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Research article

Existence of solutions for impulsive wave equations

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Abstract: We study a class of initial value problems for impulsive nonlinear wave equations. A new topological approach is applied to prove the existence of at least one and at least two nonnegative classical solutions. To prove our main results we give a suitable integral representation of the solutions of the considered problem. Then, we construct two operators so that any fixed point of their sum is a solution.

Keywords: wave equations; impulsive wave equations; positive solution; fixed point; cone; sum of operators
Mathematics Subject Classification: 35L05, 35R12, 55M20

1. Introduction

Mechanical systems with impact, heart beats, blood flows, population dynamics, industrial robotics, biotechnology, economics, etc are real world and applied science phenomena which are abruptly changed in their states at some time instants due to short time perturbations whose duration is negligible in comparison with the duration of these phenomena, please see [18]. They are called impulsive phenomena. A natural framework for mathematical simulation of such phenomena are impulsive differential equations or impulsive partial differential equations when more factors are taking into account.

Whereas impulsive differential equations are well studied, see for example the books [4, 8, 39, 42] and the references therein, the literature concerning impulsive partial differential equations does not seem to be very rich. The history of impulsive partial differential equations starts at the end of the 20th century with the pioneering work [17], in which, impulsive partial differential systems have been showed to be a natural framework for the mathematical modeling of processes in ecology and biology, like population growth, see also [10, 11, 13, 31]. We can find studies of first order partial differential equations with impulses in [5, 23, 32, 40]. Higher-order linear and nonlinear impulsive partial parabolic equations were considered in [22]. An initial boundary value problem for a nonlinear parabolic partial differential equation was discussed in [9]. The approximate controllability of an impulsive semilinear heat equation was proved in [1]. A class of impulsive wave equations was investigated in [21]. In [29] a class of impulsive semilinear evolution equations with delays is investigated for existence and uniqueness of solutions. The investigations in [29] includes several important partial differential equations such as the Burgers equation and the Benjamin-Bona-Mohany equation with impulses, delays and nonlocal conditions. A class of semilinear neutral evolution equations with impulses and nonlocal conditions in a Banach space is investigated in [2] for existence and uniqueness of solutions. To prove the main results in [2] the authors use a Karakostas fixed point theorem. In [2] an example involving Burger's equation is provided to illustrate the application of the main results. Some studies concerning impulsive Burgers equation can be found in [16, 27, 34].

Many classical methods have been successfully applied for solving impulsive partial differential equations. By using variational method, the existence of solutions for a fourth-order impulsive partial differential equations with periodic boundary conditions was obtained in [30]. The Krasnoselskii theorem is used to prove existence and uniqueness of solutions for impulsive Hamilton-Jacobi equation in [35]. Some other references on impulsive partial differential equations are: [3, 6, 12, 19, 20, 24–26, 28, 33, 36, 38, 41].

In this paper, we investigate the following impulsive wave equation

$$\begin{array}{lcl} u_{tt} - \Delta_x u & = & f(t, x, u, u_t, u_x), \quad t \in [0, T], \quad x \in \mathbb{R}^n, \\ u_t(0, x) & = & \lambda u_t(T, x), \quad x \in \mathbb{R}^n, \\ u(0, x) & = & u_0(x), \quad x \in \mathbb{R}^n, \\ u(t_k+, x) & = & u(t_k, x) + J_k(u(t_k, x)), \quad x \in \mathbb{R}^n, \end{array} \tag{1.1}$$

where

(H1) $T = t_{m+1} > t_m > \ldots > t_1 > t_0 = 0$, $J = [0, T]$, $J_0 = J \backslash \{t_k\}_{k=1}^m$, $m \in \mathbb{N}$, $\lambda \neq 1$.

(H2) $J_k \in C([0, T] \times \mathbb{R}^n)$, $|J_k(v)| \leq A|v|^{r_k}$, $v \in \mathbb{R}$, $r_k > 0$, $k \in \{1, \ldots, m\}$, $A > 0$ is a constant.

(H3) $u_0 \in C^1(\mathbb{R}^n)$, $|u_0| \leq B$ on $\mathbb{R}^n$.

(H4) $f \in C([0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n)$,

$$|f(t, x, u, v, w)| \leq \sum_{j=1}^{r} \left(a_j(t, x)|u|^{p_j} + b_j(t, x)|v|^{q_j} + \sum_{i=1}^{n} c_{ji}(t, x)|w_i|^{r_{ji}} \right),$$

$t \in [0, T]$, $x \in \mathbb{R}^n$, $u, v, w \in \mathbb{R}$, $a_j, b_j, c_{ji} \in C([0, T] \times \mathbb{R}^n)$, $0 \leq a_j, b_j, c_{ji} \leq C$ on $[0, T] \times \mathbb{R}^n$, $C > 0$, $p_j, q_j, r_{ji} > 0$, $j \in \{1, \ldots, r\}$, $i \in \{1, \ldots, n\}$, $r \in \mathbb{N}$.

Here

$$u_x = (u_{x_1}, \ldots, u_{x_n}),$$
$$u(t_k, x) = \lim_{t \to t_k -} u,$$
$$u(t_k +, x) = \lim_{t \to t_k +} u.$$

The main aim of this paper is to investigate the problem (1.1) for existence and non-uniqueness of solutions. The main objective in this article is to show and develop some topological methods in order to prove the existence of solutions to the problem from many points of view and angles under different conditions to give a good and useful data and information on the solutions to the implementers to be exploited in the best way.

The work is organized as follows. In the next section, we give some preliminary results and tools. In Section 3, we prove existence of at least one solution for the problem (1.1). In Section 4, we prove existence of at least one nonnegative solution of the problem (1.1). In Section 5, we prove existence of at least two nonnegative solutions of the problem (1.1). In Section 6, we give an example that illustrates our main results.

2. Tools

Some preliminary tools are needed to prove the main results. The following fixed point theorem for a sum of two operators is used to prove existence of at least one solution to the problem (1.1).

Theorem 2.1. *Let $\epsilon \in (0,1)$, $B > 0$, E be a Banach space and $X = \{x \in E\colon \|x\| \le B\}$. Let also, $Tx = -\epsilon x$, $x \in X$, $S\colon X \to E$ is continuous, $(I - S)(X)$ resides in a compact subset of E and*

$$\{x \in E : x = \lambda(I - S)x, \quad \|x\| = B\} = \emptyset, \tag{2.1}$$

for any $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then there exists a $x^* \in X$ so that

$$Tx^* + Sx^* = x^*.$$

Here $\mu X = \{\mu x\colon x \in X\}$ for any $\mu \in \mathbb{R}$.

Proof. Define

$$r\left(-\frac{1}{\epsilon}x\right) = \begin{cases} -\frac{1}{\epsilon}x, & \text{if} \quad \|x\| \le B\epsilon, \\ \frac{Bx}{\|x\|}, & \text{if} \quad \|x\| > B\epsilon. \end{cases}$$

Then $r\left(-\frac{1}{\epsilon}(I - S)\right)\colon X \to X$ is continuous and compact. Now, we apply the Schauder fixed point theorem and we conclude that there exists $x^* \in X$ so that

$$r\left(-\frac{1}{\epsilon}(I - S)x^*\right) = x^*.$$

Assume that $-\frac{1}{\epsilon}(I - S)x^* \notin X$. Then

$$\left\|(I - S)x^*\right\| > B\epsilon, \quad \frac{B}{\|(I - S)x^*\|} < \frac{1}{\epsilon},$$

and

$$x^* = \frac{B}{\|(I-S)x^*\|}(I-S)x^* = r\left(-\frac{1}{\epsilon}(I-S)x^*\right),$$

and hence, $\|x^*\| = B$. This contradicts with (2.1). Therefore $-\frac{1}{\epsilon}(I-S)x^* \in X$ and

$$x^* = r\left(-\frac{1}{\epsilon}(I-S)x^*\right) = -\frac{1}{\epsilon}(I-S)x^*,$$

or

$$-\epsilon x^* + Sx^* = x^*,$$

or

$$Tx^* + Sx^* = x^*.$$

This completes the proof.

Let X be a real Banach space.

Definition 2.1. *A mapping K: $X \to X$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

The concept for l-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 2.2. *Let Ω_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness α: $\Omega_X \to [0,\infty)$ is defined by*

$$\alpha(Y) = \inf\left\{\delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad and \quad diam(Y_j) \le \delta, \quad j \in \{1,\dots,m\}\right\},$$

where $diam(Y_j) = \sup\{\|x-y\|_X$: $x,y \in Y_j\}$ is the diameter of Y_j, $j \in \{1,\dots,m\}$.

For the main properties of measure of noncompactness we refer the reader to [7].

Definition 2.3. *A mapping K: $X \to X$ is said to be l-set contraction if it is continuous, bounded and there exists a constant $l \ge 0$ such that*

$$\alpha(K(Y)) \le l\alpha(Y),$$

for any bounded set $Y \subset X$. The mapping K is said to be a strict set contraction if $l < 1$.

Obviously, if K: $X \to X$ is a completely continuous mapping, then K is 0-set contraction (see [15]).

Definition 2.4. *Let X and Y be real Banach spaces. A mapping K: $X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that*

$$\|Kx - Ky\|_Y \ge h\|x-y\|_X,$$

for any $x, y \in X$.

Definition 2.5. *A closed, convex set $\mathcal{P}$ in X is said to be cone if*

(1) $\alpha x \in \mathcal{P}$ *for any* $\alpha \geq 0$ *and for any* $x \in \mathcal{P}$,

(2) $x, -x \in \mathcal{P}$ *implies* $x = 0$.

Denote $\mathcal{P}^* = \mathcal{P} \backslash \{0\}$.

Lemma 2.1. *Let* X *be a closed convex subset of a Banach space* E *and* $U \subset X$ *a bounded open subset with* $0 \in U$. *Assume there exists* $\varepsilon > 0$ *small enough and that* K: $\overline{U} \to X$ *is a strict k-set contraction that satisfies the boundary condition:*

$$Kx \notin \{x,\ \lambda x\}\ \textit{for all}\ x \in \partial U \textit{ and } \lambda \geq 1 + \varepsilon.$$

Then $i\,(K, U, X) = 1$.

Proof. Consider the homotopic deformation H: $[0, 1] \times \overline{U} \to X$ defined by

$$H(t, x) = \frac{1}{\varepsilon + 1}\, tKx.$$

The operator H is continuous and uniformly continuous in t for each x, and the mapping $H(t, .)$ is a strict set contraction for each $t \in [0, 1]$. In addition, $H(t, .)$ has no fixed point on ∂U. On the contrary,
• If $t = 0$, there exists some $x_0 \in \partial U$ such that $x_0 = 0$, contradicting $x_0 \in U$.
• If $t \in (0, 1]$, there exists some $x_0 \in \mathcal{P} \cap \partial U$ such that $\frac{1}{\varepsilon+1}\, tKx_0 = x_0$; then $Kx_0 = \frac{1+\varepsilon}{t} x_0$ with $\frac{1+\varepsilon}{t} \geq 1 + \varepsilon$, contradicting the assumption. From the invariance under homotopy and the normalization properties of the index, we deduce

$$i\,(\frac{1}{\varepsilon + 1}\, K, U, X) = i\,(0, U, X) = 1.$$

Now, we show that

$$i\,(K, U, X) = i\,(\frac{1}{\varepsilon + 1}\, K, U, X).$$

We have

$$\frac{1}{\varepsilon + 1} Kx \neq x,\ \ \forall\, x \in \partial U. \tag{2.2}$$

Then there exists $\gamma > 0$ such that

$$\|x - \frac{1}{\varepsilon + 1}\, Kx\| \geq \gamma,\ \ \forall\, x \in \partial U.$$

In other hand, we have $\frac{1}{\epsilon+1} Kx \to Kx$ as $\epsilon \to 0$, for $x \in \overline{U}$. So for ε small enough

$$\|Kx - \frac{1}{\varepsilon + 1}\, Kx\| < \frac{\gamma}{2},\ \ \forall\, x \in \partial U.$$

Define the convex deformation G: $[0, 1] \times \overline{U} \to X$ by

$$G(t, x) = tKx + (1 - t)\frac{1}{\varepsilon + 1} Kx.$$

The operator G is continuous and uniformly continuous in t for each x, and the mapping $G(t,.)$ is a strict set contraction for each $t \in [0,1]$ (since $t + \frac{1}{\varepsilon+1}(1-t) < t + 1 - t = 1$). In addition, $G(t,.)$ has no fixed point on ∂U. In fact, for all $x \in \partial U$, we have

$$\begin{array}{rcl} \|x - G(t,x)\| & = & \|x - tKx - (1-t)\frac{1}{\varepsilon+1}Kx\| \\ & \geq & \|x - \frac{1}{\varepsilon+1}Kx\| - t\|Kx - \frac{1}{\varepsilon+1}Kx\| \\ & > & \gamma - \frac{\gamma}{2} > \frac{\gamma}{2}. \end{array}$$

Then our claim follows from the invariance property by homotopy of the index.

Proposition 2.1. *Let $\mathcal{P}$ be a cone in a Banach space E. Let also, U be a bounded open subset of $\mathcal{P}$ with $0 \in U$. Assume that $T\colon \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $S\colon \overline{U} \to E$ is a l-set contraction with $0 \leq l < h - 1$, and $S(\overline{U}) \subset (I - T)(\Omega)$. If there exists $\varepsilon \geq 0$ such that*

$$Sx \neq \{(I-T)(x), \quad (I-T)(\lambda x)\} \text{ for all } x \in \partial U \cap \Omega \text{ and } \lambda \geq 1 + \varepsilon,$$

then the fixed point index $i_*(T + S, U \cap \Omega, \mathcal{P}) = 1$.

Proof. The mapping $(I - T)^{-1}S\colon \overline{U} \to \mathcal{P}$ is a strict set contraction and it is readily seen that the following condition is satisfied

$$(I-T)^{-1}Sx \notin \{x, \lambda x\}, \text{ for all } x \in \partial U \text{ and } \lambda \geq 1 + \epsilon.$$

Our claim then follows from the definition of i_* and the following Lemma 2.1.

The following result will be used to prove existence of at least two nonnegative solutions to the problem (1.1).

Theorem 2.2. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T\colon \Omega \to \mathcal{P}$ is an expansive mapping with constant $h > 1$, $S\colon \overline{U}_3 \to E$ is a k-set contraction with $0 \leq k < h - 1$ and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) $Sx \neq (I-T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$.

(ii) There exists $\epsilon \geq 0$ such that $Sx \neq (I-T)(\lambda x)$, for all $\lambda \geq 1 + \epsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$.

(iii) $Sx \neq (I-T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega,$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Proof. If

$$Sx = (I - T)x,$$

for $x \in \partial U_2 \cap \Omega$, then we get a fixed point $x_1 \in \partial U_2 \cap \Omega$ of the operator $T + S$.

Suppose that

$$Sx \neq (I - T)x,$$

for any $x \in \partial U_2 \cap \Omega$. Without loss of generality, assume that

$$Tx + Sx \neq x \text{ on } \partial U_1 \cap \Omega \text{ and } Tx + Sx \neq x \text{ on } \partial U_3 \cap \Omega.$$

Then, by Propositions 2.11 and 2.16 in [14] and Proposition 2.1, we have

$$i_* (T + S, U_1 \cap \Omega, \mathcal{P}) = i_* (T + S, U_3 \cap \Omega, \mathcal{P}) = 0 \text{ and } i_* (T + S, U_2 \cap \Omega, \mathcal{P}) = 1.$$

The additivity property of the index yields

$$i_* (T + S, (U_2 \setminus \overline{U}_1) \cap \Omega, \mathcal{P}) = 1 \text{ and } i_* (T + S, (U_3 \setminus \overline{U}_2) \cap \Omega, \mathcal{P}) = -1.$$

Consequently, by the existence property of the index, $T + S$ has at least two fixed points $x_1 \in (U_2 \setminus U_1) \cap \Omega$ and $x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$.

For $l, s \in \mathbb{N} \cup \{0\}$, define

$$PC^l([0, T]) = \{g : g^{(l-1)} \in PC^{l-1}([0, T]),\ \exists g^{(l)},$$

and it is continuous on $(t_k, t_{k+1})\exists \lim\limits_{t\to t_k+} g^{(l)}(t)$ and $\lim\limits_{t\to t_k} g^{(l)}(t)\}$,
and

$$PC^l([0, T], C^s(\mathbb{R}^n)) = \{u : u(\cdot, x) \in PC^l([0, T]),\ x \in \mathbb{R}^n, u(t, \cdot) \in C^s(\mathbb{R}^n),\ t \in [0, T]\}.$$

In $E = PC^2([0, T], C^2(\mathbb{R}^n))$, define a norm

$$\begin{aligned} \|u\| \quad = \quad & \max\{ \max_{j\in\{0,1,\dots,m\}} \sup_{(t,x)\in[t_j,t_{j+1}]\times\mathbb{R}^n} |u(t, x)|, \\ & \max_{j\in\{0,1,\dots,m\}} \sup_{(t,x)\in[t_j,t_{j+1}]\times\mathbb{R}^n} |u_t(t, x)|, \\ & \max_{j\in\{0,1,\dots,m\}} \sup_{(t,x)\in[t_j,t_{j+1}]\times\mathbb{R}^n} |u_{tt}(t, x)|, \\ & \max_{j\in\{0,1,\dots,m\}} \sup_{(t,x)\in[t_j,t_{j+1}]\times\mathbb{R}^n} |u_{x_i}(t, x)|, \\ & \max_{j\in\{0,1,\dots,m\}} \sup_{(t,x)\in[t_j,t_{j+1}]\times\mathbb{R}^n} |u_{x_i x_i}(t, x)|, \quad i \in \{1, \dots, n\}\}, \end{aligned}$$

provided it exists. Note that E is a Banach space. For $u \in E$, set

$$\begin{aligned} f_1(t, x, u(t, x), u_t(t, x), u_x(t, x)) \quad = \quad & \frac{\lambda}{1-\lambda} \int_0^T (\Delta_x u(s, x) + f(s, x, u(s, x), u_t(s, x), u_x(s, x)))\, ds \\ + \quad & \int_0^t (\Delta_x u(s, x) + f(s, x, u(s, x), u_t(s, x), u_x(s, x)))\, ds, \end{aligned}$$

$t \in [0, T]$, $x \in \mathbb{R}^n$.

Lemma 2.2. *Suppose* $(H1)$ *and* $(H4)$*. Then for any* $u \in E$ *with* $\|u\| \le B$*, we have*

$$|f(t,x,u(t,x),u_t(t,x),u_x(t,x))| \le C \sum_{j=1}^{r} \left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}} \right),$$

and

$$|f_1(t,x,u(t,x),u_t(t,x),u_x(t,x))| \le \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T \left(nB + C \sum_{j=1}^{r} \left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}} \right) \right),$$

for $t \in [0,T]$ *and* $x \in \mathbb{R}^n$.

Proof. We have

$$\begin{aligned} |f(t,x,u(t,x),u_t(t,x),u_x(t,x))| &\le \sum_{j=1}^{r} \Big(a_j(t,x)|u(t,x)|^{p_j} + b_j(t,x)|u_t(t,x)|^{q_j} \\ &+ \sum_{i=1}^{n} c_{ji}(t,x)|u_{x_i}(t,x)|^{r_{ji}} \Big) \\ &\le C \sum_{j=1}^{r} \left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}} \right), \end{aligned}$$

for $t \in [0,T]$, $x \in \mathbb{R}^n$. Next,

$$\begin{aligned} & |f_1(t,x,u(t,x),u_t(t,x),u_x(t,x))| \\ = & \left| \frac{\lambda}{1-\lambda} \int_0^T (\Delta_x u(s,x) + f(s,x,u(s,x),u_t(s,x),u_x(s,x)))\, ds \right. \\ + & \left. \int_0^t (\Delta_x u(s,x) + f(s,x,u(s,x),u_t(s,x),u_x(s,x)))\, ds \right| \\ \le & \left|\frac{\lambda}{1-\lambda}\right| \int_0^T (|\Delta_x u(s,x)| + |f(s,x,u(s,x),u_t(s,x),u_x(s,x))|)\, ds \\ + & \int_0^t (|\Delta_x u(s,x)| + |f(s,x,u(s,x),u_t(s,x),u_x(s,x))|)\, ds \\ \le & \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) \int_0^T (|\Delta_x u(s,x)| + |f(s,x,u(s,x),u_t(s,x),u_x(s,x))|)\, ds \\ \le & \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T \left(nB + C \sum_{j=1}^{r} \left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}} \right) \right), \end{aligned}$$

for $t \in [0,T]$, $x \in \mathbb{R}^n$. This completes the proof.

For $u \in E$, define

$$S_1 u(t,x) = u(t,x) - \int_0^t f_1(s,x,u(s,x),u_t(s,x),u_x(s,x))ds - u_0(x) - \sum_{0<t_k<t} J_k(u(t_k,x)),\ \ t \in [0,T],\ \ x \in \mathbb{R}^n.$$

Lemma 2.3. *Suppose* $(H1)$–$(H4)$*. If* $u \in E$ *satisfies the equation* $S_1 u = 0$ *on* $[0, T] \times \mathbb{R}^n$*, then* u *satisfies the problem* (1.1).

Proof. We have

$$u(t,x) = \int_0^t f_1(s,x,u(s,x),u_t(s,x),u_x(s,x))ds + u_0(x) + \sum_{0<t_k<t} J_k(u(t_k,x)),\ t \in [0,T],\ x \in \mathbb{R}^n.$$

Then

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^n,$$

and

$$\begin{aligned} u(t_j+,x) &= \int_0^{t_j} f_1(s,x,u(s,x),u_t(s,x),u_x(s,x))ds \\ &+ u_0(x) + \sum_{0<t_k<t_j+} J_k(u(t_k,x)), \quad x \in \mathbb{R}^n,\ j \in \{1,\ldots,m\}, \end{aligned}$$

and

$$\begin{aligned} u(t_j,x) &= \int_0^{t_j} f_1(s,x,u(s,x),u_t(s,x),u_x(s,x))ds \\ &+ u_0(x) + \sum_{0<t_k<t_j} J_k(u(t_k,x)), \quad x \in \mathbb{R}^n,\ j \in \{1,\ldots,m\}. \end{aligned}$$

By the last two equations, we find

$$u(t_j+,x) = u(t_j,x) + J_j(u(t_j,x)), \quad x \in \mathbb{R}^n,\ j \in \{1,\ldots,m\}.$$

Next,

$$\begin{aligned} u_t(t,x) &= f_1(t,x,u(t,x),u_t(t,x),u_x(t,x)) \\ &= \frac{\lambda}{1-\lambda}\int_0^T \left(\Delta_x u(s,x) + f(s,x,u(s,x),u_t(s,x),u_x(s,x))\right) ds \\ &+ \int_0^t \left(\Delta_x u(s,x) + f(s,x,u(s,x),u_t(s,x),u_x(s,x))\right) ds, \end{aligned}$$

and

$$u_{tt}(t,x) = \Delta_x u(t,x) + f(t,x,u(t,x),u_t(t,x),u_x(t,x)),$$

for $t \in [0,T]$, $x \in \mathbb{R}^n$. Hence,

$$\begin{aligned} u_t(0,x) &= \frac{\lambda}{1-\lambda}\int_0^T \left(\Delta_x u(s,x) + f(s,x,u(s,x),u_t(s,x),u_x(s,x))\right) ds, \\ u_t(T,x) &= \frac{1}{1-\lambda}\int_0^T \left(\Delta_x u(s,x) + f(s,x,u(s,x),u_t(s,x),u_x(s,x))\right) ds,\ x \in \mathbb{R}^n. \end{aligned}$$

Therefore

$$u_t(0,x) = \lambda u_t(T,x),\ x \in \mathbb{R}^n.$$

This completes the proof.

Lemma 2.4. *Suppose* $(H1)$–$(H4)$. *For* $u \in E$, $\|u\| \le B$, *we have*

$$|S_1 u(t,x)| \le 2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nB + C \sum_{j=1}^{r} \left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}} \right) \right) + A \sum_{k=1}^{m} B^{r_k}, \quad t \in [0,T], \ x \in \mathbb{R}^n.$$

Proof. We have, using Lemma 2.2,

$$\begin{aligned}
|S_1 u(t,x)| &= \left| u(t,x) - \int_0^t f_1(s,x,u(s,x),u_t(s,x),u_x(s,x))ds - u_0(x) - \sum_{0<t_k<t} J_k(u(t_k,x)) \right| \\
&\le |u(t,x)| + \int_0^t |f_1(s,x,u(s,x),u_t(s,x),u_x(s,x))|ds + |u_0(x)| + \sum_{0<t_k<t} |J_k(u(t_k,x))| \\
&\le |u(t,x)| + \int_0^t |f_1(s,x,u(s,x),u_t(s,x),u_x(s,x))|ds + |u_0(x)| + A \sum_{0<t_k<t} |u(t_k,x)|^{r_k} \\
&\le B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nB + C \sum_{j=1}^{r} \left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}} \right) \right) + B + A \sum_{k=1}^{m} B^{r_k} \\
&= 2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nB + C \sum_{j=1}^{r} \left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}} \right) \right) \\
&+ A \sum_{k=1}^{m} B^{r_k}, \quad t \in [0,T], \ x \in \mathbb{R}^n.
\end{aligned}$$

This completes the proof.

(H5) Now, suppose that D is a positive constant and g is a nonnegative continuous function on $\mathbb{R}^n$. Moreover $g > 0$ on $\mathbb{R}^n \backslash \{\cup_{i=1}^n \{x_i = 0\}\}$,

$$g(0,x_2,\ldots,x_n) = g(x_1,0,x_3,\ldots,x_n) = \ldots = g(x_1,\ldots,x_{n-1},0) = 0, \quad x_1,\ldots,x_n \in \mathbb{R},$$

and

$$216(1+T+T^2+T^3) \prod_{j=1}^{n} \left(1 + |x_j| + x_j^2 + |x_j|^3\right) \left| \int_0^x g(y)dy \right| \le D, \quad x \in \mathbb{R}^n,$$

where

$$\int_0^x = \int_0^{x_1} \ldots \int_0^{x_n}, \quad dy = dy_n \ldots dy_1.$$

In the last section, we will give an example for the constant D and the function g that satisfy $(H5)$. For $u \in E$, define the operator

$$S_2 u(t,x) = \int_0^t (t-s)^3 \int_0^x \prod_{j=1}^{n} (x_j - y_j)^3 g(y) S_1 u(s,y) dy ds, \quad t \in [0,T], x \in \mathbb{R}^n.$$

Lemma 2.5. *Suppose* $(H1)$–$(H4)$. *If* $u \in E$ *satisfies the equation*

$$S_2u(t,x) = a, \quad t \in [0,T],\ x \in \mathbb{R}^n, \tag{2.3}$$

for some constant a, then u satisfies the problem (1.1).

Proof. We differentiate four times in t and four times in $x_1, \ldots, x_n$ the Eq (2.3) and we get

$$g(x)S_1u(t,x) = 0, \quad t \in [0,T],\ x \in \mathbb{R}^n,$$

whereupon

$$S_1u(t,x) = 0, \quad t \in [0,T],\ x \in \mathbb{R}^n \backslash \{\cup_{i=1}^n \{x_i = 0\}\}.$$

Since S_1 is continuous, we get

$$\begin{aligned} S_1u(t,0,x_2,\ldots,x_n) &= \lim_{x_1\to 0} S_1u(t,x_1,x_2,\ldots,x_n) \\ &= \ldots \\ &= S_1u(t,x_1,\ldots,x_{n-1},0) \\ &= \lim_{x_n\to 0} S_1u(t,x_1,\ldots,x_n) = 0, \quad x_1,\ldots,x_n \in \mathbb{R}. \end{aligned}$$

Thus,

$$S_1u(t,x) = 0, \quad t \in [0,T],\ x \in \mathbb{R}^n.$$

Hence and Lemma 2.3, we conclude that u satisfies the problem (1.1). This completes the proof.

Lemma 2.6. *Suppose* $(H1)$–$(H5)$. *If* $u \in E$ *and* $\|u\| \le B$, *then*

$$\|S_2u\| \le D\left(2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB + C\sum_{j=1}^r\left(B^{p_j} + B^{q_j} + \sum_{i=1}^n B^{r_{ji}}\right)\right) + A\sum_{k=1}^m B^{r_k}\right).$$

Proof. We have

$$\begin{aligned} |S_2u(t,x)| &= \left|\int_0^t (t-s)^3 \int_0^x \prod_{j=1}^n (x_j-y_j)^3 g(y) S_1u(s,y)dyds\right| \\ &\le \left|\int_0^t (t-s)^3 \int_0^x \prod_{j=1}^n (x_j-y_j)^3 g(y) |S_1u(s,y)|dyds\right| \\ &\le \left(2B + T\left(C\sum_{j=1}^r\left(B^{p_j} + B^{q_j} + \sum_{i=1}^n B^{r_{ji}}\right) + nB\right) + A\sum_{k=1}^m B^{r_k}\right) \\ &\quad \times\left|\int_0^t (t-s)^3 \int_0^x \prod_{j=1}^n (x_j-y_j)^3 g(y)dyds\right| \\ &\le \left(2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB + C\sum_{j=1}^r\left(B^{p_j} + B^{q_j} + \sum_{i=1}^n B^{r_{ji}}\right)\right) + A\sum_{k=1}^m B^{r_k}\right) \end{aligned}$$

$$\times 216(1+T+T^2+T^3)\prod_{j=1}^{n}\left(1+|x_j|+x_j^2+|x_j|^3\right)\left|\int_0^x g(y)dy\right|$$
$$\leq D\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right),$$

$t\in[0,T]$, $x\in\mathbb{R}^n$, and

$$\begin{aligned}
|(S_2u)_t(t,x)| &= \left|3\int_0^t(t-s)^2\int_0^x\prod_{j=1}^{n}(x_j-y_j)^3g(y)S_1u(s,y)dyds\right| \\
&\leq \left|3\int_0^t(t-s)^2\int_0^x\prod_{j=1}^{n}(x_j-y_j)^3g(y)|S_1u(s,y)|dyds\right| \\
&\leq 3\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right) \\
&\quad\times\left|\int_0^t(t-s)^2\int_0^x\prod_{j=1}^{n}(x_j-y_j)^3g(y)dyds\right| \\
&\leq \left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right) \\
&\quad\times 216(1+T+T^2+T^3)\prod_{j=1}^{n}\left(1+|x_j|+x_j^2+|x_j|^3\right)\left|\int_0^x g(y)dy\right| \\
&\leq D\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right),
\end{aligned}$$

$t\in[0,T]$, $x\in\mathbb{R}^n$, and

$$\begin{aligned}
|(S_2u)_{tt}(t,x)| &= \left|6\int_0^t(t-s)\int_0^x\prod_{j=1}^{n}(x_j-y_j)^3g(y)S_1u(s,y)dyds\right| \\
&\leq \left|6\int_0^t(t-s)\int_0^x\prod_{j=1}^{n}(x_j-y_j)^3g(y)|S_1u(s,y)|dyds\right| \\
&\leq 6\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right) \\
&\quad\times\left|\int_0^t(t-s)\int_0^x\prod_{j=1}^{n}(x_j-y_j)^3g(y)dyds\right| \\
&\leq \left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times 216(1+T+T^2+T^3)\prod_{j=1}^{n}\left(1+|x_j|+x_j^2+|x_j|^3\right)\left|\int_0^x g(y)dy\right| \\
\leq\ & D\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right),
\end{aligned}
$$

$t\in[0,T]$, $x\in\mathbb{R}^n$, and

$$
\begin{aligned}
|(S_2u)_{x_l}(t,x)| &= \left|3\int_0^t (t-s)^3\int_0^x \prod_{j=1,j\neq l}^{n}(x_j-y_j)^3(x_l-y_l)^2 g(y)S_1u(s,y)dyds\right| \\
&\leq 3\left|\int_0^t (t-s)^3\int_0^x \prod_{j=1,j\neq l}^{n}(x_j-y_j)^3(x_l-y_l)^2 g(y)|S_1u(s,y)|dyds\right| \\
&\leq 3\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right) \\
&\quad\times\left|\int_0^t (t-s)^3\int_0^x \prod_{j=1,j\neq l}^{n}(x_j-y_j)^3(x_l-y_l)^2 g(y)dyds\right| \\
&\leq \left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right) \\
&\quad\times 216(1+T+T^2+T^3)\prod_{j=1}^{n}\left(1+|x_j|+x_j^2+|x_j|^3\right)\left|\int_0^x g(y)dy\right| \\
&\leq D\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right),
\end{aligned}
$$

$t\in[0,T]$, $x\in\mathbb{R}^n$, $l\in\{1,\ldots,n\}$, and

$$
\begin{aligned}
|(S_2u)_{x_lx_l}(t,x)| &= \left|6\int_0^t (t-s)^3\int_0^x \prod_{j=1,j\neq l}^{n}(x_j-y_j)^3(x_l-y_l)g(y)S_1u(s,y)dyds\right| \\
&\leq 6\left|\int_0^t (t-s)^3\int_0^x \prod_{j=1,j\neq l}^{n}(x_j-y_j)^3(x_l-y_l)g(y)|S_1u(s,y)|dyds\right| \\
&\leq 6\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right) \\
&\times\left|\int_0^t (t-s)^3\int_0^x \prod_{j=1,j\neq l}^{n}(x_j-y_j)^3(x_l-y_l)g(y)dyds\right| \\
&\leq \left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right)
\end{aligned}
$$

$$\begin{aligned}&\times\ \ 216(1+T+T^2+T^3)\prod_{j=1}^{n}\left(1+|x_j|+x_j^2+|x_j|^3\right)\left|\int_0^x g(y)dy\right|\\&\le\ \ D\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right),\end{aligned}$$

$t\in[0,T]$, $x\in\mathbb{R}^n$, $l\in\{1,\ldots,n\}$. Consequently

$$\|S_2u\|\le D\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right).$$

This completes the proof.

3. Existence of at least one solution

Now, suppose that

(H6) $D\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right)<B.$

(H7) $\epsilon\left(B+D\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right)\right)\le B.$

Let X_2 be the set of all equi-continuous families in E, $X_3=X_2\cup\{u_0\}$, $\widetilde{X}=X_3$. Let also,

$$X=\{u\in\widetilde{X}:\|u\|\le B\}.$$

For $u\in E$, define the operators

$$\begin{aligned}Tu(t,x)&=-\epsilon u(t,x),\\Su(t,x)&=(1+\epsilon)u(t,x)+\epsilon S_2u(t,x),\quad t\in[0,T],\quad x\in\mathbb{R}^n.\end{aligned}$$

By Lemma 2.5, it follows that any fixed point of the operator $T+S$ is a solution to the problem (1.1).

Lemma 3.1. *Suppose that* $(H1)$–$(H7)$ *hold. For* $u\in X$*, we have*

$$\|(I-S)u\|\le B\ \text{ and }\ \|((1+\epsilon)I-S)u\|<\epsilon B.$$

Proof. By Lemma 2.6, we get

$$\begin{aligned}\|(I-S)u\|&=\|-\epsilon u-\epsilon S_2u\|\le\epsilon\|u\|+\epsilon\|S_2u\|\\&\le\epsilon\left(B+D\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right)\right)\\&\le B,\end{aligned}$$

and

$$\begin{aligned}
\|((1+\epsilon)I - S)u\| &= \|\epsilon S_2 u\| = \epsilon \|S_2 u\| \\
&\leq \epsilon D\left(2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nB + C\sum_{j=1}^{r}\left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}}\right)\right) + A\sum_{k=1}^{m} B^{r_k}\right) \\
&< \epsilon B.
\end{aligned}$$

This completes the proof.

Our main result in this section is as follows.

Theorem 3.1. *Suppose that* $(H1)$–$(H7)$ *hold. Then the problem* (1.1) *has at least one solution.*

Proof. By Lemma 3.1, it follows that $I - S\colon X \to X$ and it is continuous. Since the continuous map of equi-continuous families are equi-continuous families, we conclude that $(I - S)(X)$ resides in a compact subset of E. Now, assume that there is an $u \in \partial X$ and $\lambda \in \left(0, \frac{1}{\epsilon}\right)$ so that

$$\lambda(I - S)u = u,$$

or

$$\frac{1}{\lambda}u = -\epsilon u - \epsilon S_2 u,$$

or

$$\left(\frac{1}{\lambda} + \epsilon\right)u = -\epsilon S_2 u = ((1+\epsilon)I - S)u,$$

whereupon

$$\epsilon B < \left(\frac{1}{\lambda} + \epsilon\right)B = \left(\frac{1}{\lambda} + \epsilon\right)\|u\| = \epsilon\|S_2 u\| = \|((1+\epsilon)I - S)u\| < \epsilon B.$$

This is a contradiction. Hence and Theorem 2.1, it follows that the operator $T + S$ has a fixed point and the problem (1.1) has at least one solution.

4. Existence of at least one nonnegative solution

Let

$$X_1 = \{u \in \widetilde{X} : u \geq 0, \quad \|u\| \leq B\}.$$

Below, suppose that

(H8)

$$D\left(2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nB + C\sum_{j=1}^{r}\left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}}\right)\right) + A\sum_{k=1}^{m} B^{r_k}\right) + B \leq \widetilde{r},$$

$$\epsilon\left(B + D\left(2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nB + C\sum_{j=1}^{r}\left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}}\right)\right) + A\sum_{k=1}^{m} B^{r_k}\right) + \widetilde{r}\right) \leq B,$$

$$D\left(2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nB + C\sum_{j=1}^{r}\left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}}\right)\right) + A\sum_{k=1}^{m} B^{r_k}\right) + \widetilde{r} < 2B.$$

For $u \in E$, define the operator

$$\widetilde{S}u(t,x) = (1+\epsilon)u(t,x) + \epsilon S_2 u(t,x) - \epsilon\widetilde{r}, \quad t \in [0,T], \ x \in \mathbb{R}^n.$$

Lemma 4.1. *Suppose that* $(H1)$–$(H5)$ *and* $(H8)$ *hold. If* $u \in E$ *is a fixed point of the operator* $T + \widetilde{S}$, *then it satisfies the problem* (1.1).

Proof. We have

$$u(t,x) = Tu(t,x) + \widetilde{S}u(t,x) = -\epsilon u(t,x) + (1+\epsilon)u(t,x) + \epsilon S_2 u(t,x) - \epsilon\widetilde{r}, \quad t \in [0,T], \ x \in \mathbb{R}^n,$$

whereupon

$$0 = S_2 u(t,x) - \widetilde{r}, \quad t \in [0,T], \ x \in \mathbb{R}^n.$$

Now, we apply Lemma 2.5 and we get the desired result. This completes the proof.

Lemma 4.2. *Suppose that* $(H1)$–$(H5)$ *and* $(H8)$ *hold. Then* $I - \widetilde{S} : X_1 \to X_1$,

$$\|(I - \widetilde{S})u\| \le B \quad \textit{and} \quad \|((1+\epsilon)I - \widetilde{S})u\| < 2\epsilon B, \ \ u \in X_1.$$

Proof. Take $u \in X_1$ arbitrarily. Then

$$(I - \widetilde{S})u = -\epsilon u - \epsilon S_2 u + \epsilon\widetilde{r}.$$

Since

$$\begin{aligned} \|-\epsilon u - \epsilon S_2 u\| &\le \epsilon\|u\| + \epsilon\|S_2 u\| \\ &\le \epsilon\left(B + D\left(2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB + C\sum_{j=1}^{r}\left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}}\right)\right) + A\sum_{k=1}^{m} B^{r_k}\right)\right) \\ &\le \epsilon\widetilde{r}, \end{aligned}$$

where we have used the first inequality of $(H8)$, we conclude that $(I - \widetilde{S})u \ge 0$. Next, using the second inequality of $(H8)$, we get

$$\begin{aligned} &\|(I - \widetilde{S})u\| = \|-\epsilon u - \epsilon S_2 u + \epsilon\widetilde{r}\| \\ &\le \epsilon\|u\| + \epsilon\|S_2 u\| + \epsilon\widetilde{r} \\ &\le \epsilon\left(B + D\left(2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB + C\sum_{j=1}^{r}\left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}}\right)\right) + A\sum_{k=1}^{m} B^{r_k}\right) + \widetilde{r}\right) \\ &\le B. \end{aligned}$$

Thus, $I - \widetilde{S} \colon X_1 \to X_1$. Moreover,

$$\begin{aligned} &\|((1+\epsilon)I - \widetilde{S})u\| = \|-\epsilon S_2 u + \epsilon\widetilde{r}\| \\ &\le \epsilon\|S_2 u\| + \epsilon\widetilde{r} \\ &\le \epsilon\left(D\left(2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB + C\sum_{j=1}^{r}\left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}}\right)\right) + A\sum_{k=1}^{m} B^{r_k}\right) + \widetilde{r}\right) \\ &< 2\epsilon B, \end{aligned}$$

where we have used the third inequality of $(H8)$. This completes the proof.

Our main result in this section is as follows.

Theorem 4.1. *Suppose that* $(H1)$–$(H5)$ *and* $(H8)$ *hold. Then the problem* (1.1) *has at least one nonnegative solution.*

Proof. By Lemma 4.2, we have that $I - \widetilde{S}\colon X_1 \to X_1$ and it is continuous and $(I - \widetilde{S})(X_1)$ resides in a compact subset of E. Now, assume that there are an $u \in \partial X_1$ and $\lambda \in \left(0, \frac{1}{\epsilon}\right)$ so that

$$\lambda(I - \widetilde{S})u = u,$$

or

$$\frac{1}{\lambda}u = (I - \widetilde{S})u = -\epsilon u - \epsilon S_2 u + \epsilon \widetilde{R},$$

or

$$\left(\frac{1}{\lambda} + \epsilon\right)u = -\epsilon S_2 u + \epsilon \widetilde{r} = ((I + \epsilon)I - \widetilde{S})u.$$

Hence, applying Lemma 4.2, we get

$$2\epsilon B < \left(\frac{1}{\lambda} + \epsilon\right)B = \left(\frac{1}{\lambda} + \epsilon\right)\|u\| = \|((1 + \epsilon)I - \widetilde{S})u\| < 2\epsilon B.$$

This is a contradiction. From here, applying Lemma 4.1 and Theorem 2.1, we get that the problem (1.1) has at least one nonnegative solution. This completes the proof.

5. Existence of at least two nonnegative solutions

Suppose

(H9) Let $m > 0$ be large enough and A, B, $\widetilde{\widetilde{r}}$, L, R_1 be positive constants that satisfy the following conditions

$$\widetilde{\widetilde{r}} < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right)L,$$

$$D\left(2R_1 + \left(1 + \left|\frac{\lambda}{1 - \lambda}\right|\right)T^2\left(nR_1 + C\sum_{j=1}^{r}\left(R_1^{p_j} + R_1^{q_j} + \sum_{i=1}^{n} R_1^{r_{ji}}\right)\right) + A\sum_{k=1}^{m} R_1^{r_k}\right) < \frac{L}{5}.$$

For $u \in E$, define the operators

$$\begin{aligned} T_1 u(t, x) &= (1 + m\epsilon)u(t, x) - \epsilon\frac{L}{10}, \\ S_3 u(t, x) &= -\epsilon S_2 u(t, x) - m\epsilon u(t, x) - \epsilon\frac{L}{10}, \quad t \in [0, T], \quad x \in \mathbb{R}^n. \end{aligned}$$

Our main result in this section is as follows.

Theorem 5.1. *Suppose that* $(H1)$–$(H5)$ *and* $(H9)$ *hold. Then the problem* (1.1) *has at least two nonnegative solutions.*

Proof. Define

$$
\begin{aligned}
\mathcal{P} &= \{u \in E : u \geq 0 \quad \text{on} \quad [0,T] \times \mathbb{R}^n\}, \\
\mathcal{P}^* &= \mathcal{P} \backslash \{0\}, \\
U_1 &= \mathcal{P}_{\widetilde{\widetilde{r}}} = \{v \in \mathcal{P} : \|v\| < \widetilde{\widetilde{r}}\}, \\
U_2 &= \mathcal{P}_L = \{v \in \mathcal{P} : \|v\| < L\}, \\
U_3 &= \mathcal{P}_{R_1} = \{v \in \mathcal{P} : \|v\| < R_1\}, \\
R_2 &= R_1 + \frac{D}{m}\left(2R_1 + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nR_1 + C \sum_{j=1}^{r} \left(R_1^{p_j} + R_1^{q_j} + \sum_{i=1}^{n} R_1^{r_{ji}}\right)\right) + A \sum_{k=1}^{m} R_1^{r_k}\right) + \frac{L}{5m}, \\
\Omega &= \overline{\mathcal{P}_{R_2}} = \{v \in \mathcal{P} : \|v\| \leq R_2\}.
\end{aligned}
$$

(1) For $v_1, v_2 \in \Omega$, we have

$$
\|T_1 v_1 - T_1 v_2\| = (1 + m\varepsilon)\|v_1 - v_2\|,
$$

whereupon $T_1 \colon \Omega \to E$ is an expansive operator with a a constant $\frac{1}{m\epsilon}$.

(2) For $v \in \overline{\mathcal{P}}_{R_1}$, we get

$$
\begin{aligned}
\|S_3 v\| &\leq \varepsilon\|S_2 v\| + m\varepsilon\|v\| + \varepsilon\frac{L}{10} \\
&\leq \varepsilon\Bigg(D\left(2R_1 + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nR_1 + C \sum_{j=1}^{r} \left(R_1^{p_j} + R_1^{q_j} + \sum_{i=1}^{n} R_1^{r_{ji}}\right)\right) + A \sum_{k=1}^{m} R_1^{r_k}\right) \\
&+ mR_1 + \frac{L}{10}\Bigg).
\end{aligned}
$$

Therefore $S_3(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}}_{R_1} \to E$ is continuous, we have that $S_3(\overline{\mathcal{P}}_{R_1})$ is equi-continuous. Consequently $S_3 \colon \overline{\mathcal{P}}_{R_1} \to E$ is a 0-set contraction.

(3) Let $v_1 \in \overline{\mathcal{P}}_{R_1}$. Set

$$
v_2 = v_1 + \frac{1}{m} S_2 v_1 + \frac{L}{5m}.
$$

Note that $S_2 v_1 + \frac{L}{5} \geq 0$ on $[a,b]$. We have $v_2 \geq 0$ on $[a,b]$ and

$$
\begin{aligned}
\|v_2\| &\leq \|v_1\| + \frac{1}{m}\|S_2 v_1\| + \frac{L}{5m} \\
&\leq R_1 + \frac{D}{m}\left(2R_1 + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nR_1 + C \sum_{j=1}^{r} \left(R_1^{p_j} + R_1^{q_j} + \sum_{i=1}^{n} R_1^{r_{ji}}\right)\right) + A \sum_{k=1}^{m} R_1^{r_k}\right) + \frac{L}{5m} \\
&= R_2.
\end{aligned}
$$

Therefore $v_2 \in \Omega$ and

$$
-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon S_2 v_1 - \varepsilon\frac{L}{10} - \varepsilon\frac{L}{10},
$$

or

$$(I - T_1)v_2 = -\varepsilon m v_2 + \varepsilon \frac{L}{10} = S_3 v_1.$$

Consequently $S_3(\overline{\mathcal{P}}_{R_1}) \subset (I - T_1)(\Omega)$.

(4) Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda \geq 0$ and $u \in \partial\mathcal{P}_r \cap (\Omega + \lambda u_0)$ or $u \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda u_0)$ such that

$$S_3 u = (I - T_1)(u - \lambda u_0).$$

Then

$$-\epsilon S_2 u - m\epsilon u - \epsilon \frac{L}{10} = -m\epsilon(u - \lambda u_0) + \epsilon \frac{L}{10},$$

or

$$-S_2 u = \lambda m u_0 + \frac{L}{5}.$$

Hence,

$$\|S_2 u\| = \left\| \lambda m u_0 + \frac{L}{5} \right\| > \frac{L}{5}.$$

This is a contradiction.

(5) Suppose that for any $\epsilon_1 \geq 0$ small enough there exist a $u_1 \in \partial\mathcal{P}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that $\lambda_1 u_1 \in \overline{\mathcal{P}}_{R_1}$ and

$$S_3 u_1 = (I - T_1)(\lambda_1 u_1). \tag{5.1}$$

In particular, for $\epsilon_1 > \frac{2}{5m}$, we have $u_1 \in \partial\mathcal{P}_L$, $\lambda_1 u_1 \in \overline{\mathcal{P}}_{R_1}$, $\lambda_1 \geq 1 + \epsilon_1$ and (5.1) holds. Since $u_1 \in \partial\mathcal{P}_L$ and $\lambda_1 u_1 \in \overline{\mathcal{P}}_{R_1}$, it follows that

$$\left(\frac{2}{5m} + 1 \right) L < \lambda_1 L = \lambda_1 \|u_1\| \leq R_1.$$

Moreover,

$$-\epsilon S_2 u_1 - m\epsilon u_1 - \epsilon \frac{L}{10} = -\lambda_1 m\epsilon u_1 + \epsilon \frac{L}{10},$$

or

$$S_2 u_1 + \frac{L}{5} = (\lambda_1 - 1) m u_1.$$

From here,

$$2\frac{L}{5} \geq \left\| S_2 u_1 + \frac{L}{5} \right\| = (\lambda_1 - 1) m \|u_1\| = (\lambda_1 - 1) m L$$

and

$$\frac{2}{5m} + 1 \geq \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 2.2 hold. Hence, the problem (1.1) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1,$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1.$$

This completes the proof.

6. An example

Let $T = 1$, $n = 1$, $m = 3$, $t_1 = \frac{1}{4}$, $t_2 = \frac{1}{3}$, $t_3 = \frac{1}{2}$. Consider the problem

$$\begin{aligned} u_{tt} - u_{xx} &= u^2, \quad t \in \left[0, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right], \quad x \in \mathbb{R}, \\ u_t(0,x) &= 2u_t(1,x), \quad x \in \mathbb{R}^n, \\ u(0,x) &= \frac{1}{1+x^2}, \quad x \in \mathbb{R}, \\ u(t_k+, x) &= u(t_k, x) + (u(t_k, x))^4, \quad x \in \mathbb{R}, \quad k \in \{1,2,3\}. \end{aligned}$$

Here

$$A = 1, \quad B = 1, \quad C = 1, \quad r = 1, \quad \lambda = 2.$$

Then

$$2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nB + C \sum_{j=1}^{r} \left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}}\right)\right) + A \sum_{k=1}^{m} B^{r_k} = 2 + 3(1+3) + 3 = 17.$$

Take

$$D = \epsilon = \frac{1}{10^{50}}, \quad \widetilde{r} = \frac{3}{2}, \quad R_1 = 1, \quad \widetilde{\widetilde{r}} = \frac{1}{8}, \quad L + \frac{1}{2}, \quad m = 10^{50}.$$

Then,

$$D\left(2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nB + C \sum_{j=1}^{r} \left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}}\right)\right) + A \sum_{k=1}^{m} B^{r_k}\right) = \frac{17}{10^{50}} < 1 = B,$$

and

$$\begin{aligned} &\epsilon\left(B + D\left(2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nB + C \sum_{j=1}^{r} \left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}}\right)\right) + A \sum_{k=1}^{m} B^{r_k}\right)\right) \\ &= \frac{1}{10^{50}}\left(1 + \frac{17}{10^{50}}\right) < 1 = B. \end{aligned}$$

Thus, $(H6)$ and $(H7)$ hold. Hence and Theorem 3.1, it follows that the considered problem has at least one solution. Next,

$$D\left(2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nB + C \sum_{j=1}^{r} \left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}}\right)\right) + A \sum_{k=1}^{m} B^{r_k}\right) + B = \frac{17}{10^{50}} + 1 < \frac{3}{2} = \widetilde{r},$$

and

$$\epsilon\left(B + D\left(2B + \left(1 + \left|\frac{\lambda}{1-\lambda}\right|\right) T^2 \left(nB + C \sum_{j=1}^{r} \left(B^{p_j} + B^{q_j} + \sum_{i=1}^{n} B^{r_{ji}}\right)\right) + A \sum_{k=1}^{m} B^{r_k}\right) + \widetilde{r}\right)$$

$$= \frac{1}{10^{50}}\left(1+\frac{17}{10^{50}}+\frac{3}{2}\right) < 1 = B,$$

and

$$D\left(2B+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nB+C\sum_{j=1}^{r}\left(B^{p_j}+B^{q_j}+\sum_{i=1}^{n}B^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}B^{r_k}\right)+\frac{3}{2}=\frac{17}{10^{50}}+\frac{3}{2}<2=2B.$$

Therefore ($H8$) holds. By Theorem 4.1, it follows that the considered problem has at least one nonnegative solution. Moreover,

$$\widetilde{\widetilde{r}} < L < r_1, \quad R_1 = 1 > \left(\frac{2}{5\cdot 10^{50}}+1\right)\frac{1}{2} = \left(\frac{2}{5m}+1\right)L,$$

and

$$D\left(2R_1+\left(1+\left|\frac{\lambda}{1-\lambda}\right|\right)T^2\left(nR_1+C\sum_{j=1}^{r}\left(R_1^{p_j}+R_1^{q_j}+\sum_{i=1}^{n}R_1^{r_{ji}}\right)\right)+A\sum_{k=1}^{m}R_1^{r_k}\right)=\frac{17}{10^{50}}<\frac{1}{10}=\frac{L}{5}.$$

Consequently ($H9$) holds. By Theorem 5.1, it follows that the considered problem has at least two nonnegative solutions.

Now, we will construct a function g for arbitrary n. Let

$$h(x) = \log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}, \quad l(s) = \arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}, \quad s\in\mathbb{R}.$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1-s^{22})}{(1-s^{11}\sqrt{2}+s^{22})(1+s^{11}\sqrt{2}+s^{22})}, \\ l'(s) &= \frac{11\sqrt{2}s^{10}(1+s^{20})}{1+s^{40}}, \quad s\in\mathbb{R}. \end{aligned}$$

Therefore

$$\begin{aligned} -\infty &< \lim_{s\to\pm\infty}(1+|s|+s^2+|s|^3)h(s) < \infty, \\ -\infty &< \lim_{s\to\pm\infty}(1+|s|+s^2+|s|^3)l(s) < \infty. \end{aligned}$$

Hence, there exists a positive constant C_1 so that

$$\begin{aligned} (1+|s|+s^2+|s|^3)\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}+\frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right) &\le C_1, \\ (1+|s|+s^2+|s|^3)\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}+\frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right) &\le C_1, \end{aligned}$$

$s \in \mathbb{R}$. Note that by [37], we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}} \log \frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^2)^4(1+s^{44})(1+s+s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(x) = Q(x_1)Q(x_2)\dots Q(x_n), \quad x \in \mathbb{R}^n.$$

Then there exists a positive constant C_2 so that

$$216(1+T+T^2+T^3)\prod_{j=1}^{n}\left(1+|x_j|+x_j^2+|x_j|^3\right)\left|\int_0^x g_1(y)dy\right| \le C_2, \quad x \in \mathbb{R}^n.$$

Take $g(x) = \frac{D}{C_2} g_1(x)$, $x \in \mathbb{R}^n$. Hence,

$$216(1+T+T^2+T^3)\prod_{j=1}^{n}\left(1+|x_j|+x_j^2+|x_j|^3\right)\left|\int_0^x g(y)dy\right| \le D, \quad x \in \mathbb{R}^n.$$

7. Conclusions

In this paper, we investigate initial value problems for impulsive nonlinear wave equations. We reduce the considered problem to a suitable integral equation. Then, we define two operators and show that any fixed point of their sum is a solution of the considered problem. After this, we apply recent fixed point theorems and we show that the considered problem has at least one and at least two classical solutions. The proposed approach can be applied for other classes impulsive partial differential equations.

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Conflict of interest

The authors declare there are no conflicts of interest.

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