AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 8772–8792.
DOI:10.3934/math.2023440
Received: 24 March 2022
Revised: 27 May 2022
Accepted: 31 May 2022
Published: 08 February 2023

Research article

Iteration changes discontinuity into smoothness (I): Removable and jumping cases

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Abstract: It has been proved that a self-mapping with exact one discontinuity may have a continuous iterate of the second order. It actually shows that iteration can change discontinuity into continuity. Further, we can also find some examples with exact one discontinuity which have C^1 smooth iterate of the second order, indicating that iteration can change discontinuity into smoothness. In this paper we investigate piecewise C^1 self-mappings on the open interval $(0, 1)$ having only one removable or jumping discontinuity. We give necessary and sufficient conditions for those self-mappings to have a C^1 smooth iterate of the second order.

Keywords: iteration; removable discontinuity; jumping discontinuity; C^1 smooth; piecewise smooth
Mathematics Subject Classification: 37E05, 39B12

1. Introduction

Iteration may be considered as many repetitions of the same operation. Concretely, the n-th iterate f^n of a function $f : E \to E$ is defined by $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) = x$ for all $x \in E$ inductively, where E is a nonempty set and $n > 0$ is a fixed integer.

Iteration is an extensive phenomenon in nature and human life. It causes many complexities such as bifurcations, chaos and fractals [12–14]. The computation of iteration of a general order or a higher order in the one-dimensional case is also a complicated work although efforts have been made to polynomials [1–4, 21], quasi-polynomials [17, 19], linear fractions [20] and rational fractions [11]. Along with the development of iteration theory on C^0 mappings (see e.g., Chapter 11 of [7] and Chapter 1 of [18]), attentions were also paid to iteration of "bad mappings" such as discontinuous mappings or set-valued mappings and their iterative roots [5, 6, 15, 16].

Usually, one considers that a "bad mapping" may be changed by iteration to worse with more complicated properties, but it was shown in [8] and [10] that a discontinuous self-mapping can be converted by iteration to a continuous one. In [8] and [10] necessary and sufficient conditions are found for such convertion. Further, there are found in [9] all nonsmooth continuous self-mappings whose second order iterates are C^1 smooth. Combining [8](or [10]) with [9], one can judge whether the fourth order iterate of a discontinuous self-mapping is C^1 smooth. However, it is still hard to answer: *What discontinuous self-mappings can be smoothed but not only be made continuous by the second order iteration?* For example, the self-mapping

$$f(x)=\begin{cases}-(x-\frac{1}{4})^2+\frac{3}{16}, & 0<x<\frac{1}{2},\\ \frac{1}{8}, & x=\frac{1}{2},\\ -(x-\frac{3}{4})^2+\frac{7}{16}, & \frac{1}{2}<x<1,\end{cases}$$

has a jumping discontinuity at $x_0 = 1/2$ (shown in Figure 1), but its second order iterate

$$f^2(x)=\begin{cases}-[(x-\frac{1}{4})^2+\frac{1}{16}]^2+\frac{3}{16}, & 0<x<\frac{1}{2},\\ \frac{11}{64}, & x=\frac{1}{2},\\ -[(x-\frac{3}{4})^2-\frac{3}{16}]^2+\frac{3}{16}, & \frac{1}{2}<x<1,\end{cases}$$

is a C^1 smooth self-mapping on the whole interval (0,1) (shown in Figure 2).

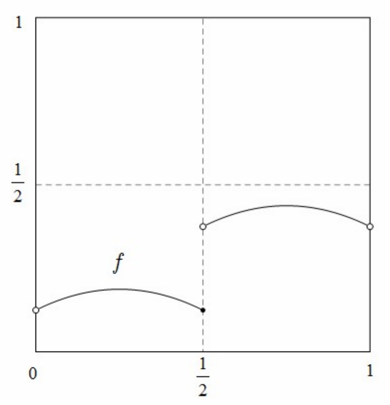

Figure 1. f is discontinuous at $x_0 = 1/2$.

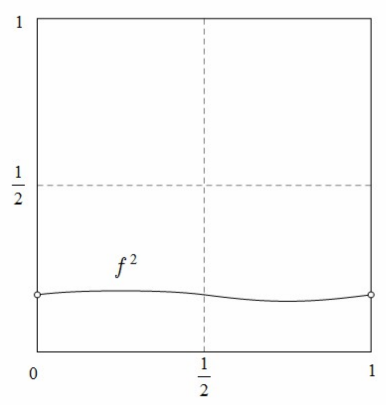

Figure 2. f^2 is C^1 smooth on $(0, 1)$.

In this paper we answer this question for $V(I, I)$, the class of C^1 self-mappings with exactly one discontinuity on the open interval $I := (0, 1)$. Each $f \in V(I, I)$ can be presented as

$$f(x)=\begin{cases}f_1(x), & x\in I_1:=(0,x_0),\\ c, & x=x_0,\\ f_2(x), & x\in I_2:=(x_0,1),\end{cases} \tag{1.1}$$

where $x_0 \in (0, 1)$ is the unique discontinuity, f_i is C^1 smooth on I_i for each $i \in \{1, 2\}$ and $c \in (0, 1)$ is a constant. Usually, there are four classes of discontinuities: removable discontinuity, jumping discontinuity, oscillating discontinuity and infinite discontinuity, but there are only the first three classes in $V(I, I)$, i.e., $V(I, I) = V_r(I, I) \cup V_j(I, I) \cup V_o(I, I)$, where

- $f \in V_r(I, I)$ has a unique removable discontinuity;
- $f \in V_j(I, I)$ has a unique jumping discontinuity;
- $f \in V_o(I, I)$ has a unique oscillating discontinuity.

In this paper we discuss C^1 smoothness of the second order iterates of mappings in $V_r(I, I)$ or $V_j(I, I)$ and leave the discussion on $V_o(I, I)$ to a continued paper. In order to investigate C^1 smoothness of the second order iterates of mappings in $V_r(I, I)$ or $V_j(I, I)$, we also need to divide them into some subclasses, i.e., $V_\tau(I, I) = V_{\tau r}(I, I) \cup V_{\tau j}(I, I) \cup V_{\tau o}(I, I) \cup V_{\tau\infty}(I, I)$ for $\tau \in \{r, j\}$, where

- $f \in V_{\tau r}(I, I)$ if the derivative f' has a removable discontinuity at x_0, i.e., both $\lim_{x\to x_0-0} f_1'(x)$ and $\lim_{x\to x_0+0} f_2'(x)$ exist and they are equal;
- $f \in V_{\tau j}(I, I)$ if the derivative f' has a jumping discontinuity at x_0, i.e., both $\lim_{x\to x_0-0} f_1'(x)$ and $\lim_{x\to x_0+0} f_2'(x)$ exist but they are not equal;
- $f \in V_{\tau o}(I, I)$ if the derivative f' has a oscillating discontinuity at x_0, i.e., both f_1' and f_2' are bounded but either $\lim_{x\to x_0-0} f_1'(x)$ or $\lim_{x\to x_0+0} f_2'(x)$ does not exist;
- $f \in V_{\tau\infty}(I, I)$ if either f_1' or f_2' is unbounded, i.e., either $\lim_{x\to x_0-0} f_1'(x) = \infty$ or $\lim_{x\to x_0+0} f_2'(x) = \infty$.

In this paper, we discuss C^1 smoothness of the second order iterates of mappings in $V_{r\mu}(I, I)$ and $V_{j\mu}(I, I)$ respectively, where $\mu \in \{r, j, o\}$. Necessary and sufficient conditions for C^1 smoothness of the second order iterates of mappings in $V_{r\mu}(I, I)$ and $V_{j\mu}(I, I)$ are obtained in Sections 2 and 3 respectively. Moreover, we also give necessary conditions and point out the difficulties in finding sufficient conditions for C^1 smoothness of the second order iterates of mappings in $V_{r\infty}(I, I)$ and $V_{j\infty}(I, I)$ in Sections 2 and 3 respectively. Finally, we use examples to demonstrate our conditions of the second order C^1 smoothness in Section 4.

For convenience, let $I_0 := \{x_0\}$, then $I = I_1 \cup I_0 \cup I_2$. Moreover, for $i, j = 0, 1, 2$ we use the notations

$$\begin{aligned}
\Delta_i^- &:= \{\alpha \in I \mid f(x) \in I_i \text{ for all } x \in U_\alpha^-\},\\
\Delta_j^+ &:= \{\alpha \in I \mid f(x) \in I_j \text{ for all } x \in U_\alpha^+\},\\
\Delta_{ij} &:= \{\alpha \in I \mid f(x) \in I_i \text{ for all } x \in U_\alpha^- \text{ and } f(x) \in I_j \text{ for all } x \in U_\alpha^+\},
\end{aligned}$$

where U_α^- and U_α^+ are the left half and the right half of the sufficiently small neighborhood of α respectively. We use D_-f and D_+f to denote the left derivative of f and the right derivative of f respectively.

2. Iteration for $V_r(I, I)$

In this section, we consider C^1 smoothness of the second order iterates of $f \in V_r(I, I)$ be defined by (1.1). Let

$$y_0 := \lim_{x\to x_0-0} f_1(x) = \lim_{x\to x_0+0} f_2(x) \neq f(x_0) = c. \tag{2.1}$$

For convenience, we use the notation:

$$\Upsilon(c) := \begin{cases} f_1(c), & \text{if } c \in I_1,\\ f_2(c), & \text{if } c \in I_2.\end{cases}$$

Moreover, let $\complement_{rr}(I, I) := \{f \in V_{rr}(I, I) \mid f(y_0) = \Upsilon(c)\}$ and for $\tau = j, o, \infty$ let $\complement_{r\tau}(I, I) := \{f \in V_{r\tau}(I, I) \mid f(y_0) = \Upsilon(c) \text{ and } f'(y_0) = 0\}$.

Theorem 2.1. *Suppose that* $f \in V_{r\tau}(I, I)$ *for* $\tau \in \{r, j, o\}$ *and* f *has a unique discontinuity at* $x_0 \in (0, 1)$. *Let* y_0 *be defined by (2.1). Then the iterate* f^2 *is* C^1 *smooth on* I *if and only if* $y_0 \in I_i$, $f(I_1 \cup I_2) \subseteq I_i$ *holds for* $i = 1$ *or* 2 *and* $f \in \complement_{r\tau}(I, I)$.

Proof. **Necessity**. Suppose that the iterate f^2 is C^1 smooth on I, which implies that the iterate f^2 is continuous on I. By Theorem 1 in [8], we see that $y_0 \in I_i$, $f(I_1 \cup I_2) \subseteq I_i$ holds and

$$f(y_0) = f_i(y_0) = \Upsilon(c) \tag{2.2}$$

for $i = 1$ or 2. It suffices to discuss the situation which $i = 1$ since the other situation can be discussed similarly. It follows that

$$f^2(x) = \begin{cases} f_1(f_1(x)), & x \in I_1, \\ f_1(f_2(x)), & x \in I_2. \end{cases} \tag{2.3}$$

By the assumption that $f \in V_{r\tau}(I, I)$ for $\tau \in \{r, j, o\}$, we need to discuss three cases: $f \in V_{rr}(I, I)$, $f \in V_{rj}(I, I)$ and $f \in V_{ro}(I, I)$. For the first case that $f \in V_{rr}(I, I)$, by (2.2) and the definition of $\complement_{rr}(I, I)$, we see that $f \in \complement_{rr}(I, I)$. For the second case that $f \in V_{rj}(I, I)$, by the definition of $V_{rj}(I, I)$, we see that $\tilde{y}_1 := \lim_{x\to x_0-0} f_1'(x) \neq \tilde{y}_2 := \lim_{x\to x_0+0} f_2'(x)$. Note that $\lim_{x\to x_0-0} f_1(x) = \lim_{x\to x_0+0} f_2(x) = y_0$. It follows from (2.3) that

$$D_-f^2(x_0) = \lim_{x\to x_0-0} f_1'(f_1(x))f_1'(x) = f_1'(y_0)\tilde{y}_1, \tag{2.4}$$

$$D_+f^2(x_0) = \lim_{x\to x_0+0} f_1'(f_2(x))f_2'(x) = f_1'(y_0)\tilde{y}_2. \tag{2.5}$$

Since f^2 is C^1 smooth on I, we have $D_-f^2(x_0) = D_+f^2(x_0)$. Since $\tilde{y}_1 \neq \tilde{y}_2$ and $y_0 \in I_1$, it follows from (2.4) and (2.5) that

$$f'(y_0) = f_1'(y_0) = 0. \tag{2.6}$$

By the definition of $\complement_{rj}(I, I)$, we obtain from (2.2) and (2.6) that $f \in \complement_{rj}(I, I)$. Finally, for the third case that $f \in V_{ro}(I, I)$, by the definition of $V_{ro}(I, I)$, we see that either $\lim_{x\to x_0-0} f_1'(x)$ or $\lim_{x\to x_0+0} f_2'(x)$ does not exist. From (2.3) we obtain

$$D_-f^2(x_0) = \lim_{x\to x_0-0} f_1'(f_1(x))f_1'(x), \tag{2.7}$$

$$D_+f^2(x_0) = \lim_{x\to x_0+0} f_1'(f_2(x))f_2'(x). \tag{2.8}$$

Note that $\lim_{x\to x_0-0} f_1'(f_1(x)) = \lim_{x\to x_0+0} f_1'(f_2(x)) = f_1'(y_0)$. Since we have assumed that f^2 is C^1 smooth on I, it follows that both $D_-f^2(x_0)$ and $D_+f^2(x_0)$ exist. We claim that (2.6) holds. In fact, if the claim is not true, then $f'(y_0) = f_1'(y_0) \neq 0$. It follows from (2.7) and (2.8) that both $\lim_{x\to x_0-0} f_1'(x)$ and $\lim_{x\to x_0+0} f_2'(x)$ exist since

$$\lim_{x\to x_0-0} f_1'(x) = \lim_{x\to x_0-0} \frac{f_1'(f_1(x))f_1'(x)}{f_1'(f_1(x))} = \frac{D_-f^2(x_0)}{f_1'(y_0)},$$
$$\lim_{x\to x_0+0} f_2'(x) = \lim_{x\to x_0+0} \frac{f_1'(f_2(x))f_2'(x)}{f_1'(f_2(x))} = \frac{D_+f^2(x_0)}{f_1'(y_0)},$$

which contradicts our assumption that either $\lim_{x\to x_0-0} f_1'(x)$ or $\lim_{x\to x_0+0} f_2'(x)$ does not exist. This proves (2.6). By the definition of $\complement_{ro}(I,I)$, it follows from (2.2) and (2.6) that $f\in\complement_{ro}(I,I)$. Therefore, this completes the proof of necessity.

Sufficiency. We only deal with the situation where $y_0\in I_1$ and $f(I_1\cup I_2)\subseteq I_1$ hold since the other one can be proved similarly.

If $f(I_1\cup I_2)\subseteq I_1$ holds, we can obtain (2.3). Obviously, f^2 is C^1 smooth on $I_1\cup I_2$ since the mappings f_1 and f_2 are C^1 on I_1 and I_2 respectively. That $y_0\in I_1$ implies that $f(y_0)=f_1(y_0)$ and $f'(y_0)=f_1'(y_0)$. By the assumption that $f\in\complement_{r\tau}(I,I)$ for $\tau\in\{r,j,o\}$, we have either $f\in\complement_{rr}(I,I)$, or $f\in\complement_{rj}(I,I)$, or $f\in\complement_{ro}(I,I)$. In the case $f\in\complement_{rr}(I,I)$, from the definition of $\complement_{rr}(I,I)$, we have $f(y_0)=\Upsilon(c)$ and $f\in V_{rr}(I,I)$. By Theorem 1 in [8], we see that f^2 is C^0 on I. Moreover, (2.3) implies that (2.4) and (2.5) hold. Since $f\in V_{rr}(I,I)$, we have $\tilde{y}_1=\tilde{y}_2$. It follows from (2.4) and (2.5) that $D_-f^2(x_0)=D_+f^2(x_0)$, which implies that Df^2 is continuous at x_0. Thus f^2 is C^1 smooth on I. In the case $f\in\complement_{rj}(I,I)$, it follows from the definition of $\complement_{rj}(I,I)$ that $f(y_0)=\Upsilon(c)$, $f\in V_{rj}(I,I)$ and $f'(y_0)=0$. Similarly, we see that f^2 is C^0 on I. Moreover, one sees that (2.4) and (2.5) hold. It follows from (2.4) and (2.5) that $D_-f^2(x_0)=D_+f^2(x_0)=0$, which implies that Df^2 is continuous at x_0. Thus f^2 is C^1 smooth on I. Finally, in the case $f\in\complement_{ro}(I,I)$, it follows from the definition of $\complement_{ro}(I,I)$ that $f(y_0)=\Upsilon(c)$, $f\in V_{ro}(I,I)$ and $f'(y_0)=0$. We similarly get that f^2 is C^0 on I. Moreover, (2.3) implies that (2.7) and (2.8) hold. Note that $\lim_{x\to x_0-0} f_1'(f_1(x))=\lim_{x\to x_0+0} f_1'(f_2(x))=f_1'(y_0)=f'(y_0)=0$ and both f_1' and f_2' are bounded. By the properties of infinitely small quantities, we get from (2.7) and (2.8) that $D_-f^2(x_0)=D_+f^2(x_0)=0$. This implies that Df^2 is continuous at x_0. Therefore, f^2 is C^1 smooth on I. The proof of the theorem is completed. □

Theorem 2.2. *Let $f\in V_{r\infty}(I,I)$ with the unique discontinuity at $x_0\in(0,1)$ and let y_0 be defined by (2.1). Suppose that the iterate f^2 is C^1 smooth on I. Then, $y_0\in I_i$, $f(I_1\cup I_2)\subseteq I_i$ holds for $i=1$ or 2 and $f\in\complement_{r\infty}(I,I)$.*

Proof. We omit the proof because its proof is totally similar to the proof of the necessity of the case that $f\in V_{ro}(I,I)$ in Theorem 2.1. □

Notice that the above Theorem 2.2 does not give sufficient conditions for f^2 to be C^1 because it is hard to judge the existence of either the limit $D_-f^2(x_0)=\lim_{x\to x_0-0} f_i'(f_1(x))f_1'(x)$ or the limit $D_+f^2(x_0)=\lim_{x\to x_0+0} f_i'(f_2(x))f_2'(x)$ for $y_0\in I_i$, $i=1$ or 2. In fact, we assume that $y_0\in I_1$, $f(I_1\cup I_2)\subseteq I_1$ holds and $f\in\complement_{r\infty}(I,I)$. With a similar discussion to the proof of Theorem 2.1, we conclude that (2.7) and (2.8) hold. Note that $f\in\complement_{r\infty}(I,I)$ and $f'(y_0)=f_1'(y_0)$. By the definition of $\complement_{r\infty}(I,I)$, we see that $\lim_{x\to x_0-0} f_1'(f_1(x))=\lim_{x\to x_0+0} f_1'(f_2(x))=f_1'(y_0)=0$ and either $\lim_{x\to x_0-0} f_1'(x)=\infty$ or $\lim_{x\to x_0+0} f_2'(x)=\infty$. From (2.7) and (2.8), either the limit $\lim_{x\to x_0-0} f_1'(f_1(x))f_1'(x)$ or the limit $\lim_{x\to x_0+0} f_1'(f_2(x))f_2'(x)$ is of $0\cdot\infty$ type. Thus, it is hard to judge the existence of either $D_-f^2(x_0)$ or $D_+f^2(x_0)$. Similarly, we see difficulty in the other one.

3. Iteration for $V_j(I,I)$

In this section, we consider C^1 smoothness of the second order iterates of $f\in V_j(I,I)$ be defined by (1.1). Let

$$y_1:=\lim_{x\to x_0-0} f_1(x)\neq y_2:=\lim_{x\to x_0+0} f_2(x) \tag{3.1}$$

and let

$$\tilde{y}_1 := \lim_{x\to x_0-0} f_1'(x) \ (\text{or } \tilde{y}_2 := \lim_{x\to x_0+0} f_2'(x))$$

if the left limit $\lim_{x\to x_0-0} f_1'(x)$ exists (or the right limit $\lim_{x\to x_0+0} f_2'(x)$ exists).

For convenience, for $i, k \in \{1, 2\}$, $m \in \{1, 2\}\backslash\{k\}$ and $\nu \in \{r, j\}$ let

$$\begin{aligned}
\hat{\mathfrak{C}}^{i}_{j\nu}(I, I) &:= \{f \in V_{j\nu}(I, I) \mid f_i(y_1) = f_i(y_2) = f(c) \text{ and } f_i'(y_1)\tilde{y}_1 = f_i'(y_2)\tilde{y}_2\},\\
\check{\mathfrak{C}}^{(i,k)}_{j\nu}(I, I) &:= \{f \in V_{j\nu}(I, I) \mid f_i(y_k) = y_m = f(c) \text{ and } f_i'(y_k)\tilde{y}_k = \tilde{y}_m^2\},\\
\bar{\mathfrak{C}}^{(i,k)}_{j\nu}(I, I) &:= \{f \in V_{j\nu}(I, I) \mid f_i(y_k) = c = f(c) \text{ and } f_i'(y_k)\tilde{y}_k = 0\},\\
\tilde{\mathfrak{C}}^{(i,k)}_{j\nu}(I, I) &:= \{f \in V_{j\nu}(I, I) \mid f_i(y_k) = y_k = f(c) \text{ and } f_i'(y_k)\tilde{y}_k = \tilde{y}_1\tilde{y}_2\}.
\end{aligned}$$

In order to investigate C^1 smoothness of the second order iterates of mappings in $V_{j\tau}(I, I)$ for $\tau = o, \infty$, we need to consider three cases:

$$\begin{aligned}
V_{j\tau+}(I, I) &:= \{f \in V_{j\tau}(I, I) \mid \lim_{x\to x_0-0} f'(x) \text{ exists but } \lim_{x\to x_0+0} f'(x) \text{ does not exist}\},\\
V_{j\tau-}(I, I) &:= \{f \in V_{j\tau}(I, I) \mid \lim_{x\to x_0+0} f'(x) \text{ exists but } \lim_{x\to x_0-0} f'(x) \text{ does not exist}\},\\
V_{j\tau*}(I, I) &:= \{f \in V_{j\tau}(I, I) \mid \text{neither } \lim_{x\to x_0-0} f'(x) \text{ nor } \lim_{x\to x_0+0} f'(x) \text{ exists}\},
\end{aligned}$$

that is, $V_{j\tau}(I, I) = V_{j\tau+}(I, I) \cup V_{j\tau-}(I, I) \cup V_{j\tau*}(I, I)$. Moreover, for $i, k \in \{1, 2\}$, $m \in \{1, 2\}\backslash\{k\}$ and $\mu \in \{-, +\}$ let

$$\begin{aligned}
\mathfrak{C}^{(i,k)}_{jo\mu}(I, I) &:= \{f \in V_{jo\mu}(I, I) \mid f_i(y_1) = f_i(y_2) = f(c) \text{ and } f_i'(y_k)\tilde{y}_k = f_i'(y_m) = 0\},\\
\mathfrak{C}^{i}_{j\tau*}(I, I) &:= \{f \in V_{j\tau*}(I, I) \mid f_i(y_1) = f_i(y_2) = f(c) \text{ and } f_i'(y_1) = f_i'(y_2) = 0\},\\
\tilde{\mathfrak{C}}^{(i,k)}_{jo\mu}(I, I) &:= \{f \in V_{jo\mu}(I, I) \mid y_k = f_i(y_m) = f(c) \text{ and } \tilde{y}_k = f_i'(y_m) = 0\},\\
\bar{\mathfrak{C}}^{(i,k)}_{j\tau\mu}(I, I) &:= \{f \in V_{j\tau\mu}(I, I) \mid c = f_i(y_k) = f(c) \text{ and } f_i'(y_k) = 0\},\\
\hat{\mathfrak{C}}^{(i,k)}_{j\tau\mu}(I, I) &:= \{f \in V_{j\tau\mu}(I, I) \mid y_k = f_i(y_k) = f(c) \text{ and } f_i'(y_k) = \tilde{y}_m = 0\},\\
\check{\mathfrak{C}}^{(i,k)}_{j\tau\mu}(I, I) &:= \{f \in V_{j\tau\mu}(I, I) \mid y_k = f_i(y_k) = f(c) \text{ and } \tilde{y}_k = 0\},\\
\mathfrak{C}^{(i,k)}_{j\infty\mu}(I, I) &:= \{f \in V_{j\infty\mu}(I, I) \mid f_i(y_k) = f_i(y_m) = f(c) \text{ and } f_i'(y_m) = 0\},\\
\tilde{\mathfrak{C}}^{(i,k)}_{j\infty\mu}(I, I) &:= \{f \in V_{j\infty\mu}(I, I) \mid y_k = f_i(y_m) = f(c) \text{ and } f_i'(y_m) = 0\}.
\end{aligned}$$

In addition, for $i, p = 1, 2$, $\tau = r, j, o, \infty$ and $\xi_p \in f^{-1}(I_0) \cap I_p$, let

$$\mathfrak{C}^{ip}_{j\tau}(I, I) := \{f \in V_{j\tau}(I, I) \mid y_i = c \text{ and } f_p'(\xi_p) = 0\}.$$

Theorem 3.1. *Let $f \in V_{j\tau}(I, I)$ for $\tau \in \{r, j\}$ and $x_0 \in (0, 1)$ be the unique discontinuity. Suppose that y_1 and y_2 are defined by (3.1). Then the iterate f^2 is C^1 smooth on I if and only if the following two conditions are both fulfilled:*

(i) *When both $y_1 \neq x_0$ and $y_2 \neq x_0$, there exists $i = 1$ or 2 such that $\{y_1, y_2\} \subseteq I_i$ and $f \in \hat{\mathfrak{C}}^{i}_{j\tau}(I, I)$; When $y_1 = x_0$ and $y_2 \in I_i$ for $i = 1$ or 2, either $f \in \check{\mathfrak{C}}^{(i,2)}_{j\tau}(I, I)$ if $x_0 \in \Delta_{1i}$, or $f \in \bar{\mathfrak{C}}^{(i,2)}_{j\tau}(I, I)$ if $x_0 \in \Delta_{0i}$, or*

$f \in \tilde{\complement}^{(i,2)}_{j\tau}(I, I)$ if $x_0 \in \Delta_{2i}$; When $y_2 = x_0$ and $y_1 \in I_i$ for $i = 1$ or 2, either $f \in \tilde{\complement}^{(i,1)}_{j\tau}(I, I)$ if $x_0 \in \Delta_{i1}$, or $f \in \bar{\complement}^{(i,1)}_{j\tau}(I, I)$ if $x_0 \in \Delta_{i0}$, or $f \in \check{\complement}^{(i,1)}_{j\tau}(I, I)$ if $x_0 \in \Delta_{i2}$.
(ii) *$\xi_p \in \Delta_{00} \cup \Delta_{11} \cup \Delta_{10} \cup \Delta_{01} \cup \Delta_{22} \cup \Delta_{20} \cup \Delta_{02}$ and for $i, p \in \{1, 2\}$, $f \in \complement^{ip}_{j\tau}(I, I)$ if $\xi_p \in \Delta_{ii} \cup \Delta_{i0} \cup \Delta_{0i}$, where $\xi_p \in f^{-1}(I_0) \cap I_p$.*

Proof. According to the location of y_1 and y_2, there are three cases to be considered: **(J-1)** Both y_1 and y_2 lie in a sub-interval divided by the discontinuity x_0 of f; **(J-2)** y_1 reaches the discontinuity x_0 of f and y_2 lies in a sub-interval divided by the discontinuity x_0 of f; **(J-3)** y_1 lies in a sub-interval divided by the discontinuity x_0 of f and y_2 reaches the discontinuity x_0 of f. Since $f \in V_{j\tau}(I, I)$ for $\tau \in \{r, j\}$, we need to discuss two situations: $f \in V_{jr}(I, I)$ and $f \in V_{jj}(I, I)$. We only discuss the situation that $f \in V_{jj}(I, I)$ because the situation that $f \in V_{jr}(I, I)$ can be discussed similarly. Under condition that $f \in V_{jj}(I, I)$, knowing that $y_1 \neq y_2$ and $\tilde{y}_1 := \lim_{x\to x_0-0} f_1'(x) \neq \tilde{y}_2 := \lim_{x\to x_0+0} f_2'(x)$.

For the **necessity**, we suppose that f^2 is C^1 smooth on I. So that f^2 is continuous on I.

In case **(J-1)**, i.e., both $y_1 \neq x_0$ and $y_2 \neq x_0$, by the continuity of f^2 on I and **(i)** of Theorem 2 in [8], there exists $i = 1$ or 2 such that $\{y_1, y_2\} \subseteq I_i$ and

$$f_i(y_1) = f_i(y_2) = f(c). \tag{3.2}$$

In what follows, we only discuss the situation that $\{y_1, y_2\} \subseteq I_1$ since the other one can be discussed similarly. By the openness of I_1 and I_2 and the continuity of f_1 on I_1 and f_2 on I_2, there are a left half neighborhood $U^-_{x_0}$ of x_0 and a right half neighborhood $U^+_{x_0}$ of x_0 such that $f_1(U^-_{x_0}) \subseteq I_1$ and $f_2(U^+_{x_0}) \subseteq I_1$, i.e., $x_0 \in \Delta_{11}$. Then

$$f^2(x) = \begin{cases} f_1(f_1(x)), & x \in U^-_{x_0}, \\ f(c), & x = x_0, \\ f_1(f_2(x)), & x \in U^+_{x_0}. \end{cases} \tag{3.3}$$

It follows from (3.3) that

$$D_-f^2(x_0) = \lim_{x\to x_0-0} f_1'(f_1(x))f_1'(x) = f_1'(y_1)\tilde{y}_1, \tag{3.4}$$

$$D_+f^2(x_0) = \lim_{x\to x_0+0} f_1'(f_2(x))f_2'(x) = f_1'(y_2)\tilde{y}_2. \tag{3.5}$$

Since f^2 is C^1 on I, we have $D_-f^2(x_0) = D_+f^2(x_0)$. From (3.4) and (3.5) we get that

$$f_1'(y_1)\tilde{y}_1 = f_1'(y_2)\tilde{y}_2. \tag{3.6}$$

Note that $i = 1$ and $\tau = j$. It follows from (3.2) and (3.6) that $f \in \hat{\complement}^1_{jj}(I, I)$.

In case **(J-2)**, i.e., $y_1 = x_0$ and $y_2 \in I_i$ for $i = 1$ or 2, we only consider the situation that $y_1 = x_0$ and $y_2 \in I_2$. The other one can be discussed similarly. From the assumption that $y_1 = x_0$ and $y_2 \in I_2$, we have $\lim_{x\to x_0-0} f_1(x) = y_1 = x_0$ and $\lim_{x\to x_0+0} f_2(x) = y_2 \in I_2$. By the continuity of f_1 on I_1 and f_2 on I_2, we see that

$$x_0 \in \Delta_{12} \cup \Delta_{02} \cup \Delta_{22}. \tag{3.7}$$

From (3.7) we need to discuss three situations: $x_0 \in \Delta_{12}$, $x_0 \in \Delta_{02}$ and $x_0 \in \Delta_{22}$. In the first situation that $x_0 \in \Delta_{12}$, by the continuity of f^2 on I and **(ii-1)** of Theorem 2 in [8], we have

$$f_2(y_2) = y_1 = f(c). \tag{3.8}$$

From the definition of Δ_{12} and $x_0 \in \Delta_{12}$, we have

$$f^2(x) = \begin{cases} f_1(f_1(x)), & x \in U^-_{x_0}, \\ f(c), & x = x_0, \\ f_2(f_2(x)), & x \in U^+_{x_0}. \end{cases} \tag{3.9}$$

Note that $\lim_{x\to x_0-0} f_1(x) = y_1 = x_0$ and $\lim_{x\to x_0+0} f_2(x) = y_2 \in I_2$. Thus, $f_1(x) \to x_0 - 0$ as $x \to x_0 - 0$ and $f_2(x) \to y_2$ as $x \to x_0 + 0$. It follows from (3.9) that

$$D_- f^2(x_0) = \lim_{x\to x_0-0} f_1'(f_1(x))f_1'(x) = \tilde{y}_1 \lim_{y\to x_0-0} f_1'(y) = \tilde{y}_1^2, \tag{3.10}$$

$$D_+ f^2(x_0) = \lim_{x\to x_0+0} f_2'(f_2(x))f_2'(x) = f_2'(y_2)\tilde{y}_2. \tag{3.11}$$

Since f^2 is C^1 on I, we have $D_- f^2(x_0) = D_+ f^2(x_0)$. From (3.10) and (3.11) we get that

$$f_2'(y_2)\tilde{y}_2 = \tilde{y}_1^2. \tag{3.12}$$

Note that $i = k = 2$ and $\tau = j$. It follows from (3.8) and (3.12) that $f \in \check{\complement}^{(2,2)}_{jj}(I, I)$. In the second situation that $x_0 \in \Delta_{02}$, we get that

$$f_2(y_2) = c = f(c) \tag{3.13}$$

similarly. By the definition of Δ_{02} and $x_0 \in \Delta_{02}$, we have

$$f^2(x) = \begin{cases} c, & x \in U^-_{x_0}, \\ f(c), & x = x_0, \\ f_2(f_2(x)), & x \in U^+_{x_0}. \end{cases} \tag{3.14}$$

Note that $\lim_{x\to x_0+0} f_2(x) = y_2 \in I_2$. We obtain from (3.14) that (3.11) and

$$D_- f^2(x_0) = 0. \tag{3.15}$$

Since f^2 is C^1 on I, we have $D_- f^2(x_0) = D_+ f^2(x_0)$. From (3.11) and (3.15) we get that

$$f_2'(y_2)\tilde{y}_2 = 0. \tag{3.16}$$

Note that $i = k = 2$ and $\tau = j$. It follows from (3.13) and (3.16) that $f \in \bar{\complement}^{(2,2)}_{jj}(I, I)$. Finally, in the three situation that $x_0 \in \Delta_{22}$, we get that

$$f_2(y_2) = y_2 = f(c) \tag{3.17}$$

similarly. By the definition of Δ_{22} and $x_0 \in \Delta_{22}$, we have

$$f^2(x) = \begin{cases} f_2(f_1(x)), & x \in U^-_{x_0}, \\ f(c), & x = x_0, \\ f_2(f_2(x)), & x \in U^+_{x_0}. \end{cases} \tag{3.18}$$

Note that $\lim_{x\to x_0-0} f_1(x) = y_1 = x_0$ and $\lim_{x\to x_0+0} f_2(x) = y_2 \in I_2$. Thus, $f_1(x) \to x_0+0$ as $x \to x_0-0$ and $f_2(x) \to y_2$ as $x \to x_0 + 0$. From (3.18) we obtain (3.11) and

$$D_-f^2(x_0) = \lim_{x\to x_0-0} f_2'(f_1(x))f_1'(x) = \tilde{y}_1 \lim_{y\to x_0+0} f_2'(y) = \tilde{y}_1\tilde{y}_2. \tag{3.19}$$

Since f^2 is C^1 on I, we have $D_-f^2(x_0) = D_+f^2(x_0)$. we get from (3.11) and (3.19) that

$$f_2'(y_2)\tilde{y}_2 = \tilde{y}_1\tilde{y}_2. \tag{3.20}$$

Note that $i = k = 2$ and $\tau = j$. It follows from (3.17) and (3.20) that $f \in \tilde{\mathfrak{C}}_{jj}^{(2,2)}(I, I)$.

We omit the proof in case **(J-3)**, i.e., $y_2 = x_0$ and $y_1 \in I_i$ for $i = 1$ or 2 because its proof is totally similar to the proof in case **(J-2)**. Thus, condition **(i)** holds.

Next, we prove that condition **(ii)** holds. In fact, suppose that a point ξ_p in $f^{-1}(I_0) \cap I_p$ for $p = 1$ or 2. In the following, we only consider a point ξ_1 in $f^{-1}(I_0) \cap I_1$ since the other one can be discussed similarly. By the continuity of f_1 on I_1 we see that

$$\begin{aligned}\xi_1 \quad \in \quad & \Delta_{11} \cup \Delta_{10} \cup \Delta_{12} \cup \Delta_{01} \cup \Delta_{02} \\ & \cup\Delta_{21} \cup \Delta_{20} \cup \Delta_{22} \cup \Delta_{00}.\end{aligned} \tag{3.21}$$

From the continuity of f^2 on I, we claim that

$$\begin{aligned}\xi_1 \quad \in \quad & \Delta_{11} \cup \Delta_{10} \cup \Delta_{01} \cup \Delta_{02} \\ & \cup\Delta_{20} \cup \Delta_{22} \cup \Delta_{00}.\end{aligned} \tag{3.22}$$

By (3.21) we need to exclude two situations: $\xi_1 \in \Delta_{12}$ and $\xi_1 \in \Delta_{21}$. If $\xi_1 \in \Delta_{12}$, by the definition of Δ_{12}, we have

$$f^2(x) = \begin{cases} f_1(f_1(x)), & x \in U_{\xi_1}^-, \\ f_2(f_1(x)), & x \in U_{\xi_1}^+. \end{cases}$$

Note that $f_1(x) \to x_0 - 0$ as $x \to \xi_1 - 0$ and $f_1(x) \to x_0 + 0$ as $x \to \xi_1 + 0$. It follows that

$$\begin{aligned}\lim_{x\to\xi_1-0} f^2(x) &= \lim_{x\to\xi_1-0} f_1(f_1(x)) = \lim_{y\to x_0-0} f_1(y) = y_1, \\ \lim_{x\to\xi_1+0} f^2(x) &= \lim_{x\to\xi_1+0} f_2(f_1(x)) = \lim_{y\to x_0+0} f_2(y) = y_2.\end{aligned}$$

Thus, we get that $y_1 = y_2$ since f^2 is continuous on I, which contradicts the assumption that $y_1 \neq y_2$. Using a similar discussion to the proof of $\xi_1 \in \Delta_{12}$, one can get that $y_1 = y_2$ if $\xi_1 \in \Delta_{21}$, a contradiction to the assumption. This completes the proof of the claimed (3.22). Thus, from the continuity of f^2 on I and (3.22), if $\xi_1 \in \Delta_{ii} \cup \Delta_{i0} \cup \Delta_{0i}$ for $i = 1$ or 2, we obtain

$$y_i = c \tag{3.23}$$

by Theorem 2 of [8] and its proof. In what follows, we only discuss the situation that $\xi_1 \in \Delta_{11}\cup\Delta_{10}\cup\Delta_{01}$ since the other one can be discussed similarly. if $\xi_1 \in \Delta_{11}$, by the definition of Δ_{11}, we have

$$f^2(x) = \begin{cases} f_1(f_1(x)), & x \in U_{\xi_1}^- \cup U_{\xi_1}^+, \\ c, & x = \xi_1. \end{cases} \tag{3.24}$$

Note that $f(\xi_1) = x_0$. Thus, $f_1(x) \to x_0 - 0$ as $x \to \xi_1$. From (3.24) we get that

$$D_- f^2(\xi_1) = \lim_{x\to\xi_1-0} f_1'(f_1(x))f_1'(x) = f_1'(\xi_1) \lim_{y\to x_0-0} f_1'(y) = \tilde{y}_1 f_1'(\xi_1), \tag{3.25}$$

$$D_+ f^2(\xi_1) = \lim_{x\to\xi_1+0} f_1'(f_1(x))f_1'(x) = f_1'(\xi_1) \lim_{y\to x_0-0} f_1'(y) = \tilde{y}_1 f_1'(\xi_1). \tag{3.26}$$

Note that f_1 is C^1 smooth on I_1 and ξ_1 is a local maximum point of f_1, we obtain

$$f_1'(\xi_1) = 0. \tag{3.27}$$

If $\xi_1 \in \Delta_{10}$, by the definition of Δ_{10}, we have

$$f^2(x) = \begin{cases} f_1(f_1(x)), & x \in U_{\xi_1}^-, \\ c, & x \in U_{\xi_1}^+ \cup \{\xi_1\}. \end{cases} \tag{3.28}$$

Note that $f_1(x) \to x_0 - 0$ as $x \to \xi_1 - 0$. From (3.28) we obtain (3.25) and

$$D_+ f^2(\xi_1) = 0. \tag{3.29}$$

Since f^2 is C^1 smooth on I, we have $D_- f^2(\xi_1) = D_+ f^2(\xi_1)$. Note that f_1 is C^1 smooth on I_1. Thus, we get from (3.25) and (3.29) that (3.27) holds. Similarly to $\xi_1 \in \Delta_{10}$, we can also get that (3.27) holds when $\xi_1 \in \Delta_{01}$. Note that $i = p = 1$ and $\tau = j$. It follows from (3.23) and (3.27) that $f \in \complement_{jj}^{11}(I, I)$. Thus, condition **(ii)** holds and this completes the proof of necessity.

For the **sufficiency**, we need to prove that f^2 is C^1 smooth on I under condition **(i)** and condition **(ii)**. First, we prove that f^2 is C^1 smooth at x_0 under condition **(i)**. In fact, when both $y_1 \neq x_0$ and $y_2 \neq x_0$, by the assumption that there exists $i = 1$ or 2 such that $\{y_1, y_2\} \subseteq I_i$, which implies that (3.3) holds when $\{y_1, y_2\} \subseteq I_1$. It follows from (3.3) that (3.4) and (3.5) hold. Since we assumed that $f \in \hat{\complement}_{jj}^1(I, I)$. From the definition of $\hat{\complement}_{jj}^1(I, I)$, we see that f^2 is C^0 at x_0 by **(i)** of Theorem 2 in [8]. Moreover, we get from (3.4) and (3.5) that $D_- f^2(x_0) = D_+ f^2(x_0)$. It follows that f^2 is C^1 smooth at x_0. The situation that $\{y_1, y_2\} \subseteq I_2$ can be proved similarly. When $y_1 = x_0$ and $y_2 \in I_i$ for $i = 1$ or 2, which implies that $\lim_{x\to x_0-0} f_1(x) = y_1 = x_0$ and $\lim_{x\to x_0+0} f_2(x) = y_2 \in I_i$. For the case that $y_1 = x_0$ and $y_2 \in I_2$, by the continuity of f_1 on I_1 and f_2 on I_2, we need to discuss three situations: $x_0 \in \Delta_{12}$, $x_0 \in \Delta_{02}$ and $x_0 \in \Delta_{22}$. We only consider the situation that $x_0 \in \Delta_{12}$ since the other situations can be discussed similarly. For the situation that $x_0 \in \Delta_{12}$, we see that (3.9) holds. It follows from (3.9) that (3.10) and (3.11) hold. Since we have assumed that $f \in \check{\complement}_{jj}^{(2,2)}(I, I)$. From the definition of $\check{\complement}_{jj}^{(2,2)}(I, I)$, we see that f^2 is C^0 at x_0 by **(ii-1)** of Theorem 2 in [8]. Moreover, we get from (3.10) and (3.11) that $D_- f^2(x_0) = D_+ f^2(x_0)$. It follows that f^2 is C^1 smooth at x_0. The proof of the case that $y_1 = x_0$ and $y_2 \in I_1$ is similar to the proof of the case that $y_1 = x_0$ and $y_2 \in I_2$. We omit the proof of the case that $y_2 = x_0$ and $y_1 \in I_i$ for $i = 1$ or 2 because its proof is totally similar to the proof of the case that $y_1 = x_0$ and $y_2 \in I_i$ for $i = 1$ or 2. Next, we prove that f^2 is C^1 smooth at ξ_p under condition **(ii)**, where $\xi_p \in f^{-1}(I_0) \cap I_p$ for $p = 1$ or 2. We only discuss the case that $\xi_1 \in f^{-1}(I_0) \cap I_1$ since the other one can be discussed similarly. By the assumption that $\xi_1 \in \Delta_{00} \cup \Delta_{11} \cup \Delta_{10} \cup \Delta_{01} \cup \Delta_{22} \cup \Delta_{20} \cup \Delta_{02}$, we need to discuss seven situations: $\xi_1 \in \Delta_{00}$, $\xi_1 \in \Delta_{11}$, $\xi_1 \in \Delta_{10}$, $\xi_1 \in \Delta_{01}$, $\xi_1 \in \Delta_{22}$ $\xi_1 \in \Delta_{20}$ and $\xi_1 \in \Delta_{02}$. In the first situation that $\xi_1 \in \Delta_{00}$, by the definition of Δ_{00}, we have

$$f^2(x) = \begin{cases} c, & x \in U_{\xi_1}^- \cup U_{\xi_1}^+, \\ c, & x = \xi_1. \end{cases}$$

Obviously, $\lim_{x\to\xi_1} f^2(x) = c = f^2(\xi_1)$ and $D_- f^2(\xi_1) = D_+ f^2(\xi_1) = 0$. It follows that f^2 is C^1 smooth at ξ_1. In the second situation that $\xi_1 \in \Delta_{11}$, by the definition of Δ_{11}, we see that (3.24) holds. It follows from (3.24) that (3.25) and (3.26) hold. Since we have assumed that $f \in \complement^{11}_{jj}(I, I)$. From the definition of $\complement^{11}_{jj}(I, I)$, we see that f^2 is C^0 at ξ_1 by Theorem 2 of [8]. Moreover, we get from (3.25) and (3.26) that $D_- f^2(\xi_1) = D_+ f^2(\xi_1) = 0$. It follows that f^2 is C^1 smooth at ξ_1. Similarly, we can prove that f^2 is C^1 smooth at ξ_1 when $\xi_1 \in \Delta_{10} \cup \Delta_{01} \cup \Delta_{22} \cup \Delta_{20} \cup \Delta_{02}$. Condition **(i)** and condition **(ii)** imply that f^2 is C^1 smooth on the whole domain I. Therefore, the proof of the theorem is completed. □

Theorem 3.2. *Let $f \in V_{jo\mu}(I, I)$ for $\mu \in \{+, -, *\}$ and $x_0 \in (0, 1)$ be the unique discontinuity. Suppose that y_1 and y_2 are defined by (3.1). Then the iterate f^2 is C^1 smooth on I if and only if the following two conditions are both fulfilled:*
(i) *When both $y_1 \neq x_0$ and $y_2 \neq x_0$, there exists $i = 1$ or 2 such that $\{y_1, y_2\} \subseteq I_i$ and $f \in \complement^{(i,1)}_{jo+}(I, I) \cup \complement^{(i,2)}_{jo-}(I, I) \cup \complement^{i}_{jo*}(I, I)$; When $y_1 = x_0$ and $y_2 \in I_i$ for $i = 1$ or 2, either $f \in \tilde{\complement}^{(i,1)}_{jo+}(I, I)$ if $x_0 \in \Delta_{1i}$, or $f \in \bar{\complement}^{(i,2)}_{jo+}(I, I)$ if $x_0 \in \Delta_{0i}$, or $f \in \hat{\complement}^{(i,2)}_{jo+}(I, I) \cup \check{\complement}^{(i,2)}_{jo-}(I, I)$ if $x_0 \in \Delta_{2i}$; When $y_2 = x_0$ and $y_1 \in I_i$ for $i = 1$ or 2, either $f \in \tilde{\complement}^{(i,2)}_{jo-}(I, I)$ if $x_0 \in \Delta_{i2}$, or $f \in \bar{\complement}^{(i,1)}_{jo-}(I, I)$ if $x_0 \in \Delta_{i0}$, or $f \in \hat{\complement}^{(i,1)}_{jo-}(I, I) \cup \check{\complement}^{(i,1)}_{jo+}(I, I)$ if $x_0 \in \Delta_{i1}$.*
(ii) *$\xi_p \in \Delta_{00} \cup \Delta_{11} \cup \Delta_{10} \cup \Delta_{01} \cup \Delta_{22} \cup \Delta_{20} \cup \Delta_{02}$ and for $i, p \in \{1, 2\}$, $f \in \complement^{ip}_{jo}(I, I)$ if $\xi_p \in \Delta_{ii} \cup \Delta_{i0} \cup \Delta_{0i}$, where $\xi_p \in f^{-1}(I_0) \cap I_p$.*

Proof. For the **necessity**, we suppose that f^2 is C^1 smooth on I. From the location of y_1 and y_2, similarly to Theorem 3.1, we also need to consider three cases: **(J-1)**, **(J-2)** and **(J-3)**. In what follows, we only consider case **(J-1)** and case **(J-2)** since the proof of case **(J-3)** is totally similar to the proof of case **(J-2)**.

In case **(J-1)**, i.e., both $y_1 \neq x_0$ and $y_2 \neq x_0$. Similarly to the proof of the necessity of case **(J-1)** in Theorem 3.1, there exists $i = 1$ or 2 such that $\{y_1, y_2\} \subseteq I_i$ and (3.2) holds. In the following, we only discuss the situation that $\{y_1, y_2\} \subseteq I_1$ since the other one can be discussed similarly. Under the situation that $\{y_1, y_2\} \subseteq I_1$, we see that (3.3) holds. Note that $f \in V_{jo\mu}(I, I)$ for $\mu \in \{+, -, *\}$. Then we need to discuss three subcases: **(J-1-o+)** $f \in V_{jo+}(I, I)$; **(J-1-o-)** $f \in V_{jo-}(I, I)$; **(J-1-o*)** $f \in V_{jo*}(I, I)$.

In subcase **(J-1-o+)**, $\tilde{y}_1 := \lim_{x\to x_0-0} f_1'(x)$ exists but $\lim_{x\to x_0+0} f_2'(x)$ does not exist. It follows from (3.3) that

$$D_- f^2(x_0) = \lim_{x\to x_0-0} f_1'(f_1(x)) f_1'(x) = f_1'(y_1)\tilde{y}_1, \tag{3.30}$$

$$D_+ f^2(x_0) = \lim_{x\to x_0+0} f_1'(f_2(x)) f_2'(x). \tag{3.31}$$

Since f^2 is C^1 on I, we see that $D_+ f^2(x_0)$ exists. Note that $\lim_{x\to x_0+0} f_1'(f_2(x)) = f_1'(y_2)$. We claim that

$$f_1'(y_2) = 0. \tag{3.32}$$

In fact, if $f_1'(y_2) \neq 0$, It follows from (3.30) that $\lim_{x\to x_0+0} f_2'(x)$ exists since

$$\lim_{x\to x_0+0} f_2'(x) = \lim_{x\to x_0+0} \frac{f_1'(f_2(x)) f_2'(x)}{f_1'(f_2(x))} = \frac{D_+ f^2(x_0)}{f_1'(y_2)},$$

which contradicts to our assumption that $\lim_{x\to x_0+0} f_2'(x)$ does not exist. Thus, the claim that (3.32) is proved. By the fact that f_2' is bounded, we get from (3.30) and (3.32) that $D_+ f^2(x_0) = 0$. Since f^2 is

C^1 on I, we see that $D_-f^2(x_0) = D_+f^2(x_0)$. It follows from (3.30) and (3.31) that

$$f_1'(y_1)\tilde{y}_1 = f_1'(y_2) = 0. \tag{3.33}$$

Note that $i = k = 1$. It follows from (3.2) and (3.33) that $f \in \complement_{jo+}^{(1,1)}(I, I)$.

In subcase **(J-1-o-)**, $\tilde{y}_2 := \lim_{x\to x_0+0} f_2'(x)$ exists but $\lim_{x\to x_0-0} f_1'(x)$ does not exist. Using a similar argument to the proof of subcase **(J-1-o+)**, we can get that

$$f_1'(y_2)\tilde{y}_2 = f_1'(y_1) = 0. \tag{3.34}$$

Note that $i = 1$ and $k = 2$. It follows from (3.2) and (3.34) that $f \in \complement_{jo-}^{(1,2)}(I, I)$.

In subcase **(J-1-o*)**, neither $\lim_{x\to x_0-0} f_1'(x)$ nor $\lim_{x\to x_0+0} f_2'(x)$ exists. It follows from (3.3) that

$$D_-f^2(x_0) = \lim_{x\to x_0-0} f_1'(f_1(x))f_1'(x),$$
$$D_+f^2(x_0) = \lim_{x\to x_0+0} f_1'(f_2(x))f_2'(x).$$

Since f^2 is C^1 on I, we see that both $D_-f^2(x_0)$ and $D_+f^2(x_0)$ exist. Note that $\lim_{x\to x_0-0} f_1'(f_1(x)) = f_1'(y_1)$ and $\lim_{x\to x_0+0} f_1'(f_2(x)) = f_1'(y_2)$. Similarly to subcase **(J-1-o+)**, we can get that

$$f_1'(y_1) = f_1'(y_2) = 0. \tag{3.35}$$

Note that $i = 1$. It follows from (3.2) and (3.35) that $f \in \complement_{jo*}^{1}(I, I)$.

In case **(J-2)**, i.e., $y_1 = x_0$ and $y_2 \in I_i$ for $i = 1$ or 2. In the following, we only discuss the situation that $y_1 = x_0$ and $y_2 \in I_2$ since the other one can be discussed similarly. Using a similar discussion to the proof of the necessity of case **(J-2)** in Theorem 3.1, we obtain (3.7) when $y_1 = x_0$ and $y_2 \in I_2$. Note that $f \in V_{jo\mu}(I, I)$ for $\mu \in \{+, -, *\}$. Then we need to discuss three subcases: **(J-2-o+)** $f \in V_{jo+}(I, I)$; **(J-2-o-)** $f \in V_{jo-}(I, I)$; **(J-2-o*)** $f \in V_{jo*}(I, I)$.

In subcase **(J-2-o+)**, $\tilde{y}_1 := \lim_{x\to x_0-0} f_1'(x)$ exists but $\lim_{x\to x_0+0} f_2'(x)$ does not exist. By (3.7) we need to discuss in three situations: $x_0 \in \Delta_{12}$, $x_0 \in \Delta_{02}$ and $x_0 \in \Delta_{22}$. In the first situation that $x_0 \in \Delta_{12}$, by **(ii-1)** of Theorem 2 in [8] and its proof, we see that (3.8) holds. By the definition of Δ_{12} and $x_0 \in \Delta_{12}$, we obtain (3.9). Note that $\lim_{x\to x_0-0} f_1(x) = y_1 = x_0$ and $\lim_{x\to x_0+0} f_2(x) = y_2 \in I_2$. Thus, $f_1(x) \to x_0 - 0$ as $x \to x_0 - 0$ and $f_2(x) \to y_2$ as $x \to x_0 + 0$. It follows from (3.9) that (3.10) and

$$D_+f^2(x_0) = \lim_{x\to x_0+0} f_2'(f_2(x))f_2'(x). \tag{3.36}$$

Since f^2 is C^1 on I, we see that $D_+f^2(x_0)$ exists. Note that $\lim_{x\to x_0+0} f_2'(f_2(x)) = f_2'(y_2)$. Similarly to subcase **(J-1-o+)**, we can get that

$$f_2'(y_2) = 0. \tag{3.37}$$

By the fact that f_2' is bounded, we get from (3.36) and (3.37) that $D_+f^2(x_0) = 0$. Since f^2 is C^1 on I, we see that $D_-f^2(x_0) = D_+f^2(x_0)$. It follows from (3.10) and (3.36) that

$$\tilde{y}_1 = f_2'(y_2) = 0. \tag{3.38}$$

Note that $i = 2$ and $k = 1$. It follows from (3.8) and (3.38) that $f \in \tilde{\complement}^{(2,1)}_{jo+}(I, I)$. In the second situation that $x_0 \in \Delta_{02}$, we obtain (3.13) similarly. By the definition of Δ_{02} and $x_0 \in \Delta_{02}$, we see that (3.14) holds. Note that $\lim_{x\to x_0+0} f_2(x) = y_2 \in I_2$. We obtain from (3.14) that (3.15) and (3.36). Similarly to the proof of the first situation that $x_0 \in \Delta_{12}$, we can get that (3.37) holds. It follows from (3.13) and (3.37) that $f \in \bar{\complement}^{(2,2)}_{jo+}(I, I)$. Finally, in the three situation that $x_0 \in \Delta_{22}$. Similarly, we get that (3.17) holds. By the definition of Δ_{22} and $x_0 \in \Delta_{22}$, we obtain (3.18). Note that $\lim_{x\to x_0-0} f_1(x) = y_1 = x_0$ and $\lim_{x\to x_0+0} f_2(x) = y_2 \in I_2$. Thus, $f_1(x) \to x_0 + 0$ as $x \to x_0 - 0$ and $f_2(x) \to y_2$ as $x \to x_0 + 0$. From (3.18) we obtain (3.36) and

$$D_- f^2(x_0) = \lim_{x\to x_0-0} f_2'(f_1(x)) f_1'(x).$$

Since f^2 is C^1 on I, we see that both $D_- f^2(x_0)$ and $D_+ f^2(x_0)$ exist. Note that $\lim_{x\to x_0-0} f_2'(f_1(x)) = \lim_{x\to x_0+0} f_2'(x)$ and $\lim_{x\to x_0+0} f_2'(f_2(x)) = f_2'(y_2)$. We claim that (3.38) holds. In fact, if $\tilde{y}_1 \neq 0$ or $f_2'(y_2) \neq 0$, then $\lim_{x\to x_0+0} f_2'(x)$ exists since

$$\lim_{x\to x_0+0} f_2'(x) = \lim_{x\to x_0-0} f_2'(f_1(x)) = \lim_{x\to x_0-0} \frac{f_2'(f_1(x)) f_1'(x)}{f_1'(x)} = \frac{D_- f^2(x_0)}{\tilde{y}_1}$$

or

$$\lim_{x\to x_0+0} f_2'(x) = \lim_{x\to x_0+0} \frac{f_2'(f_2(x)) f_2'(x)}{f_2'(f_2(x))} = \frac{D_+ f^2(x_0)}{f_2'(y_2)},$$

which contradicts to our assumption that $\lim_{x\to x_0+0} f_2'(x)$ does not exist. Thus, the claim that (3.38) is proved. Note that $i = k = 2$. It follows from (3.17) and (3.38) that $f \in \hat{\complement}^{(2,2)}_{jo+}(I, I)$.

In subcase **(J-2-o-)**, $\tilde{y}_2 := \lim_{x\to x_0+0} f_2'(x)$ exists but $\lim_{x\to x_0-0} f_1'(x)$ does not exist. We claim that

$$x_0 \in \Delta_{12} \cup \Delta_{22}. \tag{3.39}$$

By (3.7) we need to deny the situation that $x_0 \in \Delta_{02}$. If $x_0 \in \Delta_{02}$, by the definition of Δ_{02} and $x_0 \in \Delta_{02}$, there are a left half neighborhood $U^-_{x_0}$ of x_0 such that $f_1(x) = x_0$ for every $x \in U^-_{x_0}$. It follows that $\lim_{x\to x_0-0} f_1'(x) = 0$, which contradicts to our assumption that $\lim_{x\to x_0-0} f_1'(x)$ does not exist. Thus, the claim that (3.39) is proved. By (3.39) we need to discuss in two situations: $x_0 \in \Delta_{12}$ and $x_0 \in \Delta_{22}$. In the first situation that $x_0 \in \Delta_{12}$, we obtain (3.9). It follows from (3.9) that

$$D_- f^2(x_0) = \lim_{x\to x_0-0} f_1'(f_1(x)) f_1'(x).$$

Note that $\lim_{x\to x_0-0} f_1'(f_1(x)) = \lim_{x\to x_0-0} f_1'(x)$ and $\lim_{x\to x_0-0} f_1'(x)$ does not exist but f_1' is bounded. It is hard to judge the existence of the limit $D_- f^2(x_0) = \lim_{x\to x_0-0} f_1'(f_1(x)) f_1'(x)$. Thus, we omit the discussion for the situation $x_0 \in \Delta_{12}$. In the two situation that $x_0 \in \Delta_{22}$. Similarly, we get that (3.17) holds. By the definition of Δ_{22} and $x_0 \in \Delta_{22}$, we obtain (3.18). Note that $\lim_{x\to x_0-0} f_1(x) = y_1 = x_0$ and $\lim_{x\to x_0+0} f_2(x) = y_2 \in I_2$. Thus, $f_1(x) \to x_0 + 0$ as $x \to x_0 - 0$ and $f_2(x) \to y_2$ as $x \to x_0 + 0$. From (3.18) we obtain (3.11) and (3.39). Since f^2 is C^1 on I, we see that $D_- f^2(x_0)$ exists. Note that $\lim_{x\to x_0-0} f_2'(f_1(x)) = \lim_{x\to x_0+0} f_2'(x) = \tilde{y}_2$. Similarly to the proof of the three situation that $x_0 \in \Delta_{22}$ of subcase **(J-2-o+)**, we can get that

$$\tilde{y}_2 = 0. \tag{3.40}$$

Note that $i = k = 2$. It follows from (3.17) and (3.40) that $f \in \check{\mathfrak{C}}^{(2,2)}_{jo-}(I, I)$.

In subcase **(J-2-o*)**, neither $\lim_{x\to x_0-0} f_1'(x)$ nor $\lim_{x\to x_0+0} f_2'(x)$ exists, using a similar discussion to the proof of the the second subcase that $f \in V_{jo-}(I, I)$, we can get that (3.39) holds and we can not judge the existence of the limit $D_- f^2(x_0) = \lim_{x\to x_0-0} f_1'(f_1(x))f_1'(x)$ or the limit $D_- f^2(x_0) = \lim_{x\to x_0-0} f_2'(f_1(x))f_1'(x)$ as $x_0 \in \Delta_{12} \cup \Delta_{22}$. Thus, condition **(i)** holds.

Next, we prove that condition **(ii)** holds. Suppose that a point ξ_p in $f^{-1}(I_0) \cap I_p$ for $p = 1$ or 2. In the following, we only consider a point ξ_1 in $f^{-1}(I_0) \cap I_1$ since the other one can be discussed similarly. Using a similar discussion to the proof of the necessity of condition **(ii)** in Theorem 3.1, we can obtain (3.22) and (3.23) when $\xi_1 \in \Delta_{ii} \cup \Delta_{i0} \cup \Delta_{0i}$ for $i = 1$ or 2. In the following, we only discuss the situation that $i = 1$, i.e.,

$$\xi_1 \in \Delta_{11} \cup \Delta_{10} \cup \Delta_{01}. \tag{3.41}$$

The other one can be discussed similarly. Since $f \in V_{jo\mu}(I, I)$ for $\mu \in \{+, -, *\}$. Then we need to discuss three subcases: **(jo+)** $f \in V_{jo+}(I, I)$; **(jo-)** $f \in V_{jo-}(I, I)$; **(jo*)** $f \in V_{jo*}(I, I)$.

In subcase **(jo+)**, $\tilde{y}_1 := \lim_{x\to x_0-0} f_1'(x)$ exists but $\lim_{x\to x_0+0} f_2'(x)$ does not exist. Similarly to the proof of the necessity of condition **(ii)** in Theorem 3.1, we can get that $f \in \mathfrak{C}^{11}_{jo}(I, I)$.

In subcase **(jo-)**, $\tilde{y}_2 := \lim_{x\to x_0+0} f_2'(x)$ exists but $\lim_{x\to x_0-0} f_1'(x)$ does not exist. By (3.41) we need to discuss in three situations: $\xi_1 \in \Delta_{11}$, $\xi_1 \in \Delta_{10}$ and $\xi_1 \in \Delta_{01}$. In the first situation that $\xi_1 \in \Delta_{11}$, we see that (3.24) holds. Note that $f_1(x) \to x_0 - 0$ as $x \to \xi_1$. From (3.24) we get that the derivative of f^2 at ξ_1

$$Df^2(\xi_1) = \lim_{x\to\xi_1} f_1'(f_1(x))f_1'(x). \tag{3.42}$$

Since f^2 is C^1 on I, we see that $Df^2(\xi_1)$ exists. Note that $\lim_{x\to\xi_1} f_1'(f_1(x)) = \lim_{y\to x_0-0} f_1'(y)$ and $\lim_{x\to\xi_1} f_1'(x) = f_1'(\xi_1)$. We claim that (3.27) is true. In fact, if (3.27) is not true, i.e., $f_1'(\xi_1) \neq 0$, then we get from (3.42) that $\lim_{x\to x_0-0} f_1'(x)$ exists since

$$\lim_{x\to x_0-0} f_1'(x) = \lim_{x\to\xi_1} f_1'(f_1(x)) = \lim_{x\to\xi_1} \frac{f_1'(f_1(x))f_1'(x)}{f_1'(x)} = \frac{Df^2(\xi_1)}{f_1'(\xi_1)},$$

which contradicts to our assumption that $\lim_{x\to x_0-0} f_1'(x)$ does not exist. This proves the claimed (3.27). Note that $i = p = 1$. From (3.23) and (3.27) we see that $f \in \mathfrak{C}^{11}_{jo}(I, I)$. In the second situation that $\xi_1 \in \Delta_{10}$, we see that (3.28) holds. Note that $f_1(x) \to x_0 - 0$ as $x \to \xi_1 - 0$. From (3.28) we obtain (3.29) and

$$D_- f^2(\xi_1) = \lim_{x\to\xi_1-0} f_1'(f_1(x))f_1'(x). \tag{3.43}$$

Since f^2 is C^1 on I, we see that $D_- f^2(\xi_1)$ exists. Note that $\lim_{x\to\xi_1-0} f_1'(f_1(x)) = \lim_{y\to x_0-0} f_1'(y)$ and $\lim_{x\to\xi_1} f_1'(x) = f_1'(\xi_1)$. We claim that (3.27) is true. In fact, if (3.27) is not true, i.e., $f_1'(\xi_1) \neq 0$, then we get from (3.43) that $\lim_{x\to x_0-0} f_1'(x)$ exists since

$$\lim_{x\to x_0-0} f_1'(x) = \lim_{x\to\xi_1-0} f_1'(f_1(x)) = \lim_{x\to\xi_1-0} \frac{f_1'(f_1(x))f_1'(x)}{f_1'(x)} = \frac{D_- f^2(\xi_1)}{f_1'(\xi_1)},$$

which contradicts to our assumption that $\lim_{x\to x_0-0} f_1'(x)$ does not exist. This proves the claimed (3.27). From (3.23) and (3.27) we see that $f \in \mathfrak{C}^{11}_{jo}(I, I)$ since $i = p = 1$ and $f \in V_{jo}(I, I)$. In the third situation that $\xi_1 \in \Delta_{01}$, using a similar discussion to the proof of the second situation that $\xi_1 \in \Delta_{10}$, we can get that $f \in \complement_{jo}^{11}(I, I)$.

In subcase **(jo*)**, neither $\lim_{x\to x_0-0} f_1'(x)$ nor $\lim_{x\to x_0+0} f_2'(x)$ exists. Using a similar discussion to the proof of subcase **(jo-)**, we can get that $f \in \complement_{jo}^{11}(I, I)$ when $\xi_1 \in \Delta_{11} \cup \Delta_{10} \cup \Delta_{01}$. It follows that condition **(ii)** holds.

For the **sufficiency**, using a similar arguments to the proof of sufficiency in Theorem 3.1, we can prove that f^2 is C^1 smooth on the whole domain I. Therefore, the theorem is proved. □

Theorem 3.3. *Let $f \in V_{j\infty\mu}(I, I)$ for $\mu \in \{+, -, *\}$ and $x_0 \in (0, 1)$ be the unique discontinuity. Suppose that y_1 and y_2 are defined by (3.1) and the iterate f^2 is C^1 smooth on I. Then the following two conditions are both fulfilled:*
(i) *When both $y_1 \neq x_0$ and $y_2 \neq x_0$, there exists $i = 1$ or 2 such that $\{y_1, y_2\} \subseteq I_i$ and $f \in \complement_{j\infty+}^{(i,1)}(I, I) \cup \complement_{j\infty-}^{(i,2)}(I, I) \cup \complement_{j\infty*}^{i}(I, I)$; When $y_1 = x_0$ and $y_2 \in I_i$ for $i = 1$ or 2, either $f \in \tilde{\complement}_{j\infty+}^{(i,1)}(I, I)$ if $x_0 \in \Delta_{1i}$, or $f \in \bar{\complement}_{j\infty+}^{(i,2)}(I, I)$ if $x_0 \in \Delta_{0i}$, or $f \in \hat{\complement}_{j\infty+}^{(i,2)}(I, I) \cup \check{\complement}_{j\infty-}^{(i,2)}(I, I)$ if $x_0 \in \Delta_{2i}$; When $y_2 = x_0$ and $y_1 \in I_i$ for $i = 1$ or 2, either $f \in \tilde{\complement}_{j\infty-}^{(i,2)}(I, I)$ if $x_0 \in \Delta_{i2}$, or $f \in \bar{\complement}_{j\infty-}^{(i,1)}(I, I)$ if $x_0 \in \Delta_{i0}$, or $f \in \hat{\complement}_{j\infty-}^{(i,1)}(I, I) \cup \check{\complement}_{j\infty+}^{(i,1)}(I, I)$ if $x_0 \in \Delta_{i1}$.*
(ii) *$\xi_p \in \Delta_{00} \cup \Delta_{11} \cup \Delta_{10} \cup \Delta_{01} \cup \Delta_{22} \cup \Delta_{20} \cup \Delta_{02}$ and for $i, p \in \{1, 2\}$, $f \in \complement_{j\infty}^{ip}(I, I)$ if $\xi_p \in \Delta_{ii} \cup \Delta_{i0} \cup \Delta_{0i}$, where $\xi_p \in f^{-1}(I_0) \cap I_p$.*

Proof. Using a similar discussion to the proof of the necessity in Theorem 3.2, one can prove that both condition **(i)** and condition **(ii)** hold if f^2 is C^1 smooth on I. Therefore, this completes the proof. □

Notice that the above Theorem 3.3 does not give sufficient conditions of f^2 to be C^1 because it is hard to determine the existence of either $\lim_{x\to x_0-0} f_i'(f_1(x))f_1'(x)$ or $\lim_{x\to x_0+0} f_i'(f_2(x))f_2'(x)$ for $i = 1$ or 2. In fact, we assume that $\{y_1, y_2\} \subseteq I_1$ and $f \in \complement_{j\infty+}^{(1,1)}(I, I)$. A similar discussion to the proof in Theorem 3.2, we can get that

$$D_-f^2(x_0) = \lim_{x\to x_0-0} f_1'(f_1(x))f_1'(x),$$
$$D_+f^2(x_0) = \lim_{x\to x_0+0} f_1'(f_2(x))f_2'(x).$$

Note that $f \in \complement_{j\infty+}^{(1,1)}(I, I)$ and $\lim_{x\to x_0+0} f_1'(f_2(x)) = f_1'(y_2)$. By the definition of $\complement_{j\infty+}^{(1,1)}(I, I)$, we see that $f_1'(y_2) = 0$ and $\lim_{x\to x_0+0} f_2'(x) = \infty$. Thus, it is hard to judge the existence of the limit $D_+f^2(x_0) = \lim_{x\to x_0+0} f_1'(f_2(x))f_2'(x)$. We similarly see difficulty in other cases.

4. Examples

We demonstrate our theorems with some examples.

Example 4.1. Consider the mapping $F_1 : (0, 1) \to (0, 1)$ (see Figure 3) defined by

$$F_1(x) = \begin{cases} -x + 1, & 0 < x < \frac{1}{3}, \\ \frac{1}{12}, & x = \frac{1}{3}, \\ -\frac{9}{4}(x - \frac{2}{3})^2 + \frac{11}{12}, & \frac{1}{3} < x < 1, \end{cases}$$

which has a unique removable discontinuity at $x_0 = \frac{1}{3}$ since

$$\lim_{x\to\frac{1}{3}-0} F_1(x) = \lim_{x\to\frac{1}{3}+0} F_1(x) = \frac{2}{3} \neq F_1(\frac{1}{3}) = \frac{1}{12}.$$

Moreover,

$$\tilde{y}_1 = \lim_{x\to\frac{1}{3}-0} f_1'(x) = -1 \neq \tilde{y}_2 = \lim_{x\to\frac{1}{3}+0} f_2'(x) = \frac{3}{2},$$

where $f_1(x) = -x + 1$, $f_2(x) = -\frac{9}{4}(x - \frac{2}{3})^2 + \frac{11}{12}$. It follows that $F_1 \in V_{rj}(I, I)$. Note that $I_1 = (0, \frac{1}{3}), I_2 = (\frac{1}{3}, 1)$. One sees that $c = \frac{1}{12} \in I_1$, $y_0 = \frac{2}{3} \in I_2$ and $F_1(I_1 \cup I_2) \subseteq I_2$ holds. It is easy to check that $F_1(y_0) = f_2(y_0) = f_1(c)$ and $F_1'(y_0) = f_2'(y_0) = 0$, i.e., $F_1 \in \complement_{rj}(I, I)$. It follows that the assumption in Theorem 2.1 is satisfied. Furthermore, one can compute

$$F_1^2(x) = \begin{cases} -\frac{9}{4}(x - \frac{1}{3})^2 + \frac{11}{12}, & 0 < x < \frac{1}{3}, \\ \frac{11}{12}, & x = \frac{1}{3}, \\ -\frac{9}{64}(9(x - \frac{2}{3})^2 - 1)^2 + \frac{11}{12}, & \frac{1}{3} < x < 1, \end{cases}$$

which is C^1 smooth on (0, 1) as shown in Figure 4.

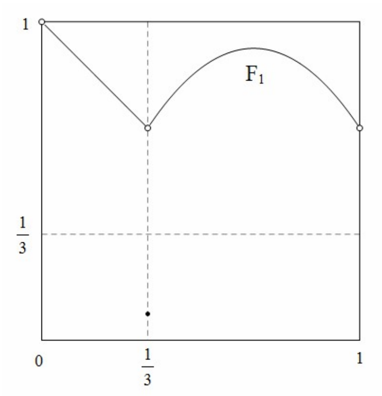

Figure 3. $F_1 \in \complement_{rj}(I, I)$.

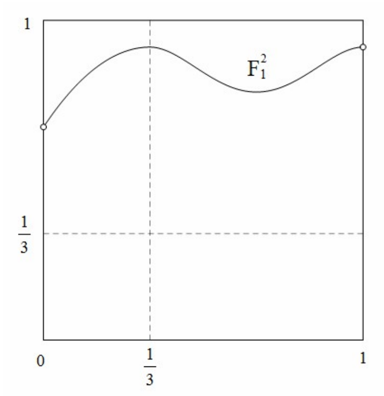

Figure 4. F_1^2 is C^1 on (0, 1).

Example 4.2. Consider the mapping $F_2 : (0, 1) \to (0, 1)$ (see Figure 5) defined by

$$F_2(x) = \begin{cases} \frac{1}{2}, & 0 < x \leq \frac{3}{8}, \\ \frac{32}{3}(x - \frac{3}{8})^2 + \frac{1}{2}, & \frac{3}{8} < x < \frac{1}{2}, \\ \frac{5}{6}, & x = \frac{1}{2}, \\ \frac{16}{3}(x - \frac{3}{4})^2 + \frac{1}{2}, & \frac{1}{2} < x < 1, \end{cases}$$

which has a unique jumping discontinuity at $x_0 = \frac{1}{2}$ since

$$y_1 = \lim_{x\to\frac{1}{2}-0} F_2(x) = \frac{2}{3} \neq \lim_{x\to\frac{1}{2}+0} F_2(x) = \frac{5}{6} = y_2.$$

Moreover,

$$\tilde{y}_1 = \lim_{x\to\frac{1}{2}-0} f_1'(x) = \frac{8}{3} \neq \tilde{y}_2 = \lim_{x\to\frac{1}{2}+0} f_2'(x) = -\frac{8}{3},$$

where

$$f_1(x) = \begin{cases} \frac{1}{2}, & 0 < x \le \frac{3}{8}, \\ \frac{32}{3}(x - \frac{3}{8})^2 + \frac{1}{2}, & \frac{3}{8} < x < \frac{1}{2}, \end{cases} \quad f_2(x) = \frac{16}{3}(x - \frac{3}{4})^2 + \frac{1}{2}.$$

It follows that $F_2 \in V_{jj}(I, I)$. Note that $I_1 = (0, \frac{1}{2}), I_2 = (\frac{1}{2}, 1)$. One can check that $\{y_1, y_2\} \subseteq I_2$, $c = \frac{5}{6} \in I_2$, $f_2(y_1) = f_2(y_2) = f_2(c)$, $f_2'(y_1)\tilde{y}_1 = f_2'(y_2)\tilde{y}_2$, which implies that $F_2 \in \hat{\complement}_{jj}^{(2,1)}(I, I)$. Moreover, one sees that $\xi_1 = \frac{3}{8} \in F_2^{-1}(I_0) \cap I_1$, $\xi_2 = \frac{3}{4} \in F_2^{-1}(I_0) \cap I_2$, $\xi_1 \in \Delta_{02}$ and $\xi_2 \in \Delta_{22}$. It is easy to check that $y_2 = c$ and $f_p'(\xi_p) = 0$, i.e., $F_2 \in \complement_{jj}^{2p}(I, I)$ for $p = 1, 2$, which implies that both assumption **(i)** and assumption **(ii)** in Theorem 3.1 are satisfied. Actually, one can compute

$$F_2^2(x) = \begin{cases} \frac{5}{6}, & 0 < x \le \frac{3}{8}, \\ \frac{16}{3}(\frac{32}{3}(x - \frac{3}{8})^2 - \frac{1}{4})^2 + \frac{1}{2}, & \frac{3}{8} < x \le \frac{1}{2}, \\ \frac{16}{3}(\frac{16}{3}(x - \frac{3}{4})^2 - \frac{1}{4})^2 + \frac{1}{2}, & \frac{1}{2} < x < 1, \end{cases}$$

which is C^1 smooth on $(0, 1)$ as shown in Figure 6.

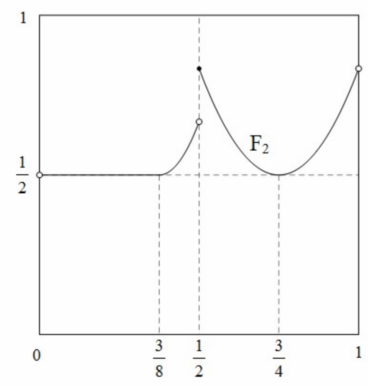

Figure 5. $F_2 \in \hat{\complement}_{jj}^{(2,1)}(I, I) \cap \complement_{jj}^{21}(I, I) \cap \complement_{jj}^{22}(I, I)$.

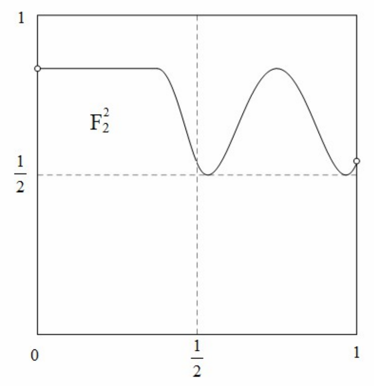

Figure 6. F_2^2 is C^1 on $(0, 1)$.

Example 4.3. Consider the mapping $F_3 : (0, 1) \to (0, 1)$ (see Figure 7) defined by

$$F_3(x) = \begin{cases} \frac{1}{3}, & 0 < x \le \frac{1}{3}, \\ \frac{1}{3} + (\frac{1}{2} - x)^2 \sin^2 \frac{\pi}{6(\frac{1}{2} - x)}, & \frac{1}{3} < x < \frac{1}{2}, \\ \frac{1}{4}, & x = \frac{1}{2}, \\ \frac{1}{6} + \frac{1}{6}(x - \frac{1}{2})^2 \cos^2 \frac{1}{x - \frac{1}{2}}, & \frac{1}{2} < x < 1, \end{cases}$$

which has a unique jumping discontinuity at $x_0 = \frac{1}{2}$ since

$$y_1 = \lim_{x \to \frac{1}{2} - 0} F_3(x) = \frac{1}{3} \neq \lim_{x \to \frac{1}{2} + 0} F_3(x) = \frac{1}{6} = y_2.$$

Moreover, neither

$$\lim_{x \to \frac{1}{2} - 0} f_1'(x) = \lim_{x \to \frac{1}{2} - 0} [-2(\frac{1}{2} - x) \sin^2 \frac{\pi}{6(\frac{1}{2} - x)} + \frac{\pi}{6} \sin \frac{\pi}{3(\frac{1}{2} - x)}]$$

nor

$$\lim_{x\to\frac{1}{2}+0} f_2'(x) = \lim_{x\to\frac{1}{2}+0} [\frac{1}{3}(x-\frac{1}{2})\cos^2\frac{1}{x-\frac{1}{2}} + \frac{1}{6}\sin\frac{2}{x-\frac{1}{2}}]$$

exists but f_1' and f_2' are both bounded, where

$$f_1(x) = \begin{cases} \frac{1}{3}, & 0 < x \le \frac{1}{3}, \\ \frac{1}{3} + (\frac{1}{2}-x)^2\sin^2\frac{\pi}{6(\frac{1}{2}-x)}, & \frac{1}{3} < x < \frac{1}{2}, \end{cases} \quad f_2(x) = \frac{1}{6} + \frac{1}{6}(x-\frac{1}{2})^2\cos^2\frac{1}{x-\frac{1}{2}}.$$

It follows that $F_3 \in V_{jo*}(I, I)$. Note that $I_1 = (0, \frac{1}{2}), I_2 = (\frac{1}{2}, 1)$. It is easy to check that $c = \frac{1}{4}$, $\{y_1, y_2\} \subseteq I_1$, $f_1(y_1) = f_1(y_2) = f_1(c)$, $f_1'(y_1) = f_1'(y_2) = 0$, which implies that $F_3 \in \complement^1_{jo*}(I, I)$. Note that $F_3^{-1}(I_0) \cap I_p = \emptyset$ for $p = 1, 2$. It follows that assumption **(i)** in Theorem 3.2 is satisfied. On the other hand, one can compute

$$F_3^2(x) = \begin{cases} \frac{1}{3}, & 0 < x \le \frac{1}{3}, \\ \frac{1}{3} + [\frac{1}{6} - (\frac{1}{2}-x)^2\sin^2\frac{\pi}{6(\frac{1}{2}-x)}]^2\sin^2\frac{\pi}{6[\frac{1}{6}-(\frac{1}{2}-x)^2\sin^2\frac{\pi}{6(\frac{1}{2}-x)}]}, & \frac{1}{3} < x < \frac{1}{2}, \\ \frac{1}{3}, & \frac{1}{2} \le x < 1, \end{cases}$$

which is C^1 smooth on $(0, 1)$ as shown in Figure 8.

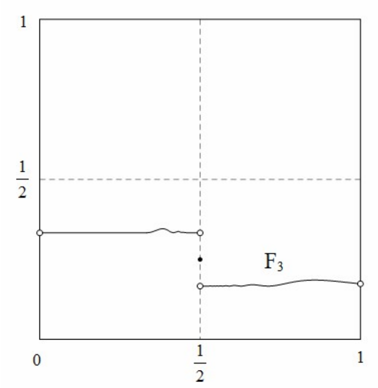

Figure 7. $F_3 \in \complement^1_{jo*}(I, I)$.

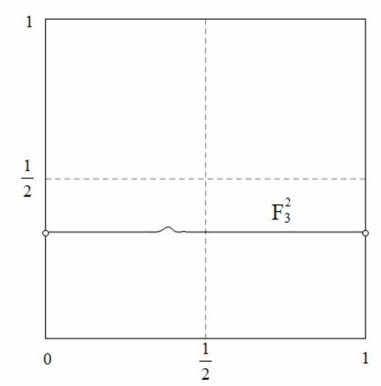

Figure 8. F_3^2 is C^1 on $(0, 1)$.

Example 4.4. Consider the mapping $F_4 : (0, 1) \to (0, 1)$ (see Figure 9) defined by

$$F_4(x) = \begin{cases} \frac{1}{2}x + \frac{3}{8}, & 0 < x < \frac{1}{2}, \\ \frac{5}{8}, & x = \frac{1}{2}, \\ \frac{1}{2}x + \frac{7}{16}, & \frac{1}{2} < x < 1, \end{cases}$$

which has a unique jumping discontinuity at $x_0 = \frac{1}{2}$ since

$$y_1 = \lim_{x\to\frac{1}{2}-0} F_4(x) = \frac{5}{8} \neq \lim_{x\to\frac{1}{2}+0} F_4(x) = \frac{11}{16} = y_2.$$

Moreover,

$$\tilde{y}_1 = \lim_{x\to\frac{1}{2}-0} f_1'(x) = \frac{1}{2} = \tilde{y}_2 = \lim_{x\to\frac{1}{2}+0} f_2'(x),$$

where $f_1(x) = \frac{1}{2}x+\frac{3}{8}$, $f_2(x) = \frac{1}{2}x+\frac{7}{16}$. It follows that $F_4 \in V_{jr}(I, I)$. Note that $I_1 = (0, \frac{1}{2})$, $I_2 = (\frac{1}{2}, 1)$. One can check that $\{y_1, y_2\} \subseteq I_2$, $c = \frac{5}{8} \in I_2$, $f_2(y_1) = \frac{3}{4} \neq f_2(y_2) = \frac{25}{32}$, which implies that $F_4 \notin \hat{\mathbb{C}}^{(2,1)}_{jr}(I, I)$, i.e., assumption **(i)** in Theorem 3.1 is not satisfied. Moreover, one sees that $\xi_1 = \frac{1}{4} \in F_4^{-1}(I_0) \cap I_1$ and $\xi_1 \in \Delta_{12}$. It follows that assumption **(ii)** in Theorem 3.1 is not satisfied. Actually, one can compute

$$F_4^2(x) = \begin{cases} \frac{1}{4}x + \frac{9}{16}, & 0 < x < \frac{1}{4}, \\ \frac{5}{8}, & x = \frac{1}{4}, \\ \frac{1}{4}x + \frac{5}{8}, & \frac{1}{4} < x < \frac{1}{2}, \\ \frac{3}{4}, & x = \frac{1}{2}, \\ \frac{1}{4}x + \frac{21}{32}, & \frac{1}{2} < x < 1, \end{cases}$$

which is not C^1 smooth on $(0, 1)$ with two nonsmooth points $\frac{1}{4}$ and $\frac{1}{2}$ as shown in Figure 10.

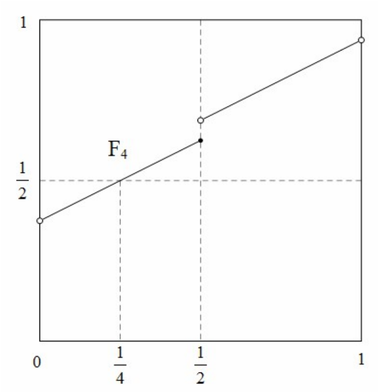

Figure 9. $F_4 \notin \hat{\mathbb{C}}^{(2,1)}_{jr}(I, I)$ and $\xi_1 \in \Delta_{12}$.

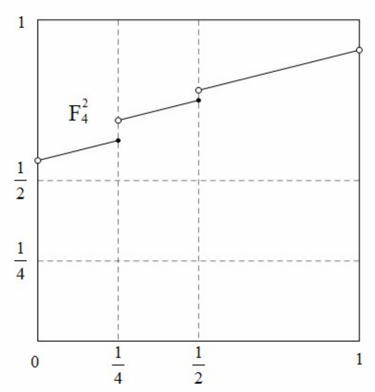

Figure 10. F_4^2 is not C^1 on $(0, 1)$.

Notice that we assumed that the mapping f is defined by (1.1) on an open interval $I = (0, 1)$. If we want to discuss on a closed interval $\bar{I} = [0, 1]$, we can turn to discuss the extension

$$\hat{f}(x) = \begin{cases} f(0), & x \in (-1, 0], \\ f(x), & x \in [0, 1], \\ f(1), & x \in [1, 2), \end{cases}$$

instead on the open interval $(-1, 2)$, where $f'_+(0) = f'_-(1) = 0$. Clearly, $\hat{f} \in V(\bar{I}, \bar{I})$.

5. Conclusions

Removable discontinuity and jumping discontinuity whose second order C^1 smoothness have be discussed in this paper, the other type of smoothness is oscillatory discontinuity, whose second order C^1 smoothness will be discussed in the next work.

Acknowledgments

This research was supported by Scientific Research Fund of Sichuan Provincial Education Department of China (No. 18ZA0242), Key project of Leshan Normal University of China (No. JG2018-1-03) and Key project of Leshan Normal University of China (No. LZD014).

Conflict of interest

The authors declare that there is no conflict of interest in this paper.

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