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Research article

Orbital stability of periodic standing waves of the coupled Klein-Gordon-Zakharov equations

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Abstract: This paper investigates the orbital stability of periodic standing waves for the following coupled Klein-Gordon-Zakharov equations

$$\begin{cases} u_{tt} - u_{xx} + u + \alpha uv + \beta |u|^2 u = 0, \\ v_{tt} - v_{xx} = (|u|^2)_{xx}, \end{cases}$$

where $\alpha > 0$ and β are two real numbers and $\alpha > \beta$. Under some suitable conditions, we show the existence of a smooth curve positive standing wave solutions of dnoidal type with a fixed fundamental period L for the above equations. Further, we obtain the stability of the dnoidal waves for the coupled Klein-Gordon-Zakharov equations by applying the abstract stability theory and combining the detailed spectral analysis given by using Lamé equation and Floquet theory. When period $L \to \infty$, dnoidal type will turn into sech-type in the sense of limit. In such case, we can obtain stability of sech-type standing waves. In particular, $\beta = 0$ is advisable, we still can show the the stability of the dnoidal type and sech-type standing waves for the classical Klein-Gordon-Zakharov equations.

Keywords: coupled Klein-Gordon-Zakharov equations; periodic standing waves; orbital stability; Floquet theory; Hamiltonian system
Mathematics Subject Classification: 35B35, 35C08, 35R10

1. Introduction

The coupled Klein-Gordon-Zakharov equations

$$\begin{cases} u_{tt} - u_{xx} + u + \alpha uv + \beta|u|^2u = 0, \\ v_{tt} - v_{xx} = (|u|^2)_{xx} \end{cases} \tag{1.1}$$

describes the interaction between the Langmuir waves and ion acoustic waves in a high frequency plasma (see [1, 2]). The complex value function $u(x, t)$ is the fast time scale component of the electric field raised by electrons, and the real value function $v(x, t)$ is the deviation of ion density from its equilibrium. In recent years, there have been many works on the research for the coupled Klein-Gordon-Zakharov Eq (1.1). In the literature [3], Guo and Yuan studied the existence and uniqueness of the global smooth solution of the Eq (1.1) without assuming small Cauchy data. Ozawa et al. [4] proved the well-posedness of the Eq (1.1) in three-dimensional space. In [5], Wang et al. considered initial and homogeneous boundary value problems of the Eq (1.1), and used the energy method to prove the existence and uniqueness of the difference solution. Chen and Zhang [6] proposed two new difference schemes for an initial-boundary-value problem of the coupled Klein-Gordon-Zakharov equations and proved stability and convergence of difference solutions. Many authors obtained some explicit exact solitary wave solutions and numerical solutions for the coupled KGZ equations by using a various of different approaches (see [7–9]). Zheng et al. [10] investigated the orbital stability of solitary waves for the Eq (1.1). Note that if $\alpha = 1$ and $\beta = 0$, the system (1.1) reduces to the classical Klein-Gordon-Zakharov equations

$$\begin{cases} u_{tt} - u_{xx} + u + uv = 0, \\ v_{tt} - v_{xx} = (|u|^2)_{xx}. \end{cases} \tag{1.2}$$

The system (1.2) arises in the study of the interaction between a Langmuir wave and an ion sound wave in plasma. Chen [11] considered orbital stability of solitary waves for the classical Klein-Gordon-Zakharov Eq (1.2). Hakkaev et al. [12] studied the orbital stability for periodic standing waves of the Eq (1.2) by applying the abstract results of Grillakis et al. [13, 14]. Hakkaev et al. [15] studied the linear stability analysis for periodic travelling waves of the Eq (1.2). In 2008, Gan [16] obtained orbital instability of standing waves for the Eq (1.2). Recently, many authors have studied the orbital stability and instability of standing waves and solitary waves (see [17–24]). The study of orbital stability for standing waves is valuable.

In this paper, we are interested in the existence and orbital stability of periodic standing wave solutions for the coupled Klein-Gordon-Zakharov equations with $\alpha > \beta$. Because here the stability refers to perturbations of the periodic wave profile itself, a study for the initial value problem of (1.1) is necessary. Similarly to Theorem in [3,4,12], we obtain the well-posedness of the initial value problem of (1.1).

Theorem 1.1. *Let $s > \frac{1}{2}$. For any fixed*

$$(u(0), u_t(0)) \in H^s(R) \times H^{s-1}(R), \quad (v(0), v_t(0)) \in H^{s-1}(R) \times H^{s-2}(R),$$

there exists a time T, depending only on the norms in the respective spaces, so that there exists a unique solution

$$u(x, t) \in C([0, T], H^s(R)), \quad u_t(x, t) \in C([0, T], H^{s-1}(R)),$$

$$v(x,t) \in C([0,T], H^{s-1}(R)), \quad v_t(x,t) \in C([0,T], H^{s-2}(R)).$$

We focus on solutions for (1.1) of the form

$$u(x,t) = e^{i\omega t}\phi(x), \quad \text{and} \;\; v(x,t) = \psi(x), \tag{1.3}$$

where $\omega \in R$, and $\phi(\xi), \psi(\xi)$: $R \to R$ are periodic smooth functions with the same arbitrary fundamental period $L > 0$.

In order to write the Eq (1.1) into Hamiltonian form and obtain our results of stability for the periodic traveling wave solutions in (1.3), we rewrite the coupled Klein-Gordon-Zakharov equations

$$\begin{cases} u_t = -\rho, \\ \rho_t = -u_{xx} + u + \alpha uv + \beta |u|^2 u, \\ v_t = n_x, \\ n_t = v_x + |u|_x^2. \end{cases} \tag{1.4}$$

Then, we have

$$\begin{aligned} u(x,t) = e^{i\omega t}\phi(x), \;\; v(x,t) = \psi(x), \\ \rho(x,t) = -i\omega u = -i\omega e^{i\omega t}\phi(x), \quad n(x,t) = 0, \end{aligned} \tag{1.5}$$

and

$$\begin{cases} \omega^2\phi = -\phi'' + \phi + \alpha\phi\psi + \beta\phi^3, \\ \psi' + (\phi^2)' = 0. \end{cases} \tag{1.6}$$

Although, the abstract orbital stability theory presented by Grillakis et al. [14] cannot be applied directly, by applying the extension version of the general theory of orbital stability presented by Grillakis et al. [13] and combining detailed spectral analysis given by using Lamé equation and Floquet theory, we obtain the orbital stability of periodic traveling waves (1.3) for the Eq (1.1).

This paper is organized as follows. In section 2, we present some remarks regarding periodic Sobolev space and Jacobi elliptic functions. Section 3 is devoted to the existence of a smooth curve of dnoidal wave solutions for the Eq (1.1). In Section 4, we study the spectral analysis of some certain self-adjoint operators necessary to obtain our stability results. In Section 5, we show orbital stability of the dnoidal waves solutions for the Eq (1.1).

2. Notation

The normal elliptic integral of the first type (see [25, 26]) is defined by

$$\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \equiv F(\varphi,k),$$

and the normal elliptic integral of the second kind

$$\int_0^y \sqrt{\frac{1-k^2t^2}{1-t^2}}dt = \int_0^\varphi \sqrt{1-k^2\sin^2\theta}d\theta \equiv E(\varphi,k),$$

where

$$y = \sin\varphi \in (0, 1], \quad k \in (0, 1).$$

The number k is called the modulus. The number $k' = \sqrt{1-k^2}$ is called the complementary modulus. $\varphi \in (0, \frac{\pi}{2}]$ is called the argument of the normal elliptic integral.

When $y = 1$, we denote $F(\frac{\pi}{2}, k)$ by $K = K(k)$ and $E(\frac{\pi}{2}, k)$ by $E = E(k)$. So

$$K(0) = E(0) = \frac{\pi}{2}, \quad E(1) = 1, \quad \text{and} \quad K(1) = +\infty.$$

For $k \in (0, 1)$,

$$K'(k) = \frac{E - k'^2 K}{k'^2 k} > 0, \quad K''(k) > 0, \quad E'(k) = \frac{E - K}{k} < 0, \quad \text{and} \quad E''(k) < 0.$$

Moreover, for $k \in (0, 1)$, we have that $E(k) < K(k)$, and $E(k) + K(k)$, $E(k)K(k)$ are strictly increasing functions.

The Jacobian elliptic functions are denoted by $sn(u; k)$, $cn(u; k)$ and $dn(u; k)$ (called snoidal, cnoidal and dnoidal, respectively) which are defined via the previous elliptic integral. Considering the elliptic integral

$$u(y; k) := u = F(\varphi, k),$$

we can define its inverse function by

$$y = \sin\varphi \equiv sn(u; k), \quad cn(u; k) = \sqrt{1 - sn^2(u; k)},$$

and

$$dn(u; k) = \sqrt{1 - k^2 sn^2(u; k)}.$$

Then, we have the asymptotic formulae

$$sn(x; 1) = \tanh(x), cn(x; 1) = sech(x) \text{ and } dn(x; 1) = sech(x).$$

3. Existence of dnoidal wave solutions for the coupled Klein-Gordon-Zakharov equations

This section is devoted to show the existence of a smooth curve of dnoidal wave solutions to the coupled Klein-Gordon-Zakharov Eq (1.1) of the form (1.3).

By the second equation in (1.6), we have

$$\psi = -\phi^2 + d_1 x + d_2. \tag{3.1}$$

Since ψ is a periodic function, it is clear that $d_1 = 0$. For simplicity, we take the other constant of integration $d_2 = 0$. So, from (3.1), it follows

$$\psi = -\phi^2. \tag{3.2}$$

Then, substituting (3.2) into the first equation of (1.6), we have

$$\phi'' - (1 - \omega^2)\phi + (\alpha - \beta)\phi^3 = 0. \tag{3.3}$$

Next, we show that the Eq (3.3) has an explicit periodic solution which will depend on Jacobian elliptic functions. In fact, multiplying (3.3) by ϕ' and integrating once, we obtain that the solution ϕ has to satisfy

$$(\phi')^2 = \frac{\alpha-\beta}{2}[-\phi^4 + \frac{2(1-\omega^2)}{\alpha-\beta}\phi^2 - \frac{4}{\alpha-\beta}A_\phi], \tag{3.4}$$

where A_ϕ is a needed integration constant different of zero. For convenience, we make

$$\varpi(t) = -t^4 + \frac{2(1-\omega^2)}{\alpha-\beta}t^2 - \frac{4}{\alpha-\beta}A_\phi.$$

For $1-\omega^2 > 0$ and $0 < A_\phi < \frac{(1-\omega^2)^2}{4(\alpha-\beta)}$, we have

$$\begin{aligned}\varpi(t) &= -(t^2 - \frac{1-\omega^2}{\alpha-\beta})^2 + \frac{(1-\omega^2)^2}{(\alpha-\beta)^2} - \frac{4A_\phi}{\alpha-\beta} \\ &= (t^2 - \frac{1-\omega^2}{\alpha-\beta} + \sqrt{\frac{(1-\omega^2)^2}{(\alpha-\beta)^2} - \frac{4A_\phi}{\alpha-\beta}}) \\ &\quad \cdot(\sqrt{\frac{(1-\omega^2)^2}{(\alpha-\beta)^2} - \frac{4A_\phi}{\alpha-\beta}} + \frac{1-\omega^2}{\alpha-\beta} - t^2),\end{aligned}$$

where

$$\sqrt{\frac{(1-\omega^2)^2}{(\alpha-\beta)^2} - \frac{4A_\phi}{\alpha-\beta}} - \frac{1-\omega^2}{\alpha-\beta} < 0,$$

and

$$\sqrt{\frac{(1-\omega^2)^2}{(\alpha-\beta)^2} - \frac{4A_\phi}{\alpha-\beta}} + \frac{1-\omega^2}{\alpha-\beta} > 0.$$

Hence, $\omega(t)$ has the real and symmetric roots $\pm\eta_1$ and $\pm\eta_2$. Without loss of generality, $0 < \eta_2 < \eta_1$. So, we can write (3.4) as

$$(\phi')^2 = \frac{\alpha-\beta}{2}(\phi^2 - \eta_2^2)(\eta_1^2 - \phi^2). \tag{3.5}$$

Assume that $1-\omega^2 > 0$, then left side of (3.5) is not negative. Therefore, we have that $\eta_2 \le \phi \le \eta_1$ and η_1, η_2 satisfy

$$\begin{cases}\eta_1^2 + \eta_2^2 = \dfrac{2(1-\omega^2)}{\alpha-\beta} > 0, \\ \eta_1^2\eta_2^2 = -\dfrac{4}{\alpha-\beta}A_\phi > 0.\end{cases} \tag{3.6}$$

Define $\rho = \frac{\phi}{\eta_1}$ and $k^2 = \frac{\eta_1^2-\eta_2^2}{\eta_1^2}$, then (3.5) becomes

$$(\rho')^2 = \frac{(\alpha-\beta)\eta_1^2}{2}(\rho^2 - \frac{\eta_2^2}{\eta_1^2})(1-\rho^2). \tag{3.7}$$

Define a new variable χ through the relation $\rho^2 = 1 - k^2 \sin^2 \chi$, from (3.7), we get

$$(\chi')^2 = \frac{(\alpha - \beta)\eta_1^2}{2}(1 - k^2 \sin^2 \chi).$$

Then, we obtain that

$$\int_0^{\chi(x)} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} = \sqrt{\frac{\alpha - \beta}{2}}\eta_1 x.$$

Via the definition of the Jacobi elliptic function, the above integral equation has the solution

$$\sin(\chi(x)) = sn(\sqrt{\frac{\alpha - \beta}{2}}\eta_1 x; k).$$

Hence, using the fact that $k^2 sn^2 + dn^2 = 1$, we obtain

$$\rho(x) = \sqrt{1 - k^2 \sin^2 \chi} = \sqrt{1 - k^2 sn^2(\sqrt{\frac{\alpha - \beta}{2}}\eta_1 x; k)} = dn(\sqrt{\frac{\alpha - \beta}{2}}\eta_1 x; k),$$

and $\rho(0) = 1$. From the relation $\rho = \frac{\phi}{\eta_1}$, we obtain the dnoidal wave solution

$$\phi(x) = \eta_1 dn(\sqrt{\frac{\alpha - \beta}{2}}\eta_1 x; k). \tag{3.8}$$

Substituting (3.8) into (3.2), we have

$$\psi(x) = -\eta_1^2 dn^2(\sqrt{\frac{\alpha - \beta}{2}}\eta_1 x; k). \tag{3.9}$$

Now, since dn has fundamental period $2K$, $dn(u; k) = dn(u + 2K; k)$, where $K = K(k)$ represents the complete elliptic integral of first kind, we obtain that ϕ and ψ have fundamental period given by

$$T_\phi = T_\psi = \frac{2\sqrt{2}}{\sqrt{\alpha - \beta}\eta_1}K(k). \tag{3.10}$$

Fix ω such that $1 - \omega^2 > 0$, and define $\nu = 1 - \omega^2$. Then, from (3.6), we get $0 < \eta_2 < \sqrt{\frac{\nu}{\alpha - \beta}} < \eta_1 < \sqrt{\frac{2\nu}{\alpha - \beta}}$ and we can see (3.10) as a function of a unique variable η_2, namely

$$T_\phi(\eta_2) = T_\psi(\eta_2) = \frac{2\sqrt{2}}{\sqrt{2\nu - (\alpha - \beta)\eta_2^2}}K(k(\eta_2)) \quad \text{with} \quad k^2(\eta_2) = \frac{2\nu - 2(\alpha - \beta)\eta_2^2}{2\nu\beta - (\alpha - \beta)\eta_2^2}.$$

Next, we will show that

$$T_\phi = T_\psi > \frac{\sqrt{2}\pi}{\sqrt{\nu}}.$$

Note that if $\eta_2 \to 0$, we have that $k(\eta_2) \to 1^-$, which implies that $K(k(\eta_2)) \to +\infty$. Therefore, $T_\phi, T_\psi \to +\infty$ as $\eta_2 \to 0$. On the other hand, if $\eta_2 \to \sqrt{\nu\beta}$, we have that $k(\eta_2) \to 0^+$, which implies that $K(k(\eta_2)) \to \frac{\pi}{2}$. Therefore, $T_\phi, T_\psi \to \frac{\sqrt{2}\pi}{\sqrt{\nu}}$ as $\eta_2 \to \sqrt{\nu\beta}$. Finally, since the function

$$\eta_2 \in (0, \sqrt{\nu\beta}) \mapsto T_\phi(\eta_2) = T_\psi(\eta_2)$$

is a strictly decreasing function (see proof of Theorem 3.2), it follows $T_\phi = T_\psi > \frac{\sqrt{2}\pi}{\sqrt{\nu}}$. Let $L > 0$, we choose $\nu > 0$ such that $\sqrt{\nu} > \frac{\sqrt{2}\pi}{L}$. By the analysis given above that there exists a unique $\eta_2 \equiv \eta_2(\nu)$ such that the fundamental period of the dnoidal waves

$$\phi_\nu = \phi(\cdot; \eta_1(\nu); \eta_2(\nu)) \text{ and } \psi_\nu = \psi(\cdot; \eta_1(\nu); \eta_2(\nu))$$

will be $T_{\phi_\nu}(\eta_2) = T_{\psi_\nu}(\eta_2) = L$.

Remark 3.1. *If $\eta_2 \to 0^+$, we obtain that $\eta_1 \to \sqrt{\frac{2\nu}{\alpha-\beta}}$, $k(\eta_2) \to 1^-$ and $dn(x,1) = sech(x)$. Consequently, the formulaes (3.8) and (3.9) lose its periodicity in this limit and we obtain a wave form with a single hump and with "infinity period"*

$$\phi(x; \sqrt{\frac{2\nu}{\alpha-\beta}}, 0) \to \sqrt{\frac{2\nu}{\alpha-\beta}} sech(\sqrt{\nu}\xi), \ \ \psi(x; \sqrt{\frac{2\nu}{\alpha-\beta}}, 0) \to -\frac{2\nu}{\alpha-\beta} sech^2(\sqrt{\nu}\xi),$$

which are exactly the classical ground state solutions for the coupled Klein-Gordon-Zakharov equations.

Theorem 3.2. *Let $L > 0$ fixed, consider $\nu_0 > \frac{2\pi^2}{L^2}$ and the unique $\eta_{2,0} = \eta_2(\nu_0) \in (0, \sqrt{\frac{\nu_0}{\alpha-\beta}})$ such that $T_{\phi_{\nu_0}} = T_{\psi_{\nu_0}} = L$. Then,*

(1) there exist intervals $I(\nu_0)$ and $B(\eta_{2,0})$ around ν_0 and $\eta_{2,0}$ respectively, and a unique smooth function $\Pi : I(\nu_0) \to B(\eta_{2,0})$ such that $\Pi(\nu_0) = \eta_{2,0}$ and

$$\frac{2\sqrt{2}}{\sqrt{2\nu - (\alpha-\beta)\eta_2^2}} K(k) = L,$$

for all $\nu \in I(\nu_0)$, $\eta_2 \in \Pi(\nu)$ and

$$k^2 = k^2(\nu) = \frac{2\nu - 2(\alpha-\beta)\eta_2^2}{2\nu - (\alpha-\beta)\eta_2^2} \in (0,1).$$

(2) The dnoidal waves $\phi(\cdot; \eta_1, \eta_2)$ and $\psi(\cdot; \eta_1, \eta_2)$ in (3.8) and (3.9) are determined by $\eta_1 \equiv \eta_1(\nu)$, $\eta_2 \equiv \eta_2(\nu) = \Pi(\nu)$, with $\eta_1^2 + \eta_2^2 = \frac{2\nu}{\alpha-\beta}$. We have fundamental period L and satisfy (3.2) and (3.5). Moreover, the mapping

$$\nu \in I(\nu_0) \mapsto (\phi(\cdot; \eta_1(\nu), \eta_2(\nu)), \psi(\cdot; \eta_1(\nu), \eta_2(\nu))) \in H^n_{per}([0, L]) \times H^n_{per}([0, L])$$

is smooth for all integer $n \geq 1$.
(3) $I(\nu_0)$ can be chosen as $(\frac{2\pi^2}{L^2}, +\infty)$.

Proof. By applying the implicit function theorem, we prove the Theorem 3.1. In fact, we consider the open set

$$\Omega = \{(\eta, \nu) \in R^2 : \nu > \frac{2\pi^2}{L^2} \; and \; \eta \in (0, \sqrt{\frac{\nu}{\alpha - \beta}})\},$$

and define $\Lambda : \Omega \to R$ by

$$\Lambda(\eta, \nu) = \frac{2\sqrt{2}}{\sqrt{2\nu - (\alpha - \beta)\eta^2}} K(k) - L, \tag{3.11}$$

where

$$k^2(\eta, \nu) = \frac{2\nu - 2(\alpha - \beta)\eta^2}{2\nu - (\alpha - \beta)\eta^2}. \tag{3.12}$$

From the hypotheses, we get $\Lambda(\eta_{2,0}, \nu_0) = 0$. Next, we show that $\partial_\eta \Lambda < 0$ in Ω. Differentiating (3.12) with respect to η, we have

$$\frac{\partial k}{\partial \eta} = -\frac{2(\alpha - \beta)\eta\nu}{k(2\nu - (\alpha - \beta)\eta^2)^2} < 0. \tag{3.13}$$

Hence, $k(\eta, \nu)$ is a strictly decreasing function of η. Then, from the relation (see [25, 26])

$$\frac{dK(k)}{dk} = \frac{E(k) - k'^2 K(k)}{kk'^2}, \tag{3.14}$$

with $E = E(k)$ being the complete elliptic integral of the second type and $k'^2 = 1 - k^2$ being the complementary modulus. Differentiating (3.11) with respect to η, and combining (3.13) with (3.14), we obtain

$$\begin{aligned}
\partial_\eta \Lambda(\eta, \nu) &= \frac{2\sqrt{2}(\alpha - \beta)\eta}{(2\nu - (\alpha - \beta)\eta^2)^{\frac{3}{2}}} K(k) + \frac{2\sqrt{2}}{\sqrt{2\nu - (\alpha - \beta)\eta^2}} \frac{dK}{dk} \frac{\partial k}{\partial \eta} \\
&= \frac{2\sqrt{2}(\alpha - \beta)\eta}{k^2 k'^2 (2\nu - (\alpha - \beta)\eta^2)^{\frac{5}{2}}} [k^2 k'^2 (2\nu - (\alpha - \beta)\eta^2) K(k) - 2\nu E(k) + 2\nu k'^2 K(k)] \\
&= \frac{2\sqrt{2}(\alpha - \beta)\eta}{k^2 k'^2 (2\nu - (\alpha - \beta)\eta^2)^{\frac{5}{2}}} [\frac{4\nu k'^2}{1 + k'^2} K(k) - 2\nu E(k)] \\
&= \frac{2\sqrt{2}(\alpha - \beta)\eta}{(1 + k'^2) k^2 k'^2 (2\nu - (\alpha - \beta)\eta^2)^{\frac{5}{2}}} [2k'^2 K(k) - (1 + k'^2) E(k)].
\end{aligned} \tag{3.15}$$

Then, from (3.15), we have

$$\frac{\partial \Lambda}{\partial \eta} < 0 \;\; \Leftrightarrow \;\; f(k') \equiv (1 + k'^2) E(\sqrt{1 - k'^2}) - 2k'^2 K(\sqrt{1 - k'^2}) > 0. \tag{3.16}$$

Since $k(\eta, \nu)$ is a strictly decreasing function of η and $k' = \sqrt{1 - k^2}$, we obtain k' is a increasing function of $\eta \in (0, \sqrt{\nu\beta})$ with $k' \in (0, 1)$. Differentiating f with respect to k' and using the relation

$$x\frac{dE(x)}{dx} = E(x) - K(x) \text{ and } E(x) < K(x),$$

we have

$$\frac{\partial f(k')}{\partial k'} = 3k'(E - K) < 0.$$

Thus, $f(k')$ is a decreasing function. Since $f(1) = 0$, we have

$$f(k') > f(1) = 0 \quad \text{for} \quad k' \in (0, 1),$$

which shows (3.16) and obtains our affirmation. Therefore, by the implicit function theorem, there exist an interval $I(\nu_0)$ around ν_0, an interval $B(\eta_{2,0})$ around $\eta_{2,0}$ and a unique smooth function Λ : $I(\nu_0) \rightarrow B(\eta_{2,0})$ such that

$$\Pi(\nu_0) = \eta_{2,0} \quad \text{and} \quad \Lambda(\Pi(\nu), \nu) = 0, \ \forall \nu \in I(\nu_0).$$

So, we can obtain (1) of Theorem 3.2.

Since ν_0 is chosen arbitrarily in the interval $I = (\frac{2\pi^2}{L^2}, +\infty)$, from the uniqueness of the function Λ, it follows that we can extend $I(\nu_0)$ to $(\frac{2\pi^2}{L^2}, +\infty)$. Using the smoothness of the function involved, we can immediately obtain part (2).

4. Spectral analysis

In order to prove the orbital stability of dnoidal wave solutions in Section 5, we need to derive two linear operators L_1, L_2 and give the spectral analysis of L_1 and L_2.

Firstly, from the Eq (3.3), we have

$$(-\partial_x^2 + \nu - (\alpha - \beta)\phi^2)\phi = 0,$$

and define

$$L_2 = -\partial_x^2 + \nu - (\alpha - \beta)\phi^2,$$

which implies $L_2\phi = 0$. Next, differentiating (3.3) with respect to x, we have

$$(-\partial_x^2 + \nu - 3(\alpha - \beta)\phi^2)\phi_x = 0.$$

Then, we define the operator

$$L_1 = -\partial_x^2 + \nu - 3(\alpha - \beta)\phi^2,$$

that is, $L_1\phi_x = 0$.

Secondly, we turn to study the spectral properties associated to the linear operators

$$L_1 = -\partial_x^2 + \nu - 3(\alpha - \beta)\phi^2, \quad L_2 = -\partial_x^2 + \nu - (\alpha - \beta)\phi^2, \tag{4.1}$$

where ϕ is the dnoidal wave solution (3.8) with the fundamental period L and

$$\nu = 1 - \omega^2 \in (\frac{2\pi^2}{L^2}, +\infty).$$

Then, according to Weyl′s essential spectrum theorem, Floquet theory [27], and the spectral analysis in section 4 of [28], we have the following theorem concerning the spectral properties of operators L_1 and L_2.

Theorem 4.1. *Let the dnoidal wave solutions ϕ and ψ given by Theorem 3.2. Then, the operator L_1 in (4.1) defined on $H^2_{per}([0,L])$ with domain $L^2_{per}([0,L])$ has exactly its first three simple eigenvalues, where zero eigenvalue is the second one with associated eigenfunction ϕ'. Moreover, the remainder of the spectrums are constituted by a discrete set of eigenvalues which are double.*

Theorem 4.2. *Let the dnoidal wave solutions ϕ and ψ given by Theorem 3.2. Then, the operator L_2 in (4.1) defined on $H^2_{per}([0,L])$ with domain $L^2_{per}([0,L])$ has zero as its first eigenvalue. The zero eigenvalue is simple and corresponds to the eigenvector ϕ_y. Moreover, the remained of the spectrums are constituted by a discrete set of eigenvalues which are double.*

Theorem 4.3. *Let $L > 0$. Consider the smooth curve of dnoidal waves ϕ, ψ determined by Theorem 3.2. Then for any real function $\varphi \in H^1([0,L])$ satisfying*

$$\langle \varphi, \Psi_0(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x)\rangle = \langle \varphi, \phi_x\rangle = 0, \tag{4.2}$$

there exists a positive number $\delta > 0$ such that

$$\langle L_1\varphi, \varphi\rangle \geq \delta \|\varphi\|^2_{H^1([0,L])}, \tag{4.3}$$

where

$$\Psi_0(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x) = 1 - (1 + k^2 - \sqrt{1-k^2+k^4})sn^2(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x) \tag{4.4}$$

is the negative eigenfunction of L_1 [29]with associated negative eigenvalue λ_0 defined in Theorem 4.1.

Proof. Combining (3.8), (4.4), $dn^2(x) + k^2 sn^2(x) = 1$, and $dn(x + 2K) = dn(x)$, we have

$$\langle \phi_x, \Psi_0(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x)\rangle = 0. \tag{4.5}$$

Hence, from (4.2), (4.5), and the theory of Lagrange multipliers, there are α, λ and θ such that

$$L_1\varphi = \alpha\varphi + \lambda\phi_x + \theta\Psi_0(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x). \tag{4.6}$$

Since

$$(L_1\varphi, \phi_x) = 0, (\varphi, \phi_x) = 0 \text{ and } (\Psi_0(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x), \phi_x) = 0,$$

we have $\lambda = 0$. From the fact

$$L_1\Psi_0(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x) = \lambda_0\Psi_0(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x), \langle \varphi, \Psi_0(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x)\rangle = 0,$$

and

$$0 = \lambda_0\langle \varphi, \Psi_0(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x)\rangle = \langle L_1\varphi, \Psi_0(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x)\rangle = \theta\langle \Psi_0(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x), \Psi_0(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x)\rangle,$$

we have $\theta = 0$. From (4.6), we have $L_1\varphi = \alpha\varphi$. Hence, if φ satisfies the condition (4.2), φ is the eigenfunction of L_1 with associated eigenvalue α. From Theorem 4.1, we know L_1 has one negative eigenvalue and one zero eigenvalue. According to the condition (4.2), we have $\alpha > 0$, and $\langle L_1\varphi, \varphi\rangle = \alpha\langle \varphi, \varphi\rangle$. This completes the proof of Theorem 4.3.

Theorem 4.4. *Let $L > 0$. Consider the smooth curve of dnoidal waves ϕ, ψ determined by Theorem 3.2. Then for any real function $\varphi \in H^1([0, L])$ satisfying*

$$\langle \varphi, \phi \rangle = 0, \tag{4.7}$$

there exists a positive number $\delta_1 > 0$ such that

$$\langle L_2\varphi, \varphi \rangle \geq \delta_1 \|\varphi\|^2_{H^1([0,L])}.$$

Proof. From Theorem 4.2, we have $L_2\phi = 0$. Hence, from (4.7), (4.5), and the theory of Lagrange multipliers, there are α, λ such that

$$L_2\varphi = \alpha\varphi + \lambda\phi. \tag{4.8}$$

Since $(L_2\varphi, \phi) = 0$ and $(\varphi, \phi) = 0$, we have $\lambda = 0$. Then, from (4.8), we get $L_2\varphi = \alpha\varphi$, that is, φ is the eigenfunction of L_2 with associated eigenvalue α. Hence, according to Theorem 4.2 and (4.7), it follows $\alpha > 0$ and

$$\langle L_2\varphi, \varphi \rangle = \alpha \langle \varphi, \varphi \rangle \geq \delta_1 \|\varphi\|^2_{H^1([0,L])}.$$

This completes the proof of Theorem 4.4.

5. Orbital stability of dnoidal waves solutions for the coupled Klein-Gordon-Zakharov equations

In this section, we will prove that the Eq (1.4) is a Hamiltonian system, and satisfies the conditions of the general orbital stability theory proposed by Grillakis [13].

Let

$$U = (u, \rho, v, n)^T.$$

The function space in which we shall work is

$$X = H^1_{complex}([0, L]) \times L^2_{complex}([0, L]) \times L^2_{real}([0, L]) \times L^2_{real}([0, L]),$$

with inner product

$$(f, g) = \int_0^L (Re(f_1\overline{g_1}) + Re(f_{1x}\overline{g_{1x}}) + Re(f_2\overline{g_2}) + f_3g_3 + f_4g_4)dx, \tag{5.1}$$

for

$$f = (f_1, f_2, f_3, f_4), \quad g = (g_1, g_2, g_3, g_4) \in X.$$

The dual space of X is

$$X^* = H^{-1}_{complex}([0, L]) \times L^2_{complex}([0, L]) \times L^2_{real}([0, L]) \times L^2_{real}([0, L]),$$

there exists a nature isomorphism I: $X \to X^*$ defined by

$$\langle f, Ig \rangle = (f, g), \tag{5.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between X and X^*

$$\langle f, g\rangle = \int_0^L (Re(f_1g_1) + Re(f_2g_2) + f_3g_3 + f_4g_4)dx. \tag{5.3}$$

By (5.1)–(5.3), we have

$$I = \begin{pmatrix} 1-\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let T be one-parameter groups of unitary operator on X defined by

$$T(s)U(\cdot) = (e^{is}u, e^{is}\rho, v, n)^T, \quad \text{for } U(\cdot)\in X, \quad s\in R. \tag{5.4}$$

Differentiating (5.4) with respect to s at $s = 0$, we have

$$T'(0) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows from Theorem 3.2 and (1.5) that there exist solitary waves $T(\omega t)\Phi(x)$ of (1.3) with $\Phi(x)$ defined by

$$\Phi(x) = (\phi(x), -i\omega\phi(x), \psi(x), 0),$$

where $\phi(x)$ and $\psi(x)$ are defined by (3.8) and (3.9), respectively. In the following we shall consider the orbital stability of periodic standing waves $T(\omega t)\Phi(x)$ of (1.4). Note that the Eq (1.4) is invariant under $T(\cdot)$, we define the orbital stability as follows.

Definition 5.1. *[13]. The solitary wave solution $T(\omega t)\Phi(x)$ is orbitally stable, if for every $\varepsilon > 0$, there exists $\delta > 0$ with the following property: If $\|U_0 - \Phi(x)\|_X < \delta$ and $U(t)$ is a solution of (1.4) in some interval $[0, t_0)$ with $U(0) = U_0$, then $U(t)$ can be continued to a solution in $0 \le t < +\infty$, and*

$$\sup_{0<t<+\infty}\inf_{s\in R}\|U(t) - T(s)\Phi\|_X < \varepsilon.$$

Otherwise $T(\omega t)\Phi(x)$ is called orbitally unstable.

Let us define a functional on X

$$E(U) = \int_0^L (|u|^2 + |\rho^2| + |u_x|^2 + \alpha|u|^2v + \frac{\beta}{2}|u|^4 + \frac{\alpha}{2}v^2 + \frac{\alpha}{2}n^2)dx. \tag{5.5}$$

By (5.4) and (5.5), we can verify that $E(U)$ is invariant under T, namely,

$$E(T(s)U) = E(U), \quad \text{for any } s\in R.$$

We also have for any $t\in R$, $U(t)$ is a flow of the Eq (1.4)

$$E(U(t)) = E(U(0)).$$

Note that the Eq (1.4) can be written as the following Hamiltonian system

$$\frac{dU}{dt} = JE'(U),$$

where $U = (u, \rho, v, n)^T$ and J is a skew-symmetrically linear operator defined by

$$J = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha}\partial x \\ 0 & 0 & \frac{1}{\alpha}\partial x & 0 \end{pmatrix},$$

and

$$E'(U) = \begin{pmatrix} -2u_{xx} + 2u + 2\alpha uv + 2\beta|u|^2 u \\ 2\rho \\ \alpha|u|^2 + \alpha v \\ \alpha n \end{pmatrix} \tag{5.6}$$

is the Fréchet derivative of E.

Differentiating (5.6) with respect to U, we have

$$E''(U)\eta = \begin{pmatrix} (-\partial_x^2 + 2 + 2\alpha v + 2\beta|u|^2)\eta_1 + 4\beta uRe(u\overline{\eta_1}) + 2\alpha u\eta_3 \\ 2\eta_2 \\ 2\alpha Re(u\overline{\eta_1}) + \alpha\eta_3 \\ \alpha\eta_4 \end{pmatrix}, \tag{5.7}$$

where E'' is the Fréchet derivative of E', and $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)^T$.

Let

$$B = \begin{pmatrix} 0 & 2i & 0 & 0 \\ -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

such that $T'(0) = JB$. Then, as in [15, 16], we define the conserved functionals $Q(U)$ as following

$$Q(U) = \frac{1}{2}\langle BU, U\rangle = -2Im\int_0^L \overline{u}\rho dx. \tag{5.8}$$

By (5.4) and (5.8), we can verify that $Q(U)$ is invariant under T, namely,

$$Q(T(s)U) = Q(U), \quad \text{for any } s \in R,$$

and for any $t \in R$, $U(t)$ is a flow of (1.4)

$$Q(U(t)) = Q(U(0)).$$

Differentiating (5.8) with respect to U, we have

$$Q'(U) = BU = \begin{pmatrix} 2i\rho \\ -2iu \\ 0 \\ 0 \end{pmatrix}, \quad Q''(U) = B = \begin{pmatrix} 0 & 2i & 0 & 0 \\ -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.9}$$

Notice that $\Phi(x) = (\phi(x), -i\omega\phi(x), \psi(x), 0)$ satisfies the Eq (1.6), combining (5.7) and (5.9), we have

$$E'(\Phi) - \omega Q'(\Phi) = \begin{pmatrix} -2\phi_{xx} + 2(1-\omega^2)\phi - 2(\alpha-\beta)\phi^3 \\ -2i\omega\phi + 2i\omega\phi \\ \alpha\phi^2 + \alpha\psi \\ 0 \end{pmatrix} = 0. \tag{5.10}$$

Now we define an operator $H_{\omega,c}$ from X to X^* by

$$H_\omega = E''(\Phi) - \omega Q''(\Phi).$$

Combining (5.7) and (5.9), we have

$$\begin{aligned} H_\omega\eta &= (E''(\Phi) - \omega Q''(\Phi))\eta \\ &= \begin{pmatrix} 2(-\partial_x^2 + 1 + \alpha\psi + \beta\phi^2)\eta_1 + 4\beta\phi Re(\phi\overline{\eta_1}) + 2\alpha\phi\eta_3 - 2i\omega\eta_2 \\ 2\eta_2 + 2i\omega\eta_1 \\ 2\alpha Re(\phi\overline{\eta_1}) + \alpha\eta_3 \\ \alpha\eta_4 \end{pmatrix}, \end{aligned} \tag{5.11}$$

where $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)^T$. Next, we consider spectral analysis of the operator H_ω. Observe that H_ω is self-adjoint in the sense that $H_\omega^* = H_\omega$. This means that $I^{-1}H_\omega$ is a bounded self-adjoint operator on X.

The spectrum of H_ω consists of the real numbers λ such that $H_\omega - \lambda I$ is not invertible. We claim that $\lambda = 0$ belongs to the spectrum of H_ω. Then, for any

$$\eta = (\eta_1, \eta_2, \eta_3, \eta_4) \in X,$$

by (5.11), we have

$$\begin{aligned} \langle H_{\omega,c}\eta, \eta\rangle &= Re\int_0^L [2(-\partial_x^2 + 1 + \alpha\psi + \beta\phi^2)\eta_1 + 4\beta\phi Re(\phi\overline{\eta_1})) + 2\alpha\phi\eta_3 - 2i\omega\eta_2]\overline{\eta_1}dx \\ &+ Re\int_0^L [(2\eta_2 + 2i\omega\eta_1)\overline{\eta_2} + (2\alpha Re(\phi\overline{\eta_1}) + \alpha\eta_3)\eta_3 + \alpha\eta_4^2]dx \\ &= \int_0^L [2(-\partial_x^2 + 1 + \alpha\psi + \beta\phi^2)y_1 \cdot y_1 + 2(-\partial_x^2 + 1 + \alpha\psi + \beta\phi^2)y_2 \cdot y_2 + 4\beta\phi^2 y_1^2 \\ &+ 2\alpha\phi y_1\eta_3 - 4\omega y_2 y_3 + 4\omega y_1 y_4 + 2y_3^2 + 2y_4^2 + 2\alpha\phi y_1\eta_3 + \alpha\eta_3^2 + \alpha\eta_4^2]dx \\ &= 2\langle L_1 y_1, y_1\rangle + 2\langle L_2 y_2, y_2\rangle \\ &+ \int_0^L [\alpha(2\phi y_1 + \eta_3)^2 + 2(\omega y_1 + y_4)^2 + 2(\omega y_2 - y_3)^2 + \alpha\eta_4^2], \end{aligned} \tag{5.12}$$

where

$$\eta_1 = y_1 + iy_2, \quad \eta_2 = y_3 + iy_4,$$
$$L_1 = -\partial_x^2 + \nu - 3(\alpha-\beta)\phi^2 \text{ and } L_2 = -\partial_x^2 + \nu - (\alpha-\beta)\phi^2.$$

Since the operators L_1 and L_2 have spectral properties in Theorems 4.1 and 4.2, we have

$$H_\omega T'(0)\Phi = 0, \quad H_\omega(i\phi, \phi, 0, 0) = 0,$$

and H_ω has a unique simple negative eigenvalue. Let $N = \{k_1\Psi^-\}$ denotes the negative eigenspace of H_ω, where Ψ^- is the negative eigenfunction of H_ω, $k_1 \in R/\{0\}$. $Z = \{k_2T'(0)\Phi + k_3\varphi\}$ denotes the kernel of the operator H_ω, where $\varphi = (i\phi, \phi, 0, 0)$, $k_2, k_3 \in R$. Furthermore, let

$$\begin{aligned} P &= \{p \in X | p = (p_{11} + ip_{12}, p_{21} + ip_{22}, p_3, p_4), \langle p_{11}, \Psi_0(\sqrt{\frac{\alpha-\beta}{2}}\eta_1 x)\rangle \\ &= \langle p_{11}, \phi_x\rangle = \langle p_{12}, \phi\rangle\} = 0. \end{aligned} \tag{5.13}$$

Combining Theorems 4.2–4.4 with (5.12), we get that the following Lemma 5.1 holds.

Lemma 5.2. *For any $\alpha > 0$ and $\zeta \in P$, there exists a constant $\delta > 0$ such that*

$$\langle H_\omega\zeta, \zeta\rangle \geq \delta\|\zeta\|_X^2,$$

where δ is independent of ζ.

According to the above analysis, when $\alpha > 0$, $\alpha > \beta$, we can get that the space X can be decomposed as a direct sum, that is, the following Assumption 5.3 holds.

Assumption 5.3. *[13] (Spectral decomposition of H_ω) The space X is decomposed as a direct sum*

$$X = N + Z + P,$$

where Z is the kernel of H_ω, N is a finite-dimensional subspace such that

$$\langle H_\omega U, U\rangle < 0, \quad \text{for } 0 \neq U \in N,$$

and P is a closed subspace such that

$$\langle H_\omega U, U\rangle \geq \delta\|U\|_X^2, \quad \text{for } U \in P,$$

with some constant $\delta > 0$ independent of U. Then, for any $\overline{U} \in X$, $\overline{U} = (u_1, u_2, u_3, u_4)^T$, choose

$$a = \frac{\langle\overline{U}, \Psi^-\rangle}{\langle\Psi^-, \Psi^-\rangle}, \quad b_1 = \frac{\langle\overline{U}, T'(0)\Phi\rangle}{\langle T'(0)\Phi, T'(0)\Phi\rangle}, \quad b_2 = \frac{\langle\overline{U}, \varphi\rangle}{\langle\varphi, \varphi\rangle},$$

and then $\overline{U}$ can be uniquely represented by

$$\overline{U} = a\Psi^- + b_1T'(0)\Phi + b_2\varphi + p_0,$$

where $p_0 \in P$.

We now define $d(\omega)$: $R \to R$ by

$$d(\omega) = E(\Phi) - \omega Q(\Phi), \tag{5.14}$$

and define $d''(\omega)$ to be function $d(\omega)$ with respect ω.

We know that J is not onto, the abstract stability theory in [14] cannot be applied directly.

But, according to the "stability theorem" in the introduction of [13] or the lines of proofs in [13], Sections 3 and 4, we can obtain the following abstract orbital stability theorem for the solitary waves of the Eq (1.4).

Theorem 5.4. *Assume that there exists the periodic standing waves $T(\omega t)\Phi(x)$ of the Eq (1.4) and Assumption 5.3 holds. If $d(\omega)$ is convex in a neighborhood of ω (in other words $d''(\omega) > 0$), then solution $T(\omega t)\Phi(x)$ is orbitally stable.*

From Theorem 5.4, the main results about orbital stability of the periodic standing waves for the Eq (1.4) can be given as follows.

Theorem 5.5. *Let $\alpha > 0$, $\alpha > \beta$, $1-\omega^2 > \frac{2\pi^2}{L^2}$, then*
(i) If $L > 2\pi$, we have $T(\omega t)\Phi(x)$ is orbital stability for some interval $|\omega| \in (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} + \varepsilon)$.
(ii) If $L > \sqrt{5}\pi$, we get that $T(\omega t)\Phi(x)$ is orbital stability for

$$\sqrt{-\frac{M(k_0)}{N(k_0)}} \leq |\omega| \leq \sqrt{1-\frac{2\pi^2}{L^2}},$$

where k_0 is the solution of $\frac{4(2-k^2)K^2(k)N(k)}{M(k)+N(k)} = L^2$.

Proof. According to Theorem 5.4, we need to find conditions that satisfy $d''(\omega) > 0$ under $\alpha > 0$, $\alpha > \beta$, $1-\omega^2 > \frac{2\pi^2}{L^2}$.

However, since J is not onto, we can't apply GSS theory [13,14] to study the orbital instability of standing waves in the case $d''(\omega) < 0$. Here, we only give conclusions of orbital stability.

Combining (5.8), (5.10) and (5.14), and using the formula

$$\int_0^K dn^2(x)dx = E(k),$$

we have

$$\begin{aligned} d'(\omega) &= -Q(\Phi) = 2Im\int_0^L \overline{\phi}\rho dx = -2\omega\int_0^L \phi^2 dx = -2\omega\eta_1^2\int_0^L dn^2(\eta_1\sqrt{\frac{\alpha-\beta}{2}}x,k)dx \\ &= -4\omega\eta_1\sqrt{\frac{2}{\alpha-\beta}}\int_0^K dn^2(x)dx = -\frac{16\omega}{L(\alpha-\beta)}KE. \end{aligned} \tag{5.15}$$

Differentiating (5.15) with respect to ω, we get

$$d''(\omega) = -\frac{16}{L(\alpha-\beta)}KE + \frac{32\omega^2}{L(\alpha-\beta)}\frac{d(KE)}{dk}\frac{dk}{d\nu}. \tag{5.16}$$

From (3.12) and $\eta_2^2 = \frac{2\nu(1-k^2)}{(2-k^2)(\alpha-\beta)}$, we have

$$\frac{\partial k}{\partial \nu} = \frac{(\alpha-\beta)\eta_2(\eta_2 - 2\frac{\partial\eta_2}{\partial\nu}\nu)}{k(2\nu-(\alpha-\beta)\eta_2^2)^2},$$

and

$$(\alpha-\beta)\eta_2\frac{\partial\eta_2}{\partial\nu} = \frac{K - \frac{dK}{dk}\frac{dk}{d\nu}(2\nu-(\alpha-\beta)\eta_2^2)}{K}.$$

Then, it follows

$$\frac{\partial k}{\partial \nu} = \frac{K}{2\nu\frac{dK}{dk} - kK(2\nu - (\alpha-\beta)\eta_2^2)}. \tag{5.17}$$

Combining (5.16) and (5.17), with the formula

$$\frac{dK}{dk} = \frac{E-(1-k^2)K}{k(1-k^2)}, \quad \frac{dE}{dk} = \frac{E-K}{k},$$

we obtain

$$\begin{aligned} d''(\omega) &= -\frac{16K}{L(\alpha-\beta)}[E - 2\omega^2\frac{E^2-(1-k^2)K^2}{k(1-k^2)} \cdot \frac{1}{2\nu(\frac{E-(1-k^2)K}{k(1-k^2)} - kK + \frac{(1-k^2)kK}{2-k^2})}] \\ &= \frac{16K}{L(\alpha-\beta)} \cdot \frac{[2(1-k^2)KE-(2-k^2)E^2] + \omega^2[2(2-k^2)E^2 - 2(1-k^2)KE - (2-k^2)(1-k^2)K^2]}{\nu[(2-k^2)E-2(1-k^2)K]}. \end{aligned}$$

For simplicity, we let

$$M(k) = 2(1-k^2)KE - (2-k^2)E^2,$$
$$N(k) = 2(2-k^2)E^2 - 2(1-k^2)KE - (2-k^2)(1-k^2)K^2,$$

then

$$d''(\omega) = \frac{16K}{L(\alpha-\beta)} \cdot \frac{M(k)+\omega^2 N(k)}{\nu[(2-k^2)E-2(1-k^2)K]}.$$

Since $\nu[(2-k^2)E - 2(1-k^2)K] > 0$, then the sign of $d''(\omega)$ depends on the sign of $M(k)+\omega^2 N(k)$. According to Theorem 5.1 and $n(H_\omega) = 1$, we have stability for $1 > \omega^2 > -\frac{M(k)}{N(k)}$, that is,

$$\frac{4(2-k^2)K^2(k)}{L^2} = 1 - \omega^2 < 1 + \frac{M(k)}{N(k)}, \tag{5.18}$$

from $\nu = 1-\omega^2$, $\frac{2\sqrt{2}}{\sqrt{2\nu-(\alpha-\beta)\eta_2^2}}K(k) = L$, and $k^2 = \frac{2\nu-2(\alpha-\beta)\eta_2^2}{2\nu-(\alpha-\beta)\eta_2^2}$. The inequality (5.18) can transfer to consider

$$f(k) = \frac{4(2-k^2)K^2(k)N(k)}{M(k)+N(k)} < L^2, \tag{5.19}$$

$f(k)$ is an increasing function. Then, using the analysis similar to [12] and the Mathematica, we get that $T(\omega t)\Phi(x)$ is orbital stability for

$$\sqrt{-\frac{M(k_0)}{N(k_0)}} \le |\omega| \le \sqrt{1-\frac{2\pi^2}{L^2}}, \tag{5.20}$$

as $L > \sqrt{5}\pi$, where k_0 is the solution of $\frac{4(2-k^2)K^2(k)N(k)}{M(k)+N(k)} = L^2$.

Remark 5.6. *When $L \to \infty$, the formulaes (3.8) and (3.9) lose its periodicity in this limit and we obtain a wave form with a single hump and with infinity periodi of the form*

$$\phi(x) = \sqrt{\frac{2(1-\omega^2)}{\alpha-\beta}} sech(\sqrt{(1-\omega^2)}x), \quad \psi(x) = -\frac{2\nu}{\alpha-\beta} sech^2(\sqrt{1-\omega^2}x),$$

and $\lim_{L\to\infty} k_0(L) = 1$, $\lim_{L\to\infty} \sqrt{-\frac{M(k_0)}{N(k_0)}} = \frac{1}{\sqrt{2}}$.

Hence, for all $1 > |\omega| > \frac{1}{\sqrt{2}}$*, the solutions* $(e^{i\omega t}\sqrt{\frac{2(1-\omega^2)}{\alpha-\beta}} sech(\sqrt{(1-\omega^2)}x), -\frac{2\nu}{\alpha-\beta} sech^2(\sqrt{1-\omega^2}x))$ *are orbitally stable. Therefore, we also obtain stability of sech-type standing waves for Eq (1.1).*

6. Conclusions

In this work, we are interested in studying the stability of the periodic standing waves for coupled Klein-Gordon-Zakharov Eq (1.1). The abstract orbital stability theory presented by Grillakis et al. [14] cannot be applied directly. However, combining the extension version of the general theory of orbital stability [13] and detailed spectral analysis and Floquet theory, and we obtain the orbital stability of periodic traveling waves of the Eq (1.4). In addition, we consider the novel situation of period $L \to \infty$, and dnoidal type turns into sech type in the case of limit. Then, we obtain stability of sech type standing waves. In our study, $\beta = 0$, $\alpha = 1$ also satisfy conditions of Theorem 5.5. So, we also can get the the stability of the dnoidal type and sech type standing waves for the classical Klein-Gordon-Zakharov Eq (1.2). Obviously, our work further extends and improves the interesting results of [11, 12].

In addition, the periodic solution $T(\omega t)\Phi(x)$ is orbital stability in the conditions of $L > \sqrt{5}\pi$ and $1 - \omega^2 > \frac{2\pi^2}{L^2}$ in this paper, but we don't obtain the concrete stable result of periodic solution for the case $L = \sqrt{5}\pi$. In future work, we will analyze such an open problem.

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Conflict of interest

All authors declare that they have no conflicts of interest.

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