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Research article

Barycentric rational collocation method for semi-infinite domain problems

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Abstract: The barycentric rational collocation method for solving semi-infinite domain problems is presented. Following the barycentric interpolation method of rational polynomial and Chebyshev polynomial, matrix equation is obtained from discrete semi-infinite domain problem. Truncation method and transformation method are presented to solve linear and nonlinear differential equation defined on the semi-infinite domain problems. At last, three numerical examples are presented to valid our theoretical analysis.

Keywords: linear barycentric rational interpolation; collocation method; semi-infinite domain problem; truncation method; barycentric interpolation method
Mathematics Subject Classification: 65D32, 65D30, 65R20

1. Introduction

The differential equations of some problems are defined on infinite intervals of some engineering problems. Because of the infinite of calculation interval, how to calculate the upper boundary value problem of infinite interval becomes an important research subject for many numerical analysts.

Semi-infinite domain problems defined on $(0, \infty)$ as

$$F(x, u(x), u'(x), u''(x)) = 0, 0 < x < \infty, \tag{1.1}$$

with the boundary condition

$$u(0) = c_0, u(\infty) = u_\infty \tag{1.2}$$

is considered, where F if continues function c_0, c_∞ are constants and $u(\infty) = \lim_{x\to\infty} u(x)$.

It is easier to get the solution of differential equation under mathematical theory in infinite interval than in finite interval by Fourier transform and Laplace transform. Numerical method cannot directly solve the differential equation problem in infinite interval. In order to solve the differential equations in infinite interval, we need to develop new methods such as truncation method and transformation method. In the paper [1], strictly monotonic transformation is transform the $[0, \infty)$ into $[-1, 1)$, two-point boundary value problem is solved by Chebyshev-Gauss collocation. In the paper [2], the method of weighted residuals is used to solve some problems involving boundary condition at infinity. In the paper [3], an original Petrov-Galerkin formulation of the Falkner-Skan equation is presented which is based on a judiciously chosen special basis function to capture the asymptotic behavior of the unknown. In the paper [4], based on the combination of Laplace transformation method and weighted residual method, an numerical method for the approximate solution of problems involving boundary condition at infinity is presented. Schrdinger-boussinesq system [5], nonlinear fractional $K(m, n)$ type equation [6], nanotechnology and fractional [7, 8], hirota-maccari system [9] and eneralized calogero-bogoyavlenskii-schiff equation [10] are studied. Generalized ϕ-convex functions [11], fractional integral operator [12], fractional-calculus theory [13], fractional inequalities [14], non-singular fractional integral operator [15] and differentiability in fractional calculus [16] are studied by Rashid and so on. In references [17, 18], infinite cell method, the method of reconstructed kernels, Howarth's numerical solution and Runge-Kutta Fehlberg method have been used to numerically solve semi-infinite problems, respectively.

Barycentrix interpolation collocation [19–23] have been developed to avoid the Runge phenomenon which is a meshfree method [24–26] to find approximate solutions to partial differential equations without integration. Meshless approach for the numerical solution of the nonlinear equal width equation, sine-Gordon system and sinh-Gordon equation were presented in [27–29], soliton wave solutions of nonlinear mathematical models, nonlinear sine-Gordon model and generalized Rosenau-KdV-RLW Equation were presented in [30–32]. In the recent paper, heat conduction equation [33], integral-differential equation [34], differential equation [35] and biharmonic equation [36] have been solved by linear barycentrix rational collocation methods. In the paper [37–39], barycentric interpolation collocation method for nonlinear problems, incompressible plane elastic problems and plane elastic problems and so on are presented.

In this paper, we first consider the boundary value problem of linear differential equation on infinite interval for certain strictly monotone differentiable functions. By transformation of algebraic and Logarithmic, the infinite interval $[0, \infty)$ is transformed to finite interval $[-1, 1)$, then the linear barycentric rational collocation methods (LBRCM) of finite interval problem is illustrated. We also give the truncation method, which is to cut the infinite interval into a finite interval solution. Thirdly, the LBRCM is extended to nonlinear problem by the linearized iterative collocation method for solving nonlinear boundary value problems on infinite interval.

This paper is organized as following: In Section 2, linear boundary problems transform from semi-infinite domain into $[-1, 1)$. In Section 3, the convergence and error analysis of LBRCM is proved. At last, three numerical examples are listed to illustrated our theorem.

2. Linear boundary problems

In order to compute the semi-infinite domain problems easily, we give the transform as

$$x = \phi(t), t \in [-1, 1), x \in (0, \infty) \tag{2.1}$$

where $\phi(t)$ is the strictly monotone differentiable functions, then we have $u(x) = u(\phi(t)) = v(t)$. As we have

$$\frac{dx}{dt} = \phi'(t), \frac{dt}{dx} = \frac{1}{\phi'(t)},$$

by the rule of derivation, we have

$$\frac{du}{dx} = \frac{1}{\phi'(t)}\frac{dv}{dt}, \tag{2.2}$$

$$\frac{d^2u}{dx^2} = \frac{1}{\phi'^2(t)}\frac{d^2v}{dt^2} - \frac{\phi''(t)}{\phi'^3(t)}\frac{dv}{dt}, \tag{2.3}$$

$$\frac{d^3u}{dx^3} = \frac{1}{\phi'^3(t)}\frac{d^3v}{dt^3} - \frac{3\phi''^2(t)}{\phi'^4(t)}\frac{d^2v}{dt^2} - \frac{\phi''(t)\phi'(t) - 3\phi''^2(t)}{\phi'^5(t)}\frac{dv}{dt}. \tag{2.4}$$

Then we transform the $(0, \infty)$ into $(-1, 1)$ as the boundary value problems

$$F(\phi(t), v(t), \frac{1}{\phi'(t)}\frac{dv}{dt}, \frac{1}{\phi'^2(t)}\frac{d^2v}{dt^2} - \frac{\phi''(t)}{\phi'^3(t)}\frac{dv}{dt}) = 0, -1 \leq t \leq 1 \tag{2.5}$$

and

$$v(-1) = c_0, v(1) = c_\infty.$$

In the following, the interval $[-1, 1]$ can be partitioned $t_0 = -1, t_1, \ldots, t_n = 1$ and its semi-infinite domain c_0, c_∞ as $x_0 = -1, x_1, \ldots, x_n = \infty$ with

$$x_k = \phi(t_k), k = 0, 1, \ldots, n$$

and

$$u(x_k) = v(t_k), -1 \leq t \leq 1,$$

then we get the value at the mesh-point

$$u'(x_k) = \frac{1}{\phi'(t_k)}v'(t_k), \tag{2.6}$$

$$u''(x_k) = \frac{1}{\phi'^2(t_k)}v''(t_k) - \frac{\phi''(t)}{\phi'^3(t)}v'(t_k), \tag{2.7}$$

and

$$u'''(x_k) = \frac{1}{\phi'^3(t)}v'''(t_k) - \frac{3\phi''^2(t)}{\phi'^4(t)}v''(t_k) - \frac{\phi''(t)\phi'(t) - 3\phi''^2(t)}{\phi'^5(t)}v'(t_k) \tag{2.8}$$

with the help of vector form

$$\mathbf{u}^{(1)} = \operatorname{diag}\left(\frac{1}{\phi'(t_k)}\right)\mathbf{v}^{(1)}, \tag{2.9}$$

$$\mathbf{u}^{(2)} = \operatorname{diag}\left(\frac{1}{\phi'^2(t_k)}\right)\mathbf{v}^{(2)} - \operatorname{diag}\left(\frac{\phi''(t)}{\phi'^3(t)}\right)\mathbf{v}^{(1)}, \tag{2.10}$$

$$\mathbf{u}^{(3)} = \operatorname{diag}\left(\frac{1}{\phi'^3(t)}\right)\mathbf{v}^{(3)} - \operatorname{diag}\left(\frac{3\phi''^2(t)}{\phi'^4(t)}\right)\mathbf{v}^{(2)} - \operatorname{diag}\left(\frac{\phi''(t)\phi'(t) - 3\phi''^2(t)}{\phi'^5(t)}\right)\mathbf{v}^{(1)}, \tag{2.11}$$

where

$$\mathbf{u} = [u(x_0), u(x_1), \ldots, u(x_n)]^{\mathrm{T}}, \mathbf{v} = [v(t_0), v(t_1), \ldots, v(t_n)]^{\mathrm{T}},$$

$$\mathbf{u}^{(1)} = [u'(x_0), u'(x_1), \ldots, u'(x_n)]^{\mathrm{T}}, \mathbf{v}^{(1)} = [v'(t_0), v'(t_1), \ldots, v'(t_n)]^{\mathrm{T}},$$

$$\mathbf{u}^{(2)} = [u''(x_0), u''(x_1), \ldots, u''(x_n)]^{\mathrm{T}}, \mathbf{v}^{(2)} = [v''(t_0), v''(t_1), \ldots, v''(t_n)]^{\mathrm{T}},$$

$$\mathbf{u}^{(3)} = [u'''(x_0), u'''(x_1), \ldots, u'''(x_n)]^{\mathrm{T}}, \mathbf{t} = [t_0, t_1, \ldots, t_n]^{\mathrm{T}}.$$

By the relationship of barycentric matrix at the meshpoint $t_0, t_1, \ldots, t_n$, we have

$$\mathbf{v}^{(1)} = \mathbf{D}^{(1)}\mathbf{v}, \mathbf{v}^{(2)} = \mathbf{D}^{(2)}\mathbf{v}, \mathbf{v}^{(3)} = \mathbf{D}^{(3)}\mathbf{v}, \tag{2.12}$$

where $\mathbf{D}^{(m)} = D_{ij}^{(m)} = R_j^{(m)}(x_i)$ is the element of the differentiation matrices and $R_j(x) = \dfrac{\frac{w_j}{x - x_j}}{\sum_{k=0}^{n} \frac{w_k}{x - x_k}}$ is basis function, see reference [37]. For $m = 2$, we have

$$R_j''(x_i) = -2\frac{w_j/w_i}{x_i - x_j}\left(\sum_{k \neq i} \frac{w_k/w_i}{x_i - x_k} + \frac{1}{x_i - x_j}\right), \quad j \neq i, \tag{2.13}$$

where

$$w_k = \sum_{i \in J_k} (-1)^i \prod_{j=i, j\neq k}^{i+d} \frac{1}{x_k - x_j}$$

is the weight function with $J_k = \{i \in I : k - d \leq i \leq k\}, 0 \leq d \leq n$ and

$$R_i''(x_i) = -\sum_{j \neq i} R_j''(x_i). \tag{2.14}$$

Then we get the differentiable matrices as

$$D_{ij}^{(1)} = R_j'(x_i), \quad D_{ij}^{(2)} = R_j''(x_i), \quad D_{ij}^{(3)} = R_j^{(3)}(x_i). \tag{2.15}$$

Combining the Eqs (2.10) and (2.12), we get

$$\mathbf{u}^{(1)} = \mathrm{diag}\left(\frac{1}{\phi'(t)}\right)\mathbf{D}^{(1)}\mathbf{v}, \tag{2.16}$$

$$\mathbf{u}^{(2)} = \left[\mathrm{diag}\left(\frac{1}{\phi'^2(t)}\right)\mathbf{D}^{(2)} - \mathrm{diag}\left(\frac{\phi''(t)}{\phi'^3(t)}\right)\mathbf{D}^{(1)}\right]\mathbf{v}, \tag{2.17}$$

$$\begin{aligned} \mathbf{u}^{(3)} &= \left[\mathrm{diag}\left(\frac{1}{\phi'^3(t)}\right)\mathbf{D}^{(3)} - \mathrm{diag}\left(\frac{3\phi''^2(t)}{\phi'^4(t_k)}\right)\mathbf{D}^{(2)}\right]\mathbf{v} \\ &- \mathrm{diag}\left(\frac{\phi''(t)\phi'(t) - 3\phi''^2(t)}{\phi'^5(t)}\right)\mathbf{D}^{(1)}\mathbf{v}. \end{aligned} \tag{2.18}$$

Taking following linear boundary value differential problems as example,

$$u''(x) + p(x)u'(x) + q(x)u(x) = f(x), 0 < x < \infty, \tag{2.19}$$

by the transformation of (2.1) and (2.2), we have

$$\frac{1}{\phi'^{(2)}}v''(t) - \frac{\phi''(t)}{\phi'^3(t)}v'(t) + \frac{p(\phi(t))}{\phi'(t)}v'(t) + q(\phi(t))u(\phi(t)) = f(\phi(t)), -1 \le t \le 1. \tag{2.20}$$

Taking the meshpoint $x_0, x_1, \ldots, x_n$ in the Eq (2.12), we have

$$u''(x_k) + p(x_k)u'(x_k) + q(x_k)u(x_k) = f(x_k), k = 0, 1, 2, \ldots, N, \tag{2.21}$$

and its matrix form can be written as

$$\mathbf{u}^{(2)} + \mathrm{diag}(p(x_k))\mathbf{u}^{(1)} + \mathrm{diag}(q(x_k))\mathbf{u} = f(\mathbf{x}), \tag{2.22}$$

Combining Eqs (2.16)–(2.18) and (2.22), we have

$$\begin{aligned} &\mathrm{diag}\left(\frac{1}{\phi'^{(2)}}\right)\mathbf{D}^{(2)}\mathbf{v} - \mathrm{diag}\left(\frac{\phi''(t)}{\phi'^3(t)}\right)\mathbf{D}^{(1)}\mathbf{v} + \mathrm{diag}\left(\frac{p(\phi(t))}{\phi'(t)}\right)\mathbf{D}^{(1)}\mathbf{v} \\ + \ &\mathrm{diag}\left(q(\phi(t))\right)\mathbf{v} = \mathbf{f}(\phi(t)), \end{aligned} \tag{2.23}$$

where $\mathbf{D}^{(2)}$ and $\mathbf{D}^{(1)}$ are the barycentrix matrix, so we can not need to get the differential equation.

In the actual calculation, we take the algebraic transformation as

$$x = L\frac{1+t}{1-t} \tag{2.24}$$

and Logarithmic transformation

$$x = -L\ln\frac{1-t}{2} = L\ln\frac{2}{1-t}, \tag{2.25}$$

where L is the positive constant called as amplification factor which determine the meshpoint of the semi-infinite domain.

3. Convergence and error analysis

In order to complete the proof of convergence rate, some lemmas are given as below. Firstly, we define the error function

$$e(x) = u(x) - r_n(x) = (x - x_i) \dots (x - x_{i+d})\, u[x_i, x_{i+1}, \dots, x_{i+d}; x] \tag{3.1}$$

and

$$e(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x)\,[u(x) - r_n(x)]}{\sum_{i=0}^{n-d} \lambda_i(x)} = \frac{A(x)}{B(x)} = O(h^{d+1}), \tag{3.2}$$

where $A(x) := \sum_{i=0}^{n-d} (-1)^i u\,[x_i, \dots, x_{i+d}; x]\,, B(x) := \sum_{i=0}^{n-d} \lambda_i(x)$, and

$$\lambda_i(x) = \frac{(-1)^i}{(x - x_i) \cdots (x - x_{i+d})}. \tag{3.3}$$

Taking the numerical scheme

$$\sum_{j=0}^{n} u_j R_j''(x) + p \sum_{j=0}^{n} u_j R_j'(x) + q \sum_{j=0}^{n} u_j R_j(x) = f(x). \tag{3.4}$$

Combining (3.4) and (2.19), we have

$$\mathcal{L}e(x) := e''(x) + pe'(x) + qe(x) - R_f(x), \tag{3.5}$$

where $R_f(x) = f(x) - f(x_k), k = 0, 1, 2, \dots, n.$

Lemma have been proved by Jean-Paul Berrut.

Lemma 1. *(see reference [19]) For $e(x)$ defined as (3.1), there holds*

$$\left|e^{(k)}(x)\right| \le Ch^{d+1-k}, u \in C^{d+k+2}[a, b], k = 0, 1, \dots. \tag{3.6}$$

Let $u(x)$ to be the solution of (2.19) and $u_n(x)$ is the numerical solution, then we have

$$\mathcal{L}u_n(x_k) = f(x_k), k = 0, 1, 2, \dots, n,$$

and

$$\lim_{n\to\infty} u_n(x) = u(x).$$

Based on the above lemma, we get the following theorem.

Theorem 1. *Let $u_n(x) : \mathcal{L}u_n(x_k) = f(x_k), f(x) \in C[a, b]$ and suppose $\mathcal{L}$ be the invertible operator, we have*

$$|u_n(x) - u(x)| \le Ch^{d-1}.$$

Proof. As

$$u_n(x) = \sum_{j=0}^{n} R_j(x)u_j.$$

Combining the Lemma 1 and Eq (3.5), we have

$$\begin{aligned} |\mathcal{L}e(x)| &= |e''(x) + qe'(x) + qe(x) - R_f(x)| \\ &\leq |e''(x)| + |qe'(x)| + |qe(x)| + |R_f(x)| \\ &\leq Ch^{d-1} + Ch^{d} + Ch^{d+1} \\ &\leq Ch^{d-1}. \end{aligned} \tag{3.7}$$

As $\mathcal{L}$ is invertible operator. Then we have

$$|u_n(x) - u(x)| \leq Ch^{d-1}.$$

The proof is completed.

4. Numerical examples

Three examples are presented to valid our theorem. All the examples were performed on personal computer by Matlab r2013a with a (Confguration: Intel(R) Core(TM) i5-8265U CPU @ 1.60GHz 1.80 GHz).

Example 1. *Consider the boundary value problems*

$$u'' + 2u' - 2u = -e^{2x}, 0 < x < \infty, \tag{4.1}$$

$$u(0) = 1, u(\infty) = 0 \tag{4.2}$$

with analysis solution

$$u(x) = \frac{1}{2}(e^{-(1+\sqrt{3})x} + e^{-2x}).$$

In Table 1, CPU running times of algebraic transformation with equidistant nodes $S = 5$ of linear barycentric rational collocation methods are presented. From Table 1, we know that the running times is less than 3 second.

Table 1. CPU running times of algebraic transformation with equidistant nodes $S = 5$.

n	$d = 2$	$d = 3$	$d = 4$	$d = 5$
10	8.1940e-03	4.7143e-03	2.0530e-03	6.5745e-04
20	2.9926e-03	8.2744e-04	1.5364e-04	3.4327e-06
40	5.8635e-04	8.0910e-05	6.7069e-06	1.5838e-07
80	9.1053e-05	6.3050e-06	2.4638e-07	4.8568e-09
160	1.2704e-05	4.4188e-07	8.4039e-09	9.7578e-11
320	1.6815e-06	2.9356e-08	2.7601e-10	1.6812e-12
time(second)	2.277	3.116	2.260	2.283

In Tables 2 and 3, the convergence of algebraic transformation with equidistant nodes and quasi-equidistant nodes $S = 5$ of linear barycentric rational collocation methods are presented. In Table 2, with $S = 5, d = 2, 3, 4, 5$, errors of equidistant nodes are $O(h^{d+1})$. In Table 3, with $S = 5, d = 2, 3, 4, 5$, errors of quasi-equidistant nodes are $O(h^{d+2})$.

Table 2. Errors of algebraic transformation with equidistant nodes $S = 5$.

n	$d = 2$		$d = 3$		$d = 4$		$d = 5$	
10	8.1940e-03		4.7143e-03		2.0530e-03		6.5745e-04	
20	2.9926e-03	1.4532	8.2744e-04	2.5103	1.5364e-04	3.7401	3.4327e-06	7.5814
40	5.8635e-04	2.3516	8.0910e-05	3.3543	6.7069e-06	4.5178	1.5838e-07	4.4379
80	9.1053e-05	2.6870	6.3050e-06	3.6817	2.4638e-07	4.7667	4.8568e-09	5.0272
160	1.2704e-05	2.8414	4.4188e-07	3.8348	8.4039e-09	4.8737	9.7578e-11	5.6373
320	1.6815e-06	2.9175	2.9356e-08	3.9119	2.7601e-10	4.9283	1.6812e-12	5.8590

Table 3. Errors of algebraic transformation with quasi-equidistant nodes $S = 5$.

n	$d = 2$		$d = 3$		$d = 4$		$d = 5$	
10	2.6753e-02		1.2130e-02		3.6934e-03		7.1962e-04	
20	2.2977e-03	3.5414	5.7573e-04	4.3971	9.2680e-05	5.3166	2.9350e-06	7.9377
40	1.7519e-04	3.7132	2.1363e-05	4.7522	1.6545e-06	5.8078	1.3132e-08	7.8042
80	1.2976e-05	3.7551	7.5147e-07	4.8293	2.8327e-08	5.8681	1.0991e-10	6.9005
160	9.3708e-07	3.7915	2.5684e-08	4.8708	4.7511e-10	5.8978	2.1025e-11	2.3862
320	6.6037e-08	3.8268	8.6168e-10	4.8976	5.7275e-11	3.0523	8.8952e-10	-

In Tables 4 and 5, errors of equidistant nodes and quasi-equidistant nodes log transformation $d = 3$ of linear barycentric rational collocation methods are presented. In Table 4, errors of equidistant nodes with $d = 3, S = 5, 15, 25, 40, 50, 60$ are $O(h^{d+1})$. In Table 5, errors of quasi-equidistant nodes with $d = 3, S = 5, 15, 25, 40, 50, 60$ are $O(h^{d+1})$.

Table 4. Errors of equidistant nodes with log transformation $d = 3$.

n	$S = 5$	$S = 15$	$S = 25$	$S = 40$	$S = 50$	$S = 60$
10	4.3819e-05	5.0803e-04	2.1984e-03	4.9277e-03	7.8668e-03	1.0289e-02
20	5.4907e-06	3.9227e-05	2.3810e-04	7.2881e-04	1.5591e-03	2.7012e-03
40	7.5950e-07	2.7109e-06	1.9208e-05	6.7990e-05	1.6734e-04	3.3250e-04
80	1.0979e-07	1.7834e-07	1.3619e-06	5.1736e-06	1.3642e-05	2.9016e-05
160	1.6204e-08	1.1450e-08	9.0754e-08	3.5710e-07	9.7447e-07	2.1444e-06
320	2.4156e-09	7.2589e-10	5.8623e-09	2.3479e-08	6.5185e-08	1.4592e-07

Table 5. Errors of quasi-equidistant nodes with log transformation $d = 3$.

n	$S = 5$	$S = 15$	$S = 25$	$S = 40$	$S = 50$	$S = 60$
10	1.9110e-05	3.0625e-04	1.9235e-03	6.5161e-03	1.5280e-02	2.8619e-02
20	1.2327e-06	7.3967e-06	5.3852e-05	2.1490e-04	5.9851e-04	1.3533e-03
40	5.5047e-08	1.7798e-07	1.4130e-06	5.9639e-06	1.7865e-05	4.2421e-05
80	2.5152e-09	4.6691e-09	3.9511e-08	1.7260e-07	5.2290e-07	1.2650e-06
160	1.1602e-10	1.3339e-10	1.1640e-09	5.1541e-09	1.5723e-08	3.8136e-08
320	5.7945e-12	6.0677e-12	3.5670e-11	1.5699e-10	4.8081e-10	1.1682e-09

In Tables 6 and 7, errors of truncation method with equidistant nodes and quasi-equidistant nodes $d = 3$ of linear barycentric rational collocation methods are presented. In Table 6, errors of equidistant nodes with $d = 3, S = 5, 15, 25, 40, 50, 60$ are $O(h^d)$. In Table 7, errors of quasi-equidistant nodes with $d = 3, S = 5, 15, 25, 40, 50, 60$ are $O(h^d)$.

Table 6. Errors of truncation method with equidistant nodes $d = 3$.

n	$S = 5$	$S = 15$	$S = 25$	$S = 40$	$S = 50$	$S = 60$
10	2.1223e-02	5.6913e-02	3.9768e-02	2.1094e-02	2.0192e-02	1.9600e-02
20	3.5188e-03	3.4810e-02	4.9520e-02	4.1346e-02	3.2513e-02	2.5230e-02
40	3.8273e-04	8.9325e-03	2.4157e-02	4.0168e-02	4.3242e-02	4.2114e-02
80	3.5450e-05	1.3067e-03	5.2897e-03	1.5028e-02	2.2010e-02	2.8100e-02
160	2.3284e-05	1.4200e-04	7.0907e-04	2.7523e-03	4.9296e-03	7.6340e-03
320	2.3284e-05	1.3380e-05	7.3950e-05	3.3526e-04	6.6710e-04	1.1485e-03

Table 7. Errors of truncation method with quasi-equidistant nodes $d = 3$.

n	$S = 5$	$S = 15$	$S = 25$	$S = 40$	$S = 50$	$S = 60$
10	2.7030e-02	3.0280e-01	4.2518e-01	3.6833e-01	3.0452e-01	2.5081e-01
20	1.8985e-03	8.9395e-02	3.5376e-01	8.9282e-01	1.1958e+00	1.4081e+00
40	1.0692e-04	7.2929e-03	4.3260e-02	1.8903e-01	3.5525e-01	5.7148e-01
80	2.3284e-05	4.4342e-04	2.9492e-03	1.5790e-02	3.3935e-02	6.2143e-02
160	2.3284e-05	2.6249e-05	1.7739e-04	9.9627e-04	2.2364e-03	4.3010e-03
320	2.3284e-05	1.5817e-06	1.0630e-05	5.9696e-05	1.3464e-04	2.6112e-04

Example 2. *Consider the boundary value problems*

$$8u'' + 2u' - u = 8e^{-\frac{1}{4}x}, 0 < x < \infty, \tag{4.3}$$

$$u(0) = 1, u(\infty) = 0 \tag{4.4}$$

and its analysis solution is

$$u(x) = -3e^{-\frac{1}{2}x} + 4e^{-\frac{3}{4}x}.$$

In Tables 8 and 9, errors of log transformation with equidistant nodes and quasi-equidistant nodes $S = 8$ of linear barycentric rational collocation methods are presented. In Table 8, errors of log transformation with equidistant nodes $S = 8, d = 2, 3, 4, 5$ are $O(h^{d+1})$. In Table 9, errors of log transformation with quasi-equidistant nodes $S = 8, d = 2, 3, 4, 5$ are $O(h^{d+2})$.

Table 8. Errors of log transformation with equidistant nodes $S = 8$.

n	$d = 2$		$d = 3$		$d = 4$		$d = 5$	
10	1.7920e-02		6.1866e-03		8.0099e-04		2.9488e-04	
20	2.6669e-03	2.7483	4.3639e-04	3.8255	3.2205e-05	4.6364	4.4835e-06	6.0394
40	3.5692e-04	2.9015	2.8675e-05	3.9277	1.1054e-06	4.8647	6.8605e-08	6.0302
80	4.5963e-05	2.9570	1.8342e-06	3.9666	3.6022e-08	4.9395	1.0604e-09	6.0156
160	5.8252e-06	2.9801	1.1594e-07	3.9837	1.1488e-09	4.9707	1.6497e-11	6.0062
320	7.3298e-07	2.9905	7.2873e-09	3.9919	3.6425e-11	4.9790	1.6509e-13	6.6428

Table 9. Errors of log transformation with quasi-equidistant nodes $S = 8$.

n	$d = 2$		$d = 3$		$d = 4$		$d = 5$	
10	1.2884e-02		3.1184e-03		3.7630e-04		5.0897e-05	
20	8.9366e-04	3.8497	6.6520e-05	5.5509	4.2951e-06	6.4530	3.8046e-07	7.0637
40	5.1689e-05	4.1118	1.4461e-06	5.5236	4.7299e-08	6.5048	2.1039e-09	7.4986
80	3.0710e-06	4.0731	3.4891e-08	5.3731	6.1916e-10	6.2553	1.9563e-11	6.7488
160	1.8483e-07	4.0545	9.4700e-10	5.2033	1.3292e-11	5.5417	3.9667e-11	-
320	1.1294e-08	4.0326	2.7845e-11	5.0879	9.5279e-11	-	1.9074e-09	-

In Tables 10 and 11, errors of equidistant nodes and quasi-equidistant nodes log transformation $d = 3$ of linear barycentric rational collocation methods are presented. In Table 10, errors of equidistant nodes log transformation with $d = 3, S = 5, 15, 25, 40, 50, 60$ are $O(h^{d+1})$. In Table 11, errors of quasi-equidistant nodes log transformation with $d = 3, S = 5, 15, 25, 40, 50, 60$ are $O(h^{d+1})$.

Table 10. Errors of equidistant nodes log transformation with $d = 3$.

n	$S = 5$	$S = 15$	$S = 25$	$S = 40$	$S = 50$	$S = 60$
10	1.2754e-01	4.7239e-03	3.3028e-03	1.1102e-15	9.6949e-04	2.0177e-03
20	1.0096e-01	1.5512e-03	1.0035e-03	1.5543e-15	6.7291e-05	1.2162e-04
40	7.7020e-02	5.2958e-04	3.2856e-04	9.7700e-15	7.1866e-06	7.4030e-06
80	5.7531e-02	1.8400e-04	1.1148e-04	1.4821e-14	1.1037e-06	4.5563e-07
160	4.2389e-02	6.4485e-05	3.8532e-05	1.2784e-13	1.8340e-07	2.8240e-08
320	3.0936e-02	2.2697e-05	1.3453e-05	5.0154e-13	3.1563e-08	1.7571e-09

Table 11. Errors of quasi-equidistant nodes log transformation with $d = 3$.

n	$S = 5$	$S = 15$	$S = 25$	$S = 40$	$S = 50$	$S = 60$
10	1.1024e-01	4.3445e-03	2.6631e-03	3.3307e-16	5.8846e-04	9.6881e-04
20	1.6316e-01	1.1904e-03	6.5698e-04	2.9421e-15	2.3458e-05	1.9189e-05
40	1.8549e-01	3.1977e-04	1.5566e-04	8.4377e-15	1.0472e-06	3.9498e-07
80	1.4828e-01	6.4136e-05	3.0793e-05	1.6739e-13	5.4456e-08	9.3150e-09
160	1.1057e-01	1.2704e-05	6.3129e-06	1.0436e-12	3.1336e-09	2.4767e-10
320	8.4925e-02	2.6665e-06	1.3904e-06	4.1613e-12	1.8558e-10	7.7586e-12

We consider the nonlinear boundary problems on semi-infinite domain with the Logarithmic transformation

$$x = -L \ln \frac{1-t}{2} = L \ln \frac{2}{1-t}, \tag{4.5}$$

where L is the positive constant called as amplification factor

$$\mathbf{u}^{(1)} = \mathrm{diag}\left(\frac{1-t}{L}\right)\mathbf{D}^{(1)}\mathbf{v}, \tag{4.6}$$

$$\mathbf{u}^{(2)} = \left[\mathrm{diag}\left(\frac{(1-t)^2}{L^2}\right)\mathbf{D}^{(2)} - \mathrm{diag}\left(\frac{1-t}{L^3}\right)\mathbf{D}^{(1)}\right]\mathbf{v}, \tag{4.7}$$

$$\mathbf{u}^{(3)} = \left[\mathrm{diag}\left(\frac{(1-t)^3}{L^3}\right)\mathbf{D}^{(3)} - \mathrm{diag}\left(\frac{3(1-t)^2}{L^3}\right)\mathbf{D}^{(2)} + \mathrm{diag}\left(\frac{1-t}{L^3}\right)\mathbf{D}^{(1)}\right]\mathbf{v}. \tag{4.8}$$

Taking the notation as below,

$$\begin{aligned} \mathbf{C}^{(1)} &= \mathrm{diag}\left(\frac{1-t}{L}\right)\mathbf{D}^{(1)}, \\ \mathbf{C}^{(2)} &= \mathrm{diag}\left(\frac{(1-t)^2}{L^2}\right)\mathbf{D}^{(2)} - \mathrm{diag}\left(\frac{1-t}{L^3}\right)\mathbf{D}^{(1)}, \\ \mathbf{C}^{(3)} &= \mathrm{diag}\left(\frac{(1-t)^3}{L^3}\right)\mathbf{D}^{(3)} - \mathrm{diag}\left(\frac{3(1-t)^2}{L^3}\right)\mathbf{D}^{(2)} + \mathrm{diag}\left(\frac{1-t}{L^3}\right)\mathbf{D}^{(1)}, \end{aligned} \tag{4.9}$$

we have

$$\mathbf{u}^{(1)} = \mathbf{C}^{(1)}\mathbf{v}, \quad \mathbf{u}^{(2)} = \mathbf{C}^{(2)}\mathbf{v}, \quad \mathbf{u}^{(3)} = \mathbf{C}^{(3)}\mathbf{v}. \tag{4.10}$$

Example 3. *Consider the nonlinear boundary value problems*

$$f^{(3)} + ff^{(2)} - \beta f^{(1)} - (1+\alpha)(f')^2 = 0, 0 < x < \infty, \tag{4.11}$$

$$f(0) = 0, f^{(1)}(0) = 0, f(\infty) = 0 \tag{4.12}$$

and its analysis solution is

$$f(\eta) = \frac{1}{\sqrt{1+\beta}}(1 - e^{-\eta\sqrt{1+\beta}}).$$

For the know function $f_0(\eta)$, (4.11) can be linearized as

$$f''' + f_0 f'' - \beta f' - (1+\alpha) f_0' f' = 0, 0 < x < \infty, \tag{4.13}$$

then we get the linearized scheme as

$$f_n^{(3)} + f_{n-1} f_n^{(2)} - \beta f_n^{(1)} - (1+\alpha)(f_{n-1}') f_n' = 0, 0 \le \eta \le \infty, \tag{4.14}$$

$$f_n(0) = 0, f_n^{(1)}(0) = 0, f_n(\infty) = 0. \tag{4.15}$$

We take the transformation as

$$\eta = -L \ln \frac{1-t}{2}.$$

Then we get the calculation of barycentrix rational interpolation formulae as

$$\left[\mathbf{C}^{(3)} + \mathrm{diag}(\mathbf{v}_{n-1})\mathbf{C}^{(2)} - \beta\mathbf{C}^{(1)} - (1+\alpha)\mathrm{diag}(\mathbf{C}^{(1)}\mathbf{v}_{n-1})\mathbf{C}^{(1)}\right]\mathbf{v}_n = 0, \tag{4.16}$$

$$e_1^{\mathbf{T}}\mathbf{v}_n = 0, (d_1^{(1)})^{\mathbf{T}}\mathbf{v}_n = \frac{L}{2}, (d_N^{(1)})^{\mathbf{T}}\mathbf{v}_n = 0. \tag{4.17}$$

In Tables 12 and 13, errors of log transformation with equidistant nodes and quasi-equidistant nodes $d = 4$ of linear barycentric rational collocation methods are presented. In Table 12, errors of log transformation with equidistant nodes $d = 4, S = 5, 15, 25, 40, 50, 60$ are $O(h^{d+1})$. In Table 13, errors of log transformation with quasi-equidistant nodes $d = 4, S = 5, 15, 25, 40, 50, 60$ are $O(h^{d+1})$.

Table 12. Errors of log transformation with equidistant nodes $d = 4$.

n	$S = 5$	$S = 15$	$S = 25$	$S = 40$	$S = 50$	$S = 60$
10	1.1417e-01	1.1643e-03	3.8685e-04	2.6218e-03	1.4407e-02	3.3166e-02
20	8.6017e-02	3.4701e-04	5.7745e-05	2.8848e-04	1.5798e-03	3.9883e-03
40	6.4617e-02	9.8838e-05	7.5309e-06	2.7478e-05	1.5133e-04	4.0728e-04
80	4.8719e-02	2.7898e-05	9.3159e-07	2.5008e-06	1.3783e-05	3.8393e-05
160	3.6883e-02	7.8585e-06	1.1180e-07	2.2369e-07	1.2320e-06	3.4918e-06
320	2.7996e-02	2.2227e-06	1.0890e-08	2.2044e-08	1.0968e-07	3.1330e-07

Table 13. Errors of log transformation with quasi-equidistant nodes $d = 4$.

n	$S = 5$	$S = 15$	$S = 25$	$S = 40$	$S = 50$	$S = 60$
10	1.5523e-01	5.4277e-04	9.8075e-05	5.4433e-04	3.0401e-03	7.5620e-03
20	1.6951e-01	6.0893e-05	1.8383e-06	4.4380e-06	2.4863e-05	7.1834e-05
40	3.6280e-01	2.0078e-05	9.6699e-08	1.1358e-07	6.0548e-07	1.7855e-06
80	3.3942e-01	1.4406e-05	1.8173e-07	4.4643e-08	1.2382e-08	4.1055e-08
160	7.0711e-01	7.0711e-01	7.0840e-01	7.1450e-01	7.2004e-01	7.2419e-01
320	7.5314e-01	7.5368e-01	7.0729e-01	9.7077e-01	8.2649e-01	7.4239e-01

In Tables 14 and 15, errors of truncation method with equidistant nodes and quasi-equidistant nodes truncation method $d = 4$ of linear barycentric rational collocation methods are presented. In Table 14, errors of truncation method with equidistant nodes $d = 4, S = 5, 15, 25, 40, 50, 60$ are $O(h^d)$. In Table 15, errors of truncation method with quasi-equidistant nodes $d = 4, S = 5, 15, 25, 40, 50, 60$ are $O(h^d)$.

Table 14. Errors of truncation method with equidistant nodes $d = 4$.

n	$S = 5$	$S = 15$	$S = 25$	$S = 40$	$S = 50$	$S = 60$
10	4.0047e-02	4.2878e-01	9.6623e-01	1.8391e+00	2.4362e+00	3.0389e+00
20	5.6161e-03	1.0508e-01	2.9263e-01	6.4704e-01	9.0921e-01	1.1798e+00
40	5.8518e-04	1.8933e-02	6.8111e-02	1.8598e-01	2.8381e-01	3.9150e-01
80	1.0416e-03	2.5582e-03	1.1535e-02	3.9660e-02	6.7434e-02	1.0125e-01
160	1.1000e-03	2.8311e-04	1.4849e-03	6.1589e-03	1.1596e-02	1.9000e-02
320	1.1055e-03	2.8011e-05	1.5973e-04	7.4542e-04	1.5086e-03	2.6428e-03

Table 15. Errors of truncation method with quasi-equidistant nodes $d = 4$.

n	$S = 5$	$S = 15$	$S = 25$	$S = 40$	$S = 50$	$S = 60$
10	8.7532e-03	1.5797e-01	3.9943e-01	8.4003e-01	1.1639e+00	1.4953e+00
20	1.0564e-03	5.2940e-03	2.1988e-02	9.7691e-02	2.0783e-01	3.7063e-01
40	1.1047e-03	1.7985e-04	1.1002e-03	5.7232e-03	1.4762e-02	3.3025e-02
80	1.1060e-03	4.8801e-06	3.3783e-05	1.8744e-04	4.1176e-04	8.0327e-04
160	1.1050e-03	1.4693e-07	9.3971e-07	5.4925e-06	1.2524e-05	2.4385e-05
320	9.5026e-04	3.3705e-06	2.3864e-06	4.9717e-07	5.3707e-07	7.2421e-07

5. Conclusions

Semi-infinite domain problems have been considered by linear barycentric interpolation method with the truncation method and transformation method. By transformation method, the semi-infinite domain $[0, \infty)$ was transformed into $[-1, 1]$ with the function become complex. The proof of the convergence rate have been presented and numerical examples conforms the theorem analysis for both the linear and nonlinear differential equation. In the future works, infinite domain problems will be considered by the linear barycentric interpolation method.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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