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Research article

On the fourth power mean of one special two-term exponential sums

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Abstract: The main purpose of this paper is using the elementary methods and the number of the solutions of some congruence equations to study the calculating problem of the fourth power mean of one special two-term exponential sums, and give an exact calculating formula for it.

Keywords: the two-term exponential sums; fourth power mean; elementary method; calculating formula
Mathematics Subject Classification: 11L03, 11L05

1. Introduction

Let p be an odd prime. For any integers $k > h \geq 1$ and integers m and n with $(n, p) = 1$, the two-term exponential sums $S(m, n, k, h; p)$ is defined as

$$S(m,n,k,h;p) = \sum_{a=0}^{p-1} e\left(\frac{ma^k + na^h}{p}\right),$$

where $e(y) = e^{2\pi iy}$ and $i^2 = -1$.

This sum play a very important role in the study of analytic number theory, so many number theorists and scholars had studied the various properties of $S(m, n, k, h; p)$, and obtained a series of meaningful research results. For example, H. Zhang and W. P. Zhang [1] proved that for any odd prime p, one has

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right) \right|^4 = \begin{cases} 2p^3 - p^2 & \text{if } 3 \nmid p-1; \\ 2p^3 - 7p^2 & \text{if } 3 \mid p-1, \end{cases} \tag{1.1}$$

where n represents any integer with $(n, p) = 1$.

W. P. Zhang and D. Han [2] used elementary and analytic methods to obtain the identity:

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{a^3+ma}{p}\right)\right|^6 = 5p^4-8p^3-p^2, \tag{1.2}$$

where p denotes an odd prime with $3 \nmid (p-1)$.

Recently, W. P. Zhang and Y. Y. Meng [3] studied the sixth power mean of $S(m,1,3,1;p)$, and proved that for any odd prime p, we have the identities

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^6 = \begin{cases} 5p^3\cdot(p-1), & \text{if } p\equiv 5 \bmod 6;\\ p^2\cdot\left(5p^2-23p-d^2\right), & \text{if } p\equiv 1 \bmod 6,\end{cases} \tag{1.3}$$

where $4p = d^2+27\cdot b^2$, and d is uniquely determined by $d\equiv 1 \bmod 3$.

On the other hand, L. Chen and X. Wang [4] studied the fourth power mean of $S(m,1,4,1;p)$, and proved that the identities

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^4 = \begin{cases} 2p^2(p-2), & \text{if } p\equiv 7 \bmod 12,\\ 2p^3, & \text{if } p\equiv 11 \bmod 12,\\ 2p(p^2-10p-2\alpha^2), & \text{if } p\equiv 1 \bmod 24,\\ 2p(p^2-4p-2\alpha^2), & \text{if } p\equiv 5 \bmod 24,\\ 2p(p^2-6p-2\alpha^2), & \text{if } p\equiv 13 \bmod 24,\\ 2p(p^2-8p-2\alpha^2), & \text{if } p\equiv 17 \bmod 24,\end{cases} \tag{1.4}$$

where $\alpha=\alpha(p)=\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)$ is an integer satisfying the following identity (see Theorem 4–11 in [5]):

$$p=\alpha^2+\beta^2=\left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)\right)^2+\left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+r\overline{a}}{p}\right)\right)^2,$$

$\left(\frac{*}{p}\right)$ denotes the Legendre's symbol, r is any quadratic non-residue modulo p and $\overline{a}$ is the solution of the congruence equation $a\cdot x\equiv 1 \bmod p$.

In addition, T. T. Wang and W. P. Zhang [6] gave calculating formulae of the fourth and sixth power mean of $S(m,n,2,1;p)$.

From the formulae (1.1)–(1.4), it is not difficult to see that the content of all these papers only involves $h=1$ in $S(m,n,k,h;p)$. Through searching literature, we have not found papers dealing with the fourth power mean of the two-term exponential sums $S(m,n,k,2;p)$ so far. Therefore, when $k>h=2$, it is difficult to obtain some ideal results.

In this paper, we use the elementary and analytic methods, and the number of the solutions of some congruence equations to study the calculating problem of the $2k$-th power mean:

$$S_{2k}(p)=\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a^2}{p}\right)\right|^{2k},$$

and give an exact calculating formula for $S_4(p)$ with $p\equiv 3 \bmod 4$.

That is, we give the following two conclusions:

Theorem 1.1. *Let p be a prime with $p \equiv 11$ mod 12, then we have the identity*

$$\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a^2}{p}\right)\right|^4 = 2p^3 - p^2\cdot\left(3+\sum_{a=1}^{p-1}\left(\frac{a-1+\overline{a}}{p}\right)\right),$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol modulo p, and $\overline{a}$ is the solution of the congruence equation $a\cdot x \equiv 1$ mod p.

Theorem 1.2. *Let p be a prime with $p \equiv 7$ mod 12, then we have the identity*

$$\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a^2}{p}\right)\right|^4 = 2p^3 - p^2\cdot\left(9-\sum_{a=1}^{p-1}\left(\frac{a-1+\overline{a}}{p}\right)\right).$$

2. Several lemmas

To complete the proofs of our theorems, we need four simple lemmas. Of course, the proofs of these lemmas need some knowledge of elementary or analytic number theory, all these can be found in references [5] and [7,8], so we do not repeat them here. First, according to W. P. Zhang and J. Y. Hu [9] or B. C. Berndt and R. J. Evans [10], we have the following:

Lemma 2.1. *Let p be an odd prime with $p \equiv 1$ mod 3. Then for any third-order character λ modulo p, we have the identity*

$$\tau^3(\lambda)+\tau^3\left(\overline{\lambda}\right) = dp,$$

where $\tau(\chi) = \sum_{a=1}^{p-1}\chi(a)e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums, $4p = d^2 + 27\cdot b^2$, and d is uniquely determined by $d \equiv 1$ mod 3.

Related results can also be found in [11–13].

Lemma 2.2. *Let p be an odd prime, then we have the identities*

$$\sum_{\substack{a=0 \\ a^3+1\equiv c^3 \bmod p \\ a^2+1\equiv c^2 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 = 2+\left(\frac{-2}{p}\right)$$

and

$$\sum_{\substack{a=0 \\ a^3+b^3\equiv c^3+1 \bmod p \\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = 3p-5-\left(\frac{-2}{p}\right)+\left(\frac{-3}{p}\right)\cdot\sum_{a=1}^{p-1}\left(\frac{a-1+\overline{a}}{p}\right),$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol modulo p, and $a\cdot\overline{a} \equiv 1$ mod p.

Proof. We only prove the second formula in Lemma 2.2. Similarly, we can deduce the first one. It is clear that from the properties of the complete residue system and the Legendre's symbol modulo p we have

$$
\begin{aligned}
\sum_{\substack{a=0\\a^3+b^3\equiv c^3+1 \bmod p\\a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 &= \sum_{\substack{a=0\\a^3+3a^2c+3ac^2+b^3+3b^2+3b\equiv 0 \bmod p\\a^2+2ac+b^2+2b\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1\\
&= \sum_{\substack{a=0\\a^3+3a^2c+3ac^2+1+3\overline{b}+3\overline{b}^2\equiv 0 \bmod p\\a^2+2ac+1+2\overline{b}\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1} 1 + \sum_{\substack{a=0\\a^3+3a^2c+3ac^2\equiv 0 \bmod p\\a^2+2ac\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1\\
&= \sum_{\substack{a=0\\a^3+3a(2c+a)^2+1+3(2b+1)^2\equiv 0 \bmod p\\a(a+2c)+1+2b\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 + p - \sum_{\substack{a=0\\a^3+3a^2c+3ac^2+1\equiv 0 \bmod p\\a^2+2ac+1\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1\\
&= \sum_{\substack{a=0\\a^3+3ac^2+1+3b^2\equiv 0 \bmod p\\ac+b\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 + p - \sum_{\substack{a=0\\a^3+3a(2c+a)^2+4\equiv 0 \bmod p\\a(2c+a)+1\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1\\
&= \sum_{\substack{a=0\\(a+1)\left(a^2-a+1+3ac^2\right)\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 + p - \sum_{\substack{a=1\\a^3+3ac^2+4\equiv 0 \bmod p\\ac+1\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} 1\\
&= 2p + \sum_{\substack{a=0\\a^2-a+1+3ac^2\equiv 0 \bmod p}}^{p-2}\sum_{c=0}^{p-1} 1 - \sum_{\substack{a=1\\a^3+3\overline{a}+4\equiv 0 \bmod p}}^{p-1} 1\\
&= 2p-2 + \sum_{\substack{a=1\\3a(a^2-a+1)+c^2\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 - \sum_{\substack{a=1\\(a+1)(a^3-a^2+a+3)\equiv 0 \bmod p}}^{p-1} 1\\
&= 2p-3 + \sum_{a=1}^{p-1}\left(1+\left(\frac{-3a(a^2-a+1)}{p}\right)\right) - \sum_{\substack{a=1\\a^3-a^2+a+3\equiv 0 \bmod p}}^{p-2} 1\\
&= 3p-4 + \left(\frac{-3}{p}\right)\cdot\sum_{a=1}^{p-1}\left(\frac{a-1+\overline{a}}{p}\right) - \sum_{\substack{a=1\\(a+1)(a^2-2a+3)\equiv 0 \bmod p}}^{p-2} 1\\
&= 3p-4 + \left(\frac{-3}{p}\right)\cdot\sum_{a=1}^{p-1}\left(\frac{a-1+\overline{a}}{p}\right) - \sum_{\substack{a=0\\(a-1)^2+2\equiv 0 \bmod p}}^{p-1} 1\\
&= 3p-5 - \left(\frac{-2}{p}\right) + \left(\frac{-3}{p}\right)\cdot\sum_{a=1}^{p-1}\left(\frac{a-1+\overline{a}}{p}\right).
\end{aligned}
$$

This proves Lemma 2.2. □

Lemma 2.3. *Let p be an odd prime, then we have the identities*

$$\sum_{\substack{a=0 \\ a^3+1\equiv c^3 \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1 = \begin{cases} p & \text{if } 3 \nmid (p-1); \\ p+d-2 & \text{if } 3 \mid (p-1), \end{cases}$$

and

$$\sum_{\substack{a=0 \\ a^3+b^3\equiv c^3+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = \begin{cases} p^2 & \text{if } 3 \nmid (p-1); \\ p^2-d+6p & \text{if } 3 \mid (p-1), \end{cases}$$

where d is the same as defined in Lemma 2.1.

Proof. We only prove the second formula in Lemma 2.3. Similarly, we can deduce the first one. If $(3, p-1) = 1$, then from the properties of the complete residue system modulo p we have

$$\sum_{\substack{a=0 \\ a^3+b^3\equiv c^3+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = \sum_{\substack{a=0 \\ a+b\equiv c+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 = p^2. \tag{2.1}$$

If $(3, p-1) = 3$, let λ denote any third-order primitive character modulo p and

$$A(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right).$$

Then for any integer m with $(m, p) = 1$, from Lemma 2.1, the definition and properties of the classical Gauss sums we have

$$A(m) = 1 + \sum_{a=1}^{p-1}\left(1 + \lambda(a) + \overline{\lambda}(a)\right)\cdot e\left(\frac{ma}{p}\right) = \overline{\lambda}(m)\cdot\tau(\lambda) + \lambda(m)\cdot\tau\left(\overline{\lambda}\right), \tag{2.2}$$

and

$$A^3(m) = \tau^3(\lambda) + \tau^3\left(\overline{\lambda}\right) + 3p\cdot A(m) = dp + 3p\cdot A(m). \tag{2.3}$$

Note that $A(m) = A(-m)$, then from (2.2), (2.3) and the properties of the trigonometric sums we have

$$\begin{aligned}
\sum_{\substack{a=0 \\ a^3+b^3\equiv c^3+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 &= \frac{1}{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{m\left(a^3+b^3-c^3-1\right)}{p}\right) \\
&= p^2 + \frac{1}{p}\sum_{m=1}^{p-1} e\left(\frac{-m}{p}\right)A^2(m)\cdot A(-m) = p^2 + \frac{1}{p}\sum_{m=1}^{p-1} e\left(\frac{-m}{p}\right)A^3(m) \\
&= p^2 + \frac{1}{p}\sum_{m=1}^{p-1}(dp + 3p\cdot A(m))\, e\left(\frac{-m}{p}\right) = p^2 - d + 6\cdot\tau(\lambda)\cdot\overline{\tau(\lambda)} \\
&= p^2 - d + 6p.
\end{aligned} \tag{2.4}$$

Now Lemma 2.3 follows from (2.1) and (2.4). □

Lemma 2.4. *Let p be an odd prime, then we have the identities*

$$\sum_{\substack{a=0 \\ a^3\equiv c^3 \bmod p}}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a^2-c^2}{p}\right) = \begin{cases} p, & \text{if } 3 \nmid (p-1); \\ p-2+\tau(\chi_2)\cdot \sum\limits_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1} \left(\frac{a^2-1}{p}\right), & \text{if } 3 \mid (p-1), \end{cases}$$

where $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre's symbol modulo p.

Proof. For any integer n with $(n,p)=1$, note the identity

$$\sum_{a=0}^{p-1} e\left(\frac{na^2}{p}\right) = 1 + \sum_{a=1}^{p-1}(1+\chi_2(a))\, e\left(\frac{na}{p}\right) = \chi_2(n)\cdot\tau(\chi_2). \tag{2.5}$$

If $3 \nmid (p-1)$, then from (2.5) and the properties of the complete residue system modulo p and Lemma 2.2, we have

$$\sum_{\substack{a=0 \\ a^3-c^3\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a^2-c^2}{p}\right) = 1 + \sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1}\sum_{c=1}^{p-1} e\left(\frac{c^2(a^2-1)}{p}\right) = 1+p-1=p. \tag{2.6}$$

If $3 \mid (p-1)$, then we have

$$\begin{aligned}
\sum_{\substack{a=0 \\ a^3-c^3\equiv 0 \bmod p}}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a^2-c^2}{p}\right) &= 1 + \sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1}\sum_{c=1}^{p-1} e\left(\frac{c^2(a^2-1)}{p}\right) \\
&= 1 + \sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{c^2(a^2-1)}{p}\right) - \sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1} 1 \\
&= 1 + p + \tau(\chi_2)\cdot \sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1}\left(\frac{a^2-1}{p}\right) - 3 \\
&= p-2+\tau(\chi_2)\cdot \sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1}\left(\frac{a^2-1}{p}\right).
\end{aligned} \tag{2.7}$$

Now Lemma 2.4 follows from (2.6) and (2.7). □

3. Results

In this paper, we prove the following two conclusions:

Theorem 3.1. *Let p be a prime with $p\equiv 11$ mod 12, then we have the identity*

$$\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a^2}{p}\right)\right|^4 = 2p^3 - p^2\cdot\left(3+\sum_{a=1}^{p-1}\left(\frac{a-1+\overline{a}}{p}\right)\right),$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol modulo p, and $\overline{a}$ is the solution of the congruence equation $a \cdot x \equiv 1 \bmod p$.

Proof. Now we apply the lemmas to complete the proofs of our theorems. For any integer n, note that the trigonometrical identities

$$\sum_{a=0}^{p-1} e\left(\frac{na}{p}\right) = \begin{cases} p & \text{if } p \mid n; \\ 0 & \text{if } p \nmid n. \end{cases}$$

From (2.5) and the properties of the reduced residue system modulo p we have,

$$\begin{aligned}
S_4(p) &= \sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a^2}{p}\right) \right|^4 = p \cdot \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3+d^3 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} e\left(\frac{a^2+b^2-c^2-d^2}{p}\right) \\
&= p \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{d^2\left(a^2+b^2-c^2-1\right)}{p}\right) + p \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^2+b^2-c^2}{p}\right) \\
&= p \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} e\left(\frac{d^2\left(a^2+b^2-c^2-1\right)}{p}\right) \\
&\quad + p \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^2+b^2-c^2}{p}\right) - p \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 \\
&= p^2 \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3+1 \bmod p \\ a^2+b^2 \equiv c^2+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 + p \cdot \tau(\chi_2) \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2+b^2-c^2-1}{p}\right) \\
&\quad + p \sum_{\substack{a=0 \\ a^3+1 \equiv c^3 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{b^2(a^2+1-c^2)}{p}\right) - p \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 + p \sum_{\substack{a=0 \\ a^3 \equiv c^3 \bmod p}}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^2-c^2}{p}\right) \\
&= p^2 \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3+1 \bmod p \\ a^2+b^2 \equiv c^2+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 + p \cdot \tau(\chi_2) \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2+b^2-c^2-1}{p}\right) \\
&\quad + p^2 \cdot \sum_{\substack{a=0 \\ a^3+1 \equiv c^3 \bmod p \\ a^2+1 \equiv c^2 \bmod p}}^{p-1} \sum_{c=0}^{p-1} 1 + p \cdot \tau(\chi_2) \sum_{\substack{a=0 \\ a^3+1 \equiv c^3 \bmod p}}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2+1-c^2}{p}\right) \\
&\quad - p \sum_{\substack{a=0 \\ a^3+1 \equiv c^3 \bmod p}}^{p-1} \sum_{c=0}^{p-1} 1 - p \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 + p \sum_{\substack{a=0 \\ a^3 \equiv c^3 \bmod p}}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^2-c^2}{p}\right).
\end{aligned} \tag{3.1}$$

It is clear that $S_4(p)$ is a real number. If $p = 12k + 11$, then $3 \nmid (p-1)$, $p \equiv 3 \bmod 4$, $\left(\frac{-1}{p}\right) = -1$

and $\left(\frac{3}{p}\right) = 1$. Note that $\tau(\chi_2) = i \cdot \sqrt{p}$ is a pure imaginary number. So from Lemma 2.2, Lemma 2.3, Lemma 2.4 and (3.1), we have

$$\begin{aligned}
S_4(p) &= \sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a^2}{p}\right) \right|^4 \\
&= p^2 \left(3p - 5 - \left(\frac{-2}{p}\right) + \left(\frac{-3}{p}\right) \cdot \sum_{a=1}^{p-1} \left(\frac{a - 1 + \overline{a}}{p}\right) \right) + p^2 \left(2 + \left(\frac{-2}{p}\right) \right) \\
&\quad + p \cdot \tau(\chi_2) \cdot \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2 + b^2 - c^2 - 1}{p}\right) \\
&\quad + p \cdot \tau(\chi_2) \cdot \sum_{\substack{a=0 \\ a^3+1 \equiv c^3 \bmod p}}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2 + 1 - c^2}{p}\right) - p\left(p + p^2\right) + p^2 \\
&= 2p^3 - p^2 \cdot \left(3 + \sum_{a=1}^{p-1} \left(\frac{a - 1 + \overline{a}}{p}\right) \right) - p \cdot \tau(\chi_2) \cdot \sum_{\substack{a=0 \\ a^3+b^3 \equiv c^3+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2 + b^2 - c^2 - 1}{p}\right) \\
&\quad - p \cdot \tau(\chi_2) \cdot \sum_{\substack{a=0 \\ a^3+1 \equiv c^3 \bmod p}}^{p-1} \sum_{c=0}^{p-1} \left(\frac{a^2 + 1 - c^2}{p}\right) \\
&= 2p^3 - p^2 \cdot \left(3 + \sum_{a=1}^{p-1} \left(\frac{a - 1 + \overline{a}}{p}\right) \right).
\end{aligned}$$

This proves Theorem 3.1. □

Theorem 3.2. *Let p be a prime with $p \equiv 7 \bmod 12$, then we have the identity*

$$\sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a^2}{p}\right) \right|^4 = 2p^3 - p^2 \cdot \left(9 - \sum_{a=1}^{p-1} \left(\frac{a - 1 + \overline{a}}{p}\right) \right).$$

Proof. Similarly, if $p = 12k + 7$, then $S_4(p)$ is also a real number, note that $3 \mid (p-1)$, $\left(\frac{-1}{p}\right) = -1$, $\left(\frac{3}{p}\right) = -1$ and $\tau(\chi_2) = i \cdot \sqrt{p}$, so from Lemma 2.2, Lemma 2.3, Lemma 2.4 and (3.1) we have

$$\begin{aligned}
S_4(p) &= \sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a^2}{p}\right) \right|^4 \\
&= p^2 \left(3p - 5 - \left(\frac{-2}{p}\right) + \left(\frac{-3}{p}\right) \cdot \sum_{a=1}^{p-1} \left(\frac{a - 1 + \overline{a}}{p}\right) \right) + p^2 \left(2 + \left(\frac{-2}{p}\right) \right) \\
&\quad - p\left(p + d - 2 + p^2 - d + 6p\right) + p(p-2) \\
&= 2p^3 - p^2 \cdot \left(9 - \sum_{a=1}^{p-1} \left(\frac{a - 1 + \overline{a}}{p}\right) \right).
\end{aligned}$$

This completes the proof of Theorem 3.2. □

From A. Weil's important works [14] and [15] we have the estimate:

$$\left|\sum_{a=1}^{p-1}\left(\frac{a-1+\overline{a}}{p}\right)\right| = \left|\sum_{a=1}^{p-1}\left(\frac{a^3-a^2+a}{p}\right)\right| \ll \sqrt{p}.$$

Combining this estimate and our theorems we can deduce the following:

Corollary 3.1. *Let a be an integer, $p-1 \geq a \geq 0$. Let p be an odd prime with $p \equiv 3 \bmod 4$. Then we have the asymptotic formula*

$$\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a^2}{p}\right)\right|^4 = 2p^3 + O\left(p^{\frac{5}{2}}\right).$$

Remark 3.1. In this paper, we only discussed the case $p \equiv 3 \bmod 4$. If $p \equiv 1 \bmod 4$, then we can not calculate the exact value of $S_4(p)$ yet. The reason is that we do not know the exact values of

$$\sum_{\substack{a=0 \\ a^3+b^3\equiv c^3+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right) + \sum_{\substack{a=0 \\ a^3+1\equiv c^3 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+1-c^2}{p}\right).$$

This will await our further study.

If $p \equiv 3 \bmod 4$, then we can easily deduce the identities

$$\sum_{\substack{a=0 \\ a^3+b^3\equiv c^3+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+b^2-c^2-1}{p}\right) + \sum_{\substack{a=0 \\ a^3+1\equiv c^3 \bmod p}}^{p-1}\sum_{c=0}^{p-1}\left(\frac{a^2+1-c^2}{p}\right) = 0.$$

4. Conclusions

The main results of this paper is to give an exact calculating formula for the fourth power mean of one special two-term exponential sums. That is,

$$S_4(p) = \sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a^2}{p}\right)\right|^4,$$

with the case $p \equiv 3 \bmod 4$.

Here, we give an example to calculate the exact results of the prime number p satisfying conditions $p \equiv 7 \bmod 12$ or $p \equiv 11 \bmod 12$. The exact results of calculation are summarised in Table 1.

At the same time, our results also provides some new and effective method for the calculating problem of the fourth power mean of the higher order two-term exponential sums. We have reasons to believe that these works will play a positive role in promoting the study of relevant problems. Furthermore, it is still an open problem for the case of $k > h \geq 3$ for $S(m, n, k, h; p)$, interested readers can continue this research.

Table 1. The calculation of $S_4(p)$.

p	$S_4(p)$	p	$S_4(p)$
7	$S_4(7) = 245$	11	$S_4(11) = 2783$
19	$S_4(19) = 11913$	23	$S_4(23) = 18515$
31	$S_4(31) = 43245$	47	$S_4(47) = 201019$
43	$S_4(43) = 134977$	59	$S_4(59) = 414239$
67	$S_4(67) = 579081$	71	$S_4(71) = 741027$
79	$S_4(79) = 979837$	83	$S_4(83) = 1095351$
103	$S_4(103) = 1920229$	107	$S_4(107) = 2278351$
127	$S_4(127) = 4080637$	131	$S_4(131) = 4376055$
139	$S_4(139) = 5429201$	167	$S_4(167) = 9900595$
151	$S_4(151) = 7045509$	179	$S_4(179) = 11759047$

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Conflict of interest

The authors declare no conflict of interest.

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