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Research article

Iteration changes discontinuity into smoothness (II): oscillating case

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Abstract: It has been shown that a self-mapping with exactly one removable or jumping discontinuity may have a C^1 smooth iterate of the second-order. However, some examples show that a self-mapping with exactly one oscillating discontinuity may also have a C^1 smooth iterate of the second-order, indicating that iteration can turn a self-mapping with exactly one oscillating discontinuity into a C^1 smooth one. In this paper, we study piecewise C^1 self-mappings on the open interval $(0, 1)$ having only one oscillating discontinuity. We give necessary and sufficient conditions for those self-mappings whose second-order iterates are C^1 smooth.

Keywords: iteration; oscillating discontinuity; C^1 smooth; piecewise C^1 smooth
Mathematics Subject Classification: 37E05, 39B12

1. Introduction

The n-th iterate f^n of a mapping $f : E \to E$ is defined by $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) = x$ for all $x \in E$ inductively, where E is a nonempty set and n is a positive integer. The research on the iteration of mappings can be traced back more than one hundred years ago at least ([1,2,7]). The iterative operation is much more complicated than the general algebraic operation, especially the iteration of nonlinear functions, so the research work is very difficult and tortuous. Iteration is a common phenomenon in nature, which has become the focus of many disciplines. Under such circumstances, dynamical system theory has developed rapidly.

It is often thought that iteration turns a bad function into a worse one. For example, the function and its iterate

$$f(x)=\begin{cases}\frac{5}{4}x+\frac{1}{8}, & 0<x<\frac{1}{2},\\ \frac{1}{8}, & x=\frac{1}{2},\\ x-\frac{1}{4}, & \frac{1}{2}<x<\frac{3}{4},\\ \frac{1}{2}, & x=\frac{3}{4},\\ -\frac{4}{5}x+\frac{11}{10}, & \frac{3}{4}<x<1,\end{cases}\quad f^2(x)=\begin{cases}\frac{25}{16}x+\frac{9}{32}, & 0<x<\frac{3}{10},\\ \frac{1}{8}, & x=\frac{3}{10},\\ \frac{5}{4}x-\frac{1}{8}, & \frac{3}{10}<x<\frac{1}{2},\\ \frac{9}{32}, & x=\frac{1}{2},\\ \frac{5}{4}x-\frac{3}{16}, & \frac{1}{2}<x<\frac{3}{4},\\ \frac{1}{8}, & x=\frac{3}{4},\\ -x+\frac{3}{2}, & \frac{3}{4}<x<1.\end{cases}$$

It is easy to see that f has exactly one discontinuity at $\frac{1}{2}$ (see Figure 1), but its iterate f^2 has exactly three discontinuities at $\frac{3}{10}$, $\frac{1}{2}$ and $\frac{3}{4}$ (see Figure 2). However, a discontinuous function may have a continuous second-order iterate as shown in [3] and [5], which shows that iteration can also convert a “bad” function to a “good” one. This encourages efforts to study of such a converting. In [3] and [5] all self-mappings on a compact interval with exactly one discontinuity and more than one but finitely many discontinuities of the same type were classified for such a converting respectively. In [4] all continuous self-mappings with exactly one nonsmooth point were classified for the converting to C^1 iterates. Recently, we investigated the C^1 smoothness iterate of the second-order for self-mappings with exactly one removable or jumping discontinuity, we obtained necessary and sufficient conditions for those self-mappings whose second-order iterates are C^1 smooth in [6]. For a continuation, we are also interested in C^1 smoothness iterate of the second-order for self-mappings with exactly one oscillatory discontinuity, the remaining case of discontinuity.

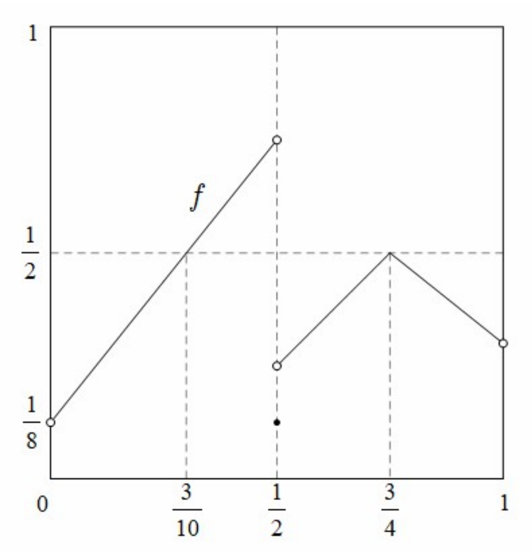

Figure 1. f is not C^0 at $\frac{1}{2}$.

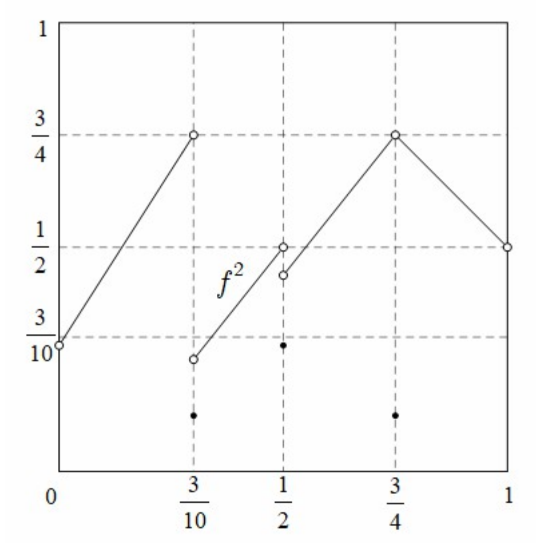

Figure 2. f^2 is not C^0 at $\frac{3}{10}$, $\frac{1}{2}$ and $\frac{3}{4}$.

It is possible to find an example with exactly one oscillatory discontinuity which is C^1 smooth by its iteration. The function and its iterate

$$f(x) = \begin{cases} \frac{1}{4}, & 0 < x \le \frac{1}{4}, \\ \frac{1}{8} + \frac{1}{8}\sin^2\frac{\pi}{4-8x}, & \frac{1}{4} < x < \frac{1}{2}, \\ \frac{1}{8}, & x = \frac{1}{2}, \\ -2(x-\frac{3}{4})^2 + \frac{3}{8}, & \frac{1}{2} < x < 1, \end{cases} \quad f^2(x) = \begin{cases} \frac{1}{4}, & 0 < x \le \frac{1}{2}, \\ \frac{1}{8} + \frac{1}{8}\sin^2\frac{\pi}{1+16(x-\frac{3}{4})^2}, & \frac{1}{2} < x < 1. \end{cases}$$

One can see that f has exactly one oscillatory discontinuity at $\frac{1}{2}$ (see Figure 3), but its iterate f^2 is C^1 smooth on the whole interval $(0, 1)$(see Figure 4).

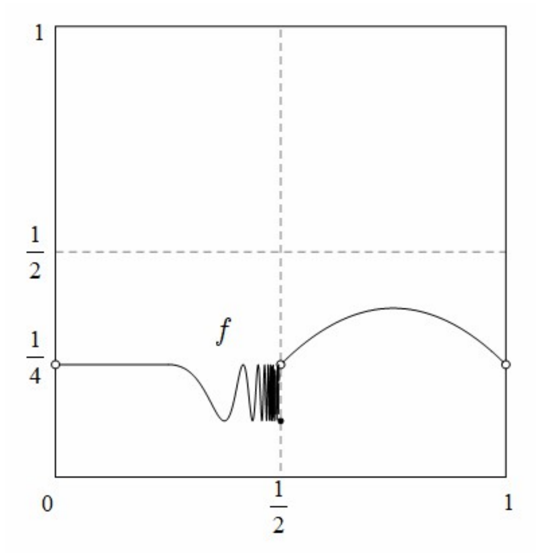

Figure 3. f is discontinuous at $\frac{1}{2}$.

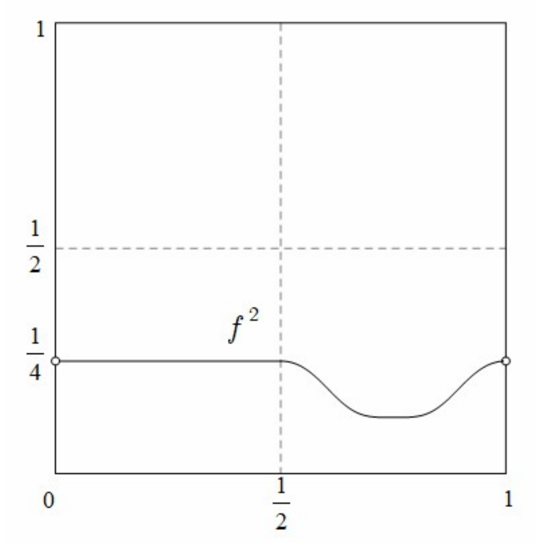

Figure 4. f^2 is C^1 smooth on $(0,1)$.

Let $I := (0,1)$ and $V_o(I,I)$ consist of all C^1 self-mappings on I with exactly one oscillatory discontinuity. Each $f \in V_o(I,I)$ can be presented as

$$f(x) = \begin{cases} f_1(x), & x \in I_1 := (0, x_0), \\ c, & x = x_0, \\ f_2(x), & x \in I_2 := (x_0, 1), \end{cases} \tag{1.1}$$

where $x_0 \in (0,1)$ is the unique oscillatory discontinuity, f_i is C^1 smooth on I_i for each $i \in \{1,2\}$ and $c \in (0,1)$ is a constant.

In this paper we continue the work of [6], investigating the second-order C^1 smoothness of mappings in $V_o(I,I)$. By the definition of oscillating discontinuities, either $\lim_{x\to x_0-0} f_1(x)$ or $\lim_{x\to x_0+0} f_2(x)$ does not exist but both f_1 and f_2 are bounded. Thus, each mapping f in $V_o(I,I)$ has 3 possibilities:

$$\begin{aligned} V_{o+}(I,I) &:= \{f \in V_o(I,I) \mid y_1 := \lim_{x\to x_0-0} f_1(x) \text{ exists but } \lim_{x\to x_0+0} f_2(x) \text{ does not exist}\}, \\ V_{o-}(I,I) &:= \{f \in V_o(I,I) \mid y_2 := \lim_{x\to x_0+0} f_2(x) \text{ exists but } \lim_{x\to x_0-0} f_1(x) \text{ does not exist}\}, \\ V_{o*}(I,I) &:= \{f \in V_o(I,I) \mid \text{neither } \lim_{x\to x_0-0} f_1(x) \text{ nor } \lim_{x\to x_0+0} f_2(x) \text{ exists}\}. \end{aligned}$$

In order to investigate C^1 smoothness of the second-order iterates of mappings in $V_o(I,I)$, we also need to consider three subclasses for $V_{o+}(I,I)$ and $V_{o-}(I,I)$ respectively, i.e., $V_{o+}(I,I) = V^E_{o+}(I,I) \cup V^O_{o+}(I,I) \cup V^\infty_{o+}(I,I)$ and $V_{o-}(I,I) = V^E_{o-}(I,I) \cup V^O_{o-}(I,I) \cup V^\infty_{o-}(I,I)$, where

$$\begin{aligned} V^E_{o+}(I,I) &:= \{f \in V_{o+}(I,I) \mid \tilde{y}_1 := \lim_{x\to x_0-0} f_1'(x) \text{ exists}\}, \\ V^O_{o+}(I,I) &:= \{f \in V_{o+}(I,I) \mid \lim_{x\to x_0-0} f_1'(x) \text{ does not exist but} f_1' \text{ is bounded}\}, \\ V^\infty_{o+}(I,I) &:= \{f \in V_{o+}(I,I) \mid \lim_{x\to x_0-0} f_1'(x) \text{ does not exist but} f_1' \text{ is unbounded}\}, \\ V^E_{o-}(I,I) &:= \{f \in V_{o-}(I,I) \mid \tilde{y}_2 := \lim_{x\to x_0+0} f_2'(x) \text{ exists}\}, \end{aligned}$$

$$V_{o-}^{O}(I,I) \ := \ \{f \in V_{o-}(I,I) \mid \lim_{x\to x_0+0} f_2'(x) \text{ does not exist but} f_2' \text{ is bounded}\},$$
$$V_{o-}^{\infty}(I,I) \ := \ \{f \in V_{o-}(I,I) \mid \lim_{x\to x_0+0} f_2'(x) \text{ does not exist but} f_2' \text{ is unbounded}\}.$$

Then $V_o(I,I) = V_{o+}^E(I,I) \cup V_{o+}^O(I,I) \cup V_{o+}^\infty(I,I) \cup V_{o-}^E(I,I) \cup V_{o-}^O(I,I) \cup V_{o-}^\infty(I,I) \cup V_{o*}(I,I)$.

In this paper, we discuss C^1 smoothness of the second-order iterates of mapping f in $V_o(I,I)\backslash V_{o-}^\infty(I,I) \cup V_{o+}^\infty(I,I)$. We give necessary and sufficient conditions for C^1 smooth f^2. We obtain necessary conditions and remark the difficulties in finding sufficient conditions for those self-mappings in $V_{o-}^\infty(I,I)$ and $V_{o+}^\infty(I,I)$ to have a C^1 smooth iterate of the second-order respectively. Moreover, we give sufficient conditions for those self-mappings in $V(I,I)$ whose second-order iterates are not C^1 smooth, where $V(I,I)$ consists of all C^1 self-mappings on I having only one discontinuity. Finally, we use examples to demonstrate our theorems.

For convenience, let $I_0 := \{x_0\}$, then $I = I_1 \cup I_0 \cup I_2$. For $i, j = 0, 1, 2$ we use the notations

$$\Delta_i^- \ := \ \{\alpha \in I \mid f(U_\alpha^-) \subset I_i\},$$
$$\Delta_j^+ \ := \ \{\alpha \in I \mid f(U_\alpha^+) \subset I_j\},$$
$$\Delta_{ij} \ := \ \{\alpha \in I \mid f(U_\alpha^-) \subset I_i \text{ and } f(U_\alpha^+) \subset I_j\},$$

where U_α^- and U_α^+ denote a sufficiently small left-half and right-half neighborhood of α respectively. We use D_-f and D_+f to denote the left derivative of f and the right derivative of f respectively.

2. Iteration for $V_o(I,I)$

In this section, we consider C^1 smoothness of the second-order iterates of $f \in V_o(I,I)$ to be defined as in (1.1). In order to discuss C^1 smoothness of the second-order iterates of mappings in $V_o(I,I)$, we need to consider constantization of a mapping near the boundary of the domain, as shown in the Fourth part of [3]. Assume that $h_1 : H_1 := (c,d) \to H_2$ and $h_2 : H_2 \to H_3$ are continuous mapping, where H_i s ($i = 1,2,3$) are all nonempty intervals. h_1 is said to be *constantized* by h_2 near c (or d) if there exist a closed interval $L \subseteq H_2$ and a vicinity (hollow neighborhood) $U \subseteq H_1$ of c (or d) such that $h_1(U) \subseteq L$ and h_2 is identical to a constant on L. For convenience, let $\theta(h_1,h_2)$ denote the constant. Moreover, we also need to define two mappings $f_{10} : I_{10} := I_1 \cup I_0 \to I$ and $f_{20} : I_{20} := I_0 \cup I_2 \to I$ such that

$$f_{10}(x) = \begin{cases} f_1(x), & x \in I_1, \\ c, & x = x_0, \end{cases}$$
$$f_{20}(x) = \begin{cases} c, & x = x_0, \\ f_2(x), & x \in I_2, \end{cases}$$

where $x_0 \in (0,1)$ is the unique oscillatory discontinuity. For convenience, for $i \in \{1,2,10,20\}$, $m, j \in \{1,2\}$, $\tau \in \{E,O,\infty\}$, $\lambda \in \{-,+\}$, $l \in \{1,2\}\backslash\{m\}$, $\mu \in \{O,\infty\}$ and $\xi_j \in f^{-1}(I_0) \cap I_j$ let

$$\complement_{o\lambda}^{Emij}(I,I) \ := \ \{f \in V_{o\lambda}^E(I,I) \mid \theta(f_m,f_i) = f_j(y_l) = f(c) \text{ and } f_j'(y_l)\tilde{y}_l = 0\},$$
$$\complement_{o\lambda}^{\mu mij}(I,I) \ := \ \{f \in V_{o\lambda}^\mu(I,I) \mid \theta(f_m,f_i) = f_j(y_l) = f(c) \text{ and } f_j'(y_l) = 0\},$$
$$\hat{\complement}_{o\lambda}^{Emi}(I,I) \ := \ \{f \in V_{o\lambda}^E(I,I) \mid \theta(f_m,f_i) = c = f(c)\},$$
$$\tilde{\complement}_{o\lambda}^{Emi}(I,I) \ := \ \{f \in V_{o\lambda}^E(I,I) \mid \theta(f_m,f_i) = y_l = f(c) \text{ and } \tilde{y}_l = 0\},$$

$$\bar{\complement}^{\tau mj}_{o\lambda}(I,I) \;:=\; \{f\in V^{\tau}_{o\lambda}(I,I)\mid y_m=c \text{ and } f'_j(\xi_j)=0\},$$
$$\complement^{j}_{0*}(I,I) \;:=\; \{f\in V_{o*}(I,I)\mid \theta(f_1,f_j)=\theta(f_2,f_j)=f(c)(c\neq x_0)\}.$$

Theorem 2.1. *Suppose that $f\in V_o(I,I)\backslash V^{\infty}_{o+}(I,I)\cup V^{\infty}_{o-}(I,I)$ with the unique oscillatory discontinuity $x_0\in(0,1)$ and that $\xi_j\in f^{-1}(I_0)\cap I_j$ for $j=1$ or 2. Let $y_1:=\lim_{x\to x_0-0}f_1(x)$ and $y_2:=\lim_{x\to x_0+0}f_2(x)$. The following results hold:*
(o-) *In the case that $f\in V^{\tau}_{o-}(I,I)$ for $\tau\in\{E,O\}$, f^2 is C^1 smooth on I if and only if there is $i\in\{1,2,20\}$ such that f_1 is constantized by f_i near x_0 and for $j\in\{1,2\}$ the following two conditions are both fulfilled:*
(o-1) *$f\in\complement^{\tau 1ij}_{o-}(I,I)$ if $y_2\in I_j$, either $f\in\hat{\complement}^{E1i}_{o-}(I,I)$ as $x_0\in\Delta^{+}_{0}$ or $f\in\tilde{\complement}^{E1i}_{o-}(I,I)$ as $x_0\in\Delta^{+}_{2}$ if $y_2=x_0$.*
(o-2) *$\xi_j\in\Delta_{00}\cup\Delta_{22}\cup\Delta_{20}\cup\Delta_{02}$ and $f\in\bar{\complement}^{\tau 2j}_{o-}(I,I)$ if $\xi_j\in\Delta_{22}\cup\Delta_{20}\cup\Delta_{02}$.*
(o+) *In the case that $f\in V^{\lambda}_{o+}(I,I)$ for $\lambda\in\{E,O\}$, f^2 is C^1 smooth on I if and only if there is $k\in\{1,2,10\}$ such that f_2 is constantized by f_k near x_0 and for $j\in\{1,2\}$ the following two conditions are both fulfilled:*
(o+1) *$f\in\complement^{\lambda 2kj}_{o+}(I,I)$ if $y_1\in I_j$, either $f\in\hat{\complement}^{E2k}_{o+}(I,I)$ as $x_0\in\Delta^{-}_{0}$ or $f\in\tilde{\complement}^{E2k}_{o+}(I,I)$ as $x_0\in\Delta^{-}_{1}$ if $y_1=x_0$.*
(o+2) *$\xi_j\in\Delta_{00}\cup\Delta_{11}\cup\Delta_{10}\cup\Delta_{01}$ and $f\in\bar{\complement}^{\lambda 1j}_{o+}(I,I)$ if $\xi_j\in\Delta_{11}\cup\Delta_{10}\cup\Delta_{01}$.*
(o*) *In the case that $f\in V_{o*}(I,I)$, f^2 is C^1 smooth on I if and only if both f_1 and f_2 are constantized by f_j near x_0, $f(I_1\cup I_2)\subseteq I_j$ holds and $f\in\complement^{j}_{0*}(I,I)$ for $j=1$ or 2.*

Proof. Since $f\in V_o(I,I)\backslash V^{\infty}_{o-}(I,I)\cup V^{\infty}_{o+}(I,I)$ and $V_o(I,I)=V^{E}_{o-}(I,I)\cup V^{O}_{o-}(I,I)\cup V^{\infty}_{o-}(I,I)\cup V^{E}_{o+}(I,I)\cup V^{O}_{o+}(I,I)\cup V^{\infty}_{o+}(I,I)\cup V_{o*}(I,I)$. Then there are five cases to be discussed: $f\in V^{E}_{o-}(I,I)$, $f\in V^{O}_{o-}(I,I)$, $f\in V^{E}_{o+}(I,I)$, $f\in V^{O}_{o+}(I,I)$ and $f\in V_{o*}(I,I)$.

For **(o-)**, i.e., $f\in V^{\tau}_{o-}(I,I)$ for $\tau\in\{E,O\}$, it implies that $y_2:=\lim_{x\to x_0+0}f_2(x)$ exists but $\lim_{x\to x_0-0}f_1(x)$ does not exist.

Sufficiency of **(o-).** Since we have assumed that there is $i\in\{1,2,20\}$ such that f_1 is constantized by f_i near x_0. In the following, we only discuss the situation that f_1 is constantized by f_1 near x_0 since the other situations can be discussed similarly. Under condition **(o-1)**, we prove that f^2 is C^1 smooth at x_0. In fact, if $y_2\in I_1$, by the definition that f_1 is constantized by f_1 near x_0 and the continuity of f_2 on I_2, there exist a sufficiently small left-half neighborhood $U^{-}_{x_0}$ of x_0 and a sufficiently small right-half neighborhood $U^{+}_{x_0}$ of x_0 such that

$$f^2(x)=\begin{cases}\theta(f_1,f_1), & x\in U^{-}_{x_0},\\ f(c), & x=x_0,\\ f_1(f_2(x)), & x\in U^{+}_{x_0}.\end{cases}\tag{2.1}$$

It follows from (2.1) that

$$\lim_{x\to x_0-0}f^2(x)=\theta(f_1,f_1),\tag{2.2}$$
$$\lim_{x\to x_0+0}f^2(x)=\lim_{x\to x_0+0}f_1(f_2(x))=f_1(y_2).\tag{2.3}$$

Since we assumed that $f\in\complement^{\tau 111}_{o-}(I,I)$ for $\tau\in\{E,O\}$. Thus, we need to discuss in two situations: $f\in\complement^{E111}_{o-}(I,I)$ and $f\in\complement^{O111}_{o-}(I,I)$. In the first situation that $f\in\complement^{E111}_{o-}(I,I)$, by the definition of $\complement^{E111}_{o-}(I,I)$

we see that $f \in V^E_{o-}(I, I)$, it implies that $\tilde{y}_2 := \lim_{x\to x_0+0} f_2'(x)$ exists. Note that $\lim_{x\to x_0+0} f_2(x) = y_2$. From (2.1) we get that

$$D_- f^2(x_0) = 0, \tag{2.4}$$

$$D_+ f^2(x_0) = \lim_{x\to x_0+0} f_1'(f_2(x)) f_2'(x) = f_1'(y_2)\tilde{y}_2. \tag{2.5}$$

By our assumption that $f \in \complement^{E111}_{o-}(I, I)$ and the definition of $\complement^{E111}_{o-}(I, I)$, we get from (2.2)–(2.5) that $\lim_{x\to x_0-0} f^2(x) = \lim_{x\to x_0+0} f^2(x) = f^2(x_0)$ and $D_- f^2(x_0) = D_+ f^2(x_0) = 0$. It implies that f^2 is C^1 smooth at x_0. In the second situation that $f \in \complement^{O111}_{o-}(I, I)$, by the definition of $\complement^{O111}_{o-}(I, I)$ we see that $f \in V^O_{o-}(I, I)$, it implies that $\lim_{x\to x_0+0} f_2'(x)$ does not exist but f_2' is bounded. From (2.1) we get that

$$D_- f^2(x_0) = 0, \tag{2.6}$$

$$D_+ f^2(x_0) = \lim_{x\to x_0+0} f_1'(f_2(x)) f_2'(x). \tag{2.7}$$

By our assumption that $f \in \complement^{O111}_{o-}(I, I)$ and the definition of $\complement^{O111}_{o-}(I, I)$, we get from (2.2) and (2.3) that $\lim_{x\to x_0-0} f^2(x) = \lim_{x\to x_0+0} f^2(x) = f^2(x_0)$. Moreover, from (2.6) and (2.7) we obtain $D_- f^2(x_0) = D_+ f^2(x_0) = 0$ since $\lim_{x\to x_0+0} f_1'(f_2(x)) = f_1'(y_2) = 0$ and f_2' is bounded. It implies that f^2 is C^1 smooth at x_0. The proof of $y_2 \in I_2$ is similar to the proof of $y_2 \in I_1$. Next, we consider that $y_2 = x_0$, we have

$$f^2(x) = \begin{cases} \theta(f_1, f_1), & x \in U^-_{x_0}, \\ f(c), & x = x_0, \\ c, & x \in U^+_{x_0}, \end{cases} \tag{2.8}$$

when $x_0 \in \Delta^+_0$. It follows from (2.8) that

$$\lim_{x\to x_0-0} f^2(x) = \theta(f_1, f_1), \tag{2.9}$$

$$\lim_{x\to x_0+0} f^2(x) = c, \tag{2.10}$$

$$D_- f^2(x_0) = D_+ f^2(x_0) = 0. \tag{2.11}$$

By our assumption that $f \in \hat{\complement}^{E11}_{o-}(I, I)$ and the definition of $\hat{\complement}^{E11}_{o-}(I, I)$, we get from (2.9)–(2.11) that $\lim_{x\to x_0-0} f^2(x) = \lim_{x\to x_0+0} f^2(x) = f^2(x_0)$ and $D_- f^2(x_0) = D_+ f^2(x_0) = 0$. It implies that f^2 is C^1 smooth at x_0. Moreover, we have

$$f^2(x) = \begin{cases} \theta(f_1, f_1), & x \in U^-_{x_0}, \\ f(c), & x = x_0, \\ f_2(f_2(x)), & x \in U^+_{x_0}, \end{cases} \tag{2.12}$$

when $x_0 \in \Delta^+_2$. By our assumption that $f \in \tilde{\complement}^{E11}_{o-}(I, I)$, we see that $f \in V^E_{o-}(I, I)$, it implies that $\tilde{y}_2 := \lim_{x\to x_0+0} f_2'(x)$ exists. Note that $f_2(x) \to x_0 + 0$ as $x \to x_0 + 0$. From (2.12) we get that

$$\lim_{x\to x_0-0} f^2(x) = \theta(f_1, f_1), \tag{2.13}$$

$$\lim_{x\to x_0+0} f^2(x) = \lim_{x\to x_0+0} f_2(f_2(x)) = \lim_{y\to x_0+0} f_2(y) = y_2, \tag{2.14}$$

$$D_-f^2(x_0) = 0, \tag{2.15}$$

$$D_+f^2(x_0) = \lim_{x\to x_0+0} f_2'(f_2(x))f_2'(x) = \tilde{y}_2^2. \tag{2.16}$$

By the assumption that $f \in \tilde{\complement}_{o-}^{E11}(I,I)$ and the definition of $\tilde{\complement}_{o-}^{E11}(I,I)$, we get from (2.13)–(2.16) that $\lim_{x\to x_0-0} f^2(x) = \lim_{x\to x_0+0} f^2(x) = f^2(x_0)$ and $D_-f^2(x_0) = D_+f^2(x_0) = 0$. It implies that f^2 is C^1 smooth at x_0. Similarly to the proof of the condition **(ii)** of the sufficiency in Theorem 3 in [6], one can prove that f^2 is C^1 smooth at ξ_j under condition **(o-2)**, where $\xi_j \in f^{-1}(I_0) \cap I_j$ for $j = 1$ or 2. Condition **(o-1)** and condition **(o-2)** imply that f^2 is C^1 smooth on the whole domain I. Therefore, the proof of sufficiency of **(o-)** is completed.

Necessity of **(o-).** Since $f \in V_{o-}^{\tau}(I,I)$ for $\tau \in \{E,O\}$, by the definition of $V_{o-}^{\tau}(I,I)$, implying that $f \in V_{o-}(I,I)$. Then we obtain from the definition of $V_{o-}(I,I)$ that $\lim_{x\to x_0-0} f_1(x)$ does not exist but $y_2 := \lim_{x\to x_0+0} f_2(x)$ exists. From the location of y_2, we need to consider two possibilities: either $y_2 \in I_j$ for $j = 1$ or 2, or $y_2 = x_0$. Assume that f^2 is C^1 smooth on I, it implies that f^2 is continuous on I. By **(i)** of Theorem 3 in reference [3], one sees that there is $i \in \{1,2,20\}$ such that f_1 is constantized by f_i near x_0. In what follows, we only consider the situation that f_1 is constantized by f_1 near x_0 since the other situations can be considered similarly. Note that $f \in V_{o-}^{\tau}(I,I)$ for $\tau \in \{E,O\}$. Thus, we need to discuss in two situations: $f \in V_{o-}^{E}(I,I)$ and $f \in V_{o-}^{O}(I,I)$. If $y_2 \in I_1$, one sees that (2.1) holds. It follows that (2.2) and (2.3) hold. In the first situation that $f \in V_{o-}^{E}(I,I)$, it implies from the definition of $V_{o-}^{E}(I,I)$ that $\tilde{y}_2 := \lim_{x\to x_0+0} f_2'(x)$ exists. It follows that (2.4) and (2.5) hold. Because we have assumed that f^2 is C^1 smooth on I, we have $\lim_{x\to x_0-0} f^2(x) = \lim_{x\to x_0+0} f^2(x) = f^2(x_0)$ and $D_-f^2(x_0) = D_+f^2(x_0)$. Then we get from (2.2)–(2.5) that $\theta(f_1, f_1) = f_1(y_2) = f(c)$ and $f_1'(y_2)\tilde{y}_2 = 0$. It implies from the definition of $\complement_{o-}^{E111}(I,I)$ that $f \in \complement_{o-}^{E111}(I,I)$. In the second situation that $f \in V_{o-}^{O}(I,I)$, it implies from the definition of $V_{o-}^{O}(I,I)$ that $\lim_{x\to x_0+0} f_2'(x)$ does not exist but f_2' is bounded. From (2.1) we obtain (2.6) and (2.7). Since f^2 is C^1 smooth on I, we see that $D_+f^2(x_0)$ exists. Note that $\lim_{x\to x_0+0} f_1'(f_2(x)) = f_1'(y_2)$. We claim that

$$f_1'(y_2) = 0. \tag{2.17}$$

In fact, if $f_1'(y_2) \neq 0$, It follows from (2.7) that $\lim_{x\to x_0+0} f_2'(x)$ exists since

$$\lim_{x\to x_0+0} f_2'(x) = \lim_{x\to x_0+0} \frac{f_1'(f_2(x))f_2'(x)}{f_1'(f_2(x))} = \frac{D_+f^2(x_0)}{f_1'(y_2)},$$

which contradicts to our assumption that $\lim_{x\to x_0+0} f_2'(x)$ does not exist. Thus, the claim that (2.17) is proved. On the other hand, by the smoothness of f^2 on I we see that f^2 is continuous on I, then we have $\lim_{x\to x_0-0} f^2(x) = \lim_{x\to x_0+0} f^2(x) = f^2(x_0)$. It follows from (2.2) and (2.3) that $\theta(f_1, f_1) = f_1(y_2) = f(c)$. It implies from (2.17) and the definition of $\complement_{o-}^{O111}(I,I)$ that $f \in \complement_{o-}^{O111}(I,I)$. We use a similar discussion to the proof of the situation $y_2 \in I_1$, One can get that $f \in \complement_{o-}^{\tau 112}(I,I)$ for $\tau \in \{E,O\}$ when $y_2 \in I_2$. Finally, if $y_2 = x_0$, by the continuity of f_2 on I_2, we see that

$$x_0 \in \Delta_1^+ \cup \Delta_0^+ \cup \Delta_2^+. \tag{2.18}$$

In the first situation that $f \in V_{o-}^{E}(I,I)$, we claim that

$$x_0 \in \Delta_0^+ \cup \Delta_2^+. \tag{2.19}$$

By (2.18) we need to deny the case $x_0 \in \Delta_1^+$. In fact, if $x_0 \in \Delta_1^+$, from the definition of Δ_1^+, we have $f^2(x) = f_1(f_2(x))$, $\forall x \in U_{x_0}^+$. Note that $f \in V_{o-}^E(I, I)$ and $f_2(x) \to x_0 - 0$ as $x \to x_0 + 0$. It follows that $\lim_{x\to x_0+0} f^2(x) = \lim_{x\to x_0+0} f_1(f_2(x)) = \lim_{y\to x_0-0} f_1(y)$ does not exist, which implies that f^2 is not continuous at x_0, it follows that f^2 is not C^1 smooth on I, a contradiction to our assumption. This proves the claimed (2.19). By (2.19), we need to discuss the two cases $x_0 \in \Delta_0^+$ and $x_0 \in \Delta_2^+$. For the case $x_0 \in \Delta_0^+$, we see that (2.8) holds. It follows from (2.8) that (2.9) and (2.10) hold. Since we have assumed that f^2 is C^1 smooth on I, we have $\lim_{x\to x_0-0} f^2(x) = \lim_{x\to x_0+0} f^2(x) = f^2(x_0)$. Then we get from (2.9) and (2.10) that $\theta(f_1, f_1) = c = f(c)$. It implies from the definition of $\hat{\mathfrak{C}}_{o-}^{E11}(I, I)$ that $f \in \hat{\mathfrak{C}}_{o-}^{E11}(I, I)$. For the case $x_0 \in \Delta_2^+$, we see that (2.12) holds. It follows from (2.12) that (2.13-2.16) hold. Since we have assumed that f^2 is C^1 smooth on I, we have $\lim_{x\to x_0-0} f^2(x) = \lim_{x\to x_0+0} f^2(x) = f^2(x_0)$ and $D_-f^2(x_0) = D_+f^2(x_0)$. Then we get from (2.13-2.16) that $\theta(f_1, f_1) = y_2 = f(c)$ and $\tilde{y}_2 = 0$. It implies from the definition of $\tilde{\mathfrak{C}}_{o-}^{E11}(I, I)$ that $f \in \tilde{\mathfrak{C}}_{o-}^{E11}(I, I)$. Thus, condition **(o-1)** holds.

Remark that condition **(o-1)** does not give the results of the second situation that $f \in V_{o-}^O(I, I)$ when $y_2 = x_0$. In fact, by (2.18), if $x_0 \in \Delta_1^+$, using a similar discussion to the proof of the first situation that $f \in V_{o-}^E(I, I)$, we can get that $\lim_{x\to x_0+0} f^2(x)$ does not exist, a contradiction to our assumption. If $x_0 \in \Delta_0^+$, from the definition of Δ_0^+, there exists a sufficiently small right-half neighborhood $U_{x_0}^+$ of x_0 such that that $f_2(x) = x_0$ for $\forall x \in U_{x_0}^+$. It implies that $\lim_{x\to x_0+0} f_2'(x) = 0$, which contradicts the fact that $f \in V_{o-}^O(I, I)$. If $x_0 \in \Delta_2^+$, from (2.12) we have $D_+f^2(x_0) = \lim_{x\to x_0+0} f_2'(f_2(x))f_2'(x)$. Note that $\lim_{x\to x_0+0} f_2'(f_2(x)) = \lim_{y\to x_0+0} f_2'(y)$ and $f \in V_{o-}^O(I, I)$, i.e., $\lim_{x\to x_0+0} f_2'(x)$ does not exist but f_2' is bounded. Thus, it is hard to determine the existence of the limit $D_+f^2(x_0) = \lim_{x\to x_0+0} f_2'(f_2(x))f_2'(x)$.

We use a similar discussion to the proof of the condition **(ii)** of the necessity in Theorem 3 in [6], one can prove that condition **(o-2)** holds. Thus, the proof of result **(o-)** is completed.

Result **(o+)** can be discussed totally in a similar way to result **(o-)**. In what follows, we consider result **(o*)**.

We first prove **necessity** of **(o*)**. Suppose that f^2 is C^1 smooth on I, it follows that f^2 is continuous on I. By **(iii)** of Theorem 3 in reference [3], we see that both f_1 and f_2 are constantized by f_j near x_0, $f(I_1 \cup I_2) \subseteq I_j$ holds and $f \in \mathfrak{C}_{0*}^j(I, I)$ for $j = 1$ or 2. Hence, the proof of necessity is completed.

Next, we prove **sufficiency** of **(o*)**, we assumed that both f_1 and f_2 are constantized by f_1 near x_0, $f(I_1 \cup I_2) \subseteq I_1$ holds and $f \in \mathfrak{C}_{0*}^1(I, I)$. By **(iii)** of Theorem 3 in reference [3], we see that f^2 is continuous on I. By the definition of constantization, there exist a sufficiently small left-half neighborhood $U_{x_0}^-$ of x_0 and a sufficiently small right-half neighborhood $U_{x_0}^+$ of x_0 such that

$$f^2(x) = \begin{cases} \theta(f_1, f_1), & x \in U_{x_0}^-, \\ f(c), & x = x_0, \\ \theta(f_2, f_1), & x \in U_{x_0}^+. \end{cases} \tag{2.20}$$

we get from (2.20) that

$$D_-f^2(x_0) = D_+f^2(x_0) = 0.$$

Thus, the derivative of f^2 is continuous at x_0. It follows that f^2 is C^1 on I because f_1 and f_2 are C^1 on I_1 and I_2 respectively. Similarly, we can prove that f^2 is C^1 on I if both f_1 and f_2 are constantized by f_2 near x_0, $f(I_1 \cup I_2) \subseteq I_2$ holds and $f \in \mathfrak{C}_{0*}^2(I, I)$. Therefore, the proof of sufficiency is completed and the theorem is proved. □

Theorem 2.2. *Suppose that $f \in V^{\infty}_{o-}(I,I) \cup V^{\infty}_{o+}(I,I)$ with the unique oscillatory discontinuity $x_0 \in (0,1)$ and that $\xi_j \in f^{-1}(I_0) \cap I_j$ for $j = 1$ or 2. Let $y_1 := \lim_{x\to x_0-0} f_1(x)$ and $y_2 := \lim_{x\to x_0+0} f_2(x)$. The following results hold:*

(o-∞) *In the case that $f \in V^{\infty}_{o-}(I,I)$, assume that f^2 is C^1 smooth on I, then $y_2 \neq x_0$ and there exists $i \in \{1,2,20\}$ such that f_1 is constantized by f_i near x_0 and for $j \in \{1,2\}$ the following two conditions are both fulfilled:*

(o-∞1) *$f \in \complement^{\infty 1ij}_{o-}(I,I)$ if $y_2 \in I_j$;*

(o-∞2) *$\xi_j \in \Delta_{00} \cup \Delta_{22} \cup \Delta_{20} \cup \Delta_{02}$ and $f \in \bar{\complement}^{\infty 2j}_{o-}(I,I)$ if $\xi_j \in \Delta_{22} \cup \Delta_{20} \cup \Delta_{02}$.*

(o+∞) *In the case that $f \in V^{\infty}_{o+}(I,I)$, assume that f^2 is C^1 smooth on I, then $y_1 \neq x_0$ and there is $k \in \{1,2,10\}$ such that f_2 is constantized by f_k near x_0 and for $j \in \{1,2\}$ the following two conditions are both fulfilled:*

(o+∞1) *$f \in \complement^{\infty 2kj}_{o+}(I,I)$ if $y_1 \in I_j$;*

(o+∞2) *$\xi_j \in \Delta_{00} \cup \Delta_{11} \cup \Delta_{10} \cup \Delta_{01}$ and $f \in \bar{\complement}^{\infty 1j}_{o+}(I,I)$ if $\xi_j \in \Delta_{11} \cup \Delta_{10} \cup \Delta_{01}$.*

Proof. For **(o-∞)**, i.e., $f \in V^{\infty}_{o-}(I,I)$, by the definition of $V^{\infty}_{o-}(I,I)$, implying that $f \in V_{o-}(I,I)$. It follows from the definition of $V_{o-}(I,I)$ that $\lim_{x\to x_0-0} f_1(x)$ does not exist but $y_2 := \lim_{x\to x_0+0} f_2(x)$ exists. Moreover, we have $\lim_{x\to x_0+0} f_2'(x) = \infty$. By the C^1 smoothness of f^2 on I, We claim that $y_2 \neq x_0$. In fact, if $y_2 = x_0$, by the continuity of f_2 on I_2, we see that (2.18) holds. From (2.18) we need to discuss in three situations: $x_0 \in \Delta_1^+$, $x_0 \in \Delta_0^+$ and $x_0 \in \Delta_2^+$. In the first situation that $x_0 \in \Delta_1^+$, from the definition of Δ_1^+, we have $f^2(x) = f_1(f_2(x))$, $\forall x \in U^+_{x_0}$. Note that $f_2(x) \to x_0 - 0$ as $x \to x_0 + 0$. It follows that $\lim_{x\to x_0+0} f^2(x) = \lim_{x\to x_0+0} f_1(f_2(x)) = \lim_{y\to x_0-0} f_1(y)$ does not exist, which implies that f^2 is not continuous at x_0, it follows that f^2 is not C^1 smooth at x_0, which contradicts to our assumption that f^2 is C^1 smooth on I. In the second situation that $x_0 \in \Delta_0^+$, from the definition of Δ_0^+, we have $f_2(x) = x_0$, $\forall x \in U^+_{x_0}$. It follows that $\lim_{x\to x_0+0} f_2'(x) = 0$, which contradicts to our assumption that $\lim_{x\to x_0+0} f_2'(x) = \infty$. Finally, in the third situation that $x_0 \in \Delta_2^+$, from the definition of Δ_2^+, we have $f^2(x) = f_2(f_2(x))$, $\forall x \in U^+_{x_0}$. It follows that $D_+ f^2(x_0) = \lim_{x\to x_0+0} f_2'(f_2(x)) f_2'(x)$. Note that $\lim_{x\to x_0+0} f_2'(f_2(x)) = \lim_{y\to x_0+0} f_2'(y) = \infty$. Thus, we obtain $D_+ f^2(x_0) = \infty$. It implies that f^2 is not C^1 smooth on I, a contradiction to our assumption. Therefore, the claim that $y_2 \neq x_0$ is proved. We use a similar discussion to the proof of the situation $f \in V^{O}_{o-}(I,I)$ of the necessity in Theorem 2.1, one can get that there exists $i \in \{1,2,20\}$ such that f_1 is constantized by f_i near x_0 and both condition **(o-∞1)** and condition **(o-∞2)** hold.

Case **(o+∞)** can be discussed totally in a similar way to case **(o-∞)**. Therefore, the theorem is proved. □

Remark that the above Theorem 2.2 does not give sufficient conditions of f^2 to be C^1 because it is hard to determine the existence of either $\lim_{x\to x_0-0} f_i'(f_1(x)) f_1'(x)$ or $\lim_{x\to x_0+0} f_i'(f_2(x)) f_2'(x)$ for $i = 1$ or 2. In fact, if $f \in V^{\infty}_{o-}(I,I)$, it follows from the definition of $V^{\infty}_{o-}(I,I)$ that $\lim_{x\to x_0-0} f_1(x)$ does not exist but $y_2 := \lim_{x\to x_0+0} f_2(x)$ exists. Moreover, we have $\lim_{x\to x_0+0} f_2'(x) = \infty$. We assume that f_1 is constantized by f_1 near x_0 and $f \in \complement^{\infty 111}_{o-}(I,I)$. A similar discussion to the proof in Theorem 2.1, we can get that

$$D_+ f^2(x_0) = \lim_{x\to x_0+0} f_1'(f_2(x)) f_2'(x). \tag{2.21}$$

By the definition of $\complement^{\infty 111}_{o-}(I,I)$, we have $\lim_{x\to x_0+0} f_1'(f_2(x)) = f_1'(y_2) = 0$. Note that $\lim_{x\to x_0+0} f_2'(x) = \infty$. It follows from (2.21) that $\lim_{x\to x_0+0} f_1'(f_2(x)) f_2'(x)$ is of $0 \cdot \infty$ type. Thus, it is hard to judge the existence of the right derivative $D_+ f^2(x_0)$. We similarly see difficulty in other cases.

The following theorem gives conditions for f^2 not to be C^1. For convenience, for $i = 1$ or 2 we use the notation

$$\lim f_i'(x) := \begin{cases} \lim_{x\to x_0-0} f_1'(x), & i = 1, \\ \lim_{x\to x_0+0} f_2'(x), & i = 2. \end{cases}$$

Theorem 2.3. *Let $f \in V(I, I)$ and $x_0 \in (0, 1)$ be the unique discontinuity. Suppose that $y_1 := \lim_{x\to x_0-0} f_1(x)$ and $y_2 := \lim_{x\to x_0+0} f_2(x)$. Then f^2 is not C^1 on I if for $i = 1$ or 2 and $j \in \{1, 2\}\backslash\{i\}$ either*
(i)*$y_i = x_0$, $f(I_i) \subseteq I_i \cup I_0$ in the case that* $\lim f_i'(x) = \infty$, *or*
(ii)*$y_i = x_0$, $f(I_i) \subseteq I_j \cup I_0$ in the case that $\tilde{y}_i := \lim f_i'(x)$ exists and $\tilde{y}_i \neq 0$ but $\lim f_j'(x)$ does not exist, or*
(iii)*$y_i = x_0$ in the case that* $\lim f_i'(x) = \infty$ *and* $\lim f_j'(x) = \infty$, *or*
(iv)*$y_i \in I_i$ in the case that* $\lim f_i'(x)$ *does not exist and $f_i'(y_i) \neq 0$, or*
(v)*$y_i \in I_j$ in the case that* $\lim f_i'(x)$ *does not exist and $f_j'(y_i) \neq 0$.*

Proof. we only prove the situation that $i = 1$ because the situation that $i = 2$ can be proved similarly.

For **(i)**, we have $y_1 = x_0$ and $f(I_1) \subseteq I_1 \cup I_0$. By $\lim_{x\to x_0-0} f_1'(x) = \infty$ and the continuity of f_1 on I_1, there exists a sufficiently small left-half neighborhood $U_{x_0}^-$ of x_0 such that $f_1(x) < x_0$ for all $x \in U_{x_0}^- \subset I_1$, which implies that $f^2(x) = f_1(f_1(x))$ for all $x \in U_{x_0}^-$. It follows that $D_-f^2(x_0) = \lim_{x\to x_0-0} f_1'(f_1(x))f_1'(x)$. Note that $f_1(x) \to x_0 - 0$ as $x \to x_0 - 0$. Then we have $\lim_{x\to x_0-0} f_1'(f_1(x)) = \lim_{y\to x_0-0} f_1'(y) = \infty$. Thus, $D_-f^2(x_0) = \lim_{x\to x_0-0} f_1'(f_1(x))f_1'(x) = \infty$. This implies that f^2 is not C^1 on I.

For **(ii)**, we have $y_1 = x_0$ and $f(I_1) \subseteq I_2 \cup I_0$. By $\tilde{y}_1 = \lim_{x\to x_0-0} f_1'(x)$ exists and $\tilde{y}_1 \neq 0$, there exists a sufficiently small left-half neighborhood $U_{x_0}^-$ of x_0 such that $f_1(x) > x_0$ for all $x \in U_{x_0}^- \subset I_1$, which implies $f^2(x) = f_2(f_1(x))$ for all $x \in U_{x_0}^-$. It follows that $D_-f^2(x_0) = \lim_{x\to x_0-0} f_2'(f_1(x))f_1'(x)$. Note that $f_1(x) \to x_0 + 0$ as $x \to x_0 - 0$ and $\lim_{x\to x_0+0} f_2'(x)$ does not exist. Then we have $\lim_{x\to x_0-0} f_2'(f_1(x)) = \lim_{y\to x_0+0} f_2'(y)$ does not exist. We claim that $D_-f^2(x_0)$ does not exists. In fact, assume that $D_-f^2(x_0) = \lim_{x\to x_0-0} f_2'(f_1(x))f_1'(x)$ exists, then

$$\lim_{x\to x_0+0} f_2'(x) = \lim_{x\to x_0-0} f_2'(f_1(x)) = \lim_{x\to x_0-0} \frac{f_2'(f_1(x))f_1'(x)}{f_1'(x)} = \frac{D_-f^2(x_0)}{\tilde{y}_1}$$

exists since $\tilde{y}_1 = \lim_{x\to x_0-0} f_1'(x)$ exists and $\tilde{y}_1 \neq 0$. However, this contradicts the fact that $\lim_{x\to x_0+0} f_2'(x)$ does not exist. This implies that f^2 is not C^1 on I.

For **(iii)**, we have both $\lim_{x\to x_0-0} f_1'(x) = \infty$ and $\lim_{x\to x_0+0} f_2'(x) = \infty$. By $\lim_{x\to x_0-0} f_1'(x) = \infty$ and $y_1 = x_0$, one sees that there exists a sufficiently small left-half neighborhood $U_{x_0}^-$ of x_0 such that either $f_1(x) < x_0$ or $f_1(x) > x_0$ for all $x \in U_{x_0}^- \subset I_1$. If $f_1(x) < x_0$ for all $x \in U_{x_0}^-$, one can prove that f^2 is not C^1 on I with a similar discussion to the proof of case **(i)**. If $f_1(x) > x_0$ for all $x \in U_{x_0}^-$, we have $f^2(x) = f_2(f_1(x))$ for all $x \in U_{x_0}^-$. It follows that $D_-f^2(x_0) = \lim_{x\to x_0-0} f_2'(f_1(x))f_1'(x)$. Note that $\lim_{x\to x_0-0} f_1'(x) = \infty$ and $\lim_{x\to x_0-0} f_2'(f_1(x)) = \lim_{y\to x_0+0} f_2'(y) = \infty$. It follows that $D_-f^2(x_0) = \lim_{x\to x_0-0} f_2'(f_1(x))f_1'(x) = \infty$. This implies that f^2 is not C^1 on I.

For **(iv)**, we have $y_1 \in I_1$. By the continuity of f_1 on I_1, there exists a sufficiently small left-half neighborhood $U_{x_0}^-$ of x_0 such that $f_1(x) < x_0$ for all $x \in U_{x_0}^- \subset I_1$. Then we get that $f^2(x) = f_1(f_1(x))$ for all $x \in U_{x_0}^-$. It follows that $D_-f^2(x_0) = \lim_{x\to x_0-0} f_1'(f_1(x))f_1'(x)$. Note that $\lim_{x\to x_0-0} f_1'(f_1(x)) =$

$f_1'(y_1) \neq 0$. We claim that $D_- f^2(x_0)$ does not exists. In fact, assume that $D_- f^2(x_0)$ exists, then

$$\lim_{x\to x_0-0} f_1'(x) = \lim_{x\to x_0-0} \frac{f_1'(f_1(x))f_1'(x)}{f_1'(f_1(x))} = \frac{D_- f^2(x_0)}{f_1'(y_1)}$$

exists. However, this contradicts the fact that $\lim_{x\to x_0-0} f_1'(x)$ does not exist. This implies that f^2 is not C^1 on I.

For **(v)**, we have $y_1 \in I_2$. By the continuity of f_1 on I_1, there exists a sufficiently small left-half neighborhood $U_{x_0}^-$ of x_0 such that $f_1(x) > x_0$ for all $x \in U_{x_0}^- \subset I_1$. Then we get that $f^2(x) = f_2(f_1(x))$ for all $x \in U_{x_0}^-$. It follows that $D_- f^2(x_0) = \lim_{x\to x_0-0} f_2'(f_1(x))f_1'(x)$. Note that $\lim_{x\to x_0-0} f_2'(f_1(x)) = f_2'(y_1) \neq 0$. Similarly to the above **(iv)**, we can also get that $D_- f^2(x_0)$ does not exists. It implies that f^2 is not C^1 on I. Therefore, this completes the proof. □

3. Examples

We demonstrate our theorems with some examples.

Example 3.1. Consider the mapping $F_1 : (0,1) \to (0,1)$ (see Figure 5) defined by

$$F_1(x) = \begin{cases} \frac{1}{4}, & 0 < x \leq \frac{1}{4}, \\ \frac{1}{4} + \frac{1}{4}\sin^2\frac{\pi}{x}, & \frac{1}{4} < x < \frac{1}{2}, \\ \frac{1}{4}, & x = \frac{1}{2}, \\ \frac{1}{8} + \frac{1}{16}\cos^2\frac{\pi}{x-\frac{1}{2}}, & \frac{1}{2} < x < 1, \end{cases}$$

which has a unique oscillating discontinuous point $x_0 = \frac{1}{2}$ since $y_1 = \lim\limits_{x\to\frac{1}{2}-0} F_1(x) = \frac{1}{4}$ exists but $\lim\limits_{x\to\frac{1}{2}+0} F_1(x)$ does not exist, i.e., $F_1 \in V_{o+}(I,I)$. Moreover, $\tilde{y}_1 = \lim_{x\to\frac{1}{2}-0} F_1'(x) = 0$ exists. It implies that $F_1 \in V_{o+}^E(I,I)$. Note that $I_1 = (0,\frac{1}{2}), I_2 = (\frac{1}{2},1)$ and $y_1 = c = \frac{1}{4} \in I_1$. It is easy to check that f_2 is constantized by f_1 near x_0, $\theta(f_2,f_1) = f_1(y_1) = f_1(c) = \frac{1}{4}$ and $f_1'(y_1)\tilde{y}_1 = 0$, i.e., $F_1 \in \complement_{o+}^{E211}(I,I)$, where

$$f_1(x) = \begin{cases} \frac{1}{4}, & 0 < x \leq \frac{1}{4}, \\ \frac{1}{4} + \frac{1}{4}\sin^2\frac{\pi}{x}, & \frac{1}{4} < x < \frac{1}{2}, \end{cases} \quad f_2(x) = \frac{1}{8} + \frac{1}{16}\cos^2\frac{\pi}{x-\frac{1}{2}}.$$

It implies that the assumption **(o+1)** of **(o+)** in Theorem 2.1 is satisfied. Furthermore, we can check that $\frac{2}{7} \in F_1^{-1}(I_0) \cap I_1$ and $\frac{2}{5} \in F_1^{-1}(I_0) \cap I_1$ since $\{\frac{2}{7},\frac{2}{5}\} \subset I_1$ and $F_1(\frac{2}{7}) = F_1(\frac{2}{5}) = x_0 = \frac{1}{2}$. Moreover, one can also check that $\frac{2}{7} \in \Delta_{11}, \frac{2}{5} \in \Delta_{11}, y_1 = c = \frac{1}{4}$ and $f_1'(\frac{2}{7}) = f_1'(\frac{2}{5}) = 0$, i.e., $F_1 \in \bar{\complement}_{o+}^{E11}(I,I)$. It implies that the assumption **(o+2)** of **(o+)** in Theorem 2.1 is satisfied. On the other hand, one can compute

$$F_1^2(x) = \begin{cases} \frac{1}{4}, & 0 < x \leq \frac{1}{4}, \\ \frac{1}{4} + \frac{1}{4}\sin^2\frac{4\pi}{1+\sin^2\frac{\pi}{x}}, & \frac{1}{4} < x < \frac{1}{2}, \\ \frac{1}{4}, & \frac{1}{2} \leq x < 1, \end{cases}$$

which is C^1 smooth on $(0,1)$ as shown in Figure 6.

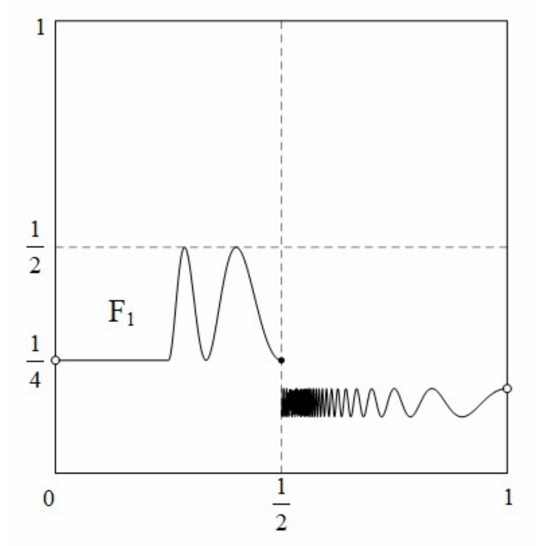

Figure 5. $F_1 \in \complement_{o+}^{E211}(I,I) \cap \bar{\complement}_{o+}^{E11}(I,I)$.

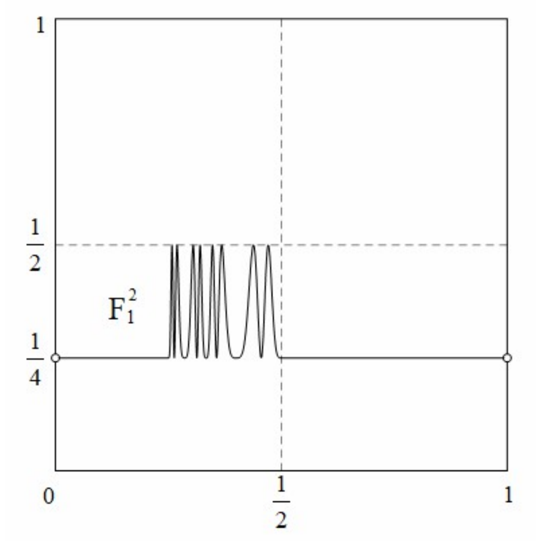

Figure 6. F_1^2 is C^1 on $(0,1)$.

Example 3.2. Consider the mapping $F_2 : (0,1) \to (0,1)$ (see Figure 7) defined by

$$F_2(x) = \begin{cases} \frac{1}{9}, & 0 < x \le \frac{1}{6}, \\ \frac{1}{18} + \frac{1}{18}\cos^2 \frac{\pi}{\frac{1}{3}-x}, & \frac{1}{6} < x < \frac{1}{3}, \\ \frac{1}{18}, & x = \frac{1}{3}, \\ \frac{1}{36} + \frac{1}{18}\sin^2 \frac{\pi}{x-\frac{1}{3}}, & \frac{1}{3} < x < 1, \end{cases}$$

which has a unique oscillating discontinuous point $x_0 = \frac{1}{3}$ since neither $\lim_{x\to\frac{1}{3}-0} F_2(x)$ nor $\lim_{x\to\frac{1}{3}+0} F_2(x)$ exists, i.e., $F_2 \in V_{o*}(I,I)$. Note that $I_1 = (0,\frac{1}{3}), I_2 = (\frac{1}{3},1), c = \frac{1}{18} \in I_1$. One can check that both f_1 and f_2 are constantized by f_1 near x_0, $F_2(I_1 \cup I_2) \subseteq I_1$ and $\theta(f_1, f_1) = \theta(f_2, f_1) = f_1(c) = \frac{1}{9}$, i.e., $F_2 \in \complement^1_{o*}(I, I)$, where

$$f_1(x) = \begin{cases} \frac{1}{9}, & 0 < x \le \frac{1}{6}, \\ \frac{1}{18} + \frac{1}{18}\cos^2 \frac{\pi}{\frac{1}{3}-x}, & \frac{1}{6} < x < \frac{1}{3}, \end{cases} \quad f_2(x) = \frac{1}{36} + \frac{1}{18}\sin^2 \frac{\pi}{x - \frac{1}{3}}.$$

It implies that the assumptions of **(o*)** in Theorem 2.1 are satisfied. Actually, one can compute

$$F_2^2(x) = \frac{1}{9}, \quad \forall\, x \in (0, 1),$$

which is C^1 smooth on $(0, 1)$ as shown in Figure 8.

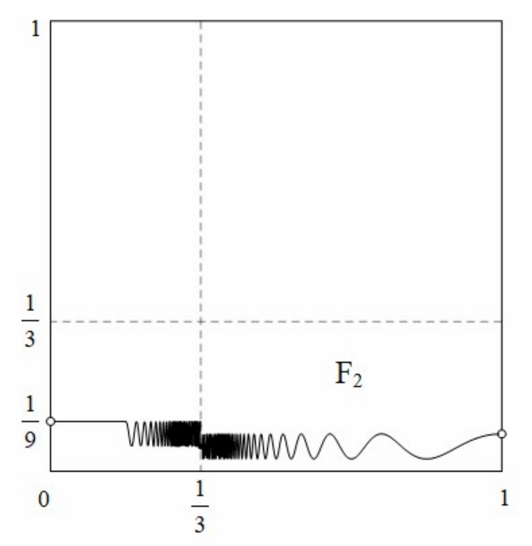

Figure 7. $F_2 \in \complement^1_{o*}(I, I)$.

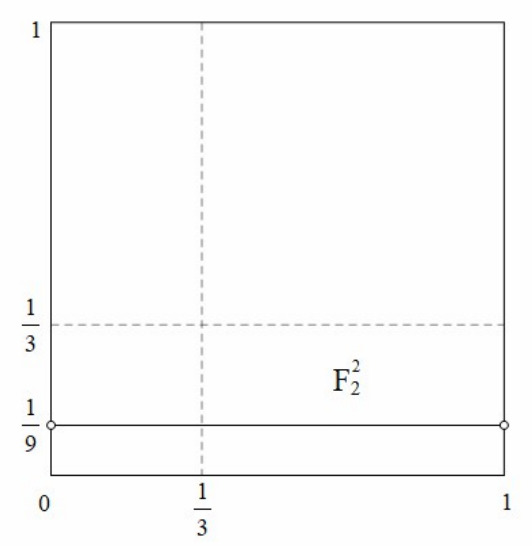

Figure 8. F_2^2 is C^1 on $(0, 1)$.

Example 3.3. Consider the mapping $F_3 : (0,1) \to (0,1)$ (see Figure 9) defined by

$$F_3(x) = \begin{cases} \frac{1}{4}, & 0 < x \le \frac{1}{4}, \\ 8(x-\frac{1}{4})^3 + \frac{1}{4}, & \frac{1}{4} < x < \frac{1}{2}, \\ \frac{1}{8}, & x = \frac{1}{2}, \\ \frac{1}{8} + \frac{1}{16}\sin^2 \frac{\pi}{x-\frac{1}{2}}, & \frac{1}{2} < x < 1, \end{cases}$$

which has a unique oscillating discontinuous point $x_0 = \frac{1}{2}$ since $y_1 = \lim_{x\to\frac{1}{2}-0} F_3(x) = \frac{3}{8}$ exists but $\lim_{x\to\frac{1}{2}+0} F_3(x)$ does not exist, i.e., $F_3 \in V_{o+}(I,I)$. Moreover, $\tilde{y}_1 = \lim_{x\to\frac{1}{2}-0} F_3'(x) = \frac{3}{2}$ exists. It implies that $F_3 \in V^E_{o+}(I,I)$. Note that $I_1 = (0,\frac{1}{2}), I_2 = (\frac{1}{2},1), c = \frac{1}{8} \in I_1$ and $y_1 = \frac{3}{8} \in I_1$. It is easy to check that f_2 is constantized by f_1 near x_0, $\theta(f_2, f_1) = \frac{1}{4} \ne f_1(y_1) = \frac{17}{64}$, i.e., $F_3 \notin \complement^{E211}_{o+}(I,I)$, where

$$f_1(x) = \begin{cases} \frac{1}{4}, & 0 < x \le \frac{1}{4}, \\ 8(x-\frac{1}{4})^3 + \frac{1}{4}, & \frac{1}{4} < x < \frac{1}{2}, \end{cases} \quad f_2(x) = \frac{1}{8} + \frac{1}{16}\sin^2 \frac{\pi}{x-\frac{1}{2}}.$$

It implies that the assumptions of **(o+)** in Theorem 2.1 are not satisfied. Actually, one can compute

$$F_3^2(x) = \begin{cases} \frac{1}{4}, & 0 < x \le \frac{1}{4}, \\ 8^4(x-\frac{1}{4})^9 + \frac{1}{4}, & \frac{1}{4} < x < \frac{1}{2}, \\ \frac{1}{4}, & \frac{1}{2} \le x < 1, \end{cases}$$

which is not C^1 smooth on $(0,1)$ with nonsmooth point $\frac{1}{2}$ as shown in Figure 10.

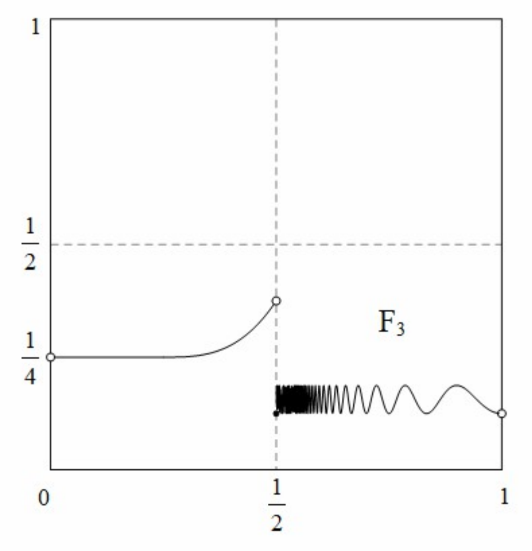

Figure 9. $F_3 \notin \complement^{E211}_{o+}(I,I)$.

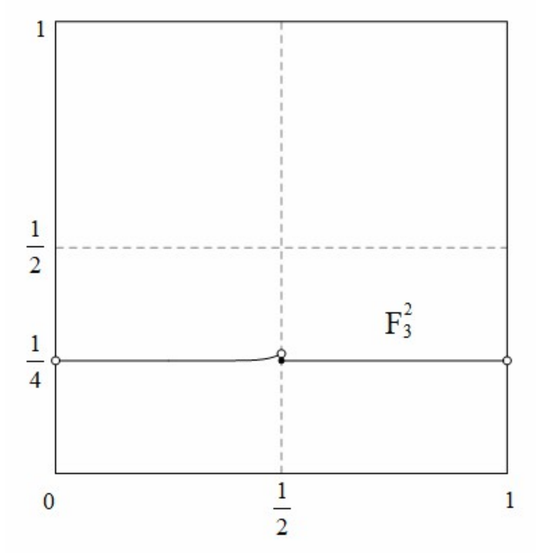

Figure 10. F_3^2 is not C^1 on $(0,1)$.

Example 3.4. Consider the mapping $F_4 : (0,1) \to (0,1)$ (see Figure 11) defined by

$$F_4(x) = \begin{cases} \frac{1}{2} - \frac{1}{4}\sqrt{1-2x}, & 0 < x < \frac{1}{2}, \\ 4(x-\frac{3}{4})^2 + \frac{1}{2}, & \frac{1}{2} \le x < 1, \end{cases}$$

which has a unique jumping discontinuous point $x_0 = \frac{1}{2}$ since

$$y_1 = \lim_{x \to \frac{1}{2}-0} F_6(x) = \frac{1}{2} \neq \lim_{x \to \frac{1}{2}+0} F_6(x) = \frac{3}{4} = y_2.$$

Note that

$$f_1(x) = \frac{1}{2} - \frac{1}{4}\sqrt{1-2x}, \quad 0 < x < \tfrac{1}{2}$$

and

$$f_2(x) = 4(x - \frac{3}{4})^2 + \frac{1}{2}, \quad \tfrac{1}{2} < x < 1.$$

Moreover, $y_1 = \frac{1}{2} = x_0$, $\lim_{x \to \frac{1}{2}-0} f_1'(x) = \infty$ and $f_1(I_1) \subseteq I_1$, i.e., the assumption **(i)** in Theorem 2.3 is satisfied. Actually, one can compute

$$F_4^2(x) = \begin{cases} \frac{1}{2} - \frac{\sqrt{2}}{8}\sqrt[4]{1-2x}, & 0 < x < \frac{1}{2}, \\ \frac{1}{2}, & x = \frac{1}{2}, \\ 4[4(x-\frac{3}{4})^2 - \frac{1}{4}]^2 + \frac{1}{2}, & \frac{1}{2} < x < 1, \end{cases}$$

which is not C^1 smooth on $(0,1)$ with nonsmooth point $\frac{1}{2}$ as shown in Figure 12.

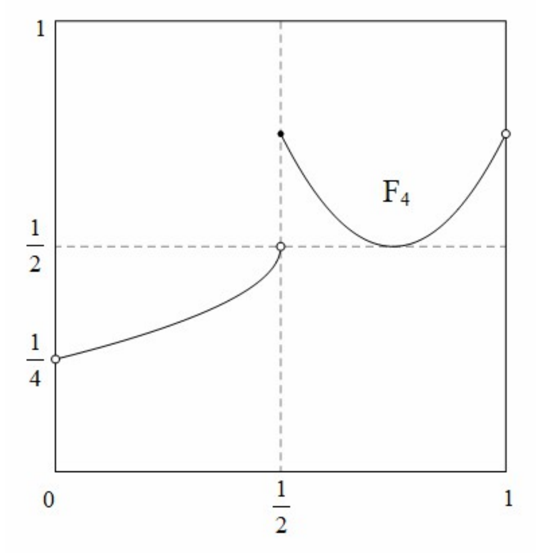

Figure 11. $F_4 \in V(I, I)$.

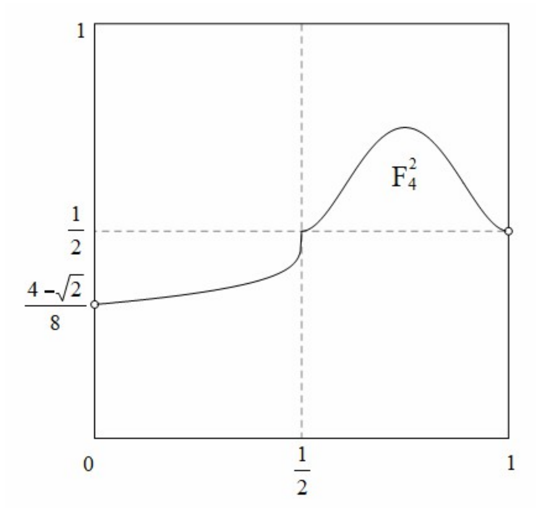

Figure 12. F_4^2 is not C^1 on $(0, 1)$.

4. Conclusions

Difference from [6], where the function has exactly a removable or a jumping discontinuity, in this paper, we show how a function with exactly one oscillating discontinuity may have a C^1 smooth iterate of second-order.

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Conflict of interest

The authors declare no conflict of interest.

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