{-1}z) - \exp(2\lambda_3 y + 0.5\lambda_3^{-1}z),\\ \Gamma_2 &= \exp(2\lambda_1^* y + 0.5\lambda_1^{*-1}z) - \exp(2\lambda_3^* y + 0.5\lambda_3^{*-1}z),\end{aligned}$$

and

$$\begin{aligned}\kappa_1 &= \exp(2\lambda_3 y + 0.5\lambda_3^{-1}z)\exp(2\lambda_1^* y + 0.5\lambda_1^{*-1}z),\\ \kappa_2 &= \exp(2\lambda_3 y + 0.5\lambda_3^{-1}z)\exp(2\lambda_3^* y + 0.5\lambda_3^{*-1}z),\\ \kappa_3 &= \exp(2\lambda_1 y + 0.5\lambda_1^{-1}z)\exp(2\lambda_1^* y + 0.5\lambda_1^{*-1}z),\\ \kappa_4 &= \exp(2\lambda_1 y + 0.5\lambda_1^{-1}z)\exp(2\lambda_3^* y + 0.5\lambda_3^{*-1}z),\\ \kappa_5 &= \exp(2\lambda_1 y + 0.5\lambda_1^{-1}z)\exp(2\lambda_3 y + 0.5\lambda_3^{-1}z)\exp(2\lambda_1^* y + 0.5\lambda_1^{*-1}z),\\ \kappa_6 &= \exp(2\lambda_1 y + 0.5\lambda_1^{-1}z)\exp(2\lambda_3 y + 0.5\lambda_3^{-1}z)\exp(2\lambda_3^* y + 0.5\lambda_3^{*-1}z),\\ \kappa_7 &= \exp(2\lambda_3 y + 0.5\lambda_3^{-1}z)\exp(2\lambda_1^* y + 0.5\lambda_1^{*-1}z)\exp(2\lambda_3^* y + 0.5\lambda_3^{*-1}z),\\ \kappa_8 &= \exp(2\lambda_1 y + 0.5\lambda_1^{-1}z)\exp(2\lambda_1^* y + 0.5\lambda_1^{*-1}z)\exp(2\lambda_3^* y + 0.5\lambda_3^{*-1}z).\end{aligned}$$



(a) (b) (c)

Figure 1. Single soliton solutions having seed solution $\chi_y = I_{2\times 2}$ and $P = O_{2\times 2}$ with parameters; (a) dark soliton: $\lambda_1 = 4i$, $\lambda_3 = 3.5i$, $c_1 = 4.5$ and $c_3 = 2.5$; (b) dark-bright soliton: $\lambda_1 = -4 - 2.4i$, $\lambda_3 = 3 + 2.6i$, $c_1 = 2.5$ and $c_3 = 3.5$; (c) dark-bright soliton: $\lambda_1 = -4 - 5.6i$, $\lambda_3 = 2 + 5.5i$, $c_1 = 2.5$ and $c_3 = 3.5$.

Similarly, the reduction requirements for $N = 2$ will also be realized for the two-component c-SP equation as

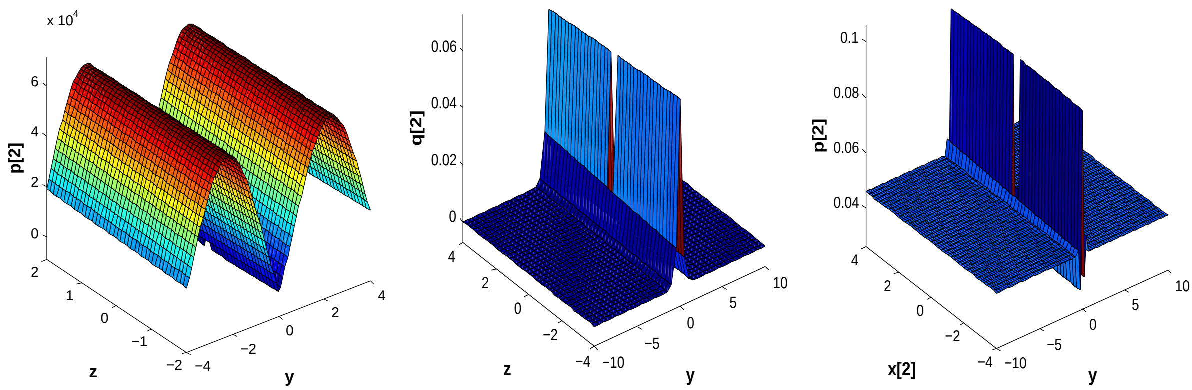
$$g_{11}^{(2)} = -g_{12}^{*(1)}, \quad g_{12}^{(2)} = g_{11}^{*(1)}, \quad g_{13}^{(2)} = -g_{14}^{*(1)} \quad g_{14}^{(2)} = g_{13}^{*(1)}, \quad \lambda_2 = -\lambda_1^*, \tag{4.15}$$

$$g_{11}^{(4)} = -g_{12}^{*(3)}, \quad g_{12}^{(4)} = g_{11}^{*(3)}, \quad g_{13}^{(4)} = -g_{14}^{*(3)}, \quad g_{14}^{(4)} = g_{13}^{*(3)}, \quad \lambda_4 = -\lambda_3^*, \tag{4.16}$$

$$g_{11}^{(6)} = -g_{12}^{*(5)}, \quad g_{12}^{(6)} = g_{11}^{*(5)}, \quad g_{13}^{(6)} = -g_{14}^{*(5)}, \quad g_{14}^{(6)} = g_{13}^{*(5)}, \quad \lambda_6 = -\lambda_5^*, \tag{4.17}$$

$$g_{11}^{(8)} = -g_{12}^{*(7)}, \quad g_{12}^{(8)} = g_{11}^{*(7)}, \quad g_{13}^{(8)} = -g_{14}^{*(7)}, \quad g_{14}^{(8)} = g_{13}^{*(7)}, \quad \lambda_8 = -\lambda_7^*. \tag{4.18}$$

Using the reduction constraints from the Eqs (4.20)–(4.23) into pair of Eq (3.35) for $N = 2$ leads us to find the exact soliton solutions for two-fold DT. The dynamics of double soliton solutions associated with seed solution $\chi_y = I_{2\times2}$ and $P = O_{2\times2}$ are shown in Figure 2. In explicit form, the exact double soliton solutions have double peaks with the same dynamic characteristics. Thus, when we compare single-soliton and double-soliton solutions graphically, both show symmetrical behavior.



(a) (b) (c)

Figure 2. Double soliton solutions having seed solution $\chi = I_{2\times2}$ and $P = O_{2\times2}$ with parameters; (a) dark soliton: $\lambda_1 = 4.2i$, $\lambda_3 = 3.5i$, $c_1 = 4.5$ and $c_3 = 2.5$; (b) dark-bright soliton: $\lambda_1 = 5-4.5i$, $\lambda_3 = 2+5i$, $c_1 = 2.5$ and $c_3 = 3.5$; (c) dark-bright soliton: $\lambda_1 = 3+4.2i$, $\lambda_3 = 10.5i$, $c_1 = 2.5$ and $c_3 = 3.5$.

Case II. In the second case, the seed solution $\chi = I_2 siny$ and $P = O_{2\times2}$ satisfy the corresponding Lax pair Eqs (2.12) and (2.13), then the particular column solution becomes

$$g_{11}^{(i)} = c_1 \exp(\lambda_i siny + 0.25\lambda_i^{-1} z), \quad g_{12}^{(i)} = c_2 \exp(\lambda_i siny - 0.25\lambda_i^{-1} z), \tag{4.19}$$

$$g_{13}^{(i)} = c_3 \exp(-\lambda_i siny + 0.25\lambda_i^{-1}), \quad g_{14}^{(i)} = c_4 \exp(-\lambda_i siny - 0.25\lambda_i^{-1} z). \tag{4.20}$$

Similarly, for N=1 and using the particular column solutions from the Eqs (4.15) and (4.16) into pair of Eq (3.35), the exact soliton solutions of c-SP component for one-fold DT in the explicit form are

$$x^{[1]} = x + \frac{\eta_1(\lambda_3^*|\lambda_1|^2 - \lambda_1|\lambda_3|^2) + \eta_2(\lambda_3|\lambda_1|^2 - \lambda_3^*|\lambda_1|^2) + \lambda_1|\lambda_3|^2\eta_3 - \lambda_3|\lambda_1|^2\eta_4}{2|\lambda_1|^2|\lambda_3|^2\Upsilon_1\Upsilon_2}, \tag{4.21}$$

$$p^{[1]} = \frac{c_1\left(\eta_5(\lambda_1^*|\lambda_3|^2 - \lambda_3^*|\lambda_1|^2) + \eta_6(\lambda_3^*|\lambda_1|^2 - \lambda_1^*|\lambda_3|^2)\right)}{2c_3|\lambda_1|^2|\lambda_3|^2\Upsilon_1\Upsilon_2} + \frac{c_1\left(\eta_7(\lambda_1|\lambda_3|^2 - \lambda_3|\lambda_1|^2) + \eta_8(\lambda_3|\lambda_1|^2 - \lambda_1|\lambda_3|^2)\right)}{2c_3|\lambda_1|^2|\lambda_3|^2\Upsilon_1\Upsilon_2}, \tag{4.22}$$

$$q^{[1]} = \frac{c_1\left(\eta_5(\lambda_1^*|\lambda_3|^2 - \lambda_3^*|\lambda_1|^2) + \eta_6(\lambda_3^*|\lambda_1|^2 - \lambda_1^*|\lambda_3|^2)\right)}{2c_3|\lambda_1|^2|\lambda_3|^2\Upsilon_1\Upsilon_2} + \frac{c_1\left(\eta_7(\lambda_3|\lambda_1|^2 - \lambda_1|\lambda_3|^2) + \eta_8(\lambda_1|\lambda_3|^2 - \lambda_3|\lambda_1|^2)\right)}{2c_3|\lambda_1|^2|\lambda_3|^2\Upsilon_1\Upsilon_2}, \tag{4.23}$$

where

$$\Upsilon_1 = \exp(2\lambda_1 siny + 0.5\lambda_1^{-1}z) - \exp(2\lambda_3 siny + 0.5\lambda_3^{-1}z),$$
$$\Upsilon_2 = \exp(2\lambda_1^* siny + 0.5\lambda_1^{*-1}z) - \exp(2\lambda_3^* siny + 0.5\lambda_3^{*-1}z),$$

and

$$\eta_1 = \exp(2\lambda_3 siny + 0.5\lambda_3^{-1}z)\exp(2\lambda_1^* siny + 0.5\lambda_1^{*-1}z),$$
$$\eta_2 = \exp(2\lambda_3 siny + 0.5\lambda_3^{-1}z)\exp(2\lambda_3^* siny + 0.5\lambda_3^{*-1}z),$$
$$\eta_3 = \exp(2\lambda_1 siny + 0.5\lambda_1^{-1}z)\exp(2\lambda_1^* siny + 0.5\lambda_1^{*-1}z),$$
$$\eta_4 = \exp(2\lambda_1 siny + 0.5\lambda_1^{-1}z)\exp(2\lambda_3^* siny + 0.5\lambda_3^{*-1}z),$$
$$\eta_5 = \exp(2\lambda_1 siny + 0.5\lambda_1^{-1}z)\exp(2\lambda_3 siny + 0.5\lambda_3^{-1}z)\exp(2\lambda_1^* siny + 0.5\lambda_1^{*-1}z),$$
$$\eta_6 = \exp(2\lambda_1 siny + 0.5\lambda_1^{-1}z)\exp(2\lambda_3 siny + 0.5\lambda_3^{-1}z)\exp(2\lambda_3^* siny + 0.5\lambda_3^{*-1}z),$$
$$\eta_7 = \exp(2\lambda_3 siny + 0.5\lambda_3^{-1}z)\exp(2\lambda_1^* siny + 0.5\lambda_1^{*-1}z)\exp(2\lambda_3^* siny + 0.5\lambda_3^{*-1}z),$$
$$\eta_8 = \exp(2\lambda_1 siny + 0.5\lambda_1^{-1}z)\exp(2\lambda_1^* siny + 0.5\lambda_1^{*-1}z)\exp(2\lambda_3^* siny + 0.5\lambda_3^{*-1}z).$$

The interesting dynamical phenomena of loop soliton, and loop-breather soliton solutions are shown in Figure 3.

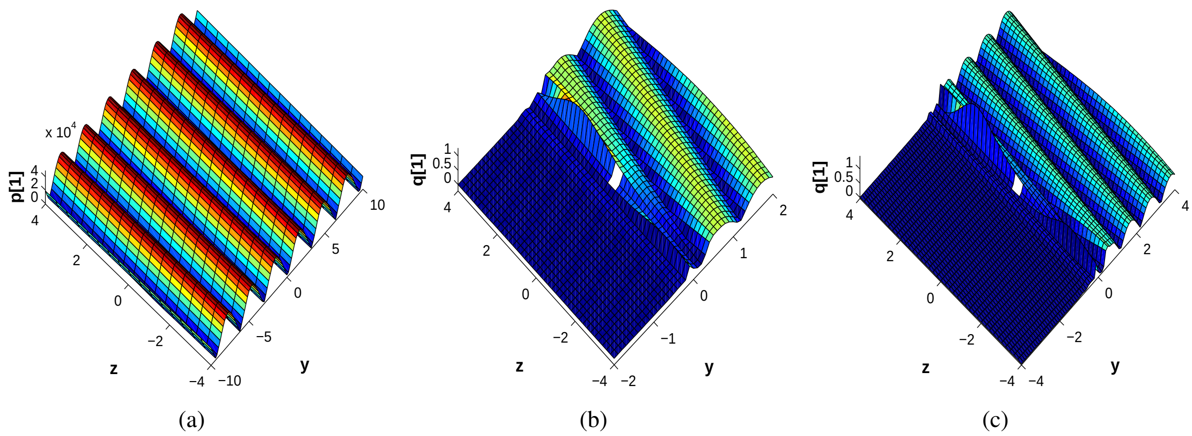


Figure 3. Single soliton solutions having seed solution $\chi = I_2 siny$ and $P = O_{2\times2}$ with parameters; (a) loop soliton: $\lambda_1 = -4i$, $\lambda_3 = -3i$, $c_1 = 4.5$ and $c_3 = 2.5$; (b) loop-breather soliton: $\lambda_1 = 3-4i$, $\lambda_3 = 2.5i$, $c_1 = 2.5$ and $c_3 = 3.5$; (c) loop-breather soliton: $\lambda_1 = 3-4.5i$, $\lambda_3 = 5i$, $c_1 = 2.5$ and $c_3 = 3.5$.

As a result, loop-soliton solutions can easily be found whenever exponential functions exist. Meanwhile, a breather solution is a bound state soliton and anti-soliton solution within a particular parameter domain. The existence of breather solutions in a particular equation depends on the specific form of the equation and the nonlinearity it contains. In general, breather solutions can exist in equations that have a balance between dispersion and nonlinearity. In addition, we have analyzed graphically that the two-component c-SP equation has diverse dynamical characteristics and interactional properties. The exact soliton solutions for one-fold DT with seed solution give peak soliton solutions as well when we change the parameters from $\lambda_1 = 4i$, $\lambda_3 = 3.5i$, $c_1 = 4.5$, $c_3 = 2.5$ to $\lambda_1 = 3.5i$, $\lambda_3 = 2.5i$, $c_1 = 8.5$, $c_3 = 8.5$. We can also figure out and analyze up to Nth-order exact soliton solutions in terms of explicit expression using the above iterative scheme.

5. Concluding remarks

In the present paper, the classical DT is employed to analyze the two-component c-SP equation. Initially, we have shown the Lax integrability of the two-component c-SP equation and have found a convenient Lax pair in the form of matrix generalization using the existing Lax pair. In addition, the multi-soliton solutions for one-fold DT have been thoroughly evaluated in terms of a ratio of ordinary determinants using DT and generalized for N-fold DT, as well as their corresponding scalar solutions. In order to find exact soliton solutions, we have tested and chosen suitable seed solutions that verify the transformed two-component c-SP equation. The particular column solutions are determined via integral calculations, and after that appropriate reduction constraints have been taken into consideration for constructing multi-soliton exact solutions in explicit form. Lastly, the one-soliton and two-soliton solutions have also been investigated via graphs, which have shown many interesting properties. This work might be helpful to formulate soliton solutions for multi-component c-SP equations in the future.

Acknowledgments

In addition, the second author wishes to thank Prof. Dr. Usman Saleem, Department of Physics, University of Punjab, for their helpful discussion and especially thank the whole team who provided software support. The work was partially supported by the National Nature Science Foundation of China (No. 11701334) and the "Jingying" Project of Shandong University of Science and Technology.

Conflict of interest

The authors declare no conflict of interest.

References

1. Y. Song, X. Shi, C. Wu, D. Tang, H. Zhang, Recent progress of study on optical solitons in fiber lasers, *Appl. Phys. Rev.*, **6** (2019), 021313. http://dx.doi.org/10.1063/1.5091811
2. K. Rajitha, C. Mishra, T. Dey, P. Panigrahi, Phase-controlled stable solitons in nonlinear fibers, *J. Opt. Soc. Am. B*, **36** (2019), 1–6. http://dx.doi.org/10.1364/JOSAB.36.000001

3. G. Agrawal, Nonlinear fiber optics, In: *Nonlinear science at the dawn of the 21st century*, Berlin: Springer, 2000. http://dx.doi.org/10.1007/3-540-46629-0_9

4. Y. Chung, C. Jones, T. Schäfer, C. Wayne, Ultra-short pulses in linear and nonlinear media, *Nonlinearity,* **18** (2005), 1351. http://dx.doi.org/10.1088/0951-7715/18/3/021

5. S. Shen, Z. Yang, Z. Pang, Y. Ge, The complex-valued astigmatic cosine-Gaussian soliton solution of the nonlocal nonlinear Schrödinger equation and its transmission characteristics, *Appl. Math. Lett.*, **125** (2022), 107755. http://dx.doi.org/10.1016/j.aml.2021.107755

6. J. Guo, Z. Yang, L. Song, Z. Pang, Propagation dynamics of tripole breathers in nonlocal nonlinear media, *Nonlinear Dyn.*, **101** (2020), 1147–1157. http://dx.doi.org/10.1007/s11071-020-05829-7

7. S. Shen, Z. Yang, X. Li, S. Zhang, Periodic propagation of complex-valued hyperbolic-cosine-Gaussian solitons and breathers with complicated light field structure in strongly nonlocal nonlinear media, *Commun. Nonlinear Sci.*, **103** (2021), 106005. http://dx.doi.org/10.1016/j.cnsns.2021.106005

8. V. Zakharov, A. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Soviet Journal of Experimental and Theoretical Physics*, **34** (1972), 118–134.

9. B. Feng, Complex short pulse and coupled complex short pulse equations, *Physica D*, **297** (2015), 62–75. http://dx.doi.org/10.1016/j.physd.2014.12.002

10. T. Schäfer, C. Wayne, Propagation of ultra-short optical pulses in cubic nonlinear media, *Physica D*, **196** (2004), 90–105. http://dx.doi.org/10.1016/j.physd.2004.04.007

11. J. Brunelli, The bi-hamiltonian structure of the short pulse equation, *Phys. Lett. A*, **353** (2006), 475–478. http://dx.doi.org/10.1016/j.physleta.2006.01.009

12. A. Sakovich, S. Sakovich, Solitary wave solutions of the short pulse equation, *J. Phys. A: Math. Gen.*, **39** (2006), 361. http://dx.doi.org/10.1088/0305-4470/39/22/L03

13. M. Rabelo, On equations which describe pseudospherical surfaces, *Stud. Appl. Math.*, **81** (1989), 221–248. http://dx.doi.org/10.1002/sapm1989813221

14. A. Sakovich, S. Sakovich, The short pulse equation is integrable, *J. Phys. Soc. Jpn.*, **74** (2005), 239–241. http://dx.doi.org/10.1143/jpsj.74.239

15. Y. Matsuno, Multiloop soliton and multibreather solutions of the short pulse model equation, *J. Phys. Soc. Jpn.*, **76** (2007), 084003. http://dx.doi.org/10.1143/JPSJ.76.084003

16. Q. Zhang, Y. Xia, Discontinuous Galerkin methods for short pulse type equations via hodograph transformations, *J. Comput. Phys.*, **399** (2019), 108928. http://dx.doi.org/10.1016/j.jcp.2019.108928

17. R. Hirota, *The direct method in soliton theory*, Cambridge: Cambridge University Press, 2004. http://dx.doi.org/10.1017/cbo9780511543043

18. Z. Li, S. Tian, J. Yang, On the soliton resolution and the asymptotic stability of N-soliton solution for the Wadati-Konno-Ichikawa equation with finite density initial data in space-time solitonic regions, *Adv. Math.*, **409** (2022), 108639. http://dx.doi.org/10.1016/j.aim.2022.108639

19. Z. Li, S. Tian, J. Yang, Soliton resolution for the Wadati-Konno-Ichikawa equation with weighted Sobolev initial data, *Ann. Henri Poincaré*, **23** (2022), 2611–2655. http://dx.doi.org/10.1007/s00023-021-01143-z

20. B. Feng, L. Ling, Darboux transformation and solitonic solution to the coupled complex short pulse equation, *Physica D*, **437** (2022), 133332. http://dx.doi.org/10.1016/j.physd.2022.133332

21. H. Sun, Z. Zhu, Darboux transformation and soliton solutions of the spatial discrete coupled complex short pulse equation, *Physica D*, **436** (2022), 133312. http://dx.doi.org/10.1016/j.physd.2022.133312

22. Z. Li, S. Tian, J. Yang, E. Fan, Soliton resolution for the complex short pulse equation with weighted Sobolev initial data in space-time solitonic regions, *J. Differ. Equations*, **329** (2022), 31–88. http://dx.doi.org/10.1016/j.jde.2022.05.003

23. H. Wang, X. Wen, Soliton elastic interactions and dynamical analysis of a reduced integrable nonlinear schrödinger system on a triangular-lattice ribbon, *Nonlinear Dyn.*, **100** (2020), 1571–1587. http://dx.doi.org/10.1007/s11071-020-05587-6

24. L. Ling, L. Zhao, B. Guo, Darboux transformation and multi-dark soliton for N-component nonlinear Schrödinger equations, *Nonlinearity*, **28** (2015), 3243. http://dx.doi.org/10.1088/0951-7715/28/9/3243

25. V. Matveev, M. Salle, *Darboux transformations and solitons*, Berlin: Springer-Verlag, 1991.

26. Z. Lin, X. Wen, Dynamical analysis of position-controllable loop rogue wave and mixed interaction phenomena for the complex short pulse equation in optical fiber, *Nonlinear Dyn.*, **108** (2022), 2573–2593. http://dx.doi.org/10.1007/s11071-022-07315-8

27. M. Manas, L. Alonso, A hodograph transformation which applies to the heavenly equation, arXiv:nlin/0209050.

28. Q. Zhao, M. Amin, Explicit solutions of rational integrable differential-difference equations, *Partial Differential Equations in Applied Mathematics*, **5** (2022), 100338. http://dx.doi.org/10.1016/j.padiff.2022.100338

29. Y. Yang, E. Fan, Soliton resolution for the short-pulse equation, *J. Differ. Equations*, **280** (2021), 644–689. http://dx.doi.org/10.1016/j.jde.2021.01.036

30. U. Saleem, M. ul Hassan, Darboux transformation and multisoliton solutions of the short pulse equation, *J. Phys. Soc. Jpn.*, **81** (2012), 094008. http://dx.doi.org/10.1143/JPSJ.81.094008

31. J. He, L. Zhang, Y. Cheng, Y. Li, Determinant representation of darboux transformation for the AKNS system, *Sci. China Ser. A*, **49** (2006), 1867–1878. http://dx.doi.org/10.1007/s11425-006-2025-1

32. H. Sarfraz, U. Saleem, Y. Hanif, Loop dynamics of a fully discrete short pulse equation, arXiv:2209.00738.

AIMS Press