Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 8702–8730.
DOI: 10.3934/math.2023437
Received: 29 September 2022
Revised: 31 December 2022
Accepted: 10 January 2023
Published: 07 February 2023

Research article

A study of the time fractional Navier-Stokes equations for vertical flow

Abdelkader Moumen[1], Ramsha Shafqat[2], Azmat Ullah Khan Niazi[2], Nuttapol Pakkaranang[3,*], Mdi Begum Jeelani[4] and Kiran Saleem[2]

[1] Department of Mathematics, Faculty of Sciences, University of Hail, Hail 55425, Saudi Arabia
[2] Department of Mathematics and Statistics, The University of Lahore, Sargodha 40100, Pakistan
[3] Mathematics and Computing Science Program, Faculty of Science and Technology, Phetchabun Rajabhat University, Phetchabun 67000, Thailand
[4] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University, Riyadh 13314, Saudi Arabia

* **Correspondence:** Email: nuttapol.pak@pcru.ac.th.

Abstract: Navier-Stokes (NS) equations dealing with gravitational force with time-fractional derivatives are discussed in this paper. These equations can be used to predict fluid velocity and pressure for a given geometry. This paper investigates the local and global existence and uniqueness of mild solutions to NS equations for the time fractional differential operator. We also work on the regularity effects of such types of equations were caused by orthogonal flow.

Keywords: Navier-Stokes equations; mild solution; existence and uniqueness; Caputo fractional derivative; Mittag-Leffler functions; regularity
Mathematics Subject Classification: 34A07, 34A08, 60G22

1. Introduction

To explain the flow of incompressible fluids in fluid mechanics, Navier-Stokes (NS) equations are applied as partial differential equations in order to discuss such fluids. Such equations constitute a modification of the equations composed by the Swiss mathematician Leonhard. In the 18th century, Euler illustrated the movement of frictionless and incompressible fluids. For the most realistic and challenging viscous fluid issue in 1821 Claude-Louis Navier, a French engineer, presented the element of viscosity (friction). Sir George Gabriel Stokes, a British physicist and mathematician, expanded this work in the mid-nineteenth century, however, comprehensive solutions have only been achieved for the case of 2-D flows. The complex vortexes and disturbance, or disruption, that arise in 3-D fluid flows (including gas) as velocities rise are demonstrated to be uncontrolled for every numerical analysis approach except approximation.

Because they characterize the physics of many scientific and engineering events, NS equations are useful. They might be models of climate, ocean waves, water flow in a pump, or air flow. The NS equations, in both their complete and simplified versions are applied for assistance with aviation and automobile design, blood flow studies, the design of power plants, pollution monitoring and a variety of other activities.

The NS equation, in modern notation for vertical flow:

$$\rho\left(\frac{\partial \tilde{u}}{\partial t} + (\tilde{u}.\nabla)\tilde{u}\right) = -\nabla P + \mu\nabla^2\tilde{u} + \rho g \tag{1.1}$$

where $\tilde{u}$ is defined as the fluid velocity vector, P is the fluid pressure, ρ is the fluid density, μ is the dynamic viscosity and ∇^2 is the Laplacian operator.

The NS equations in mathematics demonstrate the conservation of speed and mass for Newtonian fluids. Sometimes the condition applied on equation associated with stress, temperature and body weight. They result from the application of Isaac Newton's second law to water movement, as well as the concept that the fluid stress is the sum of several viscous time (rate and velocity) and pressure respectively which describes the viscous flow. The difference between Euler equations and Navier equations is that the Euler equations model is just an inviscid flow whereas NS equations are viscosity sensitive. For that reason, NS equations are a parabolic with improved analytical properties at the cost of a constrained mathematical framework (for example, they are never fully integrated).

These equations are the central part of fluid flow modeling. Solving them for a set of specific boundary conditions (entrances, exits and walls) predicts fluid velocity and pressure for a given geometry. In view of the fact that complexity recognizes only a limited number of these equations analytical solutions [1], it is relatively convenient, for instance, to solve these equations for a flow between two parallel plates, or for the flow in a circular pipe. Abbas et al. [2] and Mehmood et al. [3] solved ordinary differential equations. Niazi et al. [4], Iqbal et al. [5], Shafqat et al. [6], Alnahdi et al. [7], Khan et al. [8] and Abuasbeh et al. [9–12] investigated the existence and uniqueness of the fuzzy fractional evolution equations.

The Cauchy problem for NS equations in incompressible fluids is given by

$$\begin{cases} \partial_\upsilon^\gamma\tilde{u} - w\triangle\tilde{u} + (\tilde{u}.\nabla)\tilde{u} = -\nabla p + g, \ \upsilon > 0, \\ \nabla.\tilde{u} = 0, \\ \tilde{u}|_{\partial\Omega} = 0, \\ \tilde{u}(\upsilon, x) = axsin(\gamma) + bycos(\gamma). \end{cases} \tag{1.2}$$

If the fluid strikes orthogonally, i.e., if $\gamma = \frac{\pi}{2}$ then we have

$$\tilde{u}(0) = ax.$$

Therefore we have,

$$\begin{cases} \partial_\upsilon^\gamma\tilde{u} - w\triangle\tilde{u} + (\tilde{u}.\nabla)\tilde{u} = -\nabla p + g, \ \upsilon > 0, \\ \nabla.\tilde{u} = 0, \\ \tilde{u}|_{\partial\Omega} = 0, \\ \tilde{u}(0, x) = ax, \end{cases} \tag{1.3}$$

while ∂_υ^γ denotes the fractional order Caputo derivative. At a point $x \in \omega$ and time $\upsilon > 0$ the velocity field vector is represented by $\tilde{u} = (\tilde{u}_1(\upsilon, x), \tilde{u}_2(\upsilon, x), ..., \tilde{u}_n(\upsilon, x))$, $\rho = \rho(\upsilon, x)$ denotes the pressure and υ is the viscosity. The gravitational force or body force is represented by $g = g(\upsilon, x)$ while the initial velocity is represented by ax. First of all we assume that Ω has a smooth boundary so in order to remove the pressure term we must apply, Helmholtz-Leray projector P in 1.3, which gives

$$\begin{cases} \partial_\upsilon^\gamma \tilde{u} - wP\triangle\tilde{u} + P(\tilde{u}.\nabla)\tilde{u} = Pg, \ \upsilon > 0, \\ \nabla.\tilde{u} = 0, \\ \tilde{u}|_{\partial\Omega} = 0, \\ \tilde{u}(0, x) = ax. \end{cases} \tag{1.4}$$

In the divergence-free function space under discussion, the operator $-wP\triangle$ under Dirichlet boundary conditions is mainly the Stokes operator A. So, writing 1.3 in abstract form, we have

$$\begin{cases} {}^cD_\upsilon^\gamma \tilde{u}(\upsilon) = -A\tilde{u} + F(\tilde{u}, w) + Pg, \quad \upsilon > 0, \\ \tilde{u}(0) = ax, \end{cases} \tag{1.5}$$

where $F(\tilde{u}, w)$=$-P(\tilde{u}.\nabla)w$. If the Helmholtz-Leray projection P and the Stokes operator A makes, the solution of (1.5) is also the solution of (1.3). The fundamental goal is to prove the existence and uniqueness of global and local mild solutions of problem (1.5) in $H^{\gamma,r}$. Moreover, we also establish the regularity conclusions which claim that if Pg is Hölder continuous then there is a unique classical solution $\tilde{u}(\upsilon)$ such that $A\tilde{u}$ and $cD_\upsilon^\gamma\tilde{u}(\upsilon)$ hold Hölder continuity in J_r.

2. Preliminaries

In this section, we define the Gamma function, fractional order integral, Riemann-Liouville (RL) fractional derivative, Caputo fractional derivative and some more definitions, lemmas and theorems. For a brief review of fractional calculus definitions and properties, see [13].

Let the half space in $\mathbf{R^n}$, i.e., $\Omega = \mathcal{H} = (x_1, ..., x_n) : x_n > 0$ be the open subset of $\mathbf{R^n}$, where $n \geq 3$. Let $1 < r < \infty$. Then we have the Hödge projection which is a bounded projection P on $(L^r(\Omega)^n)$, the range for which is as follows:

$$C_\sigma^\infty(\mathcal{H}) = \Big(v \ \in (C^\infty(\mathcal{H}))^n : \ \nabla.v = 0\Big), \tag{2.1}$$

and the null space of which is given as

$$v \in (C^\infty(\mathcal{H}))^n : v = \nabla.\phi, \ \ \phi \in C^\infty(\mathcal{H}). \tag{2.2}$$

For a suitable approach, let J_r=$\overline{C_\sigma^\infty(\mathcal{H})}^{|.|_r}$, which is closed subspace of $(L^r(\mathcal{H}))^n$, $A = -\upsilon P\Delta$ is the Stokes operator in J_r containing the domain $D_r(A) = D_r(\Delta) \cap J_r$. The Stokes operator, named George Gabriel Stokes is an unbounded linear operator which is used in the theory of partial differential equations and specifically in the fields of fluid dynamics and electro magnetics;

$$D_r(\Delta) = v \in (W^{2,r}(\mathcal{H})^n) : v|_{\partial\mathcal{H}} = 0.$$

We have to introduce definitions of fractional power spaces that are related to $-A$. For $\gamma > 0$ and $v \in J_r$, define the following:

$$A^{-\gamma}v = \frac{1}{\Gamma(\gamma)} \int_0^\infty \upsilon^{\gamma-1} e^{-\upsilon A} u d\upsilon.$$

Therefore, $A^{-\gamma}$ is bounded [14], just like the injective operator on J_r. Suppose that $A^{-\gamma}$ is the inverse of $A^{-\gamma}$. For $\gamma > 0$ we symbolize the space $H^{\gamma,r}$ by the extent of $A^{-\gamma}$ with the norm

$$|v|_{H^{\gamma,r}} = |A^{\gamma} v|_r.$$

Definition 2.1. The fractional integration of order $\gamma > 0$ for a function f is defined as

$$I_0^{\gamma} f(\upsilon) = \frac{1}{\Gamma(\gamma)} \int_0^{\upsilon} (\upsilon - s)^{\gamma-1} f(s) ds, \quad \upsilon > 0.$$

The RL fractional derivative for a function $v : [0, \infty) \to \mathbf{R^n}$ of order $\gamma \in \mathbf{R^n}$ is defined by

$${}_0^L D_{\upsilon}^{\gamma} v(\upsilon) = \frac{d^n}{d\upsilon^n} (g_{n-\gamma} * v)(\upsilon), \quad \upsilon \geq 0, \quad n - 1 < \gamma < n.$$

The RL fractional order integral is defined as

$$J_{\upsilon}^{\gamma} v(\upsilon) := g_{\gamma} * v(\upsilon) = \frac{1}{\Gamma(\gamma)} \int_0^{\upsilon} (\upsilon - s)^{\gamma-1} v(s) ds, \quad \upsilon \in [0, \mathfrak{I}].$$

Thus by using the definitions of the RL fractional-order integral, we construct the Caputo fractional order differential operator.

Definition 2.2. [15] The Caputo fractional order derivative is defined as follows

$${}_0^c D_{\upsilon}^{\gamma} v(\upsilon) = \frac{d}{d\upsilon}\Big(J_{\upsilon}^{1-\gamma}[v(\upsilon) - v(0)]\Big) = \frac{d}{d\upsilon}\Big(\frac{1}{\Gamma(1-\gamma)} \int_0^{\upsilon} (\upsilon - s)^{-\gamma}[v(s) - v(0)] ds\Big), \quad \upsilon > 0.$$

Definition 2.3. [16] The Mittag-Leffler function was introduced by Magnus Gustaf (Gösta) Mittag-Leffler (Swedish mathematician) in 1902. It is a simple conclusion of the exponential function. Recently, some researchers have been focusing on the Mittag-Leffler function because of its application in the analysis of fractional differential equations (FDEs). It occurs often in the solutions of FDEs and fractional integral equations. The Mittag-Leffler function with the one-parameter $E_{\gamma}(t)$ is defined as

$$E_{\gamma}(\upsilon) = \sum_{k=0}^{\infty} \frac{\upsilon^k}{\Gamma(\gamma k + 1)}, \ \upsilon \in \mathbb{C}, \ \ \Re(\gamma) > 0.$$

Now, let us consider the generalized Mittag-Leffler functions:

$$E_{\gamma}(-\upsilon^{\gamma} A) = \int_0^{\infty} M_{\gamma}(s) e^{-s\upsilon^{\gamma} A} ds$$

and

$$E_{\gamma,\gamma}(-\upsilon^{\gamma} A) = \int_0^{\infty} \gamma s M_{\gamma}(s) e^{-s\upsilon^{\gamma} A} ds,$$

where

$$M_{\gamma}(\upsilon) := \sum_{n=0}^{\infty} \frac{(-\upsilon)^n}{n!\Gamma(-\gamma(n) + 1 - \gamma)}.$$

The function M_{γ} is also known as the Mainardi function. In order to distinguish between the fundamental solutions for some standard boundary value problems, Mainardi introduced such a type of functions that is a special type of Wright function. It is impressive that the Mainardi function plays a role as a bridge between the classical abstract theories and fractional theories.

Theorem 2.1. *Suppose that $f_n : \mathbf{R^n} \to [-\infty, \infty]$ denotes (Lebesgue) measurable functions such that the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists. Assume that there is an integrable $g : \mathbf{R^n} \to [0, \infty]$ with $|f_n(x)| \le g(x)$ for each $x \in \mathbf{R^n}$. Then, f is integrable as is f_n for each n, and*

$$\lim_{n\to\infty} \int_{\mathbf{R^n}} f_n d\mu = \int_{\mathbf{R^n}} \lim_{n\to\infty} f_n d\mu = \int_{\mathbf{R^n}} f d\mu.$$

(i) $E_{\gamma,\gamma}(-\upsilon^\gamma A) = \dfrac{1}{2\pi\iota} \int_{\Gamma_\theta} E_{\gamma,\gamma}(-\mu\upsilon^\gamma)(\mu I + A)^{-1} d\mu;$

(ii) $A^\gamma E_{\gamma,\gamma}(-\upsilon^\gamma A) = \dfrac{1}{2\pi\iota} \int_{\Gamma_\theta} \mu^\gamma E_{\gamma,\gamma}(-\mu\upsilon^\gamma)(\mu I + A)^{-1} d\mu.$

Proof. See the results in [17]. □

Let $\gamma \in (0, 1)$ and $-1 < r < \infty$, $\lambda > 0$. Then Mainardi function possess the following properties:

(i) $M_\gamma(\upsilon) \ge 0$ for all $\upsilon \ge 0$;

(ii) $\int_0^\infty \upsilon^r M_\gamma(\upsilon) d\upsilon = \dfrac{\Gamma(r+1)}{\Gamma(\gamma r + 1)}$;

(iii) $\mathcal{L}\{\gamma\upsilon M_\gamma(\upsilon)\}(z) = E_{\gamma,\gamma}(-z)$;

(iv) $\mathcal{L}\{M_\gamma(\upsilon)\}(z) = E_\gamma(-z)$;

(v) $\mathcal{L}\{\gamma\upsilon^{-(1+\gamma)} M_\gamma(\upsilon^{-\gamma})\}(\lambda) = e^{-\lambda^\gamma}$.

Proof. The proof of this proposition can be found in [18]. □

Lemma 2.1. *For $\upsilon > 0$, the operators $E_\gamma(-\upsilon^\gamma A)$ and $E_{\gamma,\gamma}(-\upsilon^\gamma A)$ in the uniform operator topology are continuous and well defined from $\mathbf{X}$ to $\mathbf{X}$. Then continuity is uniform on $[r, \infty)$ for every $r > 0$.*

Lemma 2.2. *[19] Let $0 < \gamma < 1$. Then*

(i) $\forall\, v \in X, lim_{t\to 0^+} E_\gamma(-\upsilon^\gamma A)v = v$;

(ii) $\forall\, v \in D(A)$ *and* $\upsilon > 0$, ${}^C D_\upsilon^\gamma E_\gamma(-\upsilon^\gamma A)v = -A\, E_\gamma(-\upsilon^\gamma A)v$;

(iii) $\forall\, v \in X$ *and* $E'_\gamma(-\upsilon^\gamma A)v = -\upsilon^{\gamma-1} A E_{\gamma,\gamma}(-\upsilon^\gamma A)v$;

(iv) *for* $\upsilon > 0$, $E_\gamma(-\upsilon^\gamma A)v = I_\upsilon^{1-\gamma}\{(\upsilon^{\gamma-1} E_{\gamma,\gamma}(-\upsilon^\gamma A)u)\}$.

Lemma 2.3. *Suppose that $1 < r < \infty$ and $\gamma_1 \le \gamma_2$. Then, there exist a constant $C = C(\gamma_1, \gamma_2)$ such that*

$$|e^{-\upsilon A} v|_{H^{\gamma_2,r}} \le C\upsilon^{-(\gamma_2-\gamma_1)} |v|_{H^{\gamma_1,r}} \;\; as \;\; \upsilon > 0 \;\; for\ v \in H^{\gamma_1,r}.$$

Moreover, $\lim_{\upsilon\to 0} \upsilon^{(\gamma_2-\gamma_1)} |e^{-\upsilon A} v|_H^{\gamma_2,r} = 0.$

Lemma 2.4. *Suppose that $1 < r < \infty$ and $\gamma_1 \le \gamma_2$. For any $\mathfrak{I} > 0$, there exists a constant $C_1 = C_1(\gamma_1, \gamma_2)$ such that*

$$|E_\alpha(-\upsilon^\gamma A)|_{H^{\gamma_2,r}} \le C_1 \upsilon^{-\alpha(\gamma_2-\gamma_1)} |v|_{H^{\gamma_1,r}}$$

and

$$|E_{\gamma,\gamma}(-\upsilon^\gamma A)|_{H^{\gamma_2,r}} \le C_1 \upsilon^{-\gamma(\gamma_2-\gamma_1)} |v|_{H^{\gamma_1,r}}$$

for all $v \in H^{\gamma_1,r}$ and $\upsilon \in (0, T]$. Therefore, $\lim_{\upsilon\to 0} \upsilon^{\alpha(\gamma_2-\gamma_1)} |E_\gamma(-\upsilon^\gamma A)v|_{H^{\gamma_2,r}} = 0$.

Proof. The proof of this lemma can be studied in [17]. □

Theorem 2.2. *If $f(\upsilon)$ is defined on the interval $[c, d]$ is Riemann-integrable, then $|f(\upsilon)|$ is also Riemann integrable defined on the interval $[c, d]$ and*

$$\left| \int_c^d f(\upsilon)d\upsilon \right| \leq \int_c^d |f(\upsilon)|d\upsilon.$$

Theorem 2.3. *Suppose that $f : [a, b] \to \mathbf{R^n}$ is continuous and $g : I \to \mathbf{R^n}$ is continuously differentiable with image $g(I) \subset [a, b]$, where $I \subset \mathbf{R^n}$ is some open interval showing that the function*

$$F(s) = -\int_a^{g(s)} f(\upsilon)d\upsilon$$

is continuously differentiable on I.

Theorem 2.4. *Let $\Im(\upsilon) : \upsilon \geq 0 \subset \mathcal{X}$ be a C_0 semi group on X. Then, the following holds*

(i) *If $C : D(G) \subset X \to X$, then G is said to be dense and close defined by linear operators. Therefore, $\upsilon \in [0, \infty) \to \Im(\upsilon)x \in X$ is continuously differentiable for any $x \in D(G)$:*

$$\frac{d}{d\upsilon}\Im(\upsilon)x = G\Im(\upsilon)x = \Im(\upsilon)Gx, \ for \ \upsilon > 0.$$

(ii) *Then there exists $\sigma > 0$ such that $Re(\lambda) > 0$; given $\lambda \in \rho(C)$, we have*

$$(\lambda - C)^{-1}x = \int_0^\infty e^{-\lambda\upsilon}\Im(\upsilon) \ x \ d\upsilon \ for \ all \ x \in X.$$

Theorem 2.5. *Let $\gamma \in (0, 1]$ and suppose that the positive sectorial operator is $A : D(A) \subset X \to X$. Thus, the operators $\{E_\gamma(-\upsilon^\gamma A) : \upsilon \geq 0\}$ and $\{E_{\gamma,\gamma}(-\upsilon^\gamma A) : \upsilon \geq 0\}$ as follows:*

$$E_\gamma(-\upsilon^\gamma A) = \int_0^\infty M_\gamma(s)\Im^{s\upsilon^\gamma A}ds, \ \upsilon \geq 0$$

and

$$E_{\gamma,\gamma}(-\upsilon^\gamma A) = \int_0^\infty \gamma s M_\gamma(s)\Im^{s\upsilon^\gamma}ds, \ \upsilon \geq 0,$$

where $\Im(\upsilon) : \upsilon \geq 0$ defines the C_0 semigroup which is generated by $-A$.

Let $\gamma \in (0, 1)$ and consider the $A : D(A) \subset X \to X$ is a positive sectorial operator. Then for any $x \in X$, it holds that

$$\begin{aligned} \mathcal{L}\{E_\gamma(-\upsilon^\gamma A)x\}(\lambda) &= \lambda^{\gamma-1}(\lambda^\gamma + A)^{-1}x, \\ \mathcal{L}\{E_{\gamma,\gamma}(-\upsilon^\gamma A)x\}(\lambda) &= (\lambda^\gamma + A)^{-1}x. \end{aligned}$$

Proof. The first equality can be proved analogously the second equality is for any $x \in X$; observe that by Theorem 2.4,

$$\mathcal{L}\{E_{\gamma,\gamma}(-\upsilon^\gamma A)x\}(\lambda) = \int_0^\infty e^{-\lambda\upsilon}\upsilon^{\gamma-1}E_{\gamma,\gamma}(-\upsilon^\gamma A)xd\upsilon$$

$$= \int_0^\infty e^{-\lambda\upsilon}\upsilon^{\gamma-1}(\int_0^\infty \gamma s M_\gamma(s)\Im(s\upsilon^\gamma)xds)d\upsilon.$$

Now by using $s = \omega\upsilon^{-\gamma}$, we conclude the following

$$\begin{aligned}
\mathcal{L}\{E_{\gamma,\gamma}(-\upsilon^\gamma A)x\}(\lambda) &= \int_0^\infty e^{-\lambda\upsilon}\upsilon^{\gamma-1}(\int_0^\infty \gamma(\omega\upsilon^{-\gamma})M_\gamma(\omega\upsilon^{-\gamma})T(\omega)x\upsilon^{-\gamma}d\omega)d\upsilon \\
&= \int_0^\infty \omega\Big(\int_0^\infty \gamma\upsilon^{-(1+\gamma)}M_\gamma(\omega\upsilon^{-\gamma})e^{-\lambda\upsilon}d\upsilon\Big)T(\omega)xd\omega.
\end{aligned}$$

Choose

$$H^* = \int_0^\infty \gamma\upsilon^{-(1+\gamma)}M_\gamma(\omega\upsilon^{-\gamma})e^{-\lambda\upsilon}.$$

By taking $\upsilon = \tau\omega^{\frac{1}{\gamma}}$ from Lemma 2.4, we have that

$$\begin{aligned}
H^* &= \int_0^\infty \gamma(\tau\omega^{\frac{1}{\gamma}})^{-(1+\gamma)}M_\gamma(\omega(\tau\omega^{\frac{1}{\gamma}})^{-\gamma})e^{-\lambda(\tau\omega^{\frac{1}{\gamma}})}\omega^{\frac{1}{\gamma}}d\tau \\
&= \omega^{-1}\int_0^\infty \gamma\tau^{-(1+\gamma)}M_\gamma(\tau^{-\gamma})e^{-(\lambda\omega^{\frac{1}{\gamma}})}d\tau \\
&= \omega^{-1}e^{-\lambda^\gamma\omega}.
\end{aligned}$$

Therefore, by Theorem 2.5, we have

$$\mathcal{L}\{E_{\gamma,\gamma}(-\upsilon^\gamma A)x\}(\lambda) = \int_0^\infty e^{-\lambda^\gamma\omega}\Im(\omega)xd\omega = (\lambda^\gamma + A)^{-1}x.$$

□

Lemma 2.5. *If $\tilde{u}(\upsilon)$ is the solution of (1.5) for $\tilde{u}(0) = ax$, then $\tilde{u}(\upsilon)$ is given as*

$$\tilde{u}(\upsilon) = a\upsilon + \frac{1}{\Gamma(\gamma)}\int_0^\upsilon (\upsilon - s)^{\gamma-1}(A\tilde{u}(s) + h(s))ds \text{ as } \upsilon \geq 0;$$

therefore we get

$$\tilde{u}(\upsilon) = \int_0^\upsilon E_\gamma(-(\upsilon - s)^\gamma A)ads + \int_0^\upsilon (\upsilon - s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A)h(s)ds,$$

where

$$E_\gamma(\upsilon) = \int_0^\infty q(\theta)M_\gamma(\theta)Q(\upsilon^\gamma\theta)d\theta.$$

Proof. [20] By using the above lemma rewriting (1.5) and applying the RL derivative, we get

$$\begin{aligned}
\tilde{u}(\upsilon) &= \tilde{u}(0) + \frac{1}{\Gamma(\gamma)}\int_0^\upsilon (\upsilon - s)^{\gamma-1}(-A\tilde{u}(s) + F(u(s), w(s)) + Pg(s))ds \text{ as } \upsilon \geq 0 \\
\tilde{u}(\upsilon) &= a\upsilon + \frac{1}{\Gamma(\gamma)}\int_0^\upsilon (\upsilon - s)^{\gamma-1}(-A\tilde{u}(s) + F(\tilde{u}(s), w(s)) + Pg(s))ds \text{ while } \upsilon \geq 0.
\end{aligned}$$

By applying the Laplace transform, we have the following

$$\tilde{u}(\lambda) = \frac{a}{\lambda^2} + \frac{1}{\lambda^\gamma}\{-A\tilde{u}(\lambda)\} + \frac{1}{\lambda^\gamma}\{F\tilde{u}(\lambda), w(\lambda)\} + \frac{1}{\lambda^\gamma}\{Pg(\lambda)\}.$$

Simplification yields

$$\begin{aligned}(\lambda^\gamma + A)\tilde{u}(\lambda) &= a\lambda^{\gamma-2} + F(\tilde{u}(\lambda), w(\lambda)) + Pg(\lambda)\\ \tilde{u}(\lambda) &= a\lambda^{\gamma-2}(\lambda^\gamma + A)^{-1} + F(\tilde{u}(\lambda), w(\lambda))(\lambda^\gamma + A)^{-1} + Pg(\lambda)(\lambda^\gamma + A)^{-1}\\ \tilde{u}(\lambda) &= a\lambda^{-1}\lambda^{\gamma-1}(\lambda^\gamma + A)^{-1} + F(\tilde{u}(\lambda), w(\lambda))(\lambda^\gamma + A)^{-1} + Pg(\lambda)(\lambda^\gamma + A)^{-1}.\end{aligned}$$

By taking the inverse Laplace transform and applying the convolution theorem, we get

$$\begin{aligned}\tilde{u}(\upsilon) &= \int_0^\upsilon E_\gamma(-(\upsilon - s)^\gamma A)a ds + \int_0^\upsilon (\upsilon - s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A)F(\tilde{u}(s), w(s))ds\\ &\quad + \int_0^\upsilon (\upsilon - s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A)Pg(s)ds.\end{aligned}$$

□

Definition 2.4. A Problem (1.5) has global mild solution of a function $\tilde{u} : [0, \infty) \to H^{\gamma,r}$ in $H^{\gamma,r}$, if $\tilde{u} \in C([0, \infty), H^{\gamma,r})$ and, for $\upsilon \in [0, \infty)$,

$$\begin{aligned}\tilde{u}(\upsilon) &= \int_0^\upsilon E_\gamma(-(\upsilon - s)^\gamma A)a ds + \int_0^\upsilon (\upsilon - s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A)F(\tilde{u}(s), w(s))ds\\ &\quad + \int_0^\upsilon (\upsilon - s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A)Pg(s)ds.\end{aligned}$$

Definition 2.5. Let $0 < T < \infty$. A Problem (1.5) has local mild solution of a function $\tilde{u} : [0, T] \to H^{\gamma,r}$ in $H^{\gamma,r}$ if $\tilde{u} \in ([0, T], H^{\gamma,r})$ and for $\upsilon \in [0, T]$, $\tilde{u}$ satisfies the Definition [2.4]. Conveniently, we respectively define three operators $\tilde{\psi}(\upsilon)$, $\tilde{\varphi}(\upsilon)$ and $\tilde{\varpi}(\tilde{u}, w)(\upsilon)$:

$$\begin{aligned}\tilde{\psi}(\upsilon) &= \int_0^\upsilon E_\gamma(-(\upsilon - s)^\gamma A)a ds\\ \tilde{\varphi}(\upsilon) &= \int_0^t (t - s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A)Pg(s)ds\\ \tilde{\varpi}(\tilde{u}, w)(\upsilon) &= \int_0^\upsilon (\upsilon - s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A)F(\tilde{u}(s), w(s))ds.\end{aligned}$$

3. Global and local existence of mild solutions in $H^{\gamma,r}$

To provide suitable circumstances for the existence and uniqueness of mild solutions [21] to the Problem (1.5) in $H^{\gamma,r}$, we consider this section. Suppose that we have **(K)**. For $\upsilon > 0$, Pg is continuous and

$$|pg(\upsilon)|_r = 0(\upsilon^{-\gamma(1-\beta)})\ for\ 0 < \beta < 1\ and\ \upsilon \to 0. \tag{3.1}$$

3.1. Global existence in $H^{\gamma,r}$

To deal with the mild solution of Problem (1.5) with global existence in $H^{\gamma,r}$, let

$$\mathcal{M}(\upsilon) = \sup_{s\in(0,\upsilon]} \left(s^{\gamma(1-\beta)} |Pg(s)|_r \right)$$

and

$$\begin{aligned} \mathcal{B}_1 &= C_1 maxB(\gamma(1-\beta), 1-\gamma(1-\beta)), B(\gamma(1-\alpha), 1-\gamma(1-\beta)) \\ \mathcal{L} &\geq \mathcal{M}C_1 maxB(\gamma(1-\beta), 1-2\gamma(\alpha-\beta)), B(\gamma(1-\alpha), 1-2\gamma(\alpha-\beta)). \end{aligned}$$

Theorem 3.1. *Let $1 < r < \infty$, $0 < \gamma < 1$ and (3.1) hold; then, for all at $\in H^{\gamma,r}$. Suppose that*

$$C_1|a\upsilon|_{H^{\gamma,r}} + \mathcal{B}_1\mathcal{M}_\infty < \frac{1}{4L}.$$

If a unique function $\tilde{u} : [0,\infty) \to H^{\gamma,r}$ is satisfied and we choose $\frac{n}{2r} - \frac{1}{2} < \beta$, then there is $\alpha > max(\beta, \frac{1}{2})$ where $\mathcal{M}_\infty = \sup_{s\in[0,\infty)} (s^{\gamma(1-\beta)}Pg(s))$:

(i) *$\tilde{u}(0) = ax$ and $\tilde{u} : [0,\infty) \to H^{\gamma,r}$ shows continuity;*
(ii) *$\tilde{u} : [0,\infty) \to H^{\alpha,r}$ is continuous and $\lim_{\upsilon\to 0} \upsilon^{\gamma(\alpha-\beta)}|\tilde{u}(\upsilon)|_{H^{\alpha,r}} = 0$;*
(iii) *For $\upsilon \in [0,\infty)$, $\tilde{u}$ satisfies Definition 2.4.*

Proof. Here we explain X_∞ which is subspace of all of the curves and $X_\infty = X[\infty]$, $\tilde{u} : (0,\infty) \to H^{\gamma,r}$. Now suppose that $\alpha = \frac{1+\beta}{2}$ such that the following is true:

(i) $\tilde{u} : [0,\infty) \to H^{\gamma,r}$ is continuous and bounded.
(ii) $\tilde{u} : (0,\infty) \to H^{\alpha,r}$ is continuous and bounded; furthermore, $\lim_{\upsilon\to 0} \upsilon^{\gamma(\alpha-\beta)}|\tilde{u}(\upsilon)|_{H^{\alpha,r}} = 0$ and its genuine form is given by

$$\|\tilde{u}\|_{X_\infty} = \max\left(\sup_{\upsilon\geq 0} |\tilde{u}(\upsilon)|_{H^{\gamma,r}}, \sup_{t\geq 0} \upsilon^{\gamma(\alpha-\beta)}|\tilde{u}(\upsilon)|_{H^{\alpha,r}} \right).$$

Now, we know that there exists $\mathcal{M}$ such that $\tilde{u}, w \in H^{\alpha,r}$ such that $F : H^{\alpha,r} * H^{\alpha,r} \to J_r$ is bounded and bilinear mapping:

$$\begin{aligned} |F(\tilde{u}, w)|_r &\leq \mathcal{M}|\tilde{u}|_{H^{\alpha,r}}|w|_{H^{\alpha,r}} \\ |F(\tilde{u},\tilde{u}) - F(w,w)|_r &\leq \mathcal{M}(|\tilde{u}|_{H^{\alpha,r}} + |w|_{H^{\alpha,r}})|\tilde{u} - w|_{H^{\alpha,r}}. \end{aligned}$$

Step 1:
Consider that $\tilde{u}, w \in X_\infty$. The term $\tilde{\varpi}(\tilde{u}(\upsilon), w(\tau))$ is a part of $C(0,\infty), H^{\gamma,r}$ and C $([0,\mathfrak{I}], H^{\gamma,r})$. By considering $\varepsilon > 0$ that is very small and randomly fixing $\upsilon_0 \geq 0$, now again suppose that $\upsilon > \upsilon_0$ ($\upsilon < \upsilon_0$ *analogously*); we have

$$\begin{aligned} &\left|\tilde{\varpi}(\tilde{u}(\upsilon), w(\upsilon)) - \tilde{\varpi}(\tilde{u}(\upsilon_0), w(\upsilon_0))\right|_{H^{\gamma,r}} ds \\ \leq &\int_{\upsilon_0}^{\upsilon} (\upsilon - s)^{\gamma-1} \left|E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A)F(\tilde{u}(s), w(s))\right|_{H^{\gamma,r}} ds \end{aligned}$$

$$
\begin{aligned}
& + \int_0^{\upsilon_0} \Big|(\upsilon - s)^{\gamma-1} - (\upsilon_0 - s)^{\gamma-1} E_{\gamma,\gamma}(- (\upsilon - s)^{\gamma} A) F(\tilde{u}(s), w(s))\Big|_{H^{\gamma,r}} ds \\
& + \int_0^{\upsilon_0 - \epsilon} (\upsilon_0 - s)^{\gamma-1} |E_{\gamma,\gamma}(- (\upsilon - s)^{\gamma} A) - E_{\gamma,\gamma}(- (\upsilon_0 - s)^{\gamma} A) F(\tilde{u}(s), w(s))|_{H^{\gamma,r}} ds \\
& + \int_{\upsilon_0 - \epsilon}^{\upsilon_0} (\upsilon_0 - s)^{\gamma-1} \Big|E_{\gamma,\gamma}(- (\upsilon - s)^{\gamma} A) - E_{\gamma,\gamma}(- (\upsilon_0 - s)^{\gamma} A) F(\tilde{u}(s), w(s))\Big|_{H^{\gamma,r}} ds \\
= \;& \mathcal{D}_{11}(\upsilon) + \mathcal{D}_{12}(\upsilon) + \mathcal{D}_{13}(\upsilon) + \mathcal{D}_{14}(\upsilon).
\end{aligned}
$$

In the view of Lemma 2.4, we consider every term separately for $\mathcal{D}_{11}(\upsilon)$ and get the following

$$
\begin{aligned}
\mathcal{D}_{11}(\upsilon) & \leq C_1 \int_{\upsilon_0}^{\upsilon} (\upsilon - s)^{\gamma(1-\beta)-1} |F(\tilde{u}(s), w(s))|_r ds \\
& \leq \mathcal{M} C_1 \int_{\upsilon_0}^{\upsilon} (\upsilon - s)^{\gamma(1-\beta)-1} |(\tilde{u}(s)|_{H^{\alpha,r}}, |w(s))|_{H^{\alpha,r}} ds \\
& \leq \mathcal{M} C_1 \int_{\upsilon_0}^{\upsilon} (\upsilon - s)^{\gamma(1-\beta)-1} s^{-2\gamma(\alpha-\beta)} ds \sup_{s \in (0,\upsilon]} \Big(s^{2\gamma(\alpha-\beta)} |\tilde{u}(s)|_{H^{\alpha,r}} |w(s)|_{H^{\alpha,r}} \Big)
\end{aligned}
$$

$$
= \mathcal{M} C_1 \int_{\upsilon_0/\upsilon}^{1} (1 - s)^{\gamma(1-\beta)-1} s^{-2\gamma(\alpha-\beta)} ds \sup_{s \in (0,\upsilon]} \Big(s^{2\gamma(\alpha-\beta)} |\tilde{u}(s)|_{H^{\alpha,r}} |w(s)|_{H^{\alpha,r}} \Big).
$$

By applying the β function properties, $\exists\, \delta > 0$ much smaller as $0 < \upsilon - \upsilon_0 < \delta$; we have

$$
\int_{\upsilon_0/\upsilon}^{1} (1 - s)^{\gamma(1-\beta)-1} s^{-2\gamma(\alpha-\beta)} ds \to 0
$$

which follows that as $\upsilon - \upsilon_0 \to 0$, $\mathcal{D}_{11}(\upsilon)$ tends to 0.

Now for $\mathcal{D}_{12}(\upsilon)$,

$$
\begin{aligned}
\mathcal{D}_{12}(\upsilon) & = C_1 \int_0^{\upsilon_0} ((\upsilon_0 - s)^{\gamma-1} - (\upsilon - s)^{\gamma-1})(\upsilon - s)^{-\beta\gamma} |F(\tilde{u}(s), w(s))|_r ds \\
& \leq \mathcal{M} C_1 \int_0^{\upsilon_0} ((\upsilon_0 - s)^{\gamma-1} - (\upsilon - s)^{\gamma-1})(\upsilon - s)^{-\beta\gamma} s^{-2\gamma(\alpha-\beta)} ds \sup_{s \in (0,\upsilon_0]} \Big(s^{2\gamma(\alpha-\beta)} |\tilde{u}(s)|_{H^{\alpha,r}} |w(s)|_{H^{\alpha,r}} \Big).
\end{aligned}
$$

It is interesting to note that

$$
\begin{aligned}
& \int_0^{\upsilon_0} \Big|(\upsilon_0 - s)^{\gamma-1} - (\upsilon - s)^{\gamma-1}\Big| (\upsilon - s)^{-\beta\gamma} s^{-2\gamma(\alpha-\beta)} ds \\
\leq \;& \int_0^{\upsilon_0} (\upsilon - s)^{\gamma-1} (\upsilon - s)^{-\beta\gamma} s^{-2\gamma(\alpha-\beta)} ds + \int_0^{\upsilon_0} (\upsilon_0 - s)^{\gamma-1} (\upsilon - s)^{-\beta\gamma} s^{-2\gamma(\alpha-\beta)} ds \\
\leq \;& 2 \int_0^{\upsilon_0} (\upsilon_0 - s)^{\gamma(1-\beta)-1} (\upsilon - s)^{-\beta\gamma} s^{-2\gamma(\alpha-\beta)} ds \\
= \;& 2\mathcal{B}(\gamma(1-\beta)), 1 - 2\gamma(\alpha - \beta).
\end{aligned}
$$

We can show this by applying Lebesgue's dominated convergence Theorem 2.1, we get

$$
\int_0^{\upsilon_0} ((\upsilon_0 - s)^{\gamma-1} - (\upsilon - s)^{\gamma-1})(\upsilon - s)^{-\beta\gamma} s^{-2\gamma(\alpha-\beta)} ds \to 0, \quad as\ \upsilon \to \upsilon_0.
$$

We can say that $\lim_{\upsilon\to\upsilon_0} \mathcal{D}_{12}(\upsilon) = 0$. For $\mathcal{D}_{13}(\upsilon)$, we have

$$\begin{aligned}
\mathcal{D}_{13}(\upsilon) &\leq \int_0^{\upsilon_0-\epsilon} (\upsilon_0 - s)^{\gamma-1}\Big|E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A) - E_{\gamma,\gamma}(-(\upsilon_0 - s)^\gamma A)F(\tilde{u}(s), w(s))\Big|_{H^{\gamma,r}} ds \\
&\leq \int_0^{\upsilon_0-\epsilon} (\upsilon_0 - s)^{\gamma-1}\Big((\upsilon - s)^{-\beta\gamma} + (\upsilon_0 - s)^{-\beta\gamma}\Big)|F(\tilde{u}(s), w(s))|_r ds \\
&\leq 2\mathcal{M}C_1 \int_0^{\upsilon_0} (\upsilon_0 - s)^{\gamma-1} s^{-2\gamma(\alpha-\beta)} ds \sup_{s\in(0,\upsilon_0]} (s^{2\gamma(\alpha-\beta)}|\tilde{u}(s)|_{H^{\alpha,r}}|w(s)|_{H^{\alpha,r}}).
\end{aligned}$$

Considering the uniform continuity, and by using Lebesgue's dominated convergence theorem, we get $\mathcal{D}_{13}(\upsilon)$ as follows

$$\begin{aligned}
\lim_{\upsilon\to\upsilon_0} \mathcal{D}_{13}(\upsilon) &= \int_0^{\upsilon_0-\epsilon} (\upsilon_0 - s)^{\gamma-1}|E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A) - E_{\gamma,\gamma}(-(\upsilon_0 - s)^\gamma A)F(\tilde{u}(s), w(s))|_{H^{\gamma,r}} ds \\
&= 0.
\end{aligned}$$

For $\mathcal{D}_{14}(\upsilon)$, using calculations, we estimate that, according to the β function properties

$$\begin{aligned}
\mathcal{D}_{14}(\upsilon) &\leq \int_{\upsilon_0-\epsilon}^{\upsilon_0} (\upsilon_0 - s)^{\gamma-1}((\upsilon - s)^{-\beta\gamma} + (\upsilon_0 - s)^{-\beta\gamma})\Big|F(\tilde{u}(s), w(s))\Big|_r ds \\
&\leq 2\mathcal{M}C_1 \int_0^{\upsilon_0} (\upsilon_0 - s)^{\gamma-1} s^{-2\gamma(\alpha-\beta)} ds \\
&\quad \sup_{s\in[\upsilon_0-\epsilon,\upsilon_0]} (s^{2\gamma(\alpha-\beta)}|\tilde{u}(s)|_{H^{\alpha,r}}|w(s)|_{H^{\alpha,r}}) \to 0 \ for\ \varepsilon \to 0.
\end{aligned}$$

It follows that

$$|\tilde{\varpi}(\tilde{u}(\upsilon), w(\upsilon)) - \tilde{\varpi}(\tilde{u}(\upsilon_0), w(\upsilon_0))|_{H^{\gamma,r}} ds \to 0 \ while\ \upsilon \to \upsilon_0.$$

The operator's continuity $\tilde{\varpi}(\tilde{u}, w)$ can be demonstrated in $C((0,\infty), H^{\alpha,r})$ following the preceding debate.

Step 2:

To show that $\tilde{\varpi} : X_\infty * X_\infty \to X_\infty$ is a continuous bilinear operator, we consider that, according to Lemma 2.4, we have

$$\begin{aligned}
\Big|\tilde{\varpi}(\tilde{u}(\upsilon), w(\upsilon))\Big|_{H^{\gamma,r}} &= \Big|\int_0^\upsilon (\upsilon - s)^{\gamma-1}|E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A)F(\tilde{u}(s), w(s))\Big|_{H^{\gamma,r}} ds \\
&\leq C_1 \int_0^\upsilon (\upsilon - s)^{\gamma(1-\beta)-1}\Big|F(\tilde{u}(s), w(s))\Big|_r ds \\
&\leq \mathcal{M}C_1 \int_0^\upsilon (\upsilon - s)^{\gamma(1-\beta)-1} s^{-2\gamma(\alpha-\beta)} ds \sup_{s\in(0,\upsilon]} (s^{2\gamma(\alpha-\beta)}|\tilde{u}(s)|_{H^{\alpha,r}}|w(s)|_{H^{\alpha,r}}) \\
&= \mathcal{M}C_1\mathcal{B}((\gamma(1-\beta)), 1 - 2\gamma(\alpha-\beta))\|u\|_{X_\infty}\|w\|_{X_\infty}
\end{aligned}$$

and

$$\Big|\tilde{\varpi}(\tilde{u}(\upsilon), w(\upsilon))\Big|_H^{\alpha,r} = \Big|\int_0^\upsilon (\upsilon - s)^{\gamma-1}\Big|E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A)F(\tilde{u}(s), w(s))\Big|_{H^{\alpha,r}} ds$$

$$
\begin{aligned}
&\leq C_1 \int_0^{\upsilon} (\upsilon - s)^{\gamma(1-\alpha)-1} |F(\tilde{u}(s), w(s))|_r ds \\
&\leq \mathcal{M}C_1 \int_0^{\upsilon} (\upsilon - s)^{\gamma(1-\alpha)-1} s^{-2\gamma(\alpha-\beta)} ds \sup_{s\in(0,\upsilon]} (s^{2\gamma(\alpha-\beta)} |\tilde{u}(s)|_{H^{\alpha,r}} |w(s)|_{H^{\alpha,r}}) \\
&= \mathcal{M}C_1 \upsilon^{-\gamma(\alpha-\beta)} \mathcal{B}((\gamma(1-\alpha)), 1 - 2\gamma(\alpha-\beta)) \|\tilde{u}\|_{X_\infty} \|w\|_{X_\infty}.
\end{aligned}
$$

As a result of this

$$
\sup_{\upsilon\in[0,\infty)} \upsilon^{\gamma(\alpha-\beta)} |\tilde{\varpi}(\tilde{u}(\upsilon), w(\upsilon))|_{H^{\alpha,r}} \leq \mathcal{M}C_1 \mathcal{B}(\gamma(1-\alpha)), 1 - 2\gamma(\alpha-\beta) \|\tilde{u}\|_{X_\infty} \|w\|_{X_\infty}.
$$

To be more specific,

$$
\lim_{\upsilon\to 0} \upsilon^{\gamma(\alpha-\beta)} |\tilde{\varpi}(\tilde{u}(\upsilon), w(\upsilon))|_{H^{\alpha,r}} = 0.
$$

Therefore, $\tilde{\varpi}(\tilde{u}, w) \in X_\infty$ and $\|\tilde{\varpi}(\tilde{u}(\upsilon), w(\upsilon))\|_{X_\infty} \leq L \|\tilde{u}\|_{X_\infty} \|w\|_{X_\infty}$.
Step 3:
Let $0 < \upsilon_0 < \upsilon$. We have that

$$
\begin{aligned}
\left|\tilde{\varphi}(\upsilon) - \tilde{\varphi}(\upsilon_0)\right|_{H^{\gamma,r}} &\leq \int_{\upsilon_0}^{\upsilon} (\upsilon - s)^{\gamma-1} \left|E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A) Pg(s)\right|_{H^{\gamma,r}} ds \\
&+ \int_0^{\upsilon_0} ((\upsilon_0 - s)^{\gamma-1} - (\upsilon - s)^{\gamma-1}) \left|E_{\gamma,\gamma}(-(t - s)^\gamma A) Pg(s)\right|_{H^{\gamma,r}} ds \\
&+ \int_0^{\upsilon_0-\epsilon} (\upsilon_0 - s)^{\gamma-1} \left|E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A) - E_{\gamma,\gamma}(-(\upsilon_0 - s)^\gamma A) Pg(s)\right|_{H^{\gamma,r}} ds \\
&+ \int_{\upsilon_0-\epsilon}^{\upsilon_0} (\upsilon_0 - s)^{\gamma-1} \left|E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A) - E_{\gamma,\gamma}(-(\upsilon_0 - s)^\gamma A) Pg(s)\right|_{H^{\gamma,r}} ds \\
&\leq C_1 \int_{\upsilon_0}^{\upsilon} (\upsilon - s)^{\gamma(1-\beta)-1} |Pg(s)|_r ds \\
&+ C_1 \int_0^{\upsilon_0} ((\upsilon_0 - s)^{\gamma-1} - (\upsilon - s)^{\gamma-1}) (\upsilon - s)^{-\beta\gamma} |Pg(s)|_r ds \\
&+ C_1 \int_0^{\upsilon_0-\epsilon} (\upsilon_0 - s)^{\gamma-1} \left|E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A) - E_{\gamma,\gamma}(-(\upsilon_0 - s)^\gamma A) Pg(s)\right|_{H^{\gamma,r}} ds \\
&+ 2C_1 \int_{\upsilon_0-\epsilon}^{\upsilon_0} (\upsilon_0 - s)^{\gamma(1-\beta)-1} |Pg(s)|_r ds \\
&\leq C_1 M(\upsilon) \int_{\upsilon_0}^{\upsilon} (\upsilon - s)^{\gamma(1-\beta)-1} s^{-\gamma(1-\beta)} ds \\
&+ C_1 M(\upsilon) \int_0^{\upsilon_0} ((\upsilon - s)^{\gamma-1} - (\upsilon_0 - s)^{\gamma-1}) (\upsilon - s)^{-\beta\gamma} s^{-\gamma(1-\beta)} ds \\
&+ C_1 C(\upsilon) \int_0^{\upsilon_0-\epsilon} (\upsilon_0 - s)^{\gamma-1} \left|E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A) - E_{\gamma,\gamma}(-(\upsilon_0 - s)^\gamma A)\right|_{H^{\gamma,r}} ds \\
&+ 2C_1 C(\upsilon) \int_{\upsilon_0-\epsilon}^{\upsilon_0} (\upsilon_0 - s)^{\gamma(1-\beta)-1} s^{-\gamma(1-\beta)} ds.
\end{aligned}
$$

The first two integrals and last integral tend to 0 as $\upsilon \to \upsilon_0$; also, $\varepsilon \to 0$ by using the characteristics of the β function. Now the third integral also tends to 0 as $\upsilon \to \upsilon_0$ by using Lemma 2.1. This suggests that

$$|\tilde{\varphi}(\upsilon)-\tilde{\varphi}(\upsilon_0)|_{H^{\gamma,r}}\to 0 \;\; while \;\; \upsilon\to\upsilon_0.$$

In order to evaluate the continuity of $\tilde{\varphi(\upsilon)}$ in $H^{\alpha,r}$, we must go along with the same pattern as in $H^{\gamma,r}$. On the contrary,

$$\begin{aligned}
|\tilde{\varphi}(\upsilon)|_{H^{\gamma,r}}| &= \int_0^{\upsilon}(\upsilon-s)^{\gamma-1}|E_{\gamma,\gamma}\left(-(\upsilon-s)^{\gamma}A\right)Pg(s)|_{H^{\gamma,r}}ds\\
&\le C_1\int_0^{\upsilon}(\upsilon-s)^{\gamma(1-\beta)-1}|Pg(s)|_r ds\\
&\le C_1\mathcal{M}(\upsilon)\int_0^{\upsilon}(\upsilon-s)^{\gamma(1-\beta)-1}s^{-\gamma(1-\beta)}ds\\
&= C_1\mathcal{M}(\upsilon)\mathcal{B}\Big((\gamma(1-\beta)),(1-\gamma(1-\beta))\Big) \qquad (3.2)\\
|\tilde{\varphi}(\upsilon)|_{H^{\alpha,r}}| &= \int_0^{\upsilon}(\upsilon-s)^{\gamma-1}|E_{\gamma,\gamma}(-(\upsilon-s)^{\alpha}A)Pg(s)|_{H^{\alpha,r}}ds\\
&\le C_1\int_0^{\upsilon}(\upsilon-s)^{\gamma(1-\alpha)-1}|Pg(s)|_r ds\\
&\le C_1\mathcal{M}(\upsilon)\int_0^{\upsilon}(\upsilon-s)^{\gamma(1-\alpha)-1}s^{-\gamma(1-\beta)}ds\\
&= \upsilon^{-\gamma(\alpha-\beta)}C_1\mathcal{M}(\upsilon)\mathcal{B}((\gamma(1-\alpha)),(1-\gamma(1-\beta))).
\end{aligned}$$

To be more precise,

$$\upsilon^{\gamma(\alpha-\beta)}|\tilde{\varphi}(\upsilon)|_{H^{\alpha,r}}\le C_1\mathcal{M}(\upsilon)\mathcal{B}((\gamma(1-\alpha)),(1-\gamma(1-\beta)))\to 0 \; while \; \upsilon\to\upsilon_0.$$

Because $\tilde{M}(t)\to 0$ is the same as $\upsilon\to 0$ owing to assumption, this result confirms that $\|\tilde{\varphi}(\upsilon)\|_\infty\le\tilde{B}_1\tilde{M}_\infty$ as $\tilde{\varphi}(t)\in X_\infty$. Now for $a\upsilon\in H^{\gamma,r}$, according to Lemma 2.1 obvious that

$$E_\gamma(-\upsilon^\gamma A)a\in C([0,\infty),H^{\gamma,r})$$

and

$$E_\gamma(-\upsilon^\gamma A)a\in C([0,\infty),H^{\alpha,r}).$$

Therefore by Theorem 2.3, we can say that

$$\int_0^{\upsilon}E_\gamma(-(\upsilon-s)^\gamma A)ads\in C([0,\infty),H^{\gamma,r}) \; and \; \int_0^{\upsilon}E_\gamma(-(\upsilon-s)^\gamma A)ads\in C([0,\infty),H^{\alpha,r}).$$

This implies that for all $\upsilon\in(0,\mathfrak{I}]$, and together with Theorem 2.2,

$$\int_0^{\upsilon}E_\gamma(-(\upsilon-s)^\gamma A)ads\in X_\infty.$$

By using the above condition, we get

$$\upsilon^{\gamma(\alpha-\beta)}\int_0^{\upsilon}E_\gamma(-(\upsilon-s)^\gamma A)ads\in C([0,\infty),H^{\alpha,r})$$

$$\begin{aligned} \| \int_0^{\upsilon} E_\gamma(-(\upsilon-s)^\gamma A) a ds \|_{X_\infty} &\le \int_0^{\upsilon} \|E_\gamma(-(\upsilon-s)^\gamma A) a\|_{X_\infty} ds \\ &\le C_1 \int_0^{\upsilon} |a|_{H^{\gamma,r}} ds \\ &\le C_1 |a|(t)_{H^{\gamma,r}}. \end{aligned}$$

According to Theorem 3.1, the inequality

$$\begin{aligned} \| \int_0^{\upsilon} E_\gamma(-\upsilon^\gamma A) a ds + \tilde{\varphi}(\upsilon) \|_{X_\infty} &\le \| \int_0^{\upsilon} E_\gamma(-\upsilon^\gamma A) a ds \|_{X_\infty} + \|\tilde{\varphi}(\upsilon)\|_{X_\infty} \\ &\le \int_0^{\upsilon} \|E_\gamma(-\upsilon^\gamma A) a\|_{X_\infty} ds + \|\tilde{\varphi}(\upsilon)\|_{X_\infty} \\ &\le \frac{1}{4L} \end{aligned}$$

continues to hold, resulting in F having a unique fixed point.

Step 4:

In order to verify that $\tilde{u}(\upsilon) \to a\upsilon$ in $H^{\gamma,r}$ as $\upsilon \to 0$, we must show that

$$\begin{aligned} \lim_{\upsilon \to 0} \int_0^{\upsilon} E_\gamma(-(\upsilon-s)^\gamma A) a ds &= 0 \\ \lim_{\upsilon \to 0} \int_0^{\upsilon} (\upsilon-s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\upsilon-s)^\gamma A\big) P g(s) ds &= 0 \\ \lim_{\upsilon \to 0} \int_0^{\upsilon} (\upsilon-s)^{\gamma-1} E_{\gamma,\gamma}\big(-(\upsilon-s)^\gamma A\big) F(\tilde{u}(s), w(s)) ds &= 0 \end{aligned}$$

in $H^{\gamma,r}$. In fact $\lim_{\upsilon\to 0} \tilde{\psi}(\upsilon) = 0$ and $\lim_{\upsilon\to 0} \tilde{\varphi}(\upsilon) = 0 (\lim_{\upsilon\to 0} \tilde{M}(\upsilon) = 0)$ due to (3.2). So,

$$\begin{aligned} &\int_0^{\upsilon} (\upsilon-s)^{\gamma-1} \big| E_{\gamma,\gamma}\big(-(\upsilon-s)^\gamma A\big) F(\tilde{u}(s), \tilde{u}(s)) \big|_{H^{\gamma,r}} ds \\ &\le C \int_0^{\upsilon} (\upsilon-s)^{\gamma(1-\beta)-1} |F(\tilde{u}(s), \tilde{u}(s))|_r ds \\ &\le \mathcal{M}C \int_0^{\upsilon} (\upsilon-s)^{\gamma(1-\beta)-1} |\tilde{u}(s)|^2_{H^{\alpha,r}} ds \\ &\le \mathcal{M}C \int_0^{\upsilon} (\upsilon-s)^{\gamma(1-\beta)-1} s^{-2\gamma(\alpha-\beta)} ds \sup_{s\in(0,\upsilon]} \{ s^{2\gamma(\alpha-\beta)} |\tilde{u}(s)|^2_{H^{\alpha,r}} \} \\ &= \mathcal{M}C\mathcal{B}((\gamma(1-\beta)), 1-2\gamma(\alpha-\beta)) \sup_{s\in(0,\upsilon]} \{ s^{2\gamma(\alpha-\beta)} |\tilde{u}(s)|^2_{H^{\alpha,r}} \} \to 0 \; as \; \upsilon \to \upsilon_0. \end{aligned}$$

□

Theorem 3.2. *Let* $1 < r < \infty$, $0 < \gamma < 1$ *and (3.1) holds; then, suppose that*

$$\frac{n}{2r} - \frac{1}{2} < \beta.$$

Then there is a unique function $\tilde{u} : [0, \infty) \to H^{\gamma,r}$ *and* $\alpha > \max(\beta, \frac{1}{2})$ *for all at* $\in H^{\gamma,r}$ *there exists* $\mathfrak{I}_* > 0$ *satisfying the following*

(i) $\tilde{u}(0) = a\upsilon$ *and* $\tilde{u} : [0, \Im_*] \to H^{\gamma,r}$ *is continuous,*
(ii) $\tilde{u} : [0, \Im_*] \to H^{\alpha,r}$ *is continuous, and* $\lim_{\upsilon\to 0} \upsilon^{\gamma(\alpha-\beta)}|\tilde{u}(\upsilon)|_{H^{\alpha,r}} = 0$*,*
(iii) *for* $\upsilon \in [0, \Im_*]$*,* $\tilde{u}$ *satisfies Definition 2.4.*

Proof. Consider the space of all of the curves $X = X[T]$ and $\tilde{u} : (0, \Im] \to H^{\gamma,r}$; now, suppose that $\alpha = \dfrac{1+\beta}{2}$ so we have the following

(i) $\tilde{u} : [0, \Im] \to H^{\gamma,r}$ is continuous.
(ii) $\tilde{u} : (0, \Im] \to H^{\alpha,r}$ is continuous and $\lim\limits_{\upsilon\to 0} \upsilon^{\gamma(\alpha-\beta)}|\tilde{u}(\upsilon)|_{H^{\alpha,r}} = 0$,

its original form is given by
$$\|\tilde{u}\|_X = \sup_{\upsilon\in[0,\Im]} \left(\upsilon^{\gamma(\alpha-\beta)}|\tilde{u}(\upsilon)|_{H^{\alpha,r}}\right).$$
Similar to the conclusion of Theorem 3.1, it is worth noting that the function $g : X * X \to X$ is linearly mappable and continuous. By Lemma 2.1 the function $\varphi(\upsilon) \in X$ for all $\upsilon \in (0, \Im]$. It is simple to assert that
$$\begin{aligned} E_\gamma(-\upsilon^\gamma A)a &\in C([0, \Im]), H^{\gamma,r}) \\ E_\gamma(-\upsilon^\gamma A)a &\in C([0, \Im], H^{\alpha,r}). \end{aligned}$$
Therefore by Theorem 2.3, we can say that
$$\begin{aligned} \int_0^\upsilon E_\gamma(-(\upsilon-s)^\gamma A)ads &\in C([0, \Im], H^{\gamma,r}) \\ \int_0^\upsilon E_\gamma(-(\upsilon-s)^\gamma A)ads &\in C([0, \Im], H^{\alpha,r}). \end{aligned}$$
For all $\upsilon \in (0, \Im]$, using Theorem 2.2 implies that
$$\int_0^\upsilon E_\gamma(-(\upsilon-s)^\gamma A)ads \in X_\infty.$$
We have
$$\|\tilde{u}\|_X = \sup_{\upsilon\in[0,\Im]} (\upsilon^{\gamma(\alpha-\beta)}|\tilde{u}(\upsilon)|_{H^{\alpha,r}}).$$
By using the above condition, we get
$$\upsilon^{\gamma(\alpha-\beta)} \int_0^\upsilon E_\gamma(-(\upsilon-s)^\gamma A)ads \in C([0, \Im], H^{\alpha,r}).$$
Now, consider a sufficiently small $\Im_* > 0$, such that
$$\begin{aligned} \| \int_0^\upsilon E_\gamma(-\upsilon^\gamma A)ads + \tilde{\varphi}(\upsilon)\|_{X[\Im_*]} &\leq \| \int_0^\upsilon E_\gamma(-\upsilon^\gamma A)ads\|_{X[\Im_*]} + \|\tilde{\varphi}(\upsilon)\|_{X[\Im_*]} \\ &< \frac{1}{4L} \end{aligned}$$
holds, resulting in F having a unique fixed point. □

4. Local existence in $H^{\gamma,r}$

This part is devoted to the iteration method's evaluation of a local mild solution to Problem (1.5) in J_r. Consider that $\alpha = \dfrac{1+\beta}{2}$.

Theorem 4.1. *Suppose that* $1 < r < \infty$, $0 < \gamma < 1$ *and (3.1) holds. Suppose that*

$$a\upsilon \in H^{\gamma,r} \quad \textit{with} \quad \frac{n}{2r} - \frac{1}{2} < \gamma.$$

For $a\upsilon \in H^{\gamma,r}$*, Problem (1.5) has a unique mild solution* $\tilde{u}$ *in* J_r*,* $A^{\alpha}\tilde{u}$ *shows continuity on* $(0, \mathfrak{I}]$*. Furthermore,* $\upsilon^{\gamma}(\alpha-\beta)A^{\alpha}\tilde{u}(\upsilon)$ *is bounded while* $\upsilon \to 0$*. Additionally,* $\tilde{u}$ *also shows continuity in* $[0, \mathfrak{I}]$*.*

Proof. Step 1:
Let

$$\mathcal{K}(\upsilon) = \sup_{s\in(0,\upsilon]} s^{\gamma(\alpha-\beta)}|A^{\alpha}\tilde{u}(s)|_r$$

and

$$\zeta(\upsilon) = \tilde{\varpi}(\tilde{u}, \tilde{u})(\upsilon) = \int_0^{\upsilon} (\upsilon - s)^{\gamma-1} E_{\gamma,\gamma}(- (\upsilon - s)^{\gamma} A)F(\tilde{u}(s), \tilde{u}(s))ds.$$

According to the summary of Step 2 in Theorem 3.1, $\zeta(\upsilon)$ shows continuity on $[0, \mathfrak{I}]$ and $A^{\alpha}\zeta(\upsilon)$ is continuous on $(0, \mathfrak{I}]$ and also exists, then,

$$\begin{aligned} A^{\alpha}|\tilde{\varpi}(\tilde{u}(\upsilon), \tilde{u}(\upsilon))|_{H^{\alpha,r}} &= |\int_0^{\upsilon} (\upsilon - s)^{\gamma-1}|E_{\gamma,\gamma}(- (\upsilon - s)^{\gamma}A)A^{\alpha}F(\tilde{u}(s), \tilde{u}(s))|_{H^{\alpha,r}}ds \\ &\leq C_1 \int_0^{\upsilon} (\upsilon - s)^{\gamma(1-\alpha)-1}|A^{\alpha}F(\tilde{u}(s), \tilde{u}(s))|_r ds \\ &\leq \mathcal{M}C_1 \int_0^{\upsilon} (\upsilon - s)^{\gamma(1-\alpha)-1} s^{-2\gamma(\alpha-\beta)}ds \sup_{s\in[0,\upsilon]} \{s^{2\gamma(\alpha-\beta)}|A^{\alpha}u(s)^2|_{H^{\alpha,r}}\}. \end{aligned}$$

We have

$$\mathcal{K}^2(\upsilon) = \sup_{s\in(0,\upsilon]} s^{2\gamma(\alpha-\beta)}|A^{\alpha}\tilde{u}(s)^2|_r.$$

Using the above equation, it gives the final result such that

$$|A^{\alpha}\zeta(\upsilon)|_r \leq \mathcal{M}C_1\mathcal{B}(\gamma(1-\alpha), 1 - 2\gamma(\alpha-\beta))\mathcal{K}^2(\upsilon)\upsilon^{-\gamma(\alpha-\beta)}. \tag{4.1}$$

Take into account the integral $\tilde{\varphi}(\upsilon)$. As the (3.1) holds then the inequality

$$|Pg(s)|_r \leq \mathcal{M}(\upsilon)s^{\gamma(1-\beta)}$$

is accomplished by a continuous function $\mathcal{M}(\upsilon)$. Now considering the Step 3 of Theorem 3.1, we show that $A^{\alpha}\tilde{\varphi}(\upsilon)$ shows continuity on $(0, \mathfrak{I}]$. Before discussing continuity, we signify

$$\begin{aligned} \mathcal{M}(\upsilon) &= \sup_{s\in(0,\upsilon]} \{s^{\gamma(1-\beta)}|A^{\alpha}Pg(s)|_r\} \\ |A^{\alpha}\tilde{\varphi}(\upsilon)|_{H^{\alpha,r}}| &= \int_0^{\upsilon} (\upsilon - s)^{\gamma-1}|E_{\gamma,\gamma}(- (t - s)^{\alpha}A)A^{\alpha}Pg(s)|_{H^{\alpha,r}}ds \end{aligned}$$

$$
\begin{aligned}
&\le C_1 \int_0^{\upsilon} (\upsilon - s)^{\gamma(1-\alpha)-1} |A^{\alpha} Pg(s)|_r ds \\
&\le C_1 \mathcal{M}(\upsilon) \int_0^{\upsilon} (\upsilon - s)^{\gamma(1-\alpha)-1} s^{-\gamma(1-\beta)} ds \\
|A^{\alpha}\tilde{\varphi}(\upsilon)|_r| &= \upsilon^{-\gamma(\alpha-\beta)} C_1 \mathcal{M}(\upsilon) \mathcal{B}((\gamma(1-\alpha)), (1-\gamma(1-\beta))). \qquad (4.2)
\end{aligned}
$$

As $\upsilon \to 0$, we get that $\mathcal{M}(\upsilon) = 0$ and $|Pg(\upsilon)|_r = 0(\upsilon^{-\gamma(\alpha-\beta)})$. As $\upsilon \to 0$, we get that $|A^{\alpha}\zeta(\upsilon)|_r = 0(\upsilon^{-\gamma(\alpha-\beta)})$ according to (4.2). We show that the function $\tilde{\varphi}$ is continuous in J_r. In reality, considering $0 \le \upsilon_0 < \upsilon < \Im$, we have

$$
\begin{aligned}
|\tilde{\varphi}(\upsilon) - \tilde{\varphi}(\upsilon_0)|_r &\le C_3 \int_{\upsilon_0}^{\upsilon} (\upsilon - s)^{\gamma(1-\beta)-1} |Pg(s)|_r ds + C_3 \int_0^{\upsilon_0} ((\upsilon_0 - s)^{\gamma-1} - (\upsilon - s)^{\gamma-1})(\upsilon - s)^{-\beta\gamma} |Pg(s)|_r ds \\
&+ C_3 \int_0^{\upsilon_0-\epsilon} (\upsilon_0 - s)^{\gamma-1} |E_{\gamma,\gamma}(-(\upsilon - s)^{\gamma}A) - E_{\gamma,\gamma}(-(\upsilon_0 - s)^{\gamma}A) Pg(s)|_{H^{\gamma,r}} ds \\
&+ 2C_3 \int_{\upsilon_0-\epsilon}^{\upsilon_0} (\upsilon_0 - s)^{\gamma(1-\beta)-1} |Pg(s)|_r ds \\
&\le C_3 \mathcal{M}(\upsilon) \int_{\upsilon_0}^{\upsilon} (\upsilon - s)^{\gamma(1-\beta)-1} s^{-\gamma(1-\beta)} ds \\
&+ C_3 \mathcal{M}(\upsilon) \int_0^{\upsilon_0} ((\upsilon - s)^{\gamma-1} - (\upsilon_0 - s)^{\gamma-1})(\upsilon - s)^{-\beta\gamma} s^{-\gamma(1-\beta)} ds \\
&+ C_3 \mathcal{M}(\upsilon) \int_0^{\upsilon_0-\epsilon} (\upsilon_0 - s)^{\gamma-1} |E_{\gamma,\gamma}(-(\upsilon - s)^{\gamma}A) - E_{\gamma,\gamma}(-(\upsilon_0 - s)^{\gamma}A)|_r s^{-\gamma(1-\beta)} ds \\
&+ 2C_3 \mathcal{M}(\upsilon) \int_{\upsilon_0-\epsilon}^{\upsilon_0} (\upsilon_0 - s)^{\gamma(1-\beta)-1} s^{-\gamma(1-\beta)} ds.
\end{aligned}
$$

Further, we assume the function $\int_0^{\upsilon} E_{\gamma}(-(\upsilon - s)^{\gamma}A) a ds$. From Theorem 2.2, it is clear that

$$
\left|A^{\alpha} \int_0^{\upsilon} E_{\gamma}(-(\upsilon - s)^{\gamma}A) a ds\right|_r = \int_0^{\upsilon} \left|A^{\alpha} E_{\gamma}(-(\upsilon - s)^{\gamma}A) a\right|_{H^{\gamma,r}} ds.
$$

From Lemma 2.4,

$$
\le C_1 \int_0^{\upsilon} \upsilon^{-\gamma(\alpha-\beta)} |A^{\beta} a|_r = C_1 \upsilon^{-\gamma(\alpha-\beta)} |a\upsilon|_{H^{\alpha,r}}
$$

and

$$
\lim_{\upsilon \to 0} \upsilon^{\gamma(\alpha-\beta)} |A^{\alpha} \int_0^{\upsilon} E_{\gamma}(-(\upsilon - s)^{\gamma}A) a ds|_r = \lim_{\upsilon \to 0} \upsilon^{\gamma(\alpha-\beta)} \int_0^{\upsilon} |E_{\gamma}(-(\upsilon - s)^{\gamma}A) a|_{H^{\gamma,r}} ds = 0.
$$

Step 2:
We now build the result through successive approximation:

$$
\begin{aligned}
\tilde{u}_0(\upsilon) &= \int_0^{\upsilon} E_{\gamma}(-(\upsilon - s)^{\gamma}A) a ds + \tilde{\varphi}(\upsilon) \\
\tilde{u}_{n+1}(\upsilon) &= \tilde{u}_0(\upsilon) + \varpi(\tilde{u}_n, \tilde{u}_n)(\upsilon), \ for\ n = 0, 1, 2.... \qquad (4.3)
\end{aligned}
$$

We know that denotes increasing and continuous functions on $[0, \Im]$. Moreover, $\mathcal{K}_n(0) = 0$. However, according to (4.1) and (4.3), $\mathcal{K}_n(\upsilon)$ satisfies the inequality

$$
\mathcal{K}_{n+1}(\upsilon) \le \mathcal{K}_0(\upsilon) + \mathcal{M}C_1 \mathcal{B}((\gamma(1-\alpha)), 1 - 2\gamma(\alpha - \beta)) \mathcal{K}_n{}^2(\upsilon). \qquad (4.4)
$$

Select $\Im > 0$ so that

$$4\mathcal{M}C_1B((\gamma(1-\alpha)),1-2\gamma(\alpha-\beta))\mathcal{K}_0(\upsilon) < 1. \tag{4.5}$$

Then, the sequence $K_n(\Im)$ is bounded; following a simple description of (4.4), so we get

$$\mathcal{K}_n(\Im) \leq \rho(\Im),\ for\, n = 0,1,2,..., \tag{4.6}$$

while (4.4) is just like the quadratic equation; we have

$$\mathcal{M}C_1\mathcal{B}((\gamma(1-\alpha)),1-2\gamma(\alpha-\beta))\mathcal{K}_n{}^2(\upsilon) - \mathcal{K}_{n+1}(\upsilon) + \mathcal{K}_0(\upsilon) \geq 0.$$

Here, $a = \mathcal{M}C_1\mathcal{B}((\gamma(1-\alpha)),1-2\gamma(\alpha-\beta))\mathcal{K}_n{}^2(\upsilon)$, $b = -1$ and $c = \mathcal{K}_0(\upsilon)$. After applying the quadratic formula, we obtain

$$\mathcal{K}_n(\upsilon) = \frac{1-\sqrt{1-4\mathcal{M}C_1\mathcal{B}((\gamma(1-\alpha)),1-2\gamma(\alpha-\beta))\mathcal{K}_0(\upsilon)}}{2\mathcal{M}C_1\mathcal{B}((\gamma(1-\alpha)),1-2\gamma(\alpha-\beta))}.$$

Considering (4.6),

$$\rho(\upsilon) = \frac{1-\sqrt{1-4\mathcal{M}C_1\mathcal{B}((\gamma(1-\alpha)),1-2\gamma(\alpha-\beta))\mathcal{K}_0(\upsilon)}}{2\mathcal{M}C_1\mathcal{B}((\gamma(1-\alpha)),1-2\gamma(\alpha-\beta))}.$$

Conversely, $\mathcal{K}_n(\upsilon) \leq \rho(\upsilon)$ holds for any $\upsilon \in (0,\Im]$. Correspondingly, $\rho(\upsilon) \leq 2\mathcal{K}_0(\upsilon)$. Suppose the following concept of equality:

$$y_{n+1}(\upsilon) = \int_0^{\upsilon} (\upsilon-s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon-s)^{\gamma}A)F(\tilde{u}_{n+1}(s),\tilde{u}_{n+1}(s)) - F(\tilde{u}_n(s),\tilde{u}_n(s))ds.$$

For $\upsilon \in (0,\Im]$ and $n = 0,1,2,...$, we denote $y_n = \tilde{u}_{n+1} - \tilde{u}_n$:

$$Y_n(\upsilon) = \sup_{s\in(0,\upsilon]} s^{\gamma(\alpha-\beta)}|A^{\alpha}y_n(s)|_r.$$

Using Lemma 2.3, we get

$$\begin{aligned}
\Big|F(\tilde{u}_{n+1}(s),\tilde{u}_{n+1}(s)) - F(\tilde{u}_n(s),\tilde{u}_n(s))\Big|_r &\leq \mathcal{M}(|\tilde{u}_{n+1}| + |\tilde{u}_n|)|\tilde{u}_{n+1} - \tilde{u}_n| \\
&\leq \mathcal{M}(|\tilde{u}_{n+1}| + |\tilde{u}_n|)A^{\alpha}y_n \sup_{s\in(0,\upsilon]} s^{-\gamma(\alpha-\beta)}s^{\gamma(\alpha-\beta)} \\
&\leq \mathcal{M}(|A^{\alpha}\tilde{u}_{n+1}| + |A^{\alpha}\tilde{u}_n|)y_n \sup_{s\in(0,\upsilon]} s^{-\gamma(\alpha-\beta)}s^{\gamma(\alpha-\beta)} \\
&\leq \mathcal{M}(\sup_{s\in(0,\upsilon]} s^{\gamma(\alpha-\beta)}|A^{\alpha}\tilde{u}_{n+1}| + \sup_{s\in(0,\upsilon]} s^{\gamma(\alpha-\beta)}|A^{\alpha}\tilde{u}_n|)y_n s^{-\gamma(\alpha-\beta)} \\
&\leq \mathcal{M}(\mathcal{K}_{n+1} + \mathcal{K}_n)y_n s^{-\gamma(\alpha-\beta)}.
\end{aligned}$$

Then, we set

$$\leq \mathcal{M}(\mathcal{K}_{n+1} + \mathcal{K}_n)y_n s^{-\gamma(\alpha-\beta)}A^{\alpha} \sup_{s\in(0,\upsilon]} s^{-\gamma(\alpha-\beta)}s^{\gamma(\alpha-\beta)}$$

$$\leq \mathcal{M}(\mathcal{K}_{n+1} + \mathcal{K}_n)\{\sup_{s\in(0,\upsilon]} s^{\gamma(\alpha-\beta)}A^{\alpha}y_n\}s^{-2\gamma(\alpha-\beta)}$$

$$\left|F(\tilde{u}_{n+1}(s),\tilde{u}_{n+1}(s)) - F(\tilde{u}_n(s),\tilde{u}_n(s))\right|_r \leq \mathcal{M}(\mathcal{K}_{n+1}(s) + \mathcal{K}_n(\upsilon))Y_n(s)s^{-2\gamma(\alpha-\beta)}.$$

We get the results from Step 2 in Theorem 3.1:

$$\upsilon^{\gamma}(\alpha-\beta)|A^{\alpha}y_{n+1}(\upsilon)|_r \leq 2\mathcal{M}C_1\mathcal{B}((\gamma(1-\alpha)),1-\gamma(1-\beta))\rho(\upsilon)Y_n(\upsilon).$$

It provides

$$\begin{aligned} Y_{n+1}(\mathfrak{I}) &\leq 2\mathcal{M}C_1\mathcal{B}((\gamma(1-\alpha)),1-2\gamma(\alpha-\beta))\rho(\mathfrak{I})Y_n(\mathfrak{I}) \\ &\leq 4\mathcal{M}C_1\mathcal{B}(\gamma(1-\alpha),1-2\gamma(\alpha-\beta))K_0(\mathfrak{I})Y_n(\mathfrak{I}). \end{aligned} \tag{4.7}$$

In accordance with (4.5) and (4.7), we get

$$\lim_{n\to 0}\frac{Y_{n+1}(\mathfrak{I})}{Y_n(\mathfrak{I})} \leq 4MC_1B((\gamma(1-\alpha)),1-2\gamma(\alpha-\beta))K_0(\mathfrak{I}) \leq 1.$$

Hence the series $\sum_{n=0}^{\infty} Y_n(\mathfrak{I})$ is convergent. For $\upsilon \in (0,\mathfrak{I}]$ it shows that the series $\sum_{n=0}^{\infty}\upsilon^{\gamma}(\alpha-\beta)A^{\alpha}y_n(\upsilon)$ is uniformly convergent. Thus, the sequence $\{\upsilon^{\gamma}(\alpha-\beta)A^{\alpha}\tilde{u}_n(\upsilon)\}$ is uniformly convergent in $(0,\mathfrak{I}]$. This signifies that $\lim_{n\to\infty}\tilde{u}_n(\upsilon) = \tilde{u}(\upsilon) \in D(A^{\alpha})$ and $\lim_{n\to\infty}\upsilon^{\gamma(\alpha-\beta)}A^{\alpha}\tilde{u}_n(\upsilon) = \upsilon^{\gamma(\alpha-\beta)}A^{\alpha}\tilde{u}(\upsilon)$ uniformly. From the boundedness theorem, a function f continuous on a bounded and closed interval is necessarily a bounded function. Therefore, A^{α} is closed if $A^{-\alpha}$ is bounded. As a result, the function $\mathcal{K}(\upsilon) = \sup_{s\in(0,\upsilon]} s^{\beta(\alpha-\gamma)}|A^{\alpha}\tilde{u}_n(s)|_r$ satisfies

$$\mathcal{K}(\upsilon) \leq \rho(\upsilon) \leq 2\mathcal{K}_0(\upsilon),\ for\ \upsilon \in (0,\upsilon] \tag{4.8}$$

and

$$\begin{aligned} \eta_n &= \sup_{s\in(0,\mathfrak{I}]} s^{2\gamma(\alpha-\beta)}\left|\tilde{F}(\tilde{u}_n(s),\tilde{u}_n(s)) - \tilde{F}(\tilde{u}(s),\tilde{u}(s))\right|_r \\ &\leq \mathcal{M}(\mathcal{K}_n(\mathfrak{I}) + \mathcal{K}(\mathfrak{I}))\sup_{s\in(0,\mathfrak{I}]} s^{\upsilon^{\gamma}(\alpha-\beta)}|A^{\alpha}(\tilde{u}_n(s)-\tilde{u}(s))|_r \to 0\ as\ n\to\infty. \end{aligned}$$

Lastly, it is necessary to confirm that $\tilde{u}$ is a mild solution of (1.5) in $[0,\mathfrak{I}]$. Because

$$|\varpi(\tilde{u}_n,\tilde{u}_n)(\upsilon) - \varpi(\tilde{u},\tilde{u})(\upsilon)|_r \leq \int_0^{\upsilon}(\upsilon-s)^{\gamma-1}\eta_n s^{-2\gamma(\alpha-\beta)}ds = \upsilon^{\beta\gamma}\ \ \eta_n \to 0,\ as\ n\to\infty$$

we get $\varpi(\tilde{u}_n,\tilde{u}_n)(\upsilon) \to \varpi(\tilde{u},\tilde{u})(\upsilon)$. We obtain it by taking the limits of (4.2) on both sides;

$$\tilde{u}(\upsilon) = \tilde{u}_0(\upsilon) + \varpi(\tilde{u},\tilde{u})(\upsilon). \tag{4.9}$$

Suppose that $\tilde{u}(0) = a\upsilon$. For $\upsilon \in [0,\mathfrak{I}]$, we can say that (4.9) holds and $\tilde{u} \in C([0,\mathfrak{I}],J_r)$. Furthermore,the uniform convergence of $\upsilon^{\gamma(\alpha-\beta)}A^{\alpha}u_n(\upsilon)$ to $t^{\gamma}(\alpha-\beta)A^{\alpha}\tilde{u}(\upsilon)$ shows continuity of $A^{\alpha}\tilde{u}(\upsilon)$. By (4.8) and $\mathcal{K}_0(0) = 0$, we obtain that $|A^{\alpha}u(\upsilon)|_r = 0\upsilon^{(-\gamma(\alpha-\beta))}$ is clear.
Step 3:
Now, we have to suppose that $\tilde{u}$ and $\tilde{w}$ are mild solutions of (1.5) because to prove that the mild solutions are unique. Let $y = \tilde{u} - \tilde{w}$; we consider the equality

$$y(\upsilon) = \int_0^{\upsilon}(\upsilon-s)^{\gamma-1}E_{\gamma,\gamma}\big(-(\upsilon-s)^{\gamma}A\big)(F(\tilde{u}(s),\tilde{u}(s)) - F(\tilde{w}(s),\tilde{w}(s)))ds.$$

Now, we defining the following functions:

$$\mathcal{K} = \max \sup_{s\in(0,\upsilon]} s^{\gamma(\alpha-\beta)}|A^{\alpha}u(s)|_r, \sup_{s\in(0,\upsilon]} s^{\gamma(\alpha-\beta)}|A^{\alpha}v(s)|_r.$$

According to Theorem 3.1 and Lemma 2.4, we get

$$|A^{\alpha}y(\upsilon)|_r \leq \mathcal{M}C_1\mathcal{K}\int_0^{\upsilon}(\upsilon-s)^{\gamma(1-\alpha)-1}s^{-\gamma(\alpha-\beta)}|A^{\alpha}y(s)|_r ds.$$

For $\upsilon \in (0,\mathfrak{I}]$, the Gronwall inequality demonstrates that $A^{\alpha}y(\upsilon) = 0$. It shows that for $\upsilon \in [0,\mathfrak{I}]$, $y(\upsilon) = \tilde{u}(\upsilon) - \tilde{w}(\upsilon) \equiv 0$. Hence, the mild solutions are unique. □

5. Consequences of regularity for vertical flow

In this part, we evaluate the regularity's behavior [22] of a solution $\tilde{u}$ which satisfies (1.5). We will assume throughout this section that ($\mathbf{k}_1$): $Pg(\upsilon)$ has Hölder's continuity with an exponent $\theta \in (0, \gamma(1-\alpha))$, which is given as

$$|Pg(\upsilon) - Pg(s)|_r \leq L|\upsilon - s|^{\theta} \; for\ all\ 0 < \upsilon, s \leq \mathfrak{I}. \tag{5.1}$$

Definition 5.1. The problem (1.5) has a solution that is a classical solution of a function $\tilde{u} : [0,\mathfrak{I}] \to J_r$, and $\tilde{u} \in C([0,\mathfrak{I}], J_r)$ with ${}^cD_{\upsilon}^{\upsilon} \in C([0,\mathfrak{I}], J_r)$, which satisfies (1.5) for all $\upsilon \in (0,\mathfrak{I}]$ and accepts values in $D(A)$.

Lemma 5.1. *Let Definition 5.1 be satisfied if*

$$\varphi_1(\upsilon) = \int_0^{\upsilon}(\upsilon-s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon-s)^{\gamma}A)(Pg(s), Pg(\upsilon))ds, \; for\ \upsilon \in (0,\mathfrak{I}]$$

so $A\varphi_1(\upsilon) \in C^{\theta}([0,\mathfrak{I}], J_r)$ *and* $\varphi_1(\upsilon) \in D(A)$.

Proof. From Lemma 2.4 and (5.1), we fixed $\upsilon \in (0,\mathfrak{I}]$,

$$\begin{aligned}(\upsilon-s)^{\gamma-1}|AE_{\gamma,\gamma}(-(\upsilon-s)^{\gamma}A)(Pg(s), Pg(\upsilon))|_r &\leq C_1(\upsilon-s)^{-1}|Pg(s) - Pg(\upsilon)|_r \\ &\leq C_1L(\upsilon-s)^{\theta-1} \in L^1([0,\mathfrak{I}], J_r).\end{aligned}$$

Then

$$\begin{aligned}|A\varphi(\upsilon)|_r &\leq (\upsilon-s)^{\gamma-1}|AE_{\gamma,\gamma}(-(\upsilon-s)^{\gamma}A)(Pg(s), Pg(\upsilon))|_r ds \\ &\leq C_1L\int_0^{\upsilon}(\upsilon-s)^{\theta-1}ds \\ &\leq \frac{C_1L}{\theta}\upsilon^{\theta} \\ &< \infty;\end{aligned}$$

we get $\varphi_1(\upsilon) \in D(A)$ by the closeness property of A. We must demonstrate that $A\varphi_1(\upsilon)$ has Hölder continuity. Because

$$\frac{d}{d\upsilon}(\upsilon^{\gamma-1}E_{\gamma,\gamma}(-\mu\upsilon^{\gamma})) = (\upsilon^{\gamma-2}E_{\gamma,\gamma-1}(-\mu\upsilon^{\gamma}))$$

it follows that

$$\begin{aligned}
&\frac{d}{d\upsilon}(\upsilon^{\gamma-1}AE_{\gamma,\gamma}(-\upsilon^{\gamma}A))\\
&= \frac{1}{2\pi\iota}\int_{\Gamma_\theta}(\upsilon^{\gamma-2}E_{\gamma,\gamma-1}(-\mu\upsilon^{\gamma}))A(\mu I+A)^{-1}d\mu\\
&= \frac{1}{2\pi\iota}\int_{\Gamma_\theta}(^{\gamma-2}E_{\gamma,\gamma-1}(-\mu\upsilon^{\gamma}))d\mu - \frac{1}{2\pi\iota}\int_{\Gamma_\theta}(\upsilon^{\gamma-2}E_{\gamma,\gamma-1}(-\mu\upsilon^{\gamma}))A(\mu I+A)^{-1}d\mu.
\end{aligned}$$

Substituting $-\mu\upsilon^{\gamma}=\xi$ implies that $-\upsilon^{\gamma}d\mu=d\xi$. So we have that $d\mu=-\frac{1}{\upsilon^{\gamma}}d\xi$:

$$= \frac{1}{2\pi\iota}\int_{\Gamma_\theta}(-\upsilon^{\gamma-2}E_{\gamma,\gamma-1}(\xi))\frac{1}{\upsilon^{\gamma}}d\xi - \frac{1}{2\pi\iota}\int_{\Gamma_\theta}(\upsilon^{\gamma-2}E_{\gamma,\gamma-1}(\xi))\frac{\xi}{\upsilon^{\gamma}}A(-\frac{\xi}{\upsilon^{\gamma}}I+A)^{-1}\frac{1}{\upsilon^{\gamma}}d\xi.$$

Since

$$\|(\mu I+A)^{-1}\| \le \frac{C}{|\mu|}$$

and

$$\|\frac{d}{d\upsilon}(\upsilon^{\gamma-1}AE_{\gamma,\gamma}(-\upsilon^{\gamma}A))\| \le C_{\gamma}\upsilon^{-2}\ for\ 0<\upsilon\le\Im$$

for every $0<s<\upsilon\le\Im$, we apply the mean value theorem

$$\begin{aligned}
\|(\upsilon^{\gamma-1}AE_{\gamma,\gamma}(-\upsilon^{\gamma}A))-(s^{\gamma-1}AE_{\gamma,\gamma}(-s^{\gamma}A))\| &= \|\int_s^{\upsilon}\frac{d}{d\tau}(\tau^{\gamma-1}AE_{\gamma,\gamma}(-\tau^{\gamma}A))d\tau\|\\
&\le \int_s^{\upsilon}\|\frac{d}{d\tau}(\tau^{\gamma-1}AE_{\gamma,\gamma}(-\tau^{\gamma}A))\|d\tau\\
&\le \int_s^{\upsilon}\tau^{-2}d\tau\\
&= C_{\gamma}(s^{-1}-\upsilon^{-1}).
\end{aligned}\tag{5.2}$$

Now for $0<\upsilon<\upsilon+\hbar\le\Im$, we suppose that $\hbar>0$, so

$$\begin{aligned}
A\varphi_1(\upsilon+\hbar)-A\varphi_1(\upsilon) &= \int_0^{\upsilon}(\upsilon+\hbar-s)^{\gamma-1}AE_{\gamma,\gamma}(-(\upsilon+\hbar-s)^{\gamma}A)(Pg(s)-Pg(\upsilon))ds\\
&- (\upsilon-s)^{\gamma-1}AE_{\gamma,\gamma}(-(\upsilon-s)^{\gamma}A)(Pg(s)-Pg(\upsilon))ds\\
&+ \int_0^{\upsilon}(\upsilon+\hbar-s)^{\gamma-1}AE_{\gamma,\gamma}(-(\upsilon+\hbar-s)^{\gamma}A)(Pg(\upsilon)-Pg(\upsilon+\hbar))ds\\
&+ \int_{\upsilon}^{\upsilon+\hbar}(\upsilon+\hbar-s)^{\gamma-1}AE_{\gamma,\gamma}(-(\upsilon+\hbar-s)^{\gamma}A)(Pg(s)-Pg(\upsilon+\hbar))ds\\
&= \hbar_1(\upsilon)+\hbar_2(\upsilon)+\hbar_3(\upsilon).
\end{aligned}\tag{5.3}$$

For convenience, we solve each term separately. For $\hbar_1(\upsilon)$, from (5.2) and (5.1) we have

$$|\hbar_1(\upsilon)|_r \le \int_0^{\upsilon}\|(\upsilon+\hbar-s)^{\gamma-1}AE_{\gamma,\gamma}(-(\upsilon+\hbar-s)^{\gamma}A)$$

$$
\begin{aligned}
& -(\upsilon-s)^{\gamma-1}AE_{\gamma,\gamma}(-(\upsilon-s)^{\gamma}A)\|_r(Pg(s)-Pg(\upsilon))ds \\
\leq\ & C_\gamma L\hbar\int_0^\upsilon(\upsilon+\hbar-s)^{-1}(\upsilon-s)^{\theta-1}ds \\
\leq\ & C_\gamma L\hbar\int_0^\upsilon(s+\hbar)^{-1}(\upsilon-s)^{\theta-1}ds \\
\leq\ & C_\gamma L\int_0^h\frac{\hbar}{\hbar+s}s^{\theta-1}ds+C_\gamma L\hbar\int_\hbar^\infty\frac{s}{\hbar+s}s^{\theta-1}ds \\
\leq\ & C_\gamma L\hbar^\theta. \qquad (5.4)
\end{aligned}
$$

For $\hbar_2(t)$, using Lemma 2.4 and (5.1), we get

$$
\begin{aligned}
|\hbar_2(\upsilon)|_r &\leq \int_0^\upsilon(\upsilon+\hbar-s)^{\gamma-1}|AE_{\gamma,\gamma}(-(\upsilon+\hbar-s)^\gamma A)(Pg(\upsilon)-Pg(\upsilon+\hbar))|_r ds \\
&\leq C_1\int_0^\upsilon(\upsilon+\hbar-s)^{-1}|Pg(\upsilon)-Pg(\upsilon+\hbar)|_r ds \\
&\leq C_1\hbar^\theta\int_0^\upsilon(\upsilon+\hbar-s)^{-1}ds \\
&= C_1L[ln\hbar-ln(\upsilon+\hbar)]\hbar^\theta. \qquad (5.5)
\end{aligned}
$$

Now for $\hbar_3(\upsilon)$, using Lemma 2.4 and (5.1), we have

$$
\begin{aligned}
|\hbar_3(\upsilon)|_r &\leq \int_\upsilon^{\upsilon+\hbar}(\upsilon+\hbar-s)^{\gamma-1}|AE_{\gamma,\gamma}(-(\upsilon+\hbar-s)^\gamma A)(Pg(s)-Pg(\upsilon+\hbar))|_r ds \\
&\leq C_1\int_\upsilon^{\upsilon+\hbar}(\upsilon+\hbar-s)^{-1}|Pg(s)-Pg(\upsilon+\hbar)|_r ds \\
&\leq C_1L\int_\upsilon^{\upsilon+\hbar}(\upsilon+\hbar-s)^{\theta-1}ds \\
&= C_1L\frac{\hbar^\theta}{\theta}. \qquad (5.6)
\end{aligned}
$$

Hence we can say that $A\varphi_1(\upsilon)$ is Hölder continuous by combining all of the above results. □

Theorem 5.1. *Consider that the supposition of Theorem 4.1 is satisfied. If (5.1) holds, then the classical [23] mild solutions of (1.5) are obtained for every $a\upsilon\in D(A)$.*

Proof. Step 1:
We have that $a\upsilon\in D(A)$. So, $\int_0^\upsilon E_\gamma(-\upsilon^\gamma A)ads$ is said to be classical solution in regard to the following problem:

$$
\begin{cases} {}^cD_\upsilon^\gamma\tilde{u}(\upsilon)=-A\tilde{u},\ \upsilon>0, \\ \tilde{u}(0)=ax. \end{cases} \qquad (5.7)
$$

We also verify that

$$
\tilde{\varphi}(\upsilon)=\int_0^\upsilon(\upsilon-s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon-s)^\gamma A)Pg(s)ds
$$

is a classical approach to the following problem:

$$\begin{cases} {}^cD_\upsilon^\gamma \tilde{u}(\upsilon) = -A\tilde{u} + Pg(\upsilon), & \upsilon > 0, \\ \tilde{u}(0) = 0. \end{cases} \tag{5.8}$$

$\tilde{\varphi} \in C([0, \mathfrak{I}], J_r)$ follows from Theorem 4.1. We can write $\tilde{\varphi}(\upsilon) = \tilde{\varphi}_1(\upsilon) + \tilde{\varphi}_2(\upsilon)$, while

$$\begin{aligned} \tilde{\varphi}_1(\upsilon) &= \int_0^\upsilon (\upsilon - s)^{\gamma-1} E_{\gamma,\gamma}(- (\upsilon - s)^\gamma A)(Pg(s), Pg(\upsilon))ds \\ \tilde{\varphi}_2(\upsilon) &= \int_0^\upsilon (\upsilon - s)^{\gamma-1} E_{\gamma,\gamma}(- (\upsilon - s)^\gamma A)Pg(\upsilon)ds. \end{aligned}$$

We can say that $\tilde{\varphi}_1(\upsilon) \in D(A)$ according to Lemma 5.1. In order to prove the same conclusion for $\tilde{\varphi}_2(\upsilon)$, according to Lemma 2.2(iii), we realized that

$$A\tilde{\varphi}_2(\upsilon) = Pg(\upsilon) - E_\gamma(-\upsilon^\gamma A)Pg(\upsilon).$$

It follows that (5.1):

$$|A\tilde{\varphi}_2(\upsilon)| \le (1 + C_1)|Pg(\upsilon)|_r.$$

So, for $\upsilon \in (0, \mathfrak{I}]$ we can say that $\tilde{\varphi}_2(\upsilon) \in D(A)$ and $\tilde{\varphi}_2(\upsilon) \in C^\nu((0, \mathfrak{I}], J_r)$. Now we have to show that ${}^cD_\upsilon^\gamma(\tilde{\varphi}) \in C((0, \mathfrak{I}], J_r)$. Taking $\tilde{\varphi}(0) = 0$ and in view of Lemma 2.2(iv),

$${}^cD_\upsilon^\gamma \tilde{\varphi}(\upsilon) = \frac{d}{d\upsilon}(I_\upsilon^{1-\gamma}\tilde{\varphi}(\upsilon)) = \frac{d}{d\upsilon}(E_\gamma(-\upsilon^\gamma A) * Pg).$$

It is still necessary to demonstrate that $(E_\gamma(-\upsilon^\gamma A) * Pg)$ is differentiability continuous in J_r. Suppose that $0 < h \le \mathfrak{I} - \upsilon$; therefore, we have

$$\begin{aligned} &\frac{1}{\hbar}[(E_\gamma(-(\upsilon + \hbar)^\gamma A) * Pg) - (E_\gamma(-\upsilon^\gamma A) * Pg)] \\ = &\int_0^\upsilon \frac{1}{\hbar}[(E_\gamma(-(\upsilon + \hbar - s)^\gamma A)Pg(s)) - (E_\gamma(-(\upsilon - s)^\gamma A)Pg(s))]ds \\ + &\frac{1}{\hbar}\int_\upsilon^{\upsilon+\hbar} (E_\gamma(-(\upsilon + \hbar - s)^\gamma A)Pg(s))ds. \end{aligned}$$

Notice that

$$\int_0^\upsilon \frac{1}{\hbar}\Big|(E_\gamma(-(\upsilon + \hbar - s)^\gamma A)Pg(s)) - (E_\gamma(-(\upsilon - s)^\gamma A)Pg(s))\Big|_r ds$$

$$\le \frac{1}{\hbar}\int_0^\upsilon \Big|(E_\gamma(-(\upsilon + h - s)^\gamma A)Pg(s))\Big|_r ds + \frac{1}{\hbar}\int_0^\upsilon \Big|(E_\gamma(-(\upsilon - s)^\gamma A)Pg(s))\Big|_r ds.$$

In view of Lemma 2.4,

$$\le C_1\mathcal{M}(\upsilon)\frac{1}{\hbar}\int_0^\upsilon (\upsilon + \hbar - s)^{-\gamma} s^{-\gamma(1-\beta)}ds + C_1\mathcal{M}(\upsilon)\frac{1}{\hbar}\int_0^\upsilon (\upsilon - s)^{-\gamma} s^{-\gamma(1-\beta)}ds$$

$$\leq \quad C_1\mathcal{M}(\upsilon)\frac{1}{\hbar}((\upsilon+\hbar)^{1-\gamma}+\upsilon^{1-\gamma})B(1-\gamma,1-\gamma(1-\beta)),$$

the dominated convergence theorem applies; then, we get

$$\lim_{\hbar\to 0}\int_0^{\upsilon}\frac{1}{\hbar}[(E_\gamma(-(\upsilon+\hbar-s)^\gamma A)Pg(s))-(E_\gamma(-(\upsilon-s)^\gamma A)Pg(s))]ds = E'_\gamma(-(\upsilon-s)^\gamma A)Pg(s).$$

From Lemma 2.2(iii), we have

$$\begin{aligned} E'_\gamma(-\upsilon^\gamma A)u &= -\upsilon^{\gamma-1}AE_{\gamma,\gamma}(-\upsilon^\gamma A)u \\ \int E'_\gamma(-\upsilon^\gamma A)u &= -\int \upsilon^{\gamma-1}AE_{\gamma,\gamma}(-\upsilon^\gamma A)u \\ E_\gamma(-\upsilon^\gamma A)u &= -\int \upsilon^{\gamma-1}AE_{\gamma,\gamma}(-\upsilon^\gamma A)u. \end{aligned}$$

Therefore,

$$\begin{aligned} E'_\gamma(-(\upsilon-s)^\gamma A)Pg(s) &= -\int_0^{\upsilon}(\upsilon-s)^{\gamma-1}AE_{\gamma,\gamma}(-(\upsilon-s)^\gamma A)Pg(s)ds \\ &= A\tilde{\varphi}(\upsilon). \end{aligned}$$

Conversely,

$$\frac{1}{\hbar}\int_{\upsilon}^{\upsilon+h}E_\gamma(-(\upsilon+\hbar-s)^\gamma A)Pg(s)ds.$$

Let $s^* = \upsilon+\hbar-s$ so $ds^* = -ds$ and after setting the conditions [$s=\upsilon$ implies $s^*=\hbar$] and [$s=\upsilon+\hbar$ implies $s^*=0$], we have

$$\frac{1}{\hbar}\int_{\hbar}^{0}E_\gamma(-(s^*)^\gamma A)Pg(\upsilon+\hbar-s^*)(-ds^*).$$

By replacing $s^*\to s$, we get

$$\begin{aligned} \frac{1}{\hbar}\int_0^{h}E_\gamma(-s^\gamma A)Pg(\upsilon+\hbar-s)(ds) &= \frac{1}{\hbar}\int_0^{\hbar}E_\gamma(-s^\gamma A)[Pg(\upsilon+\hbar-s)-Pg(\upsilon-s) \\ &\quad +Pg(\upsilon-s)-Pg(\upsilon)+Pg(\upsilon)]ds \\ &= \frac{1}{\hbar}\int_0^{\hbar}E_\gamma(-s^\gamma A)(Pg(\upsilon+\hbar-s)-Pg(\upsilon-s))ds \\ &\quad +\frac{1}{\hbar}\int_0^{\hbar}E_\gamma(-s^\gamma A)(Pg(\upsilon-s)-Pg(\upsilon))ds \\ &\quad +\frac{1}{\hbar}\int_0^{\hbar}E_\gamma(-s^\gamma A)Pg(\upsilon)ds. \end{aligned}$$

From Lemmas 2.1–2.4 and (5.1), we have

$$\begin{aligned} \Big|\frac{1}{\hbar}\int_0^{\hbar}E_\gamma(-s^\gamma A)(Pg(\upsilon+\hbar-s)-Pg(\upsilon-s))\Big|_r ds &\leq C_1L\hbar^\theta \\ \Big|\frac{1}{\hbar}\int_0^{\hbar}E_\gamma(-s^\gamma A)(Pg(\upsilon-s)-Pg(\upsilon))\Big|_r ds &\leq C_1L\frac{\hbar^\theta}{\theta+1}. \end{aligned}$$

From Lemma 2.1(i),

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}\int_0^h E_\gamma(-s^\gamma A)Pg(\upsilon)ds &= Pg(\upsilon)\\ \lim_{h\to 0}\frac{1}{h}\int_\upsilon^{\upsilon+h} E_\gamma((\upsilon+h-s)^\gamma A)Pg(\upsilon)ds &= Pg(\upsilon);\end{aligned}$$

we deduce that $E_\gamma(\upsilon^\gamma A)*Pg$ holds differentiability $a\upsilon$ and $\frac{d}{d\upsilon}(E_\gamma(\upsilon^\gamma A)*Pg)_+ = A\tilde{\varphi}(\upsilon)+Pg(\upsilon)$. Similarly $E_\gamma(\upsilon^\gamma A)*Pg$ holds differentiability at υ and $\frac{d}{d\upsilon}(E_\gamma(\upsilon^\gamma A)*Pg)_- = A\tilde{\varphi}(\upsilon)+Pg(\upsilon)$. We can prove that $A\tilde{\varphi} = A\tilde{\varphi}_1 + A\tilde{\varphi}_2 \in C((0,\Im],J_r)$. Clearly, we can say that according to Lemma 2.1, $\tilde{\varphi}_2(\upsilon)$ shows continuity and $\tilde{\varphi}_2(\upsilon) = Pg(\upsilon)-E_\gamma(\upsilon^\gamma A)Pg(\upsilon)$ because of Lemma 2.2(iii). Consequently, $A\tilde{\varphi}_1(\upsilon)$ is continuous according to Lemma 5.1. Thus, ${}^cD_\upsilon^\gamma\tilde{\varphi}\in C((0,\Im],J_r)$.

Step 2:

Consider $\tilde{u}$ as the mild solution of (1.5). In order to draw the conclusion that $F(\tilde{u},\tilde{u})\in C^\theta((0,\Im],J_r)$, we have to show that $A^\alpha\tilde{u}$ holds Hölder continuity in J_r according to Theorem 3.1. Now for $0<\upsilon<\upsilon+\hbar$, let $\hbar>0$. We know that $\psi(\upsilon)=\int_0^\upsilon E_\gamma(-(\upsilon-s)^\gamma)ads$. Denoting $\phi(\upsilon) := \int_0^\upsilon E_\gamma(-\upsilon^\gamma A)ads$, and by Lemma 2.2(iii) we have

$$\frac{d}{d\upsilon}E_\gamma(-\upsilon^\gamma A)a = -\upsilon^{\gamma-1}AE_{\gamma,\gamma}(-\upsilon^\gamma A)a.$$

Integrating on both sides, we get

$$\int_0^\upsilon \frac{d}{ds}E_\gamma(-(\upsilon-s)^\gamma A)ads = -\int_0^\upsilon s^{\gamma-1}AE_{\gamma,\gamma}(-s^\gamma A)ads.$$

After applying the limits, we have

$$E_\gamma(-\upsilon^\gamma A)a = \int_0^\upsilon s^{\gamma-1}AE_{\gamma,\gamma}(-s^\gamma A)ads.$$

Using the above results and Lemma 2.4, then

$$\begin{aligned}|A^\alpha\phi(\upsilon+\hbar)-A^\gamma\phi(\upsilon)|_r &= \Big|\int_\upsilon^{\upsilon+\hbar} s^{\gamma-1}A^\alpha E_{\gamma,\gamma}(-s^\gamma A)ads\Big|_r\\ &\le \int_\upsilon^{\upsilon+\hbar} s^{\gamma-1}|A^{\alpha-\beta}E_{\gamma,\gamma}(-s^\gamma A)A^\beta a|_r ds\\ &\le C_1\int_\upsilon^{\upsilon+\hbar} s^{\gamma(1+\beta-\alpha)-1}ds|A^\beta a|_r\\ &= C_1\frac{|a|_{H^{\gamma,r}}}{\gamma(1+\beta-\alpha)}((\upsilon+\hbar)^{\gamma(1+\beta-\alpha)}-\upsilon^{\gamma(1+\beta-\alpha)})\\ &= C_1\frac{|a|_{H^{\gamma,r}}}{\gamma(1+\beta-\alpha)}\hbar^\gamma(1+\beta-\alpha).\end{aligned}$$

Thus $A^\alpha\phi\in C^\theta((0,\Im],J_r)$. For every small $\varepsilon>0$ we take $\hbar$ as $\varepsilon\le\upsilon<\upsilon+\hbar\le\Im$. We have

$$|A^\alpha\phi(\upsilon+\hbar)-A^\gamma\phi(\upsilon)|_r \le \Big|\int_\upsilon^{\upsilon+\hbar}(\upsilon+\hbar-s)^{\gamma-1}A^\alpha E_{\gamma,\gamma}(-(\upsilon+\hbar-s)^\gamma A)Pg(s)ds\Big|_r$$

$$
\begin{aligned}
& +|\int_0^{\upsilon} A^{\alpha}((\upsilon+\hbar-s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon+\hbar-s)^{\gamma}A) \\
& -(\upsilon-s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon-s)^{\gamma}A))Pg(s)ds|_r. \\
= \;& \phi_1(\upsilon)+\phi_2(\upsilon).
\end{aligned}
$$

Using Lemma 2.4 and (5.1), we have

$$
\begin{aligned}
\phi_1(\upsilon) &\leq C_1\int_{\upsilon}^{\upsilon+\hbar}(\upsilon+\hbar-s)^{\gamma(1-\alpha)-1}|Pg(s)|_r ds \\
&\leq C_1\mathcal{M}(\upsilon)\int_{\upsilon}^{\upsilon+\hbar}(\upsilon+\hbar-s)^{\gamma(1-\alpha)-1}s^{-\gamma(1-\alpha)-1}ds \\
&\leq \mathcal{M}(\upsilon)\frac{C_1}{\gamma(1-\alpha)}\hbar^{\gamma(1-\alpha)}\upsilon^{-\gamma(1-\alpha)-1} \\
&\leq \mathcal{M}(\upsilon)\frac{C_1}{\gamma(1-\alpha)}\hbar^{\gamma(1-\alpha)}\varepsilon^{-\gamma(1-\alpha)-1}.
\end{aligned}
$$

To estimate ϕ_2, we have

$$
\begin{aligned}
\frac{d}{d\upsilon}(\upsilon^{\gamma-1}A^{\alpha}E_{\gamma,\gamma}(-\upsilon^{\gamma}A)) &= \frac{1}{2\pi\iota}\int_{\Gamma}\mu^{\alpha}(\upsilon^{\gamma-2}E_{\gamma,\gamma-1}(-\mu\upsilon^{\gamma}))A(\mu I+A)^{-1}d\mu \\
&= \frac{1}{2\pi\iota}\int_{\Gamma'}-(\frac{-\xi}{\upsilon^{\gamma}})^{\alpha}(\upsilon^{\gamma-2}E_{\gamma,\gamma-1}(\xi))(-\frac{\xi}{\upsilon^{\gamma}}I+A)^{-1}\frac{1}{\upsilon^{\gamma}}d\xi,
\end{aligned}
$$

which yields that

$$
\frac{d}{d\upsilon}(\upsilon^{\gamma-1}A^{\alpha}E_{\gamma,\gamma}(-\upsilon^{\gamma}A))\leq C_{\gamma}\upsilon^{\gamma(1-\alpha)-2}.
$$

Now, we apply the mean value theorem:

$$
\begin{aligned}
\|(\upsilon^{\gamma-1}A^{\alpha}E_{\gamma,\gamma}(-\upsilon^{\gamma}A))-(s^{\gamma-1}A^{\alpha}E_{\gamma,\gamma}(-s^{\gamma}A))\| &= \|\int_s^{\upsilon}\frac{d}{d\tau}(\tau^{\gamma-1}A^{\alpha}E_{\gamma,\gamma}(-\tau^{\gamma}A))d\tau\| \\
&\leq \int_s^{\upsilon}\|\frac{d}{d\tau}(\tau^{\gamma-1}A^{\alpha}E_{\gamma,\gamma}(-\tau^{\gamma}A))\|d\tau \\
&\leq \int_s^{\upsilon}\tau^{\gamma(1-\alpha)-2}d\tau \\
&= C_{\gamma}(s^{\gamma(1-\alpha)-1}-\upsilon^{\gamma(1-\alpha)-1}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\phi_2(t) &\leq |\int_0^{\upsilon}A^{\alpha}((\upsilon+\hbar-s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon+\hbar-s)^{\gamma}A)-(\upsilon-s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon-s)^{\gamma}A))Pg(s)ds|_r \\
&\leq \int_0^{\upsilon}((\upsilon-s)^{\gamma(1-\alpha)-1}-(\upsilon+\hbar-s)^{\gamma(1-\alpha)-1})|Pg(s)|_r ds \\
&\leq C_{\gamma}\mathcal{M}(\upsilon)(\int_0^{\upsilon}(\upsilon-s)^{\gamma(1-\alpha)-1}s^{-\gamma(1-\beta)}ds-\int_0^{\upsilon+\hbar}(\upsilon+\hbar-s)^{\gamma(1-\alpha)-1}s^{-\gamma(1-\beta)}ds) \\
&\quad +C_{\gamma}\mathcal{M}(\upsilon)\int_{\upsilon}^{\upsilon+\hbar}(\upsilon+\hbar-s)^{\gamma(1-\alpha)-1}s^{-\gamma(1-\beta)}ds
\end{aligned}
$$

$$\leq C_\gamma \mathcal{M}(\upsilon)(\upsilon^{\gamma(\beta-\alpha)} - (\upsilon+\hbar)^{\beta-\alpha})\mathcal{B}(\gamma(1-\alpha), 1-\gamma(1-\beta)) + C_\gamma \mathcal{M}(\upsilon)\hbar^{\gamma(1-\alpha)}\upsilon^{-\gamma(1-\beta)}$$
$$\leq C_\gamma \mathcal{M}(\upsilon)\hbar^{\gamma(\alpha-\beta)}[\varepsilon(\varepsilon+h)]^{\gamma(\beta-\alpha)} + C_\gamma \mathcal{M}(\upsilon)\hbar^{\gamma(1-\alpha)}\varepsilon^{-\gamma(1-\beta)}.$$

This shows that $A^\alpha\phi \in C^\theta([\varepsilon, \Im], J_r)$. Therefore, $A^\alpha\phi \in C^\theta((0, \Im], J_r)$ due to an arbitrary ε. Try to remember that $\zeta(\upsilon) = \int_0^\upsilon (\upsilon - s)^{\gamma-1} E_{\gamma,\gamma}(-(\upsilon - s)^\gamma A) F(\tilde{u}(s), \tilde{u}(s)) ds$. We know that $|F(\tilde{u}(s), \tilde{u}(s)|_r \leq \mathcal{M}\mathcal{K}^2(\upsilon)s^{-2\gamma(\alpha-\beta)}$, while $\mathcal{K}(\upsilon) = \sup_{s\in(0,\upsilon]} s^{\gamma(\alpha-\beta)}|A^\alpha u(s)|_r$ is bounded and continuous in $(0, \Im]$. A simple reasoning allows us to take the Hölder continuity of $A^\zeta \in C^\theta((0, \Im], J_r)$. Therefore we have that $A^\alpha\tilde{u}(\upsilon)$ $=A^\alpha\phi(\upsilon) + A^\alpha\varphi(\upsilon) + A^\alpha\zeta(\upsilon) \in C^\theta((0, \Im], J_r)$. We know that $F(\tilde{u}, \tilde{u}) \in C^\theta((0, \Im], J_r)$ is verified; similarly following Step 2, this yields that ${}^cD_\upsilon^\gamma\zeta \in C^\theta((0, \Im], J_r)$,$A\zeta \in C^\theta((0, \Im], J_r)$ and ${}^cD_\upsilon^\gamma\zeta = -A\zeta + F(\tilde{u}, \tilde{u})$. So we get that ${}^cD_\upsilon^\gamma\zeta \in C^\theta((0, \Im], J_r)$, $A\tilde{u} \in C^\theta((0, \Im], J_r)$ and ${}^cD_\upsilon^\gamma\tilde{u} = -A\tilde{u} + F(\tilde{u}, \tilde{u}) + Pg$. Hence, we can say that $\tilde{u}$ has a classical solution. □

Theorem 5.2. *Suppose (5.1) holds if $\tilde{u}$ is a classical solution of (1.5); thus, ${}^cD_\upsilon^\gamma \in C^\nu((0, \Im], J_r)$ and $A\tilde{u} \in C^\nu((0, \Im], J_r)$.*

Proof. Let $\tilde{u}$ be the classical solution of Problem (1.5) and also suppose that(5.1) holds; so,$\tilde{u}(\upsilon) = \tilde{\varphi}(\upsilon) + \phi(\upsilon) + \zeta(\upsilon)$ shows that $A\tilde{\varphi} \in C^{\gamma(1-\beta)}((0, \Im], J_r)$. It is also sufficient to demonstrate that, for every $\varepsilon > 0$ there is $A\tilde{\varphi} \in C^{\gamma(1-\beta)}((\varepsilon, \Im], J_r)$. Now by using Lemma 2.2(iii) we take $\hbar$ as $\varepsilon \leq \upsilon < \upsilon + \hbar \leq \Im$;

$$\begin{aligned}
\left|A\tilde{\varphi}(\upsilon+\hbar) - A\tilde{\varphi}(\upsilon)\right|_r &= \Big|\int_\upsilon^{\upsilon+\hbar} -s^{\gamma-1}A^2E_{\gamma,\gamma}(-s^\gamma A)ads\Big|_r \\
&\leq C_1\int_\upsilon^{\upsilon+\hbar} s^{-\gamma(1-\beta)-1}ds|a|_{H^{\gamma,r}} \\
&= \frac{C_1|a|_{H^{\gamma,r}}}{\gamma}(\upsilon^{-\gamma(1-\beta)} - (\upsilon+\hbar)^{-\gamma(1-\beta)}) \\
&= \frac{C_1|a|_{H^{\gamma,r}}}{\gamma}\frac{\hbar^{\gamma(1-\beta)}}{[\tilde{\varepsilon}(\tilde{\varepsilon}+\hbar)]^{\gamma(1-\beta)}}.
\end{aligned}$$

Using (5.1) and rewriting $\tilde{\varphi}(\upsilon)$ in the form of

$$\begin{aligned}
\tilde{\varphi}(\upsilon) &= \tilde{\varphi}_1(\upsilon) + \tilde{\varphi}_2(\upsilon) \\
&= \int_0^\upsilon (\upsilon-s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon-s)^\gamma A)(Pg(s) - Pg(\upsilon))ds + \int_0^{\upsilon\upsilon} (\upsilon-s)^{\gamma-1}E_{\gamma,\gamma}(-(\upsilon-s)^\gamma A)Pg(\upsilon)ds
\end{aligned}$$

as $\upsilon \in (0, \Im]$, we can say that $A\tilde{\varphi}_1(\upsilon) \in C^\nu)((0, \Im], J_r)$ and $A\tilde{\varphi}_2(\upsilon) \in C^\theta((0, \Im], J_r)$. These results come from Lemma 5.1 and (5.8). As we know that $F(\tilde{u}, \tilde{u}) \in C^\theta((0, \Im], J_r)$ and $\zeta(\upsilon)$ is verified analogously, so we can conclude that $A\zeta(\upsilon) \in C^\nu((0, \Im], J_r)$. Consequently, ${}^cD_\upsilon^\gamma\tilde{u} = A\tilde{u} + F(\tilde{u}, \tilde{u}) + Pg \in C^\nu((0, \Im], J_r)$ as $A\tilde{u} \in C^\nu((0, \Im], J_r)$. This proof is now complete. □

6. Conclusions

This study demonstrates the existence-uniqueness of local and global mild solutions. Meanwhile, we offer a local reasonable solution in $\mathbf{S}_\wp$. The NS equations with time-fractional derivatives of order $\gamma \in (0, 1)$ were used to simulate anomaly diffusion in fractal media. We also demonstrate the existence of regular, classical solutions to these equations in $\mathbf{S}_\wp$. The concept could be expanded upon by including MHD effects, the concept put forth in this mwork could be developed further, observability could be added and other activities could be generalized in future work. This is an interesting area with a lot of study going on that could lead to a lot of different applications and theories.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author's Contributions

All authors contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions.

References

1. G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, New York: Springer, 2011. https://doi.org/10.1007/978-0-387-09620-9
2. A. Abbas, R. Shafqat, M. B. Jeelani, N. H. Alharthi, Significance of chemical reaction and Lorentz force on third-grade fluid flow and heat transfer with Darcy-Forchheimer law over an inclined exponentially stretching sheet embedded in a porous medium, *Symmetry*, **14** (2022), 779. https://doi.org/10.3390/sym14040779
3. Y. Mehmood, R. Shafqat, I. E. Sarris, M. Bilal, T. Sajid, T. Akhtar, Numerical investigation of MWCNT and SWCNT fluid flow along with the activation energy effects over quartic auto catalytic endothermic and exothermic chemical reactions, *Mathematics*, **10** (2022), 4636. https://doi.org/10.3390/math10244636
4. A. U. K. Niazi, J. He, R. Shafqat, B. Ahmed, Existence, uniqueness, and E_q-Ulam-type stability of fuzzy fractional differential equation, *Fractal Fract.*, **5** (2021), 66. https://doi.org/10.3390/fractalfract5030066
5. N. Iqbal, A. U. K. Niazi, R. Shafqat, S. Zaland, Existence and uniqueness of mild solution for fractional-order controlled fuzzy evolution equation, *J. Funct. Space*, **2021** (2021), 5795065. https://doi.org/10.1155/2021/5795065
6. R. Shafqat, A. U. K. Niazi, M. B. Jeelani, N. H. Alharthi, Existence and uniqueness of mild solution where $\alpha \in (1, 2)$ for fuzzy fractional evolution equations with uncertainty, *Fractal Fract.*, **6** (2022), 65. https://doi.org/10.3390/fractalfract6020065
7. A. S. Alnahdi, R. Shafqat, A. U. K. Niazi, M. B. Jeelani, Pattern formation induced by fuzzy fractional-order model of COVID-19, *Axioms*, **11** (2022), 313. https://doi.org/10.3390/axioms11070313

8. A. Khan, R. Shafqat, A. U. K. Niazi, Existence results of fuzzy delay impulsive fractional differential equation by fixed point theory approach, *J. Funct. Space*, **2022** (2022), 4123949. https://doi.org/10.1155/2022/4123949
9. K. Abuasbeh, R. Shafqat, A. U. K. Niazi, M. Awadalla, Local and global existence and uniqueness of solution for time-fractional fuzzy Navier-Stokes equations, *Fractal Fract.*, **6** (2022), 330. https://doi.org/10.3390/fractalfract6060330
10. K. Abuasbeh, R. Shafqat, A. U. K. Niazi, M. Awadalla, Nonlocal fuzzy fractional stochastic evolution equations with fractional Brownian motion of order (1, 2), *AIMS Mathematics*, **7** (2022), 19344–19358. https://doi.org/10.3934/math.20221062
11. K. Abuasbeh, R. Shafqat, Fractional Brownian motion for a system of fuzzy fractional stochastic differential equation, *J. Math.*, **2022** (2022), 3559035. https://doi.org/10.1155/2022/3559035
12. K. Abuasbeh, R. Shafqat, A. U. K. Niazi, M. Awadalla, Oscillatory behavior of solution for fractional order fuzzy neutral predator-prey system, *AIMS Mathematics*, **7** (2022), 20383–20400. https://doi.org/10.3934/math.20221117
13. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science, 2006.
14. H. Komatsu, Fractional powers of operators, *Pac. J. Math.*, **19** (1966), 285–346. https://doi.org/10.2140/pjm.1966.19.285
15. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci.*, **44** (2017), 460–481. https://doi.org/10.1016/j.cnsns.2016.09.006
16. A. K. Shukla, J. C. Prajapati, On a generalization of Mittag-Leffler function and its properties, *J. Math. Anal. Appl.*, **336** (2007), 797–811. https://doi.org/10.1016/j.jmaa.2007.03.018
17. Y. Zhou, L. Peng, On the time-fractional Navier-Stokes equations, *Comput. Math. Appl.*, **73** (2017), 874–891. https://doi.org/10.1016/j.camwa.2016.03.026
18. F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, Singapore: World Scientific, 2010.
19. Y. Zhou, J. R. Wang, L. Zhang, *Basic Theory of Fractional Differential Equations*, Singapore: World scientific, 2016.
20. P. M. de Carvalho Neto, Fractional differential equations: A novel study of local and global solutions in Banach spaces, *Universidade de São Paulo*, 2013. https://doi.org/10.11606/T.55.2013.tde-06062013-145531
21. N. Masmoudi, T. K. Wong, Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods, *Comm. Pure Appl. Math.*, **68** (2015), 1683–1741. https://doi.org/10.1002/cpa.21595
22. P. M. de Carvalho Neto, G. Planas, Mild solutions to the time fractional Navier-Stokes equations in RN, *J. Differ. Equ.*, **259** (2015), 2948–2980. https://doi.org/10.1016/j.jde.2015.04.008
23. H. Kozono, L1-solutions of the Navier-Stokes equations in exterior domains, *Math. Ann.*, **312** (1998), 319–340. https://doi.org/10.1007/s002080050224

AIMS Press