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Research article

Probabilistic picture hesitant fuzzy sets and their application to multi-criteria decision-making

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Abstract: The picture hesitant fuzzy sets (PHFSs), which consider neutral membership degree as well as positive and negative membership degrees, provide decision makers (DMs) a flexible attitude to evaluate criteria values in complex multi-criteria decision-making (MCDM) situations. However, existing MCDM approaches based on PHFSs still have some drawbacks in both evaluation information expression and criteria values fusion. In this paper, our aim is to overcome these shortcomings by proposing new decision-making methods. To achieve this purpose, a new fuzzy information representation tool, called probabilistic picture hesitant fuzzy sets (P-PHFSs), is first introduced by capturing the probability of each element in PHFSs. The characteristic of P-PHFSs is that they provide more freedom to DMs so that criterion values of each alternative can be adequately described. To facilitate the use of P-PHFSs, we define the basic operational rules and comparison method of P-PHFSs. Then we also propose some aggregation operators for P-PHFSs and provide information fusion process. Furthermore, some desirable properties of these operators is discussed, and the relationship between the developed operators and the existing ones is investigated. Based on the proposed operators, two MCDM methods are developed under probabilistic picture hesitant fuzzy environment. Finally, two numerical examples are given to show the application of the developed methods, and a comparison analysis is conducted to demonstrate the effectiveness of the proposed approaches.

Keywords: multi-criteria decision-making (MCDM); probabilistic picture hesitant fuzzy set (P-PHFS); generalized probabilistic picture hesitant fuzzy aggregation operators
Mathematics Subject Classification: 03E72, 91B06

1. Introduction

Multi-criteria decision-making (MCDM), whose main goal is to rank alternatives based on experts' opinions and some principles, has recently attracted much attention as a promising research topic [1, 2]. In spite of, the uncertainty and vagueness in decision-making process have made it difficult and complicated for decision makers (DMs) to evaluate the performance of all possible alternatives and to select the most suitable one in MCDM problems. For more convenient expressions of evaluation information, there has been a lot of interest and effort in fuzzy sets (FSs) [3] and their classical extensions such as interval-valued FSs (IVFSs) [4], intuitionsitic FSs (IFSs) [5], interval-valued IFSs [6], type-2 FSs (T2FSs) [7] and picture FSs (PFSs) [8]. However, these extensions cannot deal with the situation in which the membership degree of an element of a set is difficult to determine due to ambiguity among several different values. That is, the difficulty in establishing the membership degree of an element to a set does not arise from a margin of error (as in IFS or IVFS or PFS) or a specified distribution of the possible values (as in T2FS) but arises from our hesitancy among a few different values. To handle such cases, Narukawa and Torra [9, 10] introduced the concept of hesitant fuzzy sets (HFSs) which allow the membership degree of an element to a set to be represented by a set of possible values. Therefore, HFS is an effective tool for expressing DM's preference for alternatives and is used in practical problems such as decision-making, risk evaluation, and pattern recognition, but it still has its own some drawbacks [11–21]. With the help of IFSs, Zhu et al. [22, 23] proposed an extension of HFSs, dual hesitant fuzzy sets (DHFSs), which consist of two parts: the membership hesitancy function and non-membership hesitancy function. The advantages of DHFSs are reflected in two aspects, that is, they interpret fuzzy information from both positive and negative points, and effectively express DMs' hesitancy. Furthermore, Zeng et al. [24] introduced the concept of weighted DHFSs (WDHFSs), whose main characteristic is that both the possible membership degree and non-membership degree are distributed with different weights. Thus, MCDM methods based on information measures and aggregation operators of DHFSs, as well as extensions of classical decision-making methods into dual hesitant fuzzy environment, have become an active research topic. Meanwhile, to avoid the hesitant fuzzy information loss in decision-making process, Xu and Zhou [25] combined probabilistic information with HFSs, and proposed the probabilistic hesitant fuzzy sets (P-HFSs), which consist of several possible membership values with their corresponding probabilities. So far, the issue of hesitant fuzzy information loss is challenging and attracting researchers' attention. To prevent the defect of DHFSs, Hao et al. [26] proposed the probabilistic DHFSs (P-DHFSs) by adding corresponding probability of each member in DHFSs, and investigated the operations, comparison method, and aggregation operators of P-DHFSs as well as a novel decision method for practical multi-attribute decision-making (MADM) problems. As a fascinating extension of DHFSs, P-DHFSs can explain DM's evaluation values more accurately and comprehensively because they represent not only the membership and non-membership degrees, but also the corresponding probabilistic information.

Recently, Wang and Liu [27] proposed another extension of HFSs, picture hesitant fuzzy sets (PHFSs), which consist of three parts: the positive membership hesitancy, neutral membership hesitancy, and negative membership hesitancy functions. Due to the advantages of PHFSs, which interpret fuzzy information from the neutral aspect along with the positive and negative aspects, DM's preference is effectively expressed in the decision-making process. Since then, many researchers have recently investigated and studied PHFSs in various areas. However, although existing MCDM methods using PHFS have recently presented some improved decision-making approaches, it is worth pointing out that there are still limited drawbacks in use in solving practical MCMD problems. In other words, the existing methods have two shortcomings. First, PHFSs have drawbacks in presenting complex DMs' evaluation information, so there exist many situations that cannot properly handled by PHFSs. For instance, the limitation of PHFSs requires that the elements of each membership hesitancy degree have the same weight, and if those elements have different weights, PHFSs are powerless. The second drawback is that the existing methods of information fusion fail to handle complicated situations, such as wherein attributes are correlated. Therefore, aiming at the drawbacks of PHFSs, we extend PHFSs to probabilistic picture hesitant fuzzy sets (P-PHFSs) by taking the probability of each member in picture hesitant fuzzy element (PHFE) into consideration. The P-PHFSs are more powerful and useful, as they have the ability of denoting positive membership degrees, neutral membership degrees, negative membership degrees as well as their probabilities simultaneously. To circumvent the second defect of Wang and Liu's [12] method, we provide a series of compound aggregation operators of probabilistic picture hesitant fuzzy elements (P-PHFEs). The novelties and contributions of this paper are presented as follows.

(1) A new information representation tool, called P-PHFSs, is proposed. This contribution makes it easier and more convenient to depict DMs' complex and fuzzy assessment information in decision-making problems.
(2) The operations, score function, accuracy function and comparison method of P-PHFSs are presented and discussed.
(3) Novel efficient aggregation operators are put forward, which effectively aggregate integrate evaluation values under P-PHFEs.
(4) Two new MCDM methods under P-PHFSs are developed to judge the best alternative.
(5) Some practical MCDM examples are provided to show the effectiveness of our new methods.

The structure of the rest of this paper is as follows. Section 2 reviews the relevant research. Section 3 recalls basic concepts. Section 4 proposes the P-PHFSs and introduces their related notions, such as operational rules, comparison method, etc. Section 5 presents some aggregation operators of P-PHFEs and discusses their properties. Section 6 presents two new MCDM methods under P-PHFSs. Section 7 conducts numerical experiments to show the performance of the new MCDM methods. Conclusions are provided in Section 8.

2. Literature review

To further illustrate the motivations of our proposal, this section reviews the relevant research. For better explanation and explanation, we perform literature reviews in two aspects: P-HFSs and PHFSs.

2.1. P-HFSs

After Xu and Zhou [25] proposed P-HFSs to overcome hesitant fuzzy information loss in the decision-making process, researchers became very interested in the fuzzy information loss problem. Zhang et al. [28] further discussed the operations and integrations of probabilistic hesitation fuzzy information, and proposed a method to rank probabilistic hesitant fuzzy elements (P-HFEs) based on the score and deviation values, whose comparison technique only reflects the absolute priority of P-HFEs. After that, Song et al. [29] overcame the defect of Zhang et al.'s method [28] by developing a new comparison technique for P-HFEs based on the possibility degree formula. Song et al.'s formula has the advantage of realizing optimal alignment of P-HFEs and effectively reducing computational complexity. Wang and Li [30] proposed the correlation coefficients of P-HFEs, and applied them to evaluate alternatives in practical decision-making process. Song et al. [31] developed two correlation coefficient formulas to measure the relationship between P-HFSs, and applied them in cluster analysis with probabilistic hesitant fuzzy information. Su et al. [32] proposed two kinds of entropy measures for P-HFEs, the membership degree-based entropy and distance-based entropy, and applied them to decision-making case for investment. Liu et al. [33] proposed a MADM method based on probabilistic hesitant fuzzy entropy. Farhadinia et al. [34] constructed an axiomatic framework for entropy measures for P-HFEs, taking into account two aspects of uncertainty: fuzziness and non-specificity, and proposed decision-making methods based on P-HFE entropy and P-HFE entropy-based distance measures. Zhu and Xu [35] developed probability hesitant fuzzy preference relations (PHFPRs) and a consensus index to measure the consensus degrees of PHFPRs. Li and Wang [36] further investigated the consensus building among DMs on PHFPRs with expected additional consistency. Zhou and Xu [37] introduced the concept of uncertain P-HFEs, extended them to uncertain PHFPRs (UPHFPRs), and then provided a group decision-making process under UPHFPR environment. Li and Wang [38] and Zang et al. [39], respectively, extended the classical QUALIFLEX and TODIM methods for decision-making to probabilistic hesitant fuzzy environment. Furthermore, Tian et al. [40] utilized TODIM method with probabilistic hesitant fuzzy information to simulate the perceptions of DMs in prospect theory. Song and Chen [41] proposed a new MADM method based on distance and COMPRAS method under probabilistic hesitant fuzzy environment.

As an extension of DHFSs and P-HFSs, the P-DHFSs proposed by Hao et al. [26] can explain DMs' evaluation values more accurately and comprehensively, so many researchers have since studied P-DHFSs. Ren et al. [42] proposed the revised score functions and equiprobability distance measures of probabilistic dual hesitant elements (P-DHFEs), and utilized them to develop an MCDM model based on extended TODIM method. In [43], they also developed an integrated VIKOR and AHP method to solve group MCDM (MCGDM) problems under probabilistic dual hesitant fuzzy environment. Garg and Kaur [44] presented several aggregation operators to fuse P-DHFEs by using Einstein operations, and applied them to construct an MCDM method. They, in [45], also defined the correlation coefficients for P-DHFSs and developed an MCDM method based on them. Zhao et al. [46] proposed the BASD-based PROMETHEE-II method for solving group MCDM problem with probabilistic dual hesitant fuzzy preference information. Song et al. [47] discussed group decision-making (GDM) methods based on incomplete probabilistic double hesitation fuzzy preferences (PDHFPRs) and also, in [48], investigated consensus-based GDM methods with PDHFPRs. Shao and Zhang [49] presented uncertain probabilistic dual hesitant fuzzy numbers, which includes six types of DHFS, and developed goal programming models based on uncertain PDHFPRs. However, PDHFSs have drawbacks in presenting complex DMs' evaluation information and there exist many situations in which cannot be adequately handled by PDHFSs.

2.2. PHFSs

The PHFSs proposed by Wang and Liu [27], another extension of HFSs, have been investigated and studied in various areas by many researchers due to their advantages in interpreting fuzzy information in positive, negative, and neutral aspects. Ullah et al. [50] presented some picture hesitant fuzzy aggregation operators for MADM methods. Yang et al. [51] introduced picture hesitant fuzzy entropy and similarity measurements and utilized them to make MCGDM method for end-of-life vehicle management. Jan et al. [52] proposed some distance measures for PHFSs and utilized them to solve for pattern recognition and MADM problems. Ali and Mahmood [53] presented generalized dice similarity measures (GDSMs) for PHFSs and investigated the GDSM based pattern recognition models with picture hesitant fuzzy information. They [54] further proposed MCDM method based on picture hesitant fuzzy cross-entropy. Mahmood et al. [55] applied the picture hesitant fuzzy Bonferroni mean operator and picture hesitant fuzzy geometric Bonferroni mean operator to aggregate picture hesitant fuzzy information. Ali et al. [56] proposed some new similarity measures for PHFSs and applied to medical diagnosis. Mahmood et al. [57] developed TOPSIS method based on correlation coefficient and entropy measure to solve MADM problem under picture hesitant environment. Ambrin et al. [58] extended TOPSIS method for supplier selection under picture hesitant fuzzy environment. However, as mentioned in the previous section, although the MCDM methods using PHFSs present improved decision-making approaches, there are still limited drawbacks to being used to solve practical MCMD problems.

Based on the above review analysis, our goals are to make up for the aforementioned shortcomings by proposing new MCDM methods. That is, we propose a new representation tool and technique to overcome the shortcomings of PHFSs in representing fuzzy decision information.

3. Preliminaries

In this section, the concepts of P-HFSs, P-DHFSs, PHFSs and prioritized averaging (PA) operator [59] are briefly reviewed, which are the theoretical basis of the proposed method. We extend PHFSs to P-PHFSs by taking probabilities into consideration and develop some new aggregation operators by applying PA operator to P-PHFSs in the following Sections 3 and 4, respectively.

3.1. Basic definitions

To avoid the loss of hesitant fuzzy information in decision making process, Xu and Zhou [25] brought probability to the concept of HFS and introduced P-HFS.

Definition 3.1. [25] Let X be a non-empty and finite set, a probabilistic hesitant fuzzy set (P-HFS) P on X is given as

$$\tilde{P} = \left\{\langle x, \tilde{h}(x)|p(x)|x \in X\rangle\right\}, \tag{3.1}$$

where the set $h(x)|p(x)$ contains several elements, $h(x)$ denotes the membership degree of x in X to the set $\tilde{P}$ and $p(x)$ is its associated probabilistic information. For simplicity, $\tilde{h}(x)|p(x)$ is called a probabilistic hesitant fuzzy element (P-HFE), denoted as $\tilde{h}|p$ and is given as

$$\tilde{h}|p = \left\{\gamma_i|p_i|i = 1, 2, \ldots, \#\tilde{h}\right\}, \tag{3.2}$$

where p_i satisfying $\sum_{i=1}^{\#\tilde{h}} p_i \leq 1$, is probability of the possible value γ_i and $\#\tilde{h}$ is the number of all $\gamma_i|p_i$.

To fully explain the limited information provided by experts and reduce uncertainty as much as possible, Hao et al. [26] proposed P-DHFS as follows.

Definition 3.2. [26] Let X be a non-empty and finite set, a probabilistic dual hesitant fuzzy set (P-DHFS) $\tilde{D}$ on X is defined by

$$\tilde{D} = \{\langle x, \tilde{\mu}(x)|p(x), \tilde{\nu}(x)|q(x)\rangle | x \in X\}, \tag{3.3}$$

where the sets $\tilde{\mu}(x)|p(x)$ and $\tilde{\nu}(x)|q(x)$ contain several elements, $\tilde{\mu}(x)$ and $\tilde{\nu}(x)$ denote the membership and non-membership degrees of $x \in X$ to the set $\tilde{D}$, respectively, and $p(x)$ and $q(x)$ are their associated probabilistic information. In addition, $\tilde{\mu}(x)|p(x)$, $\tilde{\nu}(x)|q(x)$, $p(x)$, and $q(x)$ satisfy the following conditions:

$$0 \leq \alpha, \beta \leq 1, \;\; 0 \leq \alpha^+ + \beta^+ \leq 1, \tag{3.4}$$

and

$$p_i \in [0,1], q_j \in [0,1], \sum_{i=1}^{\#\tilde{\mu}} p_i = 1, \sum_{j=1}^{\#\tilde{\nu}} q_j = 1, \tag{3.5}$$

where $\alpha \in \tilde{\mu}(x)$, $\beta \in \tilde{\nu}(x)$, $\alpha^+ = \cup_{\alpha \in \tilde{\mu}(x)} \max\{\alpha\}$, $\beta^+ = \cup_{\beta \in \tilde{\nu}(x)} \max\{\beta\}$, $p_i \in p(x)$, and $q_j \in q(x)$. The symbol $\#\tilde{\mu}$, and $\#\tilde{\nu}$ represent the total numbers of elements in $\tilde{\mu}(x)|p(x)$, and $\tilde{\nu}(x)|q(x)$, respectively. For convenience, we call $\tilde{d} = (\tilde{\mu}(x)|p(x), \tilde{\nu}(x)|q(x))$ a probabilistic dual hesitant fuzzy element (P-DHFE), which can be denoted by $\tilde{d} = (\tilde{\mu}|p, \tilde{\nu}|q)$ for simplicity.

Wang and Li [27] proposed PHFS to describe the hesitation for all three of he following: positive membership degree, neutral membership degree, and negative membership degree.

Definition 3.3. [27] Let X be a non-empty and finite set, a picture hesitant fuzzy set (PHFS) N on X is defined by

$$N = \{\langle x, \bar{\mu}(x), \bar{\eta}(x), \bar{\nu}(x)\rangle | x \in X\}, \tag{3.6}$$

where $\bar{\mu}(x) = \{\alpha | \alpha \in \bar{\mu}(x)\}$, $\bar{\eta}(x) = \{\beta | \beta \in \bar{\eta}(x)\}$, and $\bar{\nu}(x) = \{\gamma | \gamma \in \bar{\nu}(x)\}$ are three sets of several values in $[0,1]$, representing the potential positive, neutral, and negative membership degrees. The degrees above satisfy the condition of $0 \leq \alpha^+ + \eta^+ + \gamma^+ \leq 1$, where $\alpha^+ = \cup_{\alpha \in \bar{\mu}(x)} \max\{\alpha\}$, $\beta^+ = \cup_{\beta \in \bar{\eta}(x)} \max\{\beta\}$, and $\gamma^+ = \cup_{\gamma \in \bar{\nu}(x)} \max\{\gamma\}$. For convenience, we call $\bar{n} = (\bar{\mu}(x), \bar{\eta}(x), \bar{\nu}(x))$ is a picture hesitant fuzzy element (PHFE), denoted by $\bar{n} = (\bar{\mu}, \bar{\eta}, \bar{\nu})$.

Wang and Li [27] also provided some operations of PHFEs as follows: for three PHFEs $\bar{n} = (\bar{\mu}, \bar{\eta}, \bar{\nu})$, $\bar{n}_1 = (\bar{\mu}_1, \bar{\eta}_1, \bar{\nu}_1)$ and $\bar{n}_2 = (\bar{\mu}_2, \bar{\eta}_2, \bar{\nu}_2)$,

(1) $\bar{n}^c = (\bar{\nu}, \bar{\eta}, \bar{\mu}) = \cup_{\alpha \in \bar{\mu}, \beta \in \bar{\eta}, \gamma \in \bar{\nu}} (\{\gamma\}, \{\beta\}, \{\alpha\})$;

(2) $\bar{n}_1 \oplus \bar{n}_2 = \cup_{\substack{\alpha_1 \in \bar{\mu}_1, \beta_1 \in \bar{\eta}_1, \gamma_1 \in \bar{\nu}_1, \\ \alpha_2 \in \bar{\mu}_2, \beta_2 \in \bar{\eta}_2, \gamma_2 \in \bar{\nu}_2}} (\{\alpha_1 + \alpha_2 - \alpha_1\alpha_2\}, \{\beta_1\beta_2\}, \{\gamma_1\gamma_2\})$;

(3) $\bar{n}_1 \otimes \bar{n}_2 = \cup_{\substack{\alpha_1 \in \bar{\mu}_1, \beta_1 \in \bar{\eta}_1, \gamma_1 \in \bar{\nu}_1, \\ \alpha_2 \in \bar{\mu}_2, \beta_2 \in \bar{\eta}_2, \gamma_2 \in \bar{\nu}_2}} (\{\alpha_1\alpha_2\}, \{\beta_1 + \beta_2 - \beta_1\beta_2\}, \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\})$;

(4) $\lambda\bar{n} = \cup_{\alpha \in \bar{\mu}, \beta \in \bar{\eta}, \gamma \in \bar{\nu}} \left(\{1 - (1-\alpha)^\lambda\}, \{\beta^\lambda\}, \{\gamma^\lambda\}\right)$, $\lambda > 0$;

(5) $\bar{n}^\lambda = \cup_{\alpha \in \bar{\mu}, \beta \in \bar{\eta}, \gamma \in \bar{\nu}} \left(\{\alpha^\lambda\}, \{1 - (1-\beta)^\lambda\}, \{1 - (1-\gamma)^\lambda\}\right)$, $\lambda > 0$.

3.2. Prioritized operator

The prioritized operators play also an important role in solving many MCDM problems. The prioritized averaging (PA) operator, introduced by Yager [59], is defined in the following manner:

Definition 3.4. [59] Let $C = \{C_1, C_2, \ldots, C_n\}$ be a collection of criteria and that there is a prioritization between the criteria expressed by the linear ordering $C_1 > C_2 > \cdots > C_n$, indicate criteria C_j has a higher priority than C_k if $j < k$. The value $C_j(x)$ is the performance of any alternative x under criteria C_j, and satisfies $C_j(x) \in [0, 1]$. Then PA operator is expressed by

$$PA(C_1(x), C_2(x), \ldots, C_n(x)) = \sum_{j=1}^{n} w_j C_j(x), \tag{3.7}$$

where $w_j = \frac{T_j}{\sum_{j=1}^{n} T_j}$, $T_j = \prod_{k=1}^{j-1} C_K(x)$ $(j = 2, \ldots n)$, and $T_1 = 1$.

4. Probabilistic picture hesitant fuzzy sets

In this section, we propose the concept of P-PHFSs. To do this, we first briefly introduce the motivations of proposing P-PHFSs and explain why we need P-PHFSs. Then the definition, operational rules and comparison method of P-PHFSs are further introduced.

4.1. Motivation of proposing P-PHFSs

In practical MCDM problems, it is very necessary to comprehensively express DMs' evaluation information before determining the optimal alternative(s). In other words, it is a prerequisite to accurately and appropriately describe DMs' evaluation values, which makes the final decision-making result reliable and reasonable. Since fuzziness and vagueness exist extensively in realistic decision-making problems, DMs generally express their evaluation with the help of various fuzzy sets. In addition, sometimes it is necessary to consider the probability of the fuzzy values to more accurately represent the evaluation values provided by the DMs. The following example is provided to better illustrate this phenomenon.

Example 4.1. A university's department plans to purchase liberal arts books related to majors. The department invites three decision-making experts to evaluate the performance of potential book sellers based on the on the criterion "reputation". Each DM is required to use several values to represent positive membership, neutral membership, and negative membership of his/her evaluation value. The evaluation information provided by the three DMs is listed in Table 1.

Table 1. The evaluation information provided by three DMs.

	Possible positive membership degrees	**Possible neutral membership degrees**	**Possible negative membership degrees**
The first DM	0.3, 0.4, 0.5	0.05,0,1, 0,15	0.2,0.25,0.3
The second DM	0.3, 0.5, 0.55	0.1, 0,15	0.15,0.2
The third DM	0.25, 0.45	0.15, 0,2, 0,25	0.1,0.15,0.3

If we integrate each DM's evaluation values in the form of PHFEs, it can be expressed as $\{\{0.25, 0.3, 0.4, 0.45, 0.5, 0.55\}, \{0.05, 0.1, 0.15, 0.2, 0.25\}, \{0.1, 0.15, 0.2, 0.25, 0.3\}\}$. However, it indicates that multiple appearances of the positive membership degrees 0.3 and 0.5, neutral membership degrees 0.1 and 0.15, and negative membership degrees 0.15, 0.2, and 0.3 are ignored, which means that some basic information is lost. Hence, in order to overcome this shortcoming and to describe group's evaluation opinions more accurately, it is necessary to propose a new fuzzy information representation tool. Motivated by the PHFSs and P-HFSs, we extend PHFSs to P-PHFSs, which consider both the multiple positive membership degrees, neutral membership degrees and negative membership degrees, and their probabilistic information. The definition as well as some related notions of P-PHFSs are presented in the following subsections.

4.2. The definition of P-PHFSs

Motivated by P-HFSs, P-DHFSs and PHFSs, we present the definition of P-PHFSs.

Definition 4.2. Let X be a non-empty and finite set, a probabilistic picture hesitant fuzzy set (P-PHFS) $\tilde{N}$ on X is defined by

$$\tilde{N} = \{\langle x, \tilde{\mu}(x)|p(x), \tilde{\eta}(x)|q(x), \tilde{\nu}(x)|r(x)\rangle | x \in X\}, \tag{4.1}$$

where the sets $\tilde{\mu}(x)|p(x)$, $\tilde{\eta}(x)|q(x)$, and $\tilde{\nu}(x)|r(x)$ contains several elements, $\tilde{\mu}(x)$, $\tilde{\eta}(x)$, and $\tilde{\nu}(x)$ denote the potential positive membership degrees, neutral membership degrees, and negative membership degrees of $x \in X$ to the set $\tilde{N}$, respectively, and $p(x)$, $q(x)$, and $r(x)$ are their associated probabilistic information. In addition, $\tilde{\mu}(x)|p(x)$, $\tilde{\eta}(x)|q(x)$, $\tilde{\nu}(x)|r(x)$, $p(x)$, $q(x)$, and $r(x)$ satisfy the following conditions:

$$0 \le \alpha, \beta, \gamma \le 1, \quad 0 \le \alpha^+ + \beta^+ + \gamma^+ \le 1, \tag{4.2}$$

and

$$p_i \in [0,1], q_j \in [0,1], r_k \in [0,1], \sum_{i=1}^{\#\tilde{\mu}} p_i = 1, \sum_{j=1}^{\#\tilde{\eta}} q_j = 1, \sum_{k=1}^{\#\tilde{\nu}} r_k = 1, \tag{4.3}$$

where $\alpha \in \tilde{\mu}(x)$, $\beta \in \tilde{\eta}(x)$, $\gamma \in \tilde{\nu}(x)$, $\alpha^+ = \cup_{\alpha\in\tilde{\mu}(x)} \max\{\alpha\}$, $\beta^+ = \cup_{\beta\in\tilde{\eta}(x)} \max\{\beta\}$, $\gamma^+ = \cup_{\gamma\in\tilde{\nu}(x)} \max\{\gamma\}$, $p_i \in p(x)$, $q_j \in q(x)$, and $r_k \in r(x)$. The symbol $\#\tilde{\mu}$, $\#\tilde{\eta}$, and $\#\tilde{\nu}$ represent the total numbers of elements in $\tilde{\mu}(x)|p(x)$, $\tilde{\eta}(x)|q(x)$, and $\tilde{\nu}(x)|r(x)$, respectively. For convenience, we call $\tilde{n} = (\tilde{\mu}(x)|p(x), \tilde{\eta}(x)|q(x), \tilde{\nu}(x)|r(x))$ a probabilistic picture hesitant fuzzy element (P-PHFE), which can be denoted by $\tilde{n} = (\tilde{\mu}|p, \tilde{\eta}|q, \tilde{\nu}|r)$ for simplicity.

Now, we define the score and accuracy functions of P-PHFEs because the ranking of P-PHFEs is required in the application process to the practical MCDM problems.

Definition 4.3. Let $\tilde{n} = (\tilde{\mu}|p, \tilde{\eta}|q, \tilde{\nu}|r)$ be a P-PHFE, then the score function is defined by

$$s(\tilde{n}) = \left(1 + \frac{1}{\#\tilde{\mu}} \sum_{i=1}^{\#\tilde{\mu}} \alpha_i \cdot p_i - \frac{1}{\#\tilde{\eta}} \sum_{j=1}^{\#\tilde{\eta}} \beta_j \cdot q_j - \frac{1}{\#\tilde{\nu}} \sum_{k=1}^{\#\tilde{\nu}} \gamma_k \cdot r_k\right)/2, \quad s(\tilde{n}) \in [0,1], \tag{4.4}$$

the accuracy function is defined by

$$h(\tilde{n}) = \frac{1}{\#\tilde{\mu}}\sum_{i=1}^{\#\tilde{\mu}}\alpha_i \cdot p_i + \frac{1}{\#\tilde{\eta}}\sum_{j=1}^{\#\tilde{\eta}}\beta_j \cdot q_j + \frac{1}{\#\tilde{\nu}}\sum_{k=1}^{\#\tilde{\nu}}\gamma_k \cdot r_k, \;\; h(\tilde{n}) \in [0,1]. \tag{4.5}$$

Based on the score and accuracy values of P-PHFEs, the order relation between two P-PHFEs is defined as follows.

Definition 4.4. Let $\tilde{n}_1 = (\tilde{\mu}_1|p_1, \tilde{\eta}_1|q_1, \tilde{\nu}_1|r_1)$ and $\tilde{n}_2 = (\tilde{\mu}_2|p_2, \tilde{\eta}_2|q_2, \tilde{\nu}_2|r_2)$ be two P-PHFEs, then

(1) If $s(\tilde{n}_1) > s(\tilde{n}_2)$, then $\tilde{n}_1$ is superior to $\tilde{n}_2$, denoted by $\tilde{n}_1 > \tilde{n}_2$;

(2) If $s(\tilde{n}_1) = s(\tilde{n}_2)$, then

(a) If $h(\tilde{n}_1) > h(\tilde{n}_2)$, then $\tilde{n}_1$ is superior to $\tilde{n}_2$, denoted by $\tilde{n}_1 > \tilde{n}_2$;

(b) If $h(\tilde{n}_1) = h(\tilde{n}_2)$, then $\tilde{n}_1$ is indifferent to $\tilde{n}_2$, denoted by $\tilde{n}_1 = \tilde{n}_2$;

(c) If $h(\tilde{n}_1) < h(\tilde{n}_2)$, then $\tilde{n}_1$ is inferior to $\tilde{n}_2$, denoted by $\tilde{n}_1 < \tilde{n}_2$.

For example, let $\tilde{n}_1 = (\{0.2|1\}, \{0.3|0.4, 0.4|0.6\}, \{0.3|0.6, 0.35|0.4\})$ and $\tilde{n}_2 = (\{0.3|0.4, 0.4|0.6\}, \{0.2|1\}, \{0.2|0.6, 0.3|0.4\})$ and be two P-PHFEs, from Definition 4.4, we have $s(\tilde{n}_1) = s(\tilde{n}_2) = 0.43$, $h(\tilde{n}_1) = 0.54$, and $h(\tilde{n}_2) = 0.5$, then $\tilde{n}_1 > \tilde{n}_2$.

Remark 4.5. If the values in neutral membership degrees are all zero, i.e., $\beta = 0 \; \forall \beta \in \tilde{\eta}$, then P-PHFE reduces to P-DHFE; if the values in positive membership degrees, neutral membership degrees, and negative membership degrees each have the same probability, i.e., $p_i = \frac{1}{\#\tilde{\mu}}$, $q_j = \frac{1}{\#\tilde{\eta}}$, $r_k = \frac{1}{\#\tilde{\nu}}$, then P-PHFE reduces to PHFE. It presents the generalized property of the proposed P-PHFE. Furthermore, in this case, the score function of P-PHFE is consistent with that of PHFE or P-DHFE.

The following example demonstrates the difference between P-PHFE and PHFE.

Example 4.6. A DM provides two PHFEs, $(\{0.2, 0.25, 0.3\}, \{0.1, 0.15\}, \{0.15, 0.2\})$ and $(\{0.1, 0.3\}, \{0.05, 0.2\}, \{0.1, 0.2\})$, to evaluate the purchase of Car-1 and Car-2. It is easy found that Car-1 is more affordable than Car-2 for this DM based on his/her PHFE representation. However, if the DM thinks that PHFE information with probabilities is more suitable, and presents an evaluation of the purchase of both cars as $\tilde{n}_1 = (\{0.2|p_1, 0.25|p_2, 0.3|p_3\}, \{0.1|q_1, 0.15|q_2\}, \{0.15|r_1, 0.2|r_2\})$ and $\tilde{n}_2 = (\{0.1|p_4, 0.3|p_5\}, \{0.15|q_3, 0.2|q_4\}, \{0.2|r_3, 0.3|r_4\})$, then (1) if $p_1 = p_2 = p_3$, $q_1 = q_2$, $r_1 = r_2$, $p_4 = p_5$, $q_3 = q_4$ and $r_3 = r_4$, then the result is the same as that with the PHFEs; and (2) if $p_1 = 0.4, p_2 = 0.3, p_3 = 0.3$, $q_1 = 0.4$, $q_2 = 0.6$, $r_1 = 0.4$, $r_2 = 0.6$, $p_4 = 0.2$, $p_5 = 0.8$, $q_3 = 0.6$, $q_4 = 0.4$, $r_3 = 0.6$ and $r_4 = 0.4$, then we have $s(\tilde{n}_2) > s(\tilde{n}_1)$, and Car-2 is more affordable than Car-1.

Example 4.6 shows that the compared results based on PHFEs and P-PHFEs respectively can be different, and that P-PHFEs can be used to provide the same result as PHFEs. Thus, the proposed P-PHFEs are generalized PHFEs and can be used to describe picture hesitant fuzzy information with probabilities.

4.3. Basic operational rules of P-PHFEs

Based on the operations of P-HFEs and PHFEs, we define some basic operations of P-PHFEs and discuss their properties.

Definition 4.7. Let $\tilde{n} = \left(\tilde{\mu}|p_{\tilde{\mu}}, \tilde{\eta}|q_{\tilde{\eta}}, \tilde{\nu}|r_{\tilde{\nu}}\right)$, $\tilde{n}_1 = \left(\tilde{\mu}_1|p_{\tilde{\mu}_1}, \tilde{\eta}_1|q_{\tilde{\eta}_1}, \tilde{\nu}_1|r_{\tilde{\nu}_1}\right)$, and $\tilde{n}_2 = \left(\tilde{\mu}_2|p_{\tilde{\mu}_2}, \tilde{\eta}_2|q_{\tilde{\eta}_2}, \tilde{\nu}_2|r_{\tilde{\nu}_2}\right)$ be three P-PHFEs, $\lambda > 0$, then $\tilde{n}^c$ is the complement of $\tilde{n}$, and the operations of P-PHFEs are defined as

(1) $\tilde{n}^c = \left(\tilde{\nu}|r_{\tilde{\nu}}, \tilde{\eta}|q_{\tilde{\eta}}, \tilde{\mu}|p_{\tilde{\mu}}\right) = \cup_{\alpha\in\tilde{\mu},\beta\in\tilde{\eta},\gamma\in\tilde{\nu}} \left(\{\gamma|r_\gamma\}, \{\beta|q_\beta\}, \{\alpha|p_\alpha\}\right)$;

(2) $\tilde{n}_1 \oplus \tilde{n}_2 = \cup_{\substack{\alpha_1\in\tilde{\mu}_1,\beta_1\in\tilde{\eta}_1,\gamma_1\in\tilde{\nu}_1,\\ \alpha_2\in\tilde{\mu}_2,\beta_2\in\tilde{\eta}_2,\gamma_2\in\tilde{\nu}_2}} \left(\{\alpha_1 + \alpha_2 - \alpha_1\alpha_2|p_{\alpha_1}p_{\alpha_2}\}, \{\beta_1\beta_2|q_{\beta_1}q_{\beta_2}\}, \{\gamma_1\gamma_2|r_{\gamma_1}r_{\gamma_2}\}\right)$;

(3) $\tilde{n}_1 \otimes \tilde{n}_2 = \cup_{\substack{\alpha_1\in\tilde{\mu}_1,\beta_1\in\tilde{\eta}_1,\gamma_1\in\tilde{\nu}_1,\\ \alpha_2\in\tilde{\mu}_2,\beta_2\in\tilde{\eta}_2,\gamma_2\in\tilde{\nu}_2}} \left(\{\alpha_1\alpha_2|p_{\alpha_1}p_{\alpha_2}\}, \{\beta_1 + \beta_2 - \beta_1\beta_2|q_{\beta_1}q_{\beta_2}\}, \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2|r_{\gamma_1}r_{\gamma_2}\}\right)$;

(4) $\lambda\tilde{n} = \cup_{\alpha\in\tilde{\mu},\beta\in\tilde{\eta},\gamma\in\tilde{\nu}} \left(\{1 - (1-\alpha)^\lambda|p_\alpha\}, \{\beta^\lambda|q_\beta\}, \{\gamma^\lambda|r_\gamma\}\right)$;

(5) $\tilde{n}^\lambda = \cup_{\alpha\in\tilde{\mu},\beta\in\tilde{\eta},\gamma\in\tilde{\nu}} \left(\{\alpha^\lambda|p_\alpha\}, \{1 - (1-\beta)^\lambda|q_\beta\}, \{1 - (1-\gamma)^\lambda|r_\gamma\}\right)$.

For example, let $\tilde{n}_1 = (\{0.3|0.4, 0.4|0.6\}, \{0.2|1\}, \{0.2|0.6, 0.3|0.4\})$ and $\tilde{n}_2 = (\{0.3|1\}, \{0.2|0.4, 0.3|0.6\}, \{0.1|0.6, 0.2|0.4\})$ be two P-PHFEs, $\lambda = 2$, then

(1) $\tilde{n}_1^c = (\{0.2|0.6, 0.3|0.4\}, \{0.2|1\}, \{0.3|0.4, 0.4|0.6\})$, $\tilde{n}_2^c = (\{0.1|0.6, 0.2|0.4\}, \{0.2|0.4, 0.3|0.6\}, \{0.3|1\})$;

(2) $\tilde{n}_1 \oplus \tilde{n}_2 = (\{0.51|0.4, 0.58|0.6\}, \{0.04|0.4, 0.06|0.6\}, \{0.02|0.36, 0.03|0.24, 0.04|0.24, 0.06|0.16\})$;

(3) $\tilde{n}_1 \otimes \tilde{n}_2 = (\{0.09|0.4, 0.12|0.6\}, \{0.36|0.4, 0.44|0.6\}, \{0.28|0.36, 0.36|0.24, 0.37|0.24, 0.44|0.16\})$;

(4) $\lambda\tilde{n}_1 = (\{0.51|0.4, 0.64|0.6\}, \{0.04|1\}, \{0.04|0.6, 0.09|0.4\})$, $\lambda\tilde{n}_2 = (\{0.51|1\}, \{0.04|0.4, 0.09|0.6\}, \{0.01|0.6, 0.04|0.4\})$;

(5) $\tilde{n}_1^\lambda = (\{0.09|0.4, 0.16|0.6\}, \{0.36|1\}, \{0.36|0.6, 0.51|0.4\})$, $\tilde{n}_2^\lambda = (\{0.09|1\}, \{0.36|0.4, 0.51|0.6\}, \{0.19|0.6, 0.36|0.4\})$.

Based on Definition 4.7, we can obtain the following theorem.

Theorem 4.8. Let $\tilde{n} = \left(\tilde{\mu}|p_{\tilde{\mu}}, \tilde{\eta}|q_{\tilde{\eta}}, \tilde{\nu}|r_{\tilde{\nu}}\right)$, $\tilde{n}_1 = \left(\tilde{\mu}_1|p_{\tilde{\mu}_1}, \tilde{\eta}_1|q_{\tilde{\eta}_1}, \tilde{\nu}_1|r_{\tilde{\nu}_1}\right)$, and $\tilde{n}_2 = \left(\tilde{\mu}_2|p_{\tilde{\mu}_2}, \tilde{\eta}_2|q_{\tilde{\eta}_2}, \tilde{\nu}_2|r_{\tilde{\nu}_2}\right)$ be three P-PHFEs, $\lambda, \lambda_1, \lambda_2 > 0$, then

(1) $\tilde{n}_1 \oplus \tilde{n}_2 = \tilde{n}_2 \oplus \tilde{n}_1$;

(2) $\tilde{n}_1 \otimes \tilde{n}_2 = \tilde{n}_2 \otimes \tilde{n}_1$;

(3) $\tilde{n} \oplus (\tilde{n}_1 \oplus \tilde{n}_2) = (\tilde{n} \oplus \tilde{n}_1) \oplus \tilde{n}_2$;

(4) $\tilde{n} \otimes (\tilde{n}_1 \otimes \tilde{n}_2) = (\tilde{n} \otimes \tilde{n}_1) \otimes \tilde{n}_2$;

(5) $\lambda(\tilde{n}_1 \oplus \tilde{n}_2) = \lambda\tilde{n}_1 \oplus \lambda\tilde{n}_2$;

(6) $(\tilde{n}_1 \otimes \tilde{n}_2)^\lambda = \tilde{n}_1^\lambda \otimes \tilde{n}_2^\lambda$;

(7) $(\tilde{n}^{\lambda_1})^{\lambda_2} = \tilde{n}^{\lambda_1\lambda_2}$.

Proof. See Appendix A.

5. Generalized probabilistic picture hesitant fuzzy aggregation operators

In this section, some generalized probabilistic picture hesitant fuzzy weighted aggregation operators are defined by using the operations of P-PHFEs. For it, let Ω be the set of all P-PHFEs.

5.1. Generalized probabilistic picture hesitant fuzzy weighted average/geometric operators

Definition 5.1. Let $\tilde{n}_i$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\tilde{n}_i$ $(i = 1, 2, \ldots, n)$ such that $w_i > 0$ and $\sum_{i=1}^n w_i = 1$. For $\lambda > 0$, a generalized probabilistic picture hesitant fuzzy weighted average (GPPHFWA) operator is a function $\Omega^n \to \Omega$ such that

$$\text{GPPHFWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) = \left(w_1\tilde{n}_1^\lambda \oplus w_2\tilde{n}_2^\lambda \oplus \cdots \oplus w_n\tilde{n}_n^\lambda\right)^{\frac{1}{\lambda}} = \left(\oplus_{i=1}^n w_i\tilde{n}_i^\lambda\right)^{\frac{1}{\lambda}}. \tag{5.1}$$

Based on Definition 5.1, we can obtain the following theorem.

Theorem 5.2. Let $\tilde{n}_i = \left(\tilde{\mu}_i|p_{\tilde{\mu}_i}, \tilde{\eta}_i|q_{\tilde{\eta}_i}, \tilde{v}_i|r_{\tilde{v}_i}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs, $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\tilde{n}_i$ $(i = 1, 2, \ldots, n)$ such that $w_i > 0$ and $\sum_{i=1}^n w_i = 1$, and $\lambda > 0$, then the aggregated value by GPPHFWA operator is also a P-PHFE, and

$$\begin{aligned}
&\mathrm{GPPHFWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) \\
&= \cup_{\alpha_i \in \tilde{\mu}_i, \beta_i \in \tilde{\eta}_i, \gamma_i \in \tilde{v}_i} \left(\begin{array}{c}
\left\{ \left(1 - \prod_{i=1}^n (1 - \alpha_i^\lambda)^{w_i}\right)^{\frac{1}{\lambda}} \,\middle|\, \prod_{i=1}^n p_{\alpha_i} \right\}, \\
\left\{ 1 - \left(1 - \prod_{i=1}^n \left(1 - (1 - \beta_i)^\lambda\right)^{w_i}\right)^{\frac{1}{\lambda}} \,\middle|\, \prod_{i=1}^n q_{\beta_i} \right\}, \\
\left\{ 1 - \left(1 - \prod_{i=1}^n \left(1 - (1 - \gamma_i)^\lambda\right)^{w_i}\right)^{\frac{1}{\lambda}} \,\middle|\, \prod_{i=1}^n r_{\gamma_i} \right\}
\end{array} \right).
\end{aligned} \tag{5.2}$$

Proof. See Appendix B.

Furthermore, based on Theorem 5.2, the proposed GPPHFWA operator satisfies the boundedness and monotonicity as follows.

Theorem 5.3. Let $\tilde{n}_i = \left(\tilde{\mu}_i|p_{\tilde{\mu}_i}, \tilde{\eta}_i|q_{\tilde{\eta}_i}, \tilde{v}_i|r_{\tilde{v}_i}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs, $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\tilde{n}_i$ $(i = 1, 2, \ldots, n)$ such that $w_i > 0$ and $\sum_{i=1}^n w_i = 1$, and $\lambda > 0$, then we have the followings:

(1) Boundedness: If $\tilde{n}^- = \left(\min(\tilde{\mu}_i)|\min(p_{\tilde{\mu}_i}), \max(\tilde{\eta}_i)|\max(q_{\tilde{\eta}_i}), \max(\tilde{v}_i)|\max(r_{\tilde{v}_i})\right) = (\{\alpha_{\min}|p_{\min}\}, \{\beta_{\max}|q_{\max}\}, \{\gamma_{\max}|r_{\max}\})$ and $\tilde{n}^+ = \left(\max(\tilde{\mu}_i)|\max(p_{\tilde{\mu}_i}), \min(\tilde{\eta}_i)|\min(q_{\tilde{\eta}_i}), \min(\tilde{v}_i)|\min(r_{\tilde{v}_i})\right) = (\{\alpha_{\max}|p_{\max}\}, \{\beta_{\min}|q_{\min}\}, \{\gamma_{\min}|r_{\min}\})$, then

$$\tilde{n}^- \leq \mathrm{GPPHFWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) \leq \tilde{n}^+.$$

(2) Monotonicity: Let $\tilde{n}_i' = \left(\tilde{\mu}_i'|p_{\tilde{\mu}_i'}, \tilde{\eta}_i'|q_{\tilde{\eta}_i'}, \tilde{v}_i'|r_{\tilde{v}_i'}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs, if $w = (w_1, w_2, \ldots, w_n)^T$ is also the weight vector of $\tilde{n}_i'$, $\alpha_i \leq \alpha_i'$ for each $\alpha_i \in \tilde{\mu}_i$ and $\alpha_i' \in \tilde{\mu}_i'$, $\beta_i \geq \beta_i'$ for each $\beta_i \in \tilde{\eta}_i$ and $\beta_i' \in \tilde{\eta}_i'$, $\gamma_i \geq \gamma_i'$ for each $\gamma_i \in \tilde{v}_i$ and $\gamma_i' \in \tilde{v}_i'$ $(i = 1, 2, \ldots, n)$, and the probabilities remain the same, i.e., $p_{\tilde{\mu}_i} = p_{\tilde{\mu}_i'}$, $q_{\tilde{\eta}_i} = q_{\tilde{\eta}_i'}$, $r_{\tilde{v}_i} = r_{\tilde{v}_i'}$, then

$$\mathrm{GPPHFWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) \leq \mathrm{GPPHFWA}(\tilde{n}_1', \tilde{n}_2', \ldots, \tilde{n}_n').$$

Proof. See Appendix C.

However, the GPPHFWA operator does not satisfy the idempotency and commutativity, as illustrated by the following example.

Example 5.4. Let $\tilde{n}_1 = (\{0.3|0.25, 0.4|0.75\}, \{0.2|0.5, 0.3|0.5\}, \{0.2|0.4, 0.3|0.6\})$, $\tilde{n}_2 = (\{0.3|0.3, 0.35|0.7\}, \{0.2|0.45, 0.3|0.55\}, \{0.3|0.35, 0.4|0.65\})$ and $\tilde{n}_3 = (\{0.3|0.25, 0.4|0.75\}, \{0.2|0.5, 0.3|0.5\}, \{0.2|0.4, 0.3|0.6\})$ be three P-PHFEs, $w = (0.3, 0.5, 0.2)^T$ be the weight vector of $\tilde{n}_1$,

$\tilde{n}_2$ and $\tilde{n}_3$, $\lambda = 2$. Then by Theorem 5.2, we have

$$\mathrm{GPPHFWA}(\tilde{n}_1,\tilde{n}_2,\tilde{n}_3) = \left(\begin{array}{l} \left\{ \begin{array}{l} 0.300|0.019, 0.324|0.056, 0.324|0.044, 0.334|0.056, \\ 0.345|0.131, 0.355|0.169, 0.355|0.131, 0.374|0.394 \end{array} \right\}, \\ \left\{ \begin{array}{l} 0.200|0.113, 0.220|0.113, 0.225|0.113, 0.244|0.113, \\ 0.244|0.137, 0.265|0.137, 0.276|0.137, 0.300|0.137 \end{array} \right\}, \\ \left\{ \begin{array}{l} 0.244|0.056, 0.265|0.084, 0.276|0.084, 0.279|0.104, \\ 0.300|0.126, 0.304|0.156, 0.317|0.156, 0.346|0.234 \end{array} \right\} \end{array} \right),$$

$$\mathrm{GPPHFWA}(\tilde{n}_2,\tilde{n}_3,\tilde{n}_1) = \left(\begin{array}{l} \left\{ \begin{array}{l} 0.300|0.019, 0.315|0.044, 0.323|0.056, 0.337|0.131, \\ 0.355|0.056, 0.367|0.131, 0.374|0.169, 0.385|0.394 \end{array} \right\}, \\ \left\{ \begin{array}{l} 0.200|0.113, 0.217|0.113, 0.225|0.137, 0.244|0.113, \\ 0.244|0.137, 0.265|0.113, 0.276|0.137, 0.300|0.137 \end{array} \right\}, \\ \left\{ \begin{array}{l} 0.225|0.056, 0.244|0.104, 0.244|0.084, 0.264|0.156, \\ 0.276|0.084, 0.299|0.156, 0.300|0.126, 0.326|0.234 \end{array} \right\} \end{array} \right),$$

$$\mathrm{GPPHFWA}(\tilde{n}_1,\tilde{n}_1,\tilde{n}_1) = \left(\begin{array}{l} \left\{ \begin{array}{l} 0.300|0.015, 0.323|0.047, 0.334|0.047, 0.355|0.047, \\ 0.355|0.141, 0.374|0.141, 0.383|0.141, 0.400|0.421 \end{array} \right\}, \\ \left\{ \begin{array}{l} 0.200|0.125, 0.217|0.125, 0.225|0.125, 0.244|0.125, \\ 0.245|0.125, 0.265|0.125, 0.276|0.125, 0.300|0.125 \end{array} \right\}, \\ \left\{ \begin{array}{l} 0.200|0.064, 0.217|0.096, 0.225|0.096, 0.244|0.096, \\ 0.245|0.144, 0.265|0.144, 0.276|0.144, 0.300|0.216 \end{array} \right\} \end{array} \right).$$

From Definition 4.3, we have $s(\tilde{n}_1) = 0.4663$, $s(\tilde{n}_2) = 0,4288$, $s(\tilde{n}_3) = 0.5600$, $s(\mathrm{GPPHFWA}(\tilde{n}_1,\tilde{n}_2,\tilde{n}_3)) = 0.4878$, $s(\mathrm{GPPHFWA}(\tilde{n}_2,\tilde{n}_3,\tilde{n}_1)) = 0.4441$ and $s(\mathrm{GPPHFWA}(\tilde{n}_1,\tilde{n}_1,\tilde{n}_1)) = 0.4921$. Then $s(\mathrm{GPPHFWA}(\tilde{n}_1,\tilde{n}_2,\tilde{n}_3)) \neq s(\mathrm{GPPHFWA}(\tilde{n}_2,\tilde{n}_3,\tilde{n}_1))$ and hence $\mathrm{GPPHFWA}(\tilde{n}_1,\tilde{n}_2,\tilde{n}_3) \neq \mathrm{GPPHFWA}(\tilde{n}_2,\tilde{n}_3,\tilde{n}_1)$, which implies that the GPPHFWA operator is not commutative. Furthermore, since $s(\mathrm{GPPHFWA}(\tilde{n}_1,\tilde{n}_1,\tilde{n}_1)) \neq s(\tilde{n}_1)$, we have $\mathrm{GPPHFWA}(\tilde{n}_1,\tilde{n}_1,\tilde{n}_1) \neq \tilde{n}_1$, which implies that the GPPHFWA operator is not idempotent.

Under some restricted conditions, the reduced operators of the GPPHFWA operator are obtained as follows.

Remark 5.5. (1) If $\lambda = 1$, then the GPPHFWA operator reduces to the probabilistic picture hesitant fuzzy weighted average operator (PPHFWA) operator:

$$\begin{aligned} \mathrm{PPHFWA}(\tilde{n}_1,\tilde{n}_2,\ldots,\tilde{n}_n) &= \oplus_{i=1}^{n} w_i\tilde{n}_i \\ &= \cup_{\alpha_i\in\tilde{\mu}_i,\beta_i\in\tilde{\eta}_i,\gamma_i\in\tilde{\nu}_i} \left(\begin{array}{l} \left\{1-\prod_{i=1}^{n}(1-\alpha_i)^{w_i} \middle| \prod_{i=1}^{n} p_{\alpha_i}\right\}, \\ \left\{\prod_{i=1}^{n}\beta_i^{w_i} \middle| \prod_{i=1}^{n} q_{\beta_i}\right\}, \\ \left\{\prod_{i=1}^{n}\gamma_i^{w_i} \middle| \prod_{i=1}^{n} r_{\gamma_i}\right\} \end{array} \right). \end{aligned} \tag{5.3}$$

(2) If the probabilities of positive membership values, the probabilities of neutral membership values, and the probabilities of nonmembership values are respectively the same, i.e., $p_{\alpha_i} = \frac{1}{\#\tilde{\mu}_i}$, $q_{\beta_i} = \frac{1}{\#\tilde{\eta}_i}$, $r_{\gamma_i} = \frac{1}{\#\tilde{\nu}_i}$, then the GPPHFWA operator reduces to the generalized picture hesitant fuzzy weighted average (GPHFWA) operator [27]:

$$\mathrm{GPHFWA}(\tilde{n}_1,\tilde{n}_2,\ldots,\tilde{n}_n) = \left(\oplus_{i=1}^{n} w_i\tilde{n}_i^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$= \cup_{\alpha_i\in\tilde{\mu}_i,\beta_i\in\tilde{\eta}_i,\gamma_i\in\tilde{\nu}_i}\left(\begin{array}{l}\left\{\left(1-\prod_{i=1}^n(1-\alpha_i^\lambda)^{w_i}\right)^{\frac{1}{\lambda}}\right\},\\ \left\{1-\left(1-\prod_{i=1}^n\left(1-(1-\beta_i)^\lambda\right)^{w_i}\right)^{\frac{1}{\lambda}}\right\},\\ \left\{1-\left(1-\prod_{i=1}^n\left(1-(1-\gamma_i)^\lambda\right)^{w_i}\right)^{\frac{1}{\lambda}}\right\}\end{array}\right). \quad (5.4)$$

Based on the generalized picture hesitant fuzzy weighted geometric operator [27], we also define the generalized probabilistic picture hesitant fuzzy weighted geometric operator as follows.

Definition 5.6. Let $\tilde{n}_i$ $(i=1,2,\ldots,n)$ be a collection of P-PHFEs and $w=(w_1,w_2,\ldots,w_n)^T$ be the weight vector of $\tilde{n}_i$ $(i=1,2,\ldots,n)$ such that $w_i>0$ and $\sum_{i=1}^n w_i=1$. For $\lambda>0$, a generalized probabilistic picture hesitant fuzzy weighted average (GPPHFWG) operator is a function $\Omega^n\to\Omega$ such that

$$\mathrm{GPPHFWG}(\tilde{n}_1,\tilde{n}_2,\ldots,\tilde{n}_n)=\frac{1}{\lambda}\left(\lambda\tilde{n}_1^{w_1}\otimes\lambda\tilde{n}_2^{w_2}\otimes\cdots\otimes\lambda\tilde{n}_n^{w_n}\right)=\frac{1}{\lambda}\left(\otimes_{i=1}^n\lambda\tilde{n}_i^{w_i}\right). \quad (5.5)$$

Based on Definition 5.6, we obtain the following theorems for the GPPHFWG operator similar to Theorems 5.2 and 5.3.

Theorem 5.7. Let $\tilde{n}_i=\left(\tilde{\mu}_i|p_{\tilde{\mu}_i},\tilde{\eta}_i|q_{\tilde{\eta}_i},\tilde{\nu}_i|r_{\tilde{\nu}_i}\right)$ $(i=1,2,\ldots,n)$ be a collection of P-PHFEs, $w=(w_1,w_2,\ldots,w_n)^T$ be the weight vector of $\tilde{n}_i$ $(i=1,2,\ldots,n)$ such that $w_i>0$ and $\sum_{i=1}^n w_i=1$, and $\lambda>0$, then the aggregated value by GPPHFWG operator is also a P-PHFE, and

$$\begin{aligned}&\mathrm{GPPHFWG}(\tilde{n}_1,\tilde{n}_2,\ldots,\tilde{n}_n)\\ &= \cup_{\alpha_i\in\tilde{\mu}_i,\beta_i\in\tilde{\eta}_i,\gamma_i\in\tilde{\nu}_i}\left(\begin{array}{l}\left\{1-\left(1-\prod_{i=1}^n\left(1-(1-\alpha_i)^\lambda\right)^{w_i}\right)^{\frac{1}{\lambda}}\,\middle|\,\prod_{i=1}^n p_{\alpha_i}\right\},\\ \left\{\left(1-\prod_{i=1}^n(1-\beta_i^\lambda)^{w_i}\right)^{\frac{1}{\lambda}}\,\middle|\,\prod_{i=1}^n q_{\beta_i}\right\},\\ \left\{\left(1-\prod_{i=1}^n(1-\gamma_i^\lambda)^{w_i}\right)^{\frac{1}{\lambda}}\,\middle|\,\prod_{i=1}^n r_{\gamma_i}\right\}\end{array}\right).\end{aligned} \quad (5.6)$$

Theorem 5.8. Let $\tilde{n}_i=\left(\tilde{\mu}_i|p_{\tilde{\mu}_i},\tilde{\eta}_i|q_{\tilde{\eta}_i},\tilde{\nu}_i|r_{\tilde{\nu}_i}\right)$ $(i=1,2,\ldots,n)$ be a collection of P-PHFEs, $w=(w_1,w_2,\ldots,w_n)^T$ be the weight vector of $\tilde{n}_i$ $(i=1,2,\ldots,n)$ such that $w_i>0$ and $\sum_{i=1}^n w_i=1$, and $\lambda>0$, then we have the followings:

(1) Boundedness: If $\tilde{n}^-=\left(\min(\tilde{\mu}_i)|\min(p_{\tilde{\mu}_i}),\max(\tilde{\eta}_i)|\max(q_{\tilde{\eta}_i}),\max(\tilde{\nu}_i)|\max(r_{\tilde{\nu}_i})\right)=(\{\alpha_{\min}|p_{\min}\},\{\beta_{\max}|q_{\max}\},\{\gamma_{\max}|r_{\max}\})$ and $\tilde{n}^+=\left(\max(\tilde{\mu}_i)|\max(p_{\tilde{\mu}_i}),\min(\tilde{\eta}_i)|\min(q_{\tilde{\eta}_i}),\min(\tilde{\nu}_i)|\min(r_{\tilde{\nu}_i})\right)=(\{\alpha_{\max}|p_{\max}\},\{\beta_{\min}|q_{\min}\},\{\gamma_{\min}|r_{\min}\})$, then

$$\tilde{n}^-\leq\mathrm{GPPHFWG}(\tilde{n}_1,\tilde{n}_2,\ldots,\tilde{n}_n)\leq\tilde{n}^+.$$

(2) Monotonicity: Let $\tilde{n}_i'=\left(\tilde{\mu}_i'|p_{\tilde{\mu}_i'},\tilde{\eta}_i'|q_{\tilde{\eta}_i'},\tilde{\nu}_i'|r_{\tilde{\nu}_i'}\right)$ $(i=1,2,\ldots,n)$ be a collection of P-PHFEs, if $w=(w_1,w_2,\ldots,w_n)^T$ is also the weight vector of $\tilde{n}_i'$, $\alpha_i\leq\alpha_i'$ for each $\alpha_i\in\tilde{\mu}_i$ and $\alpha_i'\in\tilde{\mu}_i'$, $\beta_i\geq\beta_i'$ for each $\beta_i\in\tilde{\eta}_i$ and $\beta_i'\in\tilde{\eta}_i'$, $\gamma_i\geq\gamma_i'$ for each $\gamma_i\in\tilde{\nu}_i$ and $\gamma_i'\in\tilde{\nu}_i'$ $(i=1,2,\ldots,n)$, and the probabilities remain the same, i.e., $p_{\tilde{\mu}_i}=p_{\tilde{\mu}_i'}$, $q_{\tilde{\eta}_i}=q_{\tilde{\eta}_i'}$, $r_{\tilde{\nu}_i}=r_{\tilde{\nu}_i'}$, then

$$\mathrm{GPPHFWG}(\tilde{n}_1,\tilde{n}_2,\ldots,\tilde{n}_n)\leq\mathrm{GPPHFWG}(\tilde{n}_1',\tilde{n}_2',\ldots,\tilde{n}_n').$$

Several reduced operators of the GPPHFWG operator are presented as follows.

Remark 5.9. (1) If $\lambda = 1$, then the GPPHFWG operator reduces to the probabilistic picture hesitant fuzzy weighted geometric operator (PPHFWG) operator:

$$\begin{aligned} \mathrm{PPHFWG}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) &= \otimes_{i=1}^{n} \tilde{n}_i^{w_i} \\ &= \cup_{\alpha_i \in \tilde{\mu}_i, \beta_i \in \tilde{\eta}_i, \gamma_i \in \tilde{\nu}_i} \left(\begin{array}{l} \left\{ \prod_{i=1}^{n} \alpha_i^{w_i} \middle| \prod_{i=1}^{n} p_{\alpha_i} \right\}, \\ \left\{ 1 - \prod_{i=1}^{n} (1-\beta_i)^{w_i} \middle| \prod_{i=1}^{n} q_{\beta_i} \right\}, \\ \left\{ 1 - \prod_{i=1}^{n} (1-\gamma_i)^{w_i} \middle| \prod_{i=1}^{n} r_{\gamma_i} \right\} \end{array} \right). \end{aligned} \tag{5.7}$$

(2) If the probabilities of membership values, the probabilities of neutral membership values, and the probabilities of nonmembership values are respectively the same, i.e., $p_{\alpha_i} = \frac{1}{\#\tilde{\mu}_i}$, $q_{\beta_i} = \frac{1}{\#\tilde{\eta}_i}$, $r_{\gamma_i} = \frac{1}{\#\tilde{\nu}_i}$, then the GPPHFWG operator reduces to the generalized picture hesitant fuzzy weighted geometric (GPHFWG) operator [27]:

$$\begin{aligned} \mathrm{GPHFWG}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) &= \frac{1}{\lambda} \left(\otimes_{i=1}^{n} \lambda \tilde{n}_i^{w_i} \right) \\ &= \cup_{\alpha_i \in \tilde{\mu}_i, \beta_i \in \tilde{\eta}_i, \gamma_i \in \tilde{\nu}_i} \left(\begin{array}{l} \left\{ 1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1-\alpha_i)^{\lambda} \right)^{w_i} \right)^{\frac{1}{\lambda}} \right\}, \\ \left\{ \left(1 - \prod_{i=1}^{n} (1-\beta_i^{\lambda})^{w_i} \right)^{\frac{1}{\lambda}} \right\}, \\ \left\{ \left(1 - \prod_{i=1}^{n} (1-\gamma_i^{\lambda})^{w_i} \right)^{\frac{1}{\lambda}} \right\} \end{array} \right). \end{aligned} \tag{5.8}$$

To demonstrate the relationship between the PPHFWA and PPHFWG operators, we first introduce the following lemma.

Lemma 5.10. [60,61] Let $x_i > 0$, $w_i > 0$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} w_i = 1$, then $\prod_{i=1}^{n} x_i^{w_i} \le \sum_{i=1}^{n} w_i x_i$, with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 5.11. Let $\tilde{n}_i = \left(\tilde{\mu}_i | p_{\tilde{\mu}_i}, \tilde{\eta}_i | q_{\tilde{\eta}_i}, \tilde{\nu}_i | r_{\tilde{\nu}_i} \right)$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\tilde{n}_i$ $(i = 1, 2, \ldots, n)$ such that $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$, then

$$\mathrm{PPHFWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) \ge \mathrm{PPHFWG}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n).$$

Proof. For $\alpha_i \in \tilde{\mu}_i$ $(i = 1, 2, \ldots, n)$, we can obtain the following inequality from Lemma 5.10:

$$\prod_{i=1}^{n} \alpha_i^{w_i} \le \sum_{i=1}^{n} w_i \alpha_i = 1 - \sum_{i=1}^{n} w_i (1-\alpha_i) \le 1 - \prod_{i=1}^{n} (1-\alpha_i)^{w_i}.$$

Also, from Lemma 5.10, for $\beta_i \in \tilde{\eta}_i$, $\gamma_i \in \tilde{u}_i$ $(i = 1, 2, \ldots, n)$, we can obtain $1 - \prod_{i=1}^{n} (1-\beta_i)^{w_i} \ge \prod_{i=1}^{n} \beta_i^{w_i}$ and $1 - \prod_{i=1}^{n} (1-\gamma_i)^{w_i} \ge \prod_{i=1}^{n} \gamma_i^{w_i}$. According to score function $s(\tilde{n}) = \left(1 + \frac{1}{\#\tilde{\mu}} \sum_{\alpha \in \tilde{\mu}} \alpha p_\alpha - \frac{1}{\#\tilde{\eta}} \sum_{\beta \in \tilde{\eta}} \beta q_\beta - \frac{1}{\#\tilde{\nu}} \sum_{\gamma \in \tilde{\nu}} \gamma r_\gamma \right) / 2$, we have $\mathrm{PPHFWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) \ge \mathrm{PPHFWG}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n)$.

Theorem 5.11 shows that the value aggregated by the PPHFWG operator are not larger than the value obtained by the PPHFWA operator. That is to say, the PPHFWG operator reflects DM's pessimistic attitude than the PPHFWA operator in the aggregation process.

5.2. *Generalized probabilistic picture hesitant fuzzy prioritized weighted average/geometric operators*

In real life, people sometimes face attributes that have different priorities, for example, when couples choose milk for their children, safety has a higher priority than price. The GPPHFWA and GPPHFWG operatrors, introduced in the previous section, cannot handle this situation. So, based on the PA operator [59], the generalized probabilistic picture hesitant fuzzy prioritized weighted average/geometric (GPPHFPWA and GPPHFPWG) operators are developed.

Definition 5.12. Let $\tilde{n}_i$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs and $\lambda > 0$, a generalized probabilistic picture hesitant fuzzy prioritized weighted average (GPPHFPWA) operator is a function $\Omega^n \to \Omega$ such that

$$\begin{aligned}\mathrm{GPPHFPWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) &= \left(\frac{T_1}{\sum_{j=1}^n T_j}\tilde{n}_1^\lambda \oplus \frac{T_2}{\sum_{j=1}^n T_j}\tilde{n}_2^\lambda \oplus \cdots \oplus \frac{T_n}{\sum_{j=1}^n T_j}\tilde{n}_n^\lambda\right)^{\frac{1}{\lambda}} \\ &= \left(\oplus_{i=1}^n \frac{T_i}{\sum_{j=1}^n T_j}\tilde{n}_i^\lambda\right)^{\frac{1}{\lambda}}, \end{aligned} \tag{5.9}$$

where $T_j = \prod_{k=1}^{j-1} s(\tilde{n}_k)$ $(j = 2, 3, \ldots, n)$, $T_1 = 1$, and $s(\tilde{n}_k)$ is the score value of PPHFE $\tilde{n}_k$.

Based on Definition 5.12, we obtain the following theorem.

Theorem 5.13. Let $\tilde{n}_i = \left(\tilde{\mu}_i|p_{\tilde{\mu}_i}, \tilde{\eta}_i|q_{\tilde{\eta}_i}, \tilde{\nu}_i|r_{\tilde{\nu}_i}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs and $\lambda > 0$, then the aggregated value by GPPHFPWA operator is also a P-PHFE, and

$$\begin{aligned}&\mathrm{GPPHFPWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) \\ &= \cup_{\alpha_i\in\tilde{\mu}_i,\beta_i\in\tilde{\eta}_i,\gamma_i\in\tilde{\nu}_i}\left(\begin{array}{l}\left\{\left(1-\prod_{i=1}^n(1-\alpha_i^\lambda)^{\frac{T_i}{\sum_{j=1}^n T_j}}\right)^{\frac{1}{\lambda}} \Big| \prod_{i=1}^n p_{\alpha_i}\right\}, \\ \left\{1-\left(1-\prod_{i=1}^n\left(1-(1-\beta_i)^\lambda\right)^{\frac{T_i}{\sum_{j=1}^n T_j}}\right)^{\frac{1}{\lambda}} \Big| \prod_{i=1}^n q_{\beta_i}\right\}, \\ \left\{1-\left(1-\prod_{i=1}^n\left(1-(1-\gamma_i)^\lambda\right)^{\frac{T_i}{\sum_{j=1}^n T_j}}\right)^{\frac{1}{\lambda}} \Big| \prod_{i=1}^n r_{\gamma_i}\right\}\end{array}\right).\end{aligned} \tag{5.10}$$

Theorem 5.14. Let $\tilde{n}_i = \left(\tilde{\mu}_i|p_{\tilde{\mu}_i}, \tilde{\eta}_i|q_{\tilde{\eta}_i}, \tilde{\nu}_i|r_{\tilde{\nu}_i}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs and $\lambda > 0$, then we have the followings:

(1) Boundedness: If $\tilde{n}^- = \left(\min(\tilde{\mu}_i)|\min(p_{\tilde{\mu}_i}), \max(\tilde{\eta}_i)|\max(q_{\tilde{\eta}_i}), \max(\tilde{\nu}_i)|\max(r_{\tilde{\nu}_i})\right) = (\{\alpha_{\min}|p_{\min}\}, \{\beta_{\max}|q_{\max}\}, \{\gamma_{\max}|r_{\max}\})$ and $\tilde{n}^+ = \left(\max(\tilde{\mu}_i)|\max(p_{\tilde{\mu}_i}), \min(\tilde{\eta}_i)|\min(q_{\tilde{\eta}_i}), \min(\tilde{\nu}_i)|\min(r_{\tilde{\nu}_i})\right) = (\{\alpha_{\max}|p_{\max}\}, \{\beta_{\min}|q_{\min}\}, \{\gamma_{\min}|r_{\min}\})$, then

$$\tilde{n}^- \leq \mathrm{GPPHFPWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) \leq \tilde{n}^+.$$

(2) Monotonicity: Let $\tilde{n}_i' = \left(\tilde{\mu}_i'|p_{\tilde{\mu}_i'}, \tilde{\eta}_i'|q_{\tilde{\eta}_i'}, \tilde{\nu}_i'|r_{\tilde{\nu}_i'}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs, if $\alpha_i \leq \alpha_i'$ for each $\alpha_i \in \tilde{\mu}_i$ and $\alpha_i' \in \tilde{\mu}_i'$, $\beta_i \geq \beta_i'$ for each $\beta_i \in \tilde{\eta}_i$ and $\beta_i' \in \tilde{\eta}_i'$, $\gamma_i \geq \gamma_i'$ for each $\gamma_i \in \tilde{\nu}_i$ and $\gamma_i' \in \tilde{\nu}_i'$ $(i = 1, 2, \ldots, n)$, and the probabilities remain the same, i.e., $p_{\tilde{\mu}_i} = p_{\tilde{\mu}_i'}$, $q_{\tilde{\eta}_i} = q_{\tilde{\eta}_i'}$, $r_{\tilde{\nu}_i} = r_{\tilde{\nu}_i'}$, then

$$\mathrm{GPPHFPWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) \leq \mathrm{GPPHFPWA}(\tilde{n}_1', \tilde{n}_2', \ldots, \tilde{n}_n').$$

Under some restricted conditions, the reduced operators of the GPPHFPWA operator are obtained.

Remark 5.15. (1) If $\lambda = 1$, then the GPPHFPWA operator reduces to the probabilistic picture hesitant fuzzy prioritized weighted average operator (PPHFPWA) operator:

$$\begin{aligned} \mathrm{PPHFPWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) &= \oplus_{i=1}^{n} \frac{T_i}{\sum_{j=1}^{n} T_j} \tilde{n}_i \\ &= \cup_{\alpha_i \in \tilde{\mu}_i, \beta_i \in \tilde{\eta}_i, \gamma_i \in \tilde{\nu}_i} \left(\begin{array}{l} \left\{ 1 - \prod_{i=1}^{n} (1-\alpha_i)^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \middle| \prod_{i=1}^{n} p_{\alpha_i} \right\}, \\ \left\{ \prod_{i=1}^{n} \beta_i^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \middle| \prod_{i=1}^{n} q_{\beta_i} \right\}, \\ \left\{ \prod_{i=1}^{n} \gamma_i^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \middle| \prod_{i=1}^{n} r_{\gamma_i} \right\} \end{array} \right). \end{aligned} \tag{5.11}$$

(2) If the criteria are of the same priority, then the GPPHFPWA operator is reduced to the GPPHFWA operator:

$$\mathrm{GPPHFWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) = \left(w_1 \tilde{n}_1^{\lambda} \oplus w_2 \tilde{n}_2^{\lambda} \oplus \cdots \oplus w_n \tilde{n}_n^{\lambda} \right)^{\frac{1}{\lambda}} = \left(\oplus_{i=1}^{n} w_i \tilde{n}_i^{\lambda} \right)^{\frac{1}{\lambda}}. \tag{5.12}$$

(3) If the probabilities of membership values, the probabilities of neutral membership values, and the probabilities of nonmembership values are respectively the same, i.e., $p_{\alpha_i} = \frac{1}{\#\tilde{\mu}_i}$, $q_{\beta_i} = \frac{1}{\#\tilde{\eta}_i}$, $r_{\gamma_i} = \frac{1}{\#\tilde{\nu}_i}$, then the GPPHFWA operator reduces to the generalized picture hesitant fuzzy prioritized weighted average (GPHFPWA) operator [27]:

$$\begin{aligned} \mathrm{GPHFPWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) &= \left(\oplus_{i=1}^{n} \frac{T_i}{\sum_{j=1}^{n} T_j} \tilde{n}_i^{\lambda} \right)^{\frac{1}{\lambda}} \\ &= \cup_{\alpha_i \in \tilde{\mu}_i, \beta_i \in \tilde{\eta}_i, \gamma_i \in \tilde{\nu}_i} \left(\begin{array}{l} \left\{ \left(1 - \prod_{i=1}^{n} (1-\alpha_i^{\lambda})^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \right)^{\frac{1}{\lambda}} \right\}, \\ \left\{ 1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1-\beta_i)^{\lambda} \right)^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \right)^{\frac{1}{\lambda}} \right\}, \\ \left\{ 1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1-\gamma_i)^{\lambda} \right)^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \right)^{\frac{1}{\lambda}} \right\} \end{array} \right). \end{aligned} \tag{5.13}$$

Similar to the GPPHFPWA operator, we also define the generalized probabilistic picture hesitant fuzzy prioritized weighted geometric (GPPHFPWG) operator as follows.

Definition 5.16. Let $\tilde{n}_i$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs and $\lambda > 0$, a generalized probabilistic picture hesitant fuzzy prioritized weighted geometric (GPPHPFWG) operator is a function $\Omega^n \to \Omega$ such that

$$\begin{aligned} \mathrm{GPPHFPWG}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) &= \frac{1}{\lambda} \left(\lambda \tilde{n}_1^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \otimes \lambda \tilde{n}_2^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \otimes \cdots \otimes \lambda \tilde{n}_n^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \right) \\ &= \frac{1}{\lambda} \left(\otimes_{i=1}^{n} \lambda \tilde{n}_i^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \right), \end{aligned} \tag{5.14}$$

where $T_j = \prod_{k=1}^{j-1} s(\tilde{n}_k)$ $(j = 2, 3, \ldots, n)$, $T_1 = 1$, and $s(\tilde{n}_k)$ is the score value of P-PHFE $\tilde{n}_k$.

Based on Definition 5.16, we obtain the following theorem for the GPPHFPWG operator.

Theorem 5.17. Let $\tilde{n}_i = \left(\tilde{\mu}_i|p_{\tilde{\mu}_i}, \tilde{\eta}_i|q_{\tilde{\eta}_i}, \tilde{\nu}_i|r_{\tilde{\nu}_i}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs and $\lambda > 0$, then the aggregated value by GPPHFPWG operator is also a P-PHFE, and

$$\mathrm{GPPHFPWG}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) = \cup_{\alpha_i\in\tilde{\mu}_i, \beta_i\in\tilde{\eta}_i, \gamma_i\in\tilde{\nu}_i} \left(\begin{array}{l} \left\{ 1-\left(1-\prod_{i=1}^{n}\left(1-(1-\alpha_i)^{\lambda}\right)^{\frac{T_i}{\sum_{j=1}^{n} T_j}}\right)^{\frac{1}{\lambda}} \Big| \prod_{i=1}^{n} p_{\alpha_i} \right\}, \\ \left\{ \left(1-\prod_{i=1}^{n}(1-\beta_i^{\lambda})^{\frac{T_i}{\sum_{j=1}^{n} T_j}}\right)^{\frac{1}{\lambda}} \Big| \prod_{i=1}^{n} q_{\beta_i} \right\}, \\ \left\{ \left(1-\prod_{i=1}^{n}(1-\gamma_i^{\lambda})^{\frac{T_i}{\sum_{j=1}^{n} T_j}}\right)^{\frac{1}{\lambda}} \Big| \prod_{i=1}^{n} r_{\gamma_i} \right\} \end{array} \right). \tag{5.15}$$

The GPPHFPWG operator also has the properties as follows.

Theorem 5.18. Let $\tilde{n}_i = \left(\tilde{\mu}_i|p_{\tilde{\mu}_i}, \tilde{\eta}_i|q_{\tilde{\eta}_i}, \tilde{\nu}_i|r_{\tilde{\nu}_i}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs and $\lambda > 0$, then we have the followings:

(1) Boundedness: If $\tilde{n}^- = \left(\min(\tilde{\mu}_i)|\min(p_{\tilde{\mu}_i}), \max(\tilde{\eta}_i)|\max(q_{\tilde{\eta}_i}), \max(\tilde{\nu}_i)|\max(r_{\tilde{\nu}_i})\right) = (\{\alpha_{\min}|p_{\min}\}, \{\beta_{\max}|q_{\max}\}, \{\gamma_{\max}|r_{\max}\})$ and $\tilde{n}^+ = \left(\max(\tilde{\mu}_i)|\max(p_{\tilde{\mu}_i}), \min(\tilde{\eta}_i)|\min(q_{\tilde{\eta}_i}), \min(\tilde{\nu}_i)|\min(r_{\tilde{\nu}_i})\right) = (\{\alpha_{\max}|p_{\max}\}, \{\beta_{\min}|q_{\min}\}, \{\gamma_{\min}|r_{\min}\})$, then

$$\tilde{n}^- \leq \mathrm{GPPHFPWG}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) \leq \tilde{n}^+.$$

(2) Monotonicity: Let $\tilde{n}'_i = \left(\tilde{\mu}'_i|p_{\tilde{\mu}'_i}, \tilde{\eta}'_i|q_{\tilde{\eta}'_i}, \tilde{\nu}'_i|r_{\tilde{\nu}'_i}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs, if $\alpha_i \leq \alpha'_i$ for each $\alpha_i \in \tilde{\mu}_i$ and $\alpha'_i \in \tilde{\mu}'_i$, $\beta_i \geq \beta'_i$ for each $\beta_i \in \tilde{\eta}_i$ and $\beta'_i \in \tilde{\eta}'_i$, $\gamma_i \geq \gamma'_i$ for each $\gamma_i \in \tilde{\nu}_i$ and $\gamma'_i \in \tilde{\nu}'_i$ $(i = 1, 2, \ldots, n)$, and the probabilities remain the same, i.e., $p_{\tilde{\mu}_i} = p_{\tilde{\mu}'_i}$, $q_{\tilde{\eta}_i} = q_{\tilde{\eta}'_i}$, $r_{\tilde{\nu}_i} = r_{\tilde{\nu}'_i}$, then

$$\mathrm{GPPHFPWG}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) \leq \mathrm{GPPHFPWG}(\tilde{n}'_1, \tilde{n}'_2, \ldots, \tilde{n}'_n).$$

Several reduced operators of the GPPHFPWG operator can be obtained as follows.

Remark 5.19. (1) If $\lambda = 1$, then the GPPHFPWG operator reduces to the probabilistic picture hesitant fuzzy weighted prioritized geometric operator (PPHFPWG) operator:

$$\mathrm{PPHFPWG}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) = \otimes_{i=1}^{n} \tilde{n}_i^{\frac{T_i}{\sum_{j=1}^{n} T_j}} = \cup_{\alpha_i\in\tilde{\mu}_i, \beta_i\in\tilde{\eta}_i, \gamma_i\in\tilde{\nu}_i} \left(\begin{array}{l} \left\{ \prod_{i=1}^{n} \alpha_i^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \Big| \prod_{i=1}^{n} p_{\alpha_i} \right\}, \\ \left\{ 1-\prod_{i=1}^{n}(1-\beta_i)^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \Big| \prod_{i=1}^{n} q_{\beta_i} \right\}, \\ \left\{ 1-\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \Big| \prod_{i=1}^{n} r_{\gamma_i} \right\} \end{array} \right). \tag{5.16}$$

(2) If the criteria are of the same priority, then the GPPHFPWG operator is reduced to the GPPHFWG operator:

$$\mathrm{GPPHFWG}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) = \frac{1}{\lambda}\left(\lambda\tilde{n}_1^{w_1} \otimes \lambda\tilde{n}_2^{w_2} \otimes \cdots \otimes \lambda\tilde{n}_n^{w_n}\right) = \frac{1}{\lambda}\left(\otimes_{i=1}^{n} \lambda\tilde{n}_i^{w_i}\right). \tag{5.17}$$

(3) If the probabilities of membership values, the probabilities of neutral membership values, and the probabilities of nonmembership values are respectively the same, i.e., $p_{\alpha_i} = \frac{1}{\#\tilde{\mu}_i}$, $q_{\beta_i} = \frac{1}{\#\tilde{\eta}_i}$, $r_{\gamma_i} = \frac{1}{\#\tilde{\nu}_i}$, then the GPPHFPWG operator reduces to the generalized picture hesitant fuzzy prioritized weighted geometric (GPHFPWG) operator [27]:

$$\mathrm{GPHFWG}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) = \frac{1}{\lambda}\left(\otimes_{i=1}^{n} \lambda \tilde{n}_i^{\frac{T_i}{\sum_{j=1}^{n} T_j}}\right)$$

$$= \cup_{\alpha_i \in \tilde{\mu}_i, \beta_i \in \tilde{\eta}_i, \gamma_i \in \tilde{\nu}_i} \left(\begin{array}{l} \left\{1 - \left(1 - \prod_{i=1}^{n}\left(1 - (1-\alpha_i)^{\lambda}\right)^{\frac{T_i}{\sum_{j=1}^{n} T_j}}\right)^{\frac{1}{\lambda}}\right\}, \\ \left\{\left(1 - \prod_{i=1}^{n}(1 - \beta_i^{\lambda})^{\frac{T_i}{\sum_{j=1}^{n} T_j}}\right)^{\frac{1}{\lambda}}\right\}, \\ \left\{\left(1 - \prod_{i=1}^{n}(1 - \gamma_i^{\lambda})^{\frac{T_i}{\sum_{j=1}^{n} T_j}}\right)^{\frac{1}{\lambda}}\right\} \end{array} \right). \tag{5.18}$$

Similar to Theorem 5.11, the PPHFPWA and PPHFPWG operators have the relationship as follows.

Theorem 5.20. Let $\tilde{n}_i = \left(\tilde{\mu}_i | p_{\tilde{\mu}_i}, \tilde{\eta}_i | q_{\tilde{\eta}_i}, \tilde{\nu}_i | r_{\tilde{\nu}_i}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of P-PHFEs, then

$$\mathrm{PPHFPWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) \geq \mathrm{PPHFPWG}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n).$$

6. MCDM methods under probabilistic picture hesitant fuzzy environment

In this section, we utilize the proposed aggregation operators to deal with different MCDM problems under probabilistic picture hesitant fuzzy environment.

Let $X = \{x_1, x_2, \ldots x_n\}$ be a set of n alternatives and $C = \{C_1, C_2, \ldots, C_m\}$ be a set of m criteria. Suppose that the group of DMs provides the evaluation value of the alternative x_j $(j = 1, 2, \ldots, n)$ concerning the criterion C_i $(i = 1, 2, \ldots, m)$ by using the P-PHFEs $\tilde{n}_{ij} = \left(\tilde{\mu}_{ij} | p_{\tilde{\mu}_{ij}}, \tilde{\eta}_{ij} | q_{\tilde{\eta}_{ij}}, \tilde{\nu}_{ij} | r_{\tilde{\nu}_{ij}}\right)$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. All these P-PHFEs are contained in the probabilistic picture hesitant fuzzy decision matrix $N = \left(\tilde{n}_{ij}\right)_{m \times n}$. In general, the criteria can be divided into two types: the cost criteria and benefit criteria. So, the evaluation values concerning the cost criteria should be transformed into the evaluation values concerning the benefit criteria to obtain the normalized probabilistic picture hesitant fuzzy decision matrix $\overline{N} = \left(\bar{n}_{ij}\right)_{m \times n}$ as

$$\bar{n}_{ij} = \begin{cases} \tilde{n}_{ij}, & \text{for benefit criterion;} \\ \tilde{n}_{ij}^{c}, & \text{for cost criterion,} \end{cases} \tag{6.1}$$

where $\tilde{n}_{ij}^{c} = \left(\tilde{\nu}_{ij} | r_{\tilde{\nu}_{ij}}, \tilde{\eta}_{ij} | q_{\tilde{\eta}_{ij}}, \tilde{\mu}_{ij} | p_{\tilde{\mu}_{ij}}\right)$ is the complement of $\tilde{n}_{ij}$.

According to the above-mentioned assumptions, when the criteria of a specific MCDM problem are in same priority level and $w = (w_1, w_2, \ldots, w_m)^T$ is the weight vector of the criteria, we can develop a novel approach to solve it based on the GPPHFWA operator or the GPPHFWG operator, which involved the following steps.

Approach 1:

Step 1: Utilize Eq (6.1) to obtain the normalized probabilistic picture hesitant fuzzy decision matrix $\overline{N} = \left(\bar{n}_{ij}\right)_{m\times n}$.

Step 2: Utilize the GPPHFWA operator (Eq (5.2))

$$\bar{n}_j = \mathrm{GPPHFWA}(\bar{n}_{1j}, \bar{n}_{2j}, \ldots, \bar{n}_{mj}) = \left(\oplus_{i=1}^{m} w_i \bar{n}_{ij}^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$= \cup_{\alpha_{ij}\in\tilde{\mu}_{ij},\beta_{ij}\in\tilde{\eta}_{ij},\gamma_{ij}\in\tilde{\nu}_{ij}} \begin{pmatrix} \left\{\left(1-\prod_{i=1}^{m}(1-\alpha_{ij}^{\lambda})^{w_i}\right)^{\frac{1}{\lambda}} \,\middle|\, \prod_{i=1}^{m} p_{\alpha_{ij}}\right\}, \\ \left\{1-\left(1-\prod_{i=1}^{m}\left(1-(1-\beta_{ij})^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}} \,\middle|\, \prod_{i=1}^{m} q_{\beta_{ij}}\right\}, \\ \left\{1-\left(1-\prod_{i=1}^{m}\left(1-(1-\gamma_{ij})^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}} \,\middle|\, \prod_{i=1}^{m} r_{\gamma_{ij}}\right\} \end{pmatrix} \tag{6.2}$$

or the GPPHFWG operator (Eq (5.6))

$$\bar{n}_j = \mathrm{GPPHFWG}(\bar{n}_{1j}, \bar{n}_{2j}, \ldots, \bar{n}_{mj}) = \frac{1}{\lambda}\left(\otimes_{i=1}^{m} \lambda \bar{n}_{ij}^{w_i}\right)$$

$$= \cup_{\alpha_{ij}\in\tilde{\mu}_{ij},\beta_{ij}\in\tilde{\eta}_{ij},\gamma_{ij}\in\tilde{\nu}_{ij}} \begin{pmatrix} \left\{1-\left(1-\prod_{i=1}^{m}\left(1-(1-\alpha_{ij})^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}} \,\middle|\, \prod_{i=1}^{n} p_{\alpha_{ij}}\right\}, \\ \left\{\left(1-\prod_{i=1}^{m}(1-\beta_{ij}^{\lambda})^{w_i}\right)^{\frac{1}{\lambda}} \,\middle|\, \prod_{i=1}^{m} q_{\beta_{ij}}\right\}, \\ \left\{\left(1-\prod_{i=1}^{m}(1-\gamma_{ij}^{\lambda})^{w_i}\right)^{\frac{1}{\lambda}} \,\middle|\, \prod_{i=1}^{m} r_{\gamma_{ij}}\right\} \end{pmatrix} \tag{6.3}$$

to aggregate all the evaluation values $\bar{n}_{ij}$ $(i = 1, 2, \ldots, m)$ of the jth column and get the overall rating value $\bar{n}_j$ corresponding to the alternative x_j $(j = 1, 2, \ldots, n)$.

Step 3: Compute the score and accuracy values of each overall rating value $\bar{n}_j$ $(j = 1, 2, \ldots, n)$ using Definition 4.3.

Step 4: Use Definition 4.4 to rank the overall rating values $\bar{n}_j$ $(j = 1, 2, \ldots, n)$ in descending order, rank all the alternatives x_j $(j = 1, 2, \ldots, n)$ in accordance with $\bar{n}_j$ $(j = 1, 2, \ldots, n)$, and finally select the most desirable alternative(s).

Step 5: End.

When the criteria are in different priorities, we can develop another novel approach to solve the MCDM problem based on the GPPHFPWA operator or the GPPHFPWG operator, which involved the following steps.

Approach 2:

Step 1: Utilize Eq (6.1) to obtain the normalized probabilistic picture hesitant fuzzy decision matrix $\overline{N} = \left(\bar{n}_{ij}\right)_{m\times n}$.

Step 2: Compute the values of T_{ij} $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ using the equation as

$$T_{ij} = \prod_{k=1}^{i-1} s(\bar{n}_{ij}),\ T_{1j} = 1,\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n. \tag{6.4}$$

Step 3: Utilize the GPPHFPWA operator (Eq (5.10))

$$\bar{n}_j = \mathrm{GPPHFPWA}(\bar{n}_{1j}, \bar{n}_{2j}, \ldots, \bar{n}_{mj}) = \left(\oplus_{i=1}^{m} \frac{T_{ij}}{\sum_{i=1}^{m} T_{ij}} \bar{n}_{ij}^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$= \cup_{\alpha_{ij}\in\tilde{\mu}_{ij},\beta_{ij}\in\tilde{\eta}_{ij},\gamma_{ij}\in\tilde{\nu}_{ij}} \begin{pmatrix} \left\{\left(1-\prod_{i=1}^{m}(1-\alpha_{ij}^{\lambda})^{\frac{T_{ij}}{\sum_{i=1}^{m}T_{ij}}}\right)^{\frac{1}{\lambda}} \Big| \prod_{i=1}^{m} p_{\alpha_{ij}}\right\}, \\ \left\{1-\left(1-\prod_{i=1}^{m}\left(1-(1-\beta_{ij})^{\lambda}\right)^{\frac{T_{ij}}{\sum_{i=1}^{m}T_{ij}}}\right)^{\frac{1}{\lambda}} \Big| \prod_{i=1}^{m} q_{\beta_{ij}}\right\}, \\ \left\{1-\left(1-\prod_{i=1}^{m}\left(1-(1-\gamma_{ij})^{\lambda}\right)^{\frac{T_{ij}}{\sum_{i=1}^{m}T_{ij}}}\right)^{\frac{1}{\lambda}} \Big| \prod_{i=1}^{m} r_{\gamma_{ij}}\right\} \end{pmatrix} \tag{6.5}$$

or the GPPHFPWG operator (Eq (5.15))

$$\bar{n}_j = \text{GPPHFPWA}(\bar{n}_{1j}, \bar{n}_{2j}, \ldots, \bar{n}_{mj}) = \frac{1}{\lambda}\left(\otimes_{i=1}^{m}\lambda\bar{n}_i^{\frac{T_{ij}}{\sum_{i=1}^{m}T_{ij}}}\right)$$

$$= \cup_{\alpha_{ij}\in\tilde{\mu}_{ij},\beta_{ij}\in\tilde{\eta}_{ij},\gamma_{ij}\in\tilde{\nu}_{ij}} \begin{pmatrix} \left\{1-\left(1-\prod_{i=1}^{m}\left(1-(1-\alpha_{ij})^{\lambda}\right)^{\frac{T_{ij}}{\sum_{i=1}^{m}T_{ij}}}\right)^{\frac{1}{\lambda}} \Big| \prod_{i=1}^{m} p_{\alpha_{ij}}\right\}, \\ \left\{\left(1-\prod_{i=1}^{m}(1-\beta_{ij}^{\lambda})^{\frac{T_{ij}}{\sum_{i=1}^{m}T_{ij}}}\right)^{\frac{1}{\lambda}} \Big| \prod_{i=1}^{m} q_{\beta_{ij}}\right\}, \\ \left\{\left(1-\prod_{i=1}^{m}(1-\gamma_{ij}^{\lambda})^{\frac{T_{ij}}{\sum_{i=1}^{m}T_{ij}}}\right)^{\frac{1}{\lambda}} \Big| \prod_{i=1}^{m} r_{\gamma_{ij}}\right\} \end{pmatrix} \tag{6.6}$$

to aggregate all the evaluation values $\bar{n}_{ij}$ $(1 = 1, 2, \ldots, m)$ of the jth column and get the overall rating value $\bar{n}_j$ corresponding to the alternative x_j $(j = 1, 2, \ldots, n)$.

Step 4: Compute the score and accuracy values of each overall rating value $\bar{n}_j$ $(j = 1, 2, \ldots, n)$ using Definition 4.3.

Step 5: Use Definition 4.4 to rank the overall rating values $\bar{n}_j$ $(j = 1, 2, \ldots, n)$ in descending order, rank all the alternatives x_j $(j = 1, 2, \ldots, n)$ in accordance with $\bar{n}_j$ $(j = 1, 2, \ldots, n)$, and finally select the most desirable alternative(s).

Step 6: End.

7. Illustrative examples and analysis

For convenience of comparison, in this section we adopt the illustrative examples in [14, 27, 62].

7.1. Numerical examples

Example 7.1. Suppose that an organization wants to construct the enterprise resource planning (ERP) system [27,62]. After investigating the existing suppliers of ERP systems on the market, it was decided that five potential ERP systems x_j $(j = 1, 2, 3, 4, 5)$ would be selected first. DMs utilize P-PHFEs to evaluate the five alternatives x_j $(j = 1, 2, 3, 4, 5)$ with respect to four criteria, namely, function and technology (C_1), strategic fitness (C_2), ability of supplier (C_3), and reputation of supplier (C_4), the corresponding weight vector is $w = (0.2, 0.1, 0.3, 0.4)^T$, and construct their probabilistic picture hesitant fuzzy decision matrix $N = (\tilde{n}_{ij})_{4\times5}$ shown in Table 2.

Then we use the above proposed Approach 1 to choose the optimal EPR system which are presented as below.

Table 2. Probabilistic picture hesitant fuzzy decision matrix N.

	x_1	x_2	x_3	x_4	x_5
C_1	$(\{0.43 \mid 0.5, 0.53 \mid 0.5\}, \{0.33 \mid 1\}, \{0.06 \mid 0.4, 0.09 \mid 0.6\})$	$(\{0.53 \mid 0.3, 0.65 \mid 0.4, 0.73 \mid 0.3\}, \{0.10 \mid 0.4, 0.12 \mid 0.6\}, \{0.08 \mid 1\})$	$(\{0.72 \mid 0.3, 0.86 \mid 0.4, 0.91 \mid 0.3\}, \{0.03 \mid 1\}, \{0.02 \mid 1\})$	$(\{0.77 \mid 0.5, 0.85 \mid 0.5\}, \{0.09 \mid 1\}, \{0.05 \mid 1\})$	$(\{0.70 \mid 0.6, 0.81 \mid 0.2, 0.90 \mid 0.2\}, \{0.05 \mid 1\}, \{0.02 \mid 1\})$
C_2	$(\{0.76 \mid 0.3, 0.89 \mid 0.7\}, \{0.05 \mid 0.4, 0.08 \mid 0.6\}, \{0.03 \mid 1\})$	$(\{0.13 \mid 1\}, \{0.53 \mid 0.6, 0.64 \mid 0.4\}, \{0.12 \mid 0.6, 0.21 \mid 0.4\})$	$(\{0.07 \mid 1\}, \{0.05 \mid 0.4, 0.09 \mid 0.6\}, \{0.05 \mid 1\})$	$(\{0.65 \mid 0.6, 0.74 \mid 0.4\}, \{0.10 \mid 0.7, 0.16 \mid 0.3\}, \{0.10 \mid 1\})$	$(\{0.68 \mid 1\}, \{0.08 \mid 1\}, \{0.13 \mid 0.6, 0.21 \mid 0.4\})$
C_3	$(\{0.42 \mid 1\}, \{0.35 \mid 1\}, \{0.12 \mid 0.6, 0.18 \mid 0.4\})$	$(\{0.03 \mid 1\}, \{0.77 \mid 0.6, 0.82 \mid 0.4\}, \{0.10 \mid 0.6, 0.13 \mid 0.4\})$	$(\{0.04 \mid 1\}, \{0.65 \mid 0.4, 0.72 \mid 0.3, 0.85 \mid 0.3\}, \{0.05 \mid 0.7, 0.10 \mid 0.3\})$	$(\{0.02 \mid 1\}, \{0.78 \mid 0.4, 0.89 \mid 0.6\}, \{0.05 \mid 1\})$	$(\{0.05 \mid 1\}, \{0.77 \mid 0.6, 0.87 \mid 0.4\}, \{0.06 \mid 1\})$
C_4	$(\{0.08 \mid 1\}, \{0.75 \mid 0.4, 0.89 \mid 0.6\}, \{0.02 \mid 1\})$	$(\{0.58 \mid 0.7, 0.73 \mid 0.3\}, \{0.15 \mid 1\}, \{0.08 \mid 1\})$	$(\{0.45 \mid 0.7, 0.68 \mid 0.3\}, \{0.18 \mid 0.7, 0.26 \mid 0.3\}, \{0.06 \mid 1\})$	$(\{0.08 \mid 1\}, \{0.65 \mid 0.4, 0.84 \mid 0.6\}, \{0.06 \mid 1\})$	$(\{0.13 \mid 1\}, \{0.65 \mid 0.4, 0.75 \mid 0.6\}, \{0.09 \mid 1\})$

Step 1: Since all of these criteria are benefit types, the normalized probabilistic picture hesitant fuzzy decision matrix $\bar{N}$ is as the same as the probabilistic picture hesitant fuzzy decision matrix N.

Step 2: Utilize the GPPHFWA operator ($\lambda = 1$) (Eq (6.2)) to aggregate all the evaluation values $\bar{n}_{ij}$ ($i = 1, 2, 3, 4$) of the jth column and get the overall rating value $\bar{n}_j$ corresponding to the alternative x_j ($j = 1, 2, 3, 4, 5$):

$$\tilde{n}_1 = \left(\begin{array}{l} \{0.3636|0.15, 0.3877|0.35, 0.4113|0.15, 0.4336|0.35\}, \\ \{0.3862|0.16, 0.4048|0.24, 0.4136|0.24, 0.4335|0.36\}, \\ \{0.0444|0.24, 0.0482|0.36, 0.0502|0.16, 0.0544|0.24\} \end{array} \right),$$

$$\tilde{n}_2 = \left(\begin{array}{l} \{0.4061|0.21, 0.4401|028, 0.4684|0.28, 0.5023|0.09, 0.5308|0.12, 0.5545|0.12\}, \\ \left\{ \begin{array}{l} 0.2563|0.14, 0.2612|0.10, 0.2612|0.10, 0.2659|0.22, \\ 0.2662|0.06, 0.2709|0.14, 0.2709|0.14, 0.2761|0.10, \end{array} \right\}, \\ \{0.0891|0.36, 0.0942|0.24, 0.0964|0.24, 0.1019|0.16\} \end{array} \right),$$

$$\tilde{n}_3 = \left(\begin{array}{l} \{0.4014|0.21, 0.4789|0.28, 0.5180|0.09, 0.5230|0.21, 0.5804|0.12, 0.6159|0.09\}, \\ \left\{ \begin{array}{l} 0.1627|0.11, 0.1677|0.08, 0.1725|0.17, 0.1763|0.08, \\ 0.1779|0.13, 0.1870|0.13, 0.1884|0.05, 0.1943|0.04, \\ 0.1999|0.07, 0.2042|0.04, 0.2061|0.05, 0.2166|0.05 \end{array} \right\}, \\ \{0.0448|0.70, 0.0551|0.30\} \end{array} \right),$$

$$\tilde{n}_4 = \left(\begin{array}{l} \{0.3549|0.30, 0.3738|0.20, 0.4077|0.30, 0.4251|0.20\}, \\ \left\{ \begin{array}{l} 0.3834|0.14, 0.3989|0.21, 0.4028|0.05, 0.4181|0.07, \\ 0.4248|0.17, 0.4420|0.25, 0.4452|0.07, 0.5632|0.11 \end{array} \right\}, \\ \{0.0576|1\} \end{array} \right),$$

$$\tilde{n}_5 = \left(\begin{array}{l} \{0.3468|0.60, 0.4038|0.20, 0.4756|0.20, \}, \\ \{0.3321|0.24, 0.3444|0.16, 0.3516|0.36, 0.3647|0.24\}, \\ \{0.0612|0.60, 0.0642|0.40\} \end{array} \right).$$

Step 3: Compute the score values of each overall rating value $\tilde{n}_j$ ($j = 1, 2, 3, 4, 5$) using Definition 4.3:

$$s(\tilde{n}_1) = 0.4925, s(\tilde{n}_2) = 0.5090, s(\tilde{n}_3) = 0.5277, s(\tilde{n}_4) = 0.4915, s(\tilde{n}_5) = 0.5116.$$

Step 4: Utilize Definition 4.4 to rank the overall rating values $\tilde{n}_j$ ($j = 1, 2, 3, 4, 5$) in descending order: $\tilde{n}_3 > \tilde{n}_5 > \tilde{n}_2 > \tilde{n}_1 > \tilde{n}_4$. Then we rank all the alternatives x_j ($j = 1, 2, 3, 4, 5$) in accordance with $\tilde{n}_j$ ($j = 1, 2, 3, 4, 5$) as follows:

$$x_3 \succ x_5 \succ x_2 \succ x_1 \succ x_4.$$

Thus, the best alternative is x_3.

If the GPPHFWA operator is replaced by the GPPHFWG operator in the above Step 2, the procedure for selecting the best alternative(s) is as follows.

Step 1': see Step 1.

Step 2': Utilize the GPPHFWG operator ($\lambda = 1$) (Eq (6.3)) to aggregate all the evaluation values $\bar{n}_{ij}$ ($i = 1, 2, 3, 4$) of the jth column and get the overall rating value $\bar{n}_j$ corresponding to the alternative x_j

$(j = 1, 2, 3, 4, 5)$:

$$\tilde{n}_1 = \begin{pmatrix} \{0.2307|0.15, 0.2343|0.35, 0.2405|0.15, 0.2443|0.35\}, \\ \{0.5365|0.16, 0.5380|0.24, 0.6663|0.24, 0.6673|0.36\}, \\ \{0.0600|0.24, 0.0661|0.36, 0.0797|0.16, 0.0856|0.24\} \end{pmatrix},$$

$$\tilde{n}_2 = \begin{pmatrix} \{0.2017|0.21, 0.2101|0.09, 0.2151|0.28, 0.2212|0.12, 0.2304|0.21, 0.2358|0.09\}, \\ \left\{ \begin{array}{l} 0.4526|0.14, 0.4550|0.10, 0.4670|0.10, 0.46939|0.06, \\ 0.4914|0.22, 0.4936|0.14, 0.5047|0.14, 0.5070|0.10 \end{array} \right\}, \\ \{0.0901|0.36, 0.0993|0.24, 0.0999|0.24, 0.1090|0.16\} \end{pmatrix},$$

$$\tilde{n}_3 = \begin{pmatrix} \{0.1986|0.21, 0.2057|0.09, 0.2081|0.28, 0.2342|0.12, 0.2427|0.21, 0.2454|0.09\}, \\ \left\{ \begin{array}{l} 0.3334|0.11, 0.3363|0.05, 0.3602|0.08, 0.3630|0.04, \\ 0.3766|0.17, 0.3792|0.07, 0.4016|0.08, 0.4042|0.04, \\ 0.4830|0.13, 0.4852|0.05, 0.5038|0.13, 0.5059|0.05 \end{array} \right\}, \\ \{0.0481|0.70, 0.0634|0.30\} \end{pmatrix},$$

$$\tilde{n}_4 = \begin{pmatrix} \{0.1024|0.30, 0.1037|0.20, 0.1044|0.30, 0.1058|0.20\}, \\ \left\{ \begin{array}{l} 0.5949|0.11, 0.5977|0.17, 0.6709|0.17, 0.6732|0.25, \\ 0.7038|0.05, 0.7058|0.07, 0.7594|0.07, 0.7611|0.11 \end{array} \right\}, \\ \{0.0591|1\} \end{pmatrix},$$

$$\tilde{n}_5 = \begin{pmatrix} \{0.1613|0.60, 0.1660|0.20, 0.1696|0.20\}, \\ \{0.5850|0.24, 0.6372|0.36, 0.6503|0.16, 0.6943|0.24\}, \\ \{0.0716|0.60, 0.0805|0.40\} \end{pmatrix}.$$

Step 3': Compute the score values of each overall rating value $\tilde{n}_j$ $(j = 1, 2, 3, 4, 5)$ using Definition 4.3:

$$s(\tilde{n}_1) = 0.4439, s(\tilde{n}_2) = 0.4760, s(\tilde{n}_3) = 0.4881, s(\tilde{n}_4) = 0.4397, s(\tilde{n}_5) = 0.4285.$$

Step 4': Utilize Definition 4.4 to rank the overall rating values $\tilde{n}_j$ $(j = 1, 2, 3, 4, 5)$ in descending order: $\tilde{n}_3 > \tilde{n}_2 > \tilde{n}_1 > \tilde{n}_4 > \tilde{n}_5$. Then we rank all the alternatives x_j $(j = 1, 2, 3, 4, 5)$ in accordance with $\tilde{n}_j$ $(j = 1, 2, 3, 4, 5)$ as follows:

$$x_3 \succ x_2 \succ x_1 \succ x_4 \succ x_5.$$

Thus, the best alternative is x_3.

Example 7.2. Suppose that a university seeks to recruit excellent foreign professors to improve the teaching level and scientific research [14, 27]. There are five foreign professors selected by the human resources department at the university. According to priority level, the criteria for investigation are morality (C_1), research ability (C_2), teaching capacity (C_3), and educational experience (C_4), and there is a priority relationship $C_1 \succ C_2 \succ C_3 \succ C_4$ between the criteria. Then, the probabilistic picture hesitant fuzzy decision matrix $N = (\tilde{n}_{ij})_{4\times 5}$ is presented in Table 3.

Then we use the proposed Approach 2 to choose the optimal foreign professor(s) which are presented as below.

Step 1: Since all of these criteria are benefit types, the normalized probabilistic picture hesitant fuzzy decision matrix $\bar{N}$ is as the same as the probabilistic picture hesitant fuzzy decision matrix N.

Table 3. Probabilistic picture hesitant fuzzy decision matrix N.

	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$
C_1	$(\{0.40\mid 0.3, 0.50\mid 0.4, 0.70\mid 0.3\}, \{0.05\mid 1\}, \{0.10\mid 0.4, 0.20\mid 0.6\})$	$(\{0.65\mid 0.4, 0.75\mid 0.6\}, \{0.02\mid 0.4, 0.04\mid 0.6\}, \{0.15\mid 1\})$	$(\{0.70\mid 1\}, \{0.06\mid 0.4, 0.10\mid 0.6\}, C_1\ \{0.10\mid 0.4, 0.15\mid 0.6\})$	$(\{0.50\mid 0.5, 0.60\mid 0.3, 0.70\mid 0.2\}, \{0.08\mid 1\}, \{0.10\mid 1\})$	$(\{0.65\mid 1\}, \{0.05\mid 0.6, 0.10\mid 0.4\}, \{0.15\mid 0.4, 0.20\mid 0.6\})$
C_2	$(\{0.65\mid 1\}, \{0.05\mid 0.4, 0.08\mid 0.6\}, \{0.15\mid 1\})$	$(\{0.60\mid 1\}, \{0.05\mid 0.6, 0.10\mid 0.4\}, \{0.10\mid 0.6, 0.20\mid 0.4\})$	$(\{0.20\mid 0.3, 0.30\mid 0.4, 0.50\mid 0.3\}, \{0.04\mid 1\}, \{0.30\mid 0.6, 0.40\mid 0.4\})$	$(\{0.40\mid 0.6, 0.50\mid 0.4\}, \{0.20\mid 1\}, \{0.10\mid 0.6, 0.20\mid 0.4\})$	$(\{0.50\mid 0.7, 0.70\mid 0.3\}, \{0.08\mid 1\}, \{0.20\mid 1\})$
C_3	$(\{0.40\mid 0.4, 0.50\mid 0.3, 0.60\mid 0.3\}, \{0.03\mid 1\}, \{0.10\mid 0.6, 0.20\mid 0.4\})$	$(\{0.75\mid 0.6, 0.80\mid 0.4\}, \{0.06\mid 1\}, \{0.05\mid 0.6, 0.08\mid 0.4\})$	$(\{0.50\mid 1\}, \{0.03\mid 0.4, 0.06\mid 0.6\}, \{0.30\mid 0.7, 0.35\mid 0.3\})$	$(\{0.85\mid 1\}, \{0.03\mid 0.4, 0.07\mid 0.6\}, \{0.05\mid 1\})$	$(\{0.70\mid 0.6, 0.80\mid 0.4\}, \{0.04\mid 1\}, \{0.10\mid 1\})$
C_4	$(\{0.55\mid 1\}, \{0.10\mid 0.4, 0.15\mid 0.6\}, \{0.15\mid 1\})$	$(\{0.40\mid 0.7, 0.50\mid 0.3\}, \{0.20\mid 1\}, \{0.15\mid 0.5, 0.25\mid 0.5\})$	$(\{0.50\mid 0.7, 0.70\mid 0.3\}, \{0.10\mid 1\}, \{0.10\mid 1\})$	$(\{0.45\mid 1\}, \{0.10\mid 0.4, 0.20\mid 0.6\}, \{0.15\mid 0.3, 0.30\mid 0.7\})$	$(\{0.35\mid 1\}, \{0.10\mid 0.4, 0.20\mid 0.6\}, \{0.30\mid 0.6, 0.40\mid 0.4\})$

Step 2: Utilize Eq (6.4) to calculate the values of T_{ij} ($i = 1,2,3,4; j = 1,2,3,4,5$), which are contained in the matrix $T = (T_{ij})_{4\times5}$:

$$T = \begin{pmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0.3420 & 0.3888 & 0.3656 & 0.4022 & 0.4125 \\ 0.0448 & 0.0778 & 0.0443 & 0.0968 & 0.0588 \\ 0.1125 & 0.0878 & 0.1231 & 0.0869 & 0.1400 \end{pmatrix}.$$

Step 3: Utilize the GPPHFPWA operator ($\lambda = 1$) (Eq (6.5)) to aggregate all the evaluation values $\bar{n}_{ij}$ ($i = 1,2,3,4$) of the jth column and get the overall rating value $\tilde{n}_j$ corresponding to the alternative x_j ($j = 1,2,3,4,5$):

$$\tilde{n}_1 = \begin{pmatrix} \left\{ \begin{array}{l} 0.4807|0.12, 0.4836|0.09, 0.4870|0.09, 0.5402|0.16, \\ 0.5427|0.09, 0.5457|0.12, 0.6730|0.12, 0.6747|0.09, 0.6769|0.09 \end{array} \right\}, \\ \{0.0519|0.16, 0.0535|0.24, 0.0577|0.24, 0.0595|0.36\}, \\ \{0.1131|0.24, 0.1154|0.16, 0.1795|0.36, 0.1833|0.24\} \end{pmatrix},$$

$$\tilde{n}_2 = \begin{pmatrix} \left\{ \begin{array}{l} 0.6332|0.17, 0.6369|0.07, 0.6372|0.11, 0.6409|0.05, \\ 0.7046|0.25, 0.7076|0.11, 0.7078|0.17, 0.7108|0.07 \end{array} \right\}, \\ \{0.0303|0.24, 0.0360|0.16, 0.0473|0.36, 0.0562|0.24\}, \\ \left\{ \begin{array}{l} 0.1283|0.18, 0.1313|0.12, 0.1320|0.18, 0.1352|0.12, \\ 0.1526|0.12, 0.1562|0.08, 0.1570|0.12, 0.1608|0.08 \end{array} \right\} \end{pmatrix},$$

$$\tilde{n}_3 = \begin{pmatrix} \{0.5992|0.21, 0.6118|0.28, 0.6153|0.09, 0.6274|0.12, 0.6417|0.21, 0.6561|0.09\}, \\ \{0.0556|0.16, 0.0568|0.24, 0.0776|0.24, 0.0792|0.36\}, \\ \left\{ \begin{array}{l} 0.1341|0.17, 0.1347|0.07, 0.1437|0.11, 0.1443|0.05, \\ 0.1748|0.25, 0.17550.11, 0.1872|0.17, 0.1880|0.07 \end{array} \right\} \end{pmatrix},$$

$$\tilde{n}_4 = \begin{pmatrix} \{0.5109|0.30, 0.5330|0.20, 0.5751|0.18, 0.5943|0.12, 0.6456|0.12, 0.6616|0.08\}, \\ \{0.0962|0.16, 0.1001|0.24, 0.1013|0.24, 0.1053|0.36\}, \\ \{0.0980|0.18, 0.1018|0.42, 0.11690|0.12, 0.1214|0.28\} \end{pmatrix},$$

$$\tilde{n}_5 = \begin{pmatrix} \{0.5976|0.42, 0.6035|0.28, 0.6469|0.18, 0.6521|0.12\}, \\ \{0.0594|0.24, 0.0631|0.36, 0.0913|0.16, 0.0970|0.24\}, \\ \{0.1690|0.24, 0.1732|0.16, 0.2020|0.36, 0.2071|0.24\} \end{pmatrix}.$$

Step 4: Compute the score values of each overall rating value $\tilde{n}_j$ ($j = 1,2,3,4,5$) using Definition 4.3:

$$s(\tilde{n}_1) = 0.5050, s(\tilde{n}_2) = 0.5271, s(\tilde{n}_3) = 0.5329, s(\tilde{n}_4) = 0.5208, s(\tilde{n}_5) = 0.5436.$$

Step 5: Utilize Definition 4.4 to rank the overall rating values $\tilde{n}_j$ ($j = 1,2,3,4,5$) in descending order: $\tilde{n}_5 > \tilde{n}_3 > \tilde{n}_2 > \tilde{n}_4 > \tilde{n}_1$. Then we rank all the alternatives x_j ($j = 1,2,3,4,5$) in accordance with $\tilde{n}_j$ ($j = 1,2,3,4,5$) as follows:

$$x_5 \succ x_3 \succ x_2 \succ x_4 \succ x_1.$$

Thus, the best alternative is x_5.

If the GPPHFPWA operator is replaced by the GPPHFPWG operator in the above Step 3, the procedure for selecting the best alternative(s) is as follows.

Step 1': see Step 1.

Step 2': see Step 2.

Step 3': Utilize the GPPHFPWG operator ($\lambda = 1$) (Eq (6.6)) to aggregate all the evaluation values $\tilde{n}_{ij}$ ($i = 1, 2, 3, 4$) of the jth column and get the overall rating value $\tilde{n}_j$ corresponding to the alternative x_j ($j = 1, 2, 3, 4, 5$):

$$\tilde{n}_1 = \left(\begin{array}{l} \left\{ \begin{array}{l} 0.4577|0.12, 0.4607|0.09, 0.4632|0.09, 0.5311|0.16, \\ 0.5346|0.12, 0.5376|0.12, 0.6647|0.12, 0.6692|0.09, 0.6728|0.09 \end{array} \right\}, \\ \{0.0533|0.16, 0.0573|0.24, 0.0602|0.24, 0.0642|0.36\}, \\ \{0.1155|0.24, 0.1186|0.16, 0.1823|0.36, 0.1852|0.24\} \end{array} \right),$$

$$\tilde{n}_2 = \left(\begin{array}{l} \left\{ \begin{array}{l} 0.6243|0.17, 0.6264|0.11, 0.6323|0.07, 0.0.6343|0.05, \\ 0.6846|0.25, 0.6868|0.17, 0.6932|0.11, 0.6955|0.07 \end{array} \right\}, \\ \{0.0407|0.24, 0.0533|0.36, 0.0536|0.16, 0.0660|0.24\}, \\ \left\{ \begin{array}{l} 0.1329|0.18, 0.1343|0.12, 0.1391|0.18, 0.1404|0.12, \\ 0.1581|0.12, 0.1595|0.08, 0.1640|0.12, 0.1654|0.08 \end{array} \right\} \end{array} \right),$$

$$\tilde{n}_3 = \left(\begin{array}{l} \{0.5005|0.21, 0.5142|0.09, 0.5513|0.28, 0.5664|0.12, 0.6227|0.21, 0.6398|0.09\}, \\ \{0.0577|0.16, 0.0586|0.24, 0.0841|0.24, 0.0849|0.36\}, \\ \left\{ \begin{array}{l} 0.1585|0.17, 0.1603|0.07, 0.1889|0.11, 0.1893|0.25, \\ 0.1906|0.05, 0.1910|0.11, 0.2185|0.17, 0.2202|0.07 \end{array} \right\} \end{array} \right),$$

$$\tilde{n}_4 = \left(\begin{array}{l} \{0.4852|0.30, 0.5135|0.20, 0.5443|0.18, 0.5760|0.12, 0.5999|0.12, 0.6348|0.08\}, \\ \{0.1102|0.16, 0.1125|0.24, 0.1160|0.24, 0.1182|0.36\}, \\ \{0.0998|0.18, 0.1094|0.42, 0.1263|0.12, 0.1356|0.28\} \end{array} \right),$$

$$\tilde{n}_5 = \left(\begin{array}{l} \{0.5775|0.42, 0.5803|0.28, 0.6295|0.18, 0.6325|0.12\}, \\ \{0.0618|0.24, 0.0714|0.36, 0.0928|0.16, 0.1020|0.24\}, \\ \{0.1754|0.24, 0.1863|0.16, 0.2058|0.36, 0.2164|0.24\} \end{array} \right).$$

Step 4': Compute the score values of each overall rating value $\tilde{n}_j$ ($j = 1, 2, 3, 4, 5$) using Definition 4.3:

$$s(\tilde{n}_1) = 0.5036, s(\tilde{n}_2) = 0.5245, s(\tilde{n}_3) = 0.5258, s(\tilde{n}_4) = 0.5158, s(\tilde{n}_5) = 0.5396.$$

Step 5': Utilize Definition 4.4 to rank the overall rating values $\tilde{n}_j$ ($j = 1, 2, 3, 4, 5$) in descending order: $\tilde{n}_5 > \tilde{n}_3 > \tilde{n}_2 > \tilde{n}_4 > \tilde{n}_1$. Then we rank all the alternatives x_j ($j = 1, 2, 3, 4, 5$) in accordance with $\tilde{n}_j$ ($j = 1, 2, 3, 4, 5$) as follows:

$$x_5 \succ x_3 \succ x_2 \succ x_4 \succ x_1.$$

Thus, the best alternative is x_5.

7.2. Analysis

To explain how the different parameter value λ plays a role in the aggregation operator, different possible values of λ, such as $0.1, 0.3, 0.5, 3, 5, 7, 10$, are used in the approaches of above two examples. Then, the corresponding score values and rankings of the alternatives are presented in Tables 4–7.

Table 4. Score values obtained by the GPPHFWA operator and rankings of alternatives in Example 4.

Parameter Value	$s(\tilde{n}_1)$	$s(\tilde{n}_2)$	$s(\tilde{n}_3)$	$s(\tilde{n}_4)$	$s(\tilde{n}_5)$	Ranking
$\lambda = 0.1$	0.4788	0.5020	0.5138	0.4712	0.4807	$x_3 > x_2 > x_5 > x_1 > x_4$
$\lambda = 0.3$	0.4819	0.5041	0.5158	0.4761	0.4860	$x_3 > x_2 > x_5 > x_1 > x_4$
$\lambda = 0.5$	0.4850	0.5061	0.5178	0.4812	0.4914	$x_3 > x_2 > x_5 > x_1 > x_4$
$\lambda = 1$	0.4925	0.5090	0.5277	0.4915	0.5116	$x_3 > x_5 > x_2 > x_1 > x_4$
$\lambda = 3$	0.5155	0.5199	0.5322	0.5222	0.5421	$x_5 > x_3 > x_4 > x_2 > x_1$
$\lambda = 5$	0.5299	0.5241	0.5378	0.5345	0.5598	$x_5 > x_3 > x_4 > x_1 > x_2$
$\lambda = 7$	0.5402	0.5264	0.5407	0.5413	0.5696	$x_5 > x_4 > x_3 > x_1 > x_2$
$\lambda = 10$	0.5511	0.5287	0.5441	0.5473	0.5783	$x_5 > x_1 > x_4 > x_3 > x_2$

Table 5. Score values obtained by the GPPHFWG operator and rankings of alternatives in Example 4.

Parameter Value	$s(\tilde{n}_1)$	$s(\tilde{n}_2)$	$s(\tilde{n}_3)$	$s(\tilde{n}_4)$	$s(\tilde{n}_5)$	Ranking
$\lambda = 0.1$	0.4528	0.4832	0.4953	0.4448	0.4404	$x_3 > x_2 > x_1 > x_4 > x_5$
$\lambda = 0.3$	0.4507	0.4816	0.4936	0.4436	0.4375	$x_3 > x_2 > x_1 > x_4 > x_5$
$\lambda = 0.5$	0.4485	0.4800	0.4920	0.4424	0.4347	$x_3 > x_2 > x_1 > x_4 > x_5$
$\lambda = 1$	0.4439	0.4760	0.4881	0.4397	0.4285	$x_3 > x_2 > x_1 > x_4 > x_5$
$\lambda = 3$	0.4275	0.4637	0.4774	0.4323	0.4125	$x_3 > x_2 > x_4 > x_1 > x_5$
$\lambda = 5$	0.4180	0.4566	0.4717	0.4271	0.4036	$x_3 > x_2 > x_4 > x_1 > x_5$
$\lambda = 7$	0.4120	0.4521	0.4684	0.4229	0.3977	$x_3 > x_2 > x_4 > x_1 > x_5$
$\lambda = 10$	0.4063	0.4479	0.4654	0.4182	0.3920	$x_3 > x_2 > x_4 > x_1 > x_5$

Table 6. Score values obtained by the GPPHFPWA operator and rankings of alternatives in Example 5.

Parameter Value	$s(\tilde{n}_1)$	$s(\tilde{n}_2)$	$s(\tilde{n}_3)$	$s(\tilde{n}_4)$	$s(\tilde{n}_5)$	Ranking
$\lambda = 0.1$	0.5047	0.5268	0.5318	0.5202	0.5431	$x_5 > x_3 > x_2 > x_4 > x_1$
$\lambda = 0.3$	0.5048	0.5269	0.5320	0.5203	0.5432	$x_5 > x_3 > x_2 > x_4 > x_1$
$\lambda = 0.5$	0.5049	0.5270	0.5323	0.5205	0.5433	$x_5 > x_3 > x_2 > x_4 > x_1$
$\lambda = 1$	0.5050	0.5271	0.5329	0.5208	0.5436	$x_5 > x_3 > x_2 > x_4 > x_1$
$\lambda = 3$	0.5056	0.5275	0.5349	0.5226	0.5449	$x_5 > x_3 > x_2 > x_4 > x_1$
$\lambda = 5$	0.5062	0.5279	0.5364	0.5248	0.5461	$x_5 > x_3 > x_2 > x_4 > x_1$
$\lambda = 7$	0.5068	0.5283	0.5374	0.5272	0.5472	$x_5 > x_3 > x_2 > x_4 > x_1$
$\lambda = 10$	0.5077	0.5288	0.5385	0.5307	0.5485	$x_5 > x_3 > x_4 > x_2 > x_1$

Table 7. Score values obtained by the GPPHFPWG operator and rankings of alternatives in Example 5.

Parameter Value	$s(\tilde{n}_1)$	$s(\tilde{n}_2)$	$s(\tilde{n}_3)$	$s(\tilde{n}_4)$	$s(\tilde{n}_5)$	**Ranking**
$\lambda = 0.1$	0.5042	0.5270	0.5284	0.5180	0.5412	$x_5 \succ x_3 \succ x_2 \succ x_4 \succ x_1$
$\lambda = 0.3$	0.5041	0.5268	0.5279	0.5176	0.5409	$x_5 \succ x_3 \succ x_2 \succ x_4 \succ x_1$
$\lambda = 0.5$	0.5040	0.5254	0.5273	0.5171	0.5406	$x_5 \succ x_3 \succ x_2 \succ x_4 \succ x_1$
$\lambda = 1$	0.5036	0.5245	0.5258	0.5158	0.5396	$x_5 \succ x_3 \succ x_2 \succ x_4 \succ x_1$
$\lambda = 3$	0.5014	0.5202	0.5185	0.5091	0.5336	$x_5 \succ x_2 \succ x_3 \succ x_4 \succ x_1$
$\lambda = 5$	0.4989	0.5146	0.5126	0.5027	0.5260	$x_5 \succ x_2 \succ x_3 \succ x_4 \succ x_1$
$\lambda = 7$	0.4968	0.5102	0.5085	0.4980	0.5189	$x_5 \succ x_2 \succ x_3 \succ x_4 \succ x_1$
$\lambda = 10$	0.4945	0.5056	0.5047	0.4935	0.5108	$x_5 \succ x_2 \succ x_3 \succ x_1 \succ x_4$

Tables 4 and 5 show that when we utilize the GPPHFWA operator to aggregate evaluation values, the best EPR system in Example 7.1 is x_3 for $0.1 \leq \lambda \leq 1$, but x_5 for $3 \leq \lambda \leq 10$; however, when the GPPHFWG operator is utilized, the best alternative is x_3. From Tables 6 and 7, we can see that the best foreign professor in Example 7.2 using each of the GPPHFPWA and GPPHFPWG operators is the same as x_5, but there are some differences between ranking results with respect to different values of λ. In addition, the score values of all alternatives depend on different values of λ because the aggregation process of the proposed operators has changed. The following results are obtained from Tables 4–7:

(1) In Example 7.1, the score values of each alternative obtained by the GPPHFWA operator are greater than those obtained by the GPPHFWG operator, and the difference between them increases along with the increasing of λ. For GPPHFWA operator, the choice of parameter value has a greater impact on the score values and the ranking of alternatives, whereas for GPPHFWG operator, parameter values slightly affect the final ranking of alternatives. It means that the GPPHFWA operator reflects the optimistic attitudes of DMs, while the GPPHFWG operator reflects the pessimistic attitudes of DMs. Furthermore, the level of optimism and pessimism are greater with the bigger value of λ.

(2) In Example 7.2, the score values of each alternative obtained by the GPPHFPWA operator are greater than those obtained by the GPPHFPWG operator, but the score values by the two operators are relatively stable when the different values of λ are used. That is, the parameter value cannot reflect the attitude of DMs. In addition, when the value of λ is relatively high, the ranking obtained by the GPPHFPWA and GPPHFPWG operators are slightly changed, but the best alternative is always the same. This means that the ranking obtained by the GPPHFWA operator is more affected by the parameter λ than by the GPPHFPWA operator, but the rankings obtained by the GPPHFWG and GPPHFPWG operators are relatively unaffected by the parameter λ.

The aforementioned analysis results show that the value of λ plays a significant role in using the GPPHFWA operator rather than using other operators in MCDM problems, especially when the value of λ is relatively high. The value of λ can be determined according to DMs' preference to obtain different ranking results and thus the proposed methods are very flexible in dealing with various situations in real MCDM problems.

7.3. Comparison

To demonstrate the feasibility of the proposed MCDM methods, we compare the rankings of Example 7.1 with those obtained by existing MCDM methods, including the PFWA and PFWG operators [62], the picture fuzzy cross-entropy [63], the PFHWA operator [64], the PFDWHM operator [65], and the GPHFWA and GPHFWG operators [27] as shown in Table 8. Furthermore, the comparison of Example 7.2 between the GPPHFPWA and GPPHFPWG operators and existing operators, including the HFPWA and HFPWG operators [14] and the GPHFPWA and GPHFPWG operators [27], is shown in Table 9.

Table 8. Comparison result of Example 4.

Method	Use Tool	Ranking
Wei [62]	PFWA operator	$x_3 \succ x_2 \succ x_1 \succ x_5 \succ x_4$
	PFWG operator	$x_3 \succ x_1 \succ x_2 \succ x_5 \succ x_4$
Wei [63]	Picture fuzzy cross-entropy	$x_3 \succ x_1 \succ x_2 \succ x_5 \succ x_4$
Wei [64]	PFHWA operator ($\lambda = 2$)	$x_3 \succ x_2 \succ x_5 \succ x_1 \succ x_4$
Zang et al. [65]	FDWHM operator ($\lambda = 2, p, q = 1$)	$x_3 \succ x_5 \succ x_4 \succ x_1 \succ x_2$
Wang and Li [27]	GPHFWA operator ($\lambda = 1$)	$x_3 \succ x_2 \succ x_5 \succ x_1 \succ x_4$
	GPHFWG operator ($\lambda = 1$)	$x_3 \succ x_2 \succ x_1 \succ x_5 \succ x_4$
Our Approach 1	GPPHFWA operator ($\lambda = 1$)	$x_3 \succ x_5 \succ x_2 \succ x_1 \succ x_4$
	GPPHFWG operator ($\lambda = 1$)	$x_3 \succ x_2 \succ x_1 \succ x_4 \succ x_5$

Table 9. Comparison result of Example 5.

Method	Use Tool	Ranking
Wei [14]	HFPWA operator	$x_5 \succ x_2 \succ x_1 \succ x_4 \succ x_3$
	HFPWG operator	$x_2 \succ x_5 \succ x_1 \succ x_4 \succ x_3$
Wang and Li [27]	GPHFPWA operator ($\lambda = 1$)	$x_2 \succ x_4 \succ x_1 \succ x_5 \succ x_3$
	GPHFPWG operator ($\lambda = 1$)	$x_2 \succ x_1 \succ x_5 \succ x_4 \succ x_3$
Our Approach 2	GPPHFPWA operator ($\lambda = 1$)	$x_5 \succ x_3 \succ x_2 \succ x_4 \succ x_1$
	GPPHFPWG operator ($\lambda = 1$)	$x_5 \succ x_3 \succ x_2 \succ x_4 \succ x_1$

Table 8 shows that the best alternative of Example 7.1 obtained by MCDM methods based on the GPPHFWA and GPPHFWG operators is the same as the existing methods, but there are some differences in the ranking order of alternatives. So, the results can demonstrate the feasibility of the proposed method. Compared with PFSs and PHFSs, P-PHFSs can describe DM's evaluation values more accurately and comprehensively, as it denotes not only the positive membership degrees, neutral-membership degrees and negative membership degrees, but also the corresponding probabilistic information. For example, the evaluation information of alternative x_1 for criterion C_1 provided by DM is represented as picture fuzzy number (PFN) $(0.53, 0.33, 0.09)$ [63-66] or PHFE $(\{0.43, 0.53\}, \{0.33\}, \{0.06, 0.09\})$ [27]. In fact, DM may doubt to determine an exact value for each membership level, or may doubt that the importance of several values for each membership level is the same. Then, PFN or PHFE cannot deal with this situation, but P-PHFE can be used to express the evaluation information as $(\{0.43|0.5, 0.53|0.5\}, \{0.33|1\}, \{0.06|0.4, 0.09|0.6\})$ as shown in Table 2. As a result, the proposed method can solve the MCDM problem when DMs feel difficulty to determine the exact values of each membership level, or the importance of several values at each membership level. However, if the number of criteria is relatively large, the aggregation process of the proposed operators becomes more complex than the existing methods [49,63–66] and the data size becomes relatively large. This is the limitation of the proposed method. In Table 9, the best alternative to Example 7.2 obtained by the GPPHFPWA and GPPHFPWG operators is x_5, but the results of other existing MCDM methods [14, 27] are x_2 and x_5, respectively. This is because the difference arises by ignoring the complex evaluation information of DMs in the MCDM methods using the GPHFPWA and GPHFPWG operators or the HFPWA and HFPWG operators. HFSs [14] allows DMs to provide several values of positive membership level, and PHFSs allows DMs to provide several values of negative and neutral membership level as well as positive membership level. For example, the evaluation information of alternative x_1 for criterion C_1 provided by DM is expressed as HFE $\{0.4, 0.5, 0.7\}$ [14], or PHFE $(\{0.40, 0.50, 0.70\}, \{0.05\}, \{0.10, 0.20\})$ [27]. However, in some situations, it is less convincing to express the evaluation information of DMs that only consider positive membership level or consider negative and neutral membership levels as well as positive membership level. This is because it does not take into account the importance of values at each membership level. Thus, the proposed method can overcome the limitations of HFSs and PHFSs. It is noteworthy that the GPPHFPWA and GPPHFPWG operators also have the same disadvantage as the GPPHFWA and GPPHFWG operators.

By the above comparison results, our proposed methods have two advantages:

(1) Since our proposed operators are reduced to other specific aggregation operators by varying the value of λ, it is very flexible to deal with MCDM problems in several different situations. Furthermore, the parameter λ can also be considered a measure of DMs' optimistic and pessimistic attitudes in the information fusion process of the GPPHFWA and GPPHFWG operators (or, the GPPHFPWA and GPPHFPWG operators), so that the value of λ can be determined by DMs according to their actual preferences.

(2) Since the weight values of criteria play important role in MCDM problems, the proposed methods can deal with cases where the criteria have the same or different priorities in the MCDM process. When the criteria have the same priority level, the proposed method based on the GPPHFWA and GPPHFWG operators with weighted vectors of the criteria can used to determine the ranking of alternatives. On the other hand, when the criteria have different priority levels, the GPPHFPWA and GPPHFPWG operators can be used to determine the ranking of alternatives. Thus, the proposed different aggregation operators can be used to handle different situations of MCDM problems. It can be seen that our proposed methods are more powerful and flexible than existing MCDM methods.

8. Conclusions

In this paper, we presented two noble MCDM methods that can be used to effectively solve practical decision-making problems. To achieve that, the main contributions of this paper are as follows. Firstly, we proposed a novel representation tool, called P-PHFSs, to describe DMs' complex evaluation information more accurately and effectively. Compared with PHFSs and P-HFSs, the proposed P-PHFSs handle DMs' fuzzy judgments more effectively because they not only describe the positive membership degree, neutral membership degree and negative membership degree, but also depict their corresponding probabilistic information. Due to this advantage, in the framework of P-PHFSs, DMs can fully express their evaluations, which can reduce information loss. Secondly, a series of aggregation operators of P-PHFEs were developed, which are useful to aggregate criteria values given in the form of probabilistic picture hesitant fuzzy information. Our proposed aggregation operators are more useful because they not only reduce the negative impact of DMs' excessively high or low evaluation values on the final decision results, but also reflects the interrelationship among various criteria. Thirdly, two new MCDM methods were derived to help DMs choose the optimal alternative(s). Numerical examples have clearly illustrated the effectiveness of our methods. From comparative analysis, the superiority of our proposed methods is that they not only provide DMs with much freedom to express their decision-making information, but also yield reasonable and reliable decision-making results. These characteristics make our methods more suitable to handle real-world MCDM problems.

In future research, we will use Einstein t-norm and t-conorm, or Hamacher t-norm and t-conorm to investigate other operations of P-PHFSs and develop different aggregation operators to aggregate P-PHFSs. Furthermore, we will study consensus-building processes based on P-PHFSs because the consensus reaching process [66–68], which is aimed at a decision-making result that are acceptable to many experts, is an important part of group decision-making.

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Conflict of interest

The authors declare no conflict of interest.

Appendix A

Proof. Since (1)–(4) and (7) are trivial, we prove (5) and (6).
(5)

$$
\begin{aligned}
&\lambda(\tilde{n}_1 \oplus \tilde{n}_2) \\
&= \lambda\left(\cup_{\substack{\alpha_1\in\tilde{\mu}_1,\beta_1\in\tilde{\eta}_1,\gamma_1\in\tilde{v}_1,\\ \alpha_2\in\tilde{\mu}_2,\beta_2\in\tilde{\eta}_2,\gamma_2\in\tilde{v}_2}} \left(\{\alpha_1+\alpha_2-\alpha_1\alpha_2|p_{\alpha_1}p_{\alpha_2}\},\{\beta_1\beta_2|q_{\beta_1}q_{\beta_2}\},\{\gamma_1\gamma_2|r_{\gamma_1}r_{\gamma_2}\}\right)\right) \\
&= \cup_{\substack{\alpha_1\in\tilde{\mu}_1,\beta_1\in\tilde{\eta}_1,\gamma_1\in\tilde{v}_1,\\ \alpha_2\in\tilde{\mu}_2,\beta_2\in\tilde{\eta}_2,\gamma_2\in\tilde{v}_2}} \left(\left\{1-(1-(\alpha_1+\alpha_2-\alpha_1\alpha_2))^{\lambda}\,|p_{\alpha_1}p_{\alpha_2}\right\},\{(\beta_1\beta_2)^{\lambda}\,|q_{\beta_1}q_{\beta_2}\},\{(\gamma_1\gamma_2)^{\lambda}\,|r_{\gamma_1}r_{\gamma_2}\}\right) \\
&= \cup_{\substack{\alpha_1\in\tilde{\mu}_1,\beta_1\in\tilde{\eta}_1,\gamma_1\in\tilde{v}_1,\\ \alpha_2\in\tilde{\mu}_2,\beta_2\in\tilde{\eta}_2,\gamma_2\in\tilde{v}_2}} \left(\left\{1-\left((1-\alpha_1)^{\lambda}(1-\alpha_2)^{\lambda}\right)|p_{\alpha_1}p_{\alpha_2}\right\},\{(\beta_1\beta_2)^{\lambda}\,|q_{\beta_1}q_{\beta_2}\},\{(\gamma_1\gamma_2)^{\lambda}\,|r_{\gamma_1}r_{\gamma_2}\}\right) \\
&= \cup_{\substack{\alpha_1\in\tilde{\mu}_1,\beta_1\in\tilde{\eta}_1,\gamma_1\in\tilde{v}_1,\\ \alpha_2\in\tilde{\mu}_2,\beta_2\in\tilde{\eta}_2,\gamma_2\in\tilde{v}_2}} \left(\left\{\begin{array}{l} 1-(1-\alpha_1)^{\lambda}+1-(1-\alpha_2)^{\lambda} \\ -(1-(1-\alpha_1)^{\lambda})(1-(1-\alpha_2)^{\lambda}) \end{array} |p_{\alpha_1}p_{\alpha_2}\right\},\{\beta_1^{\lambda}\beta_2^{\lambda}|q_{\beta_1}q_{\beta_2}\},\{\gamma_1^{\lambda}\gamma_2^{\lambda}|r_{\gamma_1}r_{\gamma_2}\}\right) \\
&= \lambda\tilde{n}_1 \oplus \lambda\tilde{n}_2;
\end{aligned}
$$

(6)

$$(\tilde{n}_1 \otimes \tilde{n}_2)^{\lambda}$$

$$
\begin{aligned}
&= \left(\cup_{\substack{\alpha_1 \in \tilde{\mu}_1, \beta_1 \in \tilde{\eta}_1, \gamma_1 \in \tilde{\nu}_1, \\ \alpha_2 \in \tilde{\mu}_2, \beta_2 \in \tilde{\eta}_2, \gamma_2 \in \tilde{\nu}_2}} \left(\{\alpha_1 \alpha_2 | p_{\alpha_1} p_{\alpha_2}\}, \{\beta_1 + \beta_2 - \beta_1 \beta_2 | q_{\beta_1} q_{\beta_2}\}, \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2 | r_{\gamma_1} r_{\gamma_2}\} \right) \right)^{\lambda} \\
&= \cup_{\substack{\alpha_1 \in \tilde{\mu}_1, \beta_1 \in \tilde{\eta}_1, \gamma_1 \in \tilde{\nu}_1, \\ \alpha_2 \in \tilde{\mu}_2, \beta_2 \in \tilde{\eta}_2, \gamma_2 \in \tilde{\nu}_2}} \left(\{(\alpha_1 \alpha_2)^{\lambda} | p_{\alpha_1} p_{\alpha_2}\}, \left\{ 1 - (1 - (\beta_1 + \beta_2 - \beta_1 \beta_2))^{\lambda} | q_{\beta_1} q_{\beta_2} \right\}, \right. \\
&\qquad \left. \left\{ 1 - (1 - (\gamma_1 + \gamma_2 - \gamma_1 \gamma_2))^{\lambda} | r_{\gamma_1} r_{\gamma_2} \right\} \right) \\
&= \cup_{\substack{\alpha_1 \in \tilde{\mu}_1, \beta_1 \in \tilde{\eta}_1, \gamma_1 \in \tilde{\nu}_1, \\ \alpha_2 \in \tilde{\mu}_2, \beta_2 \in \tilde{\eta}_2, \gamma_2 \in \tilde{\nu}_2}} \left(\{(\alpha_1 \alpha_2)^{\lambda} | p_{\alpha_1} p_{\alpha_2}\}, \left\{ 1 - \left((1 - \beta_1)^{\lambda} (1 - \beta_2)^{\lambda} \right) | q_{\beta_1} q_{\beta_2} \right\}, \right. \\
&\qquad \left. \left\{ 1 - \left((1 - \gamma_1)^{\lambda} (1 - \gamma_2)^{\lambda} \right) | r_{\gamma_1} r_{\gamma_2} \right\} \right) \\
&= \cup_{\substack{\alpha_1 \in \tilde{\mu}_1, \beta_1 \in \tilde{\eta}_1, \gamma_1 \in \tilde{\nu}_1, \\ \alpha_2 \in \tilde{\mu}_2, \beta_2 \in \tilde{\eta}_2, \gamma_2 \in \tilde{\nu}_2}} \left(\{(\alpha_1 \alpha_2)^{\lambda} | p_{\alpha_1} p_{\alpha_2}\}, \left\{ \begin{array}{l} 1 - (1 - \beta_1)^{\lambda} + 1 - (1 - \beta_2)^{\lambda} \\ -(1 - (1 - \beta_1)^{\lambda})(1 - (1 - \beta_2)^{\lambda}) \end{array} | q_{\beta_1} q_{\beta_2} \right\}, \right. \\
&\qquad \left. \left\{ \begin{array}{l} 1 - (1 - \beta_1)^{\lambda} + 1 - (1 - \beta_2)^{\lambda} \\ -(1 - (1 - \beta_1)^{\lambda})(1 - (1 - \beta_2)^{\lambda}) \end{array} | q_{\beta_1} q_{\beta_2} \right\} \right) \\
&= \tilde{n}_1^{\lambda} \otimes \tilde{n}_2^{\lambda},
\end{aligned}
$$

which complete the proof.

Appendix B

Proof. Equation (5.2) can be proved by mathematical induction on n as follows.

For $n = 2$, let $\tilde{n}_1 = \left(\tilde{\mu}_1 | p_{\tilde{\mu}_1}, \tilde{\eta}_1 | q_{\tilde{\eta}_1}, \tilde{\nu}_1 | r_{\tilde{\nu}_1} \right)$ and $\tilde{n}_2 = \left(\tilde{\mu}_2 | p_{\tilde{\mu}_2}, \tilde{\eta}_2 | q_{\tilde{\eta}_2}, \tilde{\nu}_2 | r_{\tilde{\nu}_2} \right)$, using operational laws in Definition 4.7, since

$$
\begin{aligned}
\tilde{n}_1^{\lambda} &= \cup_{\alpha_1 \in \tilde{\mu}_1, \beta_1 \in \tilde{\eta}_1, \gamma_1 \in \tilde{\nu}_1} \left(\left\{ \alpha_1^{\lambda} \middle| p_{\alpha_1} \right\}, \left\{ 1 - (1 - \beta_1)^{\lambda} \middle| q_{\beta_1} \right\}, \left\{ 1 - (1 - \gamma_1)^{\lambda} \middle| r_{\gamma_1} \right\} \right), \\
\tilde{n}_2^{\lambda} &= \cup_{\alpha_2 \in \tilde{\mu}_2, \beta_2 \in \tilde{\eta}_2, \gamma_2 \in \tilde{\nu}_2} \left(\left\{ \alpha_2^{\lambda} \middle| p_{\alpha_2} \right\}, \left\{ 1 - (1 - \beta_2)^{\lambda} \middle| q_{\beta_2} \right\}, \left\{ 1 - (1 - \gamma_2)^{\lambda} \middle| r_{\gamma_2} \right\} \right),
\end{aligned}
$$

we have

$$
\begin{aligned}
w_1 \tilde{n}_1^{\lambda} &= \cup_{\alpha_1 \in \tilde{\mu}_1, \beta_1 \in \tilde{\eta}_1, \gamma_1 \in \tilde{\nu}_1} \left(\left\{ 1 - (1 - \alpha_1^{\lambda})^{w_1} \middle| p_{\alpha_1} \right\}, \left\{ \left(1 - (1 - \beta_1)^{\lambda} \right)^{w_1} \middle| q_{\beta_1} \right\}, \left\{ \left(1 - (1 - \gamma_1)^{\lambda} \right)^{w_1} \middle| r_{\gamma_1} \right\} \right), \\
w_2 \tilde{n}_2^{\lambda} &= \cup_{\alpha_2 \in \tilde{\mu}_2, \beta_2 \in \tilde{\eta}_2, \gamma_2 \in \tilde{\nu}_2} \left(\left\{ 1 - (1 - \alpha_2^{\lambda})^{w_2} \middle| p_{\alpha_2} \right\}, \left\{ \left(1 - (1 - \beta_2)^{\lambda} \right)^{w_2} \middle| q_{\beta_2} \right\}, \left\{ \left(1 - (1 - \gamma_2)^{\lambda} \right)^{w_2} \middle| r_{\gamma_2} \right\} \right).
\end{aligned}
$$

Then, by Definition 4.7, we get

$$
\begin{aligned}
\mathrm{GPPHFWA}(\tilde{n}_1, \tilde{n}_2) &= \left(w_1 \tilde{n}_1^{\lambda} \oplus w_2 \tilde{n}_2^{\lambda} \right)^{\frac{1}{\lambda}} \\
&= \cup_{\substack{\alpha_1 \in \tilde{\mu}_1, \beta_1 \in \tilde{\eta}_1, \gamma_1 \in \tilde{\nu}_1 \\ \alpha_2 \in \tilde{\mu}_2, \beta_2 \in \tilde{\eta}_2, \gamma_2 \in \tilde{\nu}_2}} \left(\begin{array}{l} \left\{ \left(1 - \prod_{i=1}^{2} (1 - \alpha_i^{\lambda})^{w_i} \right)^{\frac{1}{\lambda}} \middle| \prod_{i=1}^{2} p_{\alpha_i} \right\}, \\ \left\{ 1 - \left(1 - \prod_{i=1}^{2} \left(1 - (1 - \beta_i)^{\lambda} \right)^{w_i} \right)^{\frac{1}{\lambda}} \middle| \prod_{i=1}^{2} q_{\beta_i} \right\}, \\ \left\{ 1 - \left(1 - \prod_{i=1}^{2} \left(1 - (1 - \gamma_i)^{\lambda} \right)^{w_i} \right)^{\frac{1}{\lambda}} \middle| \prod_{i=1}^{2} r_{\gamma_i} \right\} \end{array} \right).
\end{aligned}
$$

Thus, Eq (5.2) holds for $n = 2$.

Suppose that Eq (5.2) holds for $n = k$. When $n = k + 1$, by operational laws of P-PHFEs, we have

$$
\left(\oplus_{i=1}^{k} w_i \tilde{n}_i^{\lambda} \right) \oplus w_{k+1} \tilde{n}_{k+1}^{\lambda}
$$

$$= \cup_{\alpha_i\in\tilde{\mu}_i,\beta_i\in\tilde{\eta}_i,\gamma_i\in\tilde{\nu}_i}\begin{pmatrix}\left\{1-\prod_{i=1}^{k}(1-\alpha_i^{\lambda})^{w_i}\Big|\prod_{i=1}^{k}p_{\alpha_i}\right\},\\ \left\{\prod_{i=1}^{k}\left(1-(1-\beta_i)^{\lambda}\right)^{w_i}\Big|\prod_{i=1}^{k}q_{\beta_i}\right\},\\ \left\{\prod_{i=1}^{k}\left(1-(1-\gamma_i)^{\lambda}\right)^{w_i}\Big|\prod_{i=1}^{k}r_{\gamma_i}\right\}\end{pmatrix}$$

$$\oplus \cup_{\alpha_{k+1}\in\tilde{\mu}_{k+1},\beta_{k+1}\in\tilde{\eta}_{k+1},\gamma_{k+1}\in\tilde{\nu}_{k+1}}\begin{pmatrix}\left\{1-(1-\alpha_{k+1}^{\lambda})^{w_{k+1}}\Big|p_{\alpha_{k+1}}\right\},\\ \left\{\left(1-(1-\beta_{k+1})^{\lambda}\right)^{w_{k+1}}\Big|q_{\beta_{k+1}}\right\},\\ \left\{\left(1-(1-\gamma_{k+1})^{\lambda}\right)^{w_{k+1}}\Big|r_{\gamma_{k+1}}\right\}\end{pmatrix}$$

$$= \cup_{\alpha_i\in\tilde{\mu}_i,\beta_i\in\tilde{\eta}_i,\gamma_i\in\tilde{\nu}_i}\begin{pmatrix}\left\{1-\prod_{i=1}^{k+1}(1-\alpha_i^{\lambda})^{w_i}\Big|\prod_{i=1}^{k+1}p_{\alpha_i}\right\},\\ \left\{\prod_{i=1}^{k+1}\left(1-(1-\beta_i)^{\lambda}\right)^{w_i}\Big|\prod_{i=1}^{k+1}q_{\beta_i}\right\},\\ \left\{\prod_{i=1}^{k+1}\left(1-(1-\gamma_i)^{\lambda}\right)^{w_i}\Big|\prod_{i=1}^{k+1}r_{\gamma_i}\right\}\end{pmatrix}.$$

Then,

$$\mathrm{GPPHFWA}(\tilde{n}_1,\tilde{n}_2,\ldots,\tilde{n}_{k+1}) = \left(\oplus_{i=1}^{k+1}w_i\tilde{n}_i^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$= \cup_{\alpha_i\in\tilde{\mu}_i,\beta_i\in\tilde{\eta}_i,\gamma_i\in\tilde{\nu}_i}\begin{pmatrix}\left\{\left(1-\prod_{i=1}^{k+1}(1-\alpha_i^{\lambda})^{w_i}\right)^{\frac{1}{\lambda}}\Big|\prod_{i=1}^{k+1}p_{\alpha_i}\right\},\\ \left\{1-\left(1-\prod_{i=1}^{k+1}\left(1-(1-\beta_i)^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}}\Big|\prod_{i=1}^{k+1}q_{\beta_i}\right\},\\ \left\{1-\left(1-\prod_{i=1}^{k+1}\left(1-(1-\gamma_i)^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}}\Big|\prod_{i=1}^{k+1}r_{\gamma_i}\right\}\end{pmatrix},$$

i.e., Eq (5.2) holds for $n = k + 1$. Thus, Eq (5.2) holds for all n.

Appendix C

Proof. (1) Since $\alpha_{\min} \le \alpha_i \le \alpha_{\max}$ for all i, then $1-\alpha_{\max}^{\lambda} \le 1-\alpha_i^{\lambda} \le 1-\alpha_{\min}^{\lambda}$. Since $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of h_i satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, then for all i, we have

$$\begin{aligned}
&\left(1-\alpha_{\max}^{\lambda}\right)^{w_i} \le \left(1-\alpha_i^{\lambda}\right)^{w_i} \le \left(1-\alpha_{\min}^{\lambda}\right)^{w_i}\\
&\Rightarrow \prod_{i=1}^{n}\left(1-\alpha_{\max}^{\lambda}\right)^{w_i} \le \prod_{i=1}^{n}\left(1-\alpha_i^{\lambda}\right)^{w_i} \le \prod_{i=1}^{n}\left(1-\alpha_{\min}^{\lambda}\right)^{w_i}\\
&\Rightarrow 1-\prod_{i=1}^{n}\left(1-\alpha_{\min}^{\lambda}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\alpha_i^{\lambda}\right)^{w_i} \le 1-\prod_{i=1}^{n}\left(1-\alpha_{\max}^{\lambda}\right)^{w_i}\\
&\Rightarrow \alpha_{\min} = \left(1-\prod_{i=1}^{n}\left(1-\alpha_{\min}^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}} \le \left(1-\prod_{i=1}^{n}\left(1-\alpha_i^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}} \le \left(1-\prod_{i=1}^{n}\left(1-\alpha_{\max}^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}} = \alpha_{\max}.
\end{aligned}$$

Hence, the required result for positive membership values is obtained. Next, for neutral membership, since $\beta_{\min} \le \beta_i \le \beta_{\max}$ for all i, then $(1-\beta_{\max})^{\lambda} \le (1-\beta_i)^{\lambda} \le (1-\beta_{\min})^{\lambda}$. Since $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of h_i satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, then for all i, we have

$$\left(1-(1-\beta_{\min})^{\lambda}\right)^{w_i} \le \left(1-(1-\beta_i)^{\lambda}\right)^{w_i} \le \left(1-(1-\beta_{\max})^{\lambda}\right)^{w_i}$$

$$\Rightarrow \prod_{i=1}^{n}\left(1-(1-\beta_{\min})^{\lambda}\right)^{w_i} \leq \prod_{i=1}^{n}\left(1-(1-\beta_i)^{\lambda}\right)^{w_i} \leq \prod_{i=1}^{n}\left(1-(1-\beta_{\max})^{\lambda}\right)^{w_i}$$

$$\Rightarrow 1-\prod_{i=1}^{n}\left(1-(1-\beta_{\min})^{\lambda}\right)^{w_i} \leq 1-\prod_{i=1}^{n}\left(1-(1-\beta_i)^{\lambda}\right)^{w_i} \leq 1-\prod_{i=1}^{n}\left(1-(1-\beta_{\max})^{\lambda}\right)^{w_i}$$

$$\Rightarrow \left(1-\prod_{i=1}^{n}\left(1-(1-\beta_{\max})^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}} \leq \left(1-\prod_{i=1}^{n}\left(1-(1-\beta_i)^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}} \leq \left(1-\prod_{i=1}^{n}\left(1-(1-\beta_{\min})^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}}$$

$$\Rightarrow \beta_{\min} = 1-\left(1-\prod_{i=1}^{n}\left(1-(1-\beta_{\min})^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}} \leq 1-\left(1-\prod_{i=1}^{n}\left(1-(1-\beta_i)^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}}$$

$$\leq 1-\left(1-\prod_{i=1}^{n}\left(1-(1-\beta_{\max})^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}} = \beta_{\max}.$$

Hence, we obtain the required result for neutral membership values. Similarly, for negative membership, since $\gamma_{\min} \leq \gamma_i \leq \gamma_{\max}$ for all i, we have

$$\gamma_{\min} \leq 1-\left(1-\prod_{i=1}^{n}\left(1-(1-\gamma_i)^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}} \leq \gamma_{\max}.$$

Hence, the required result for negative membership values is obtained.

Now, for probabilities, since $p_{\min} \leq p_{\alpha_i} \leq p_{\max}$, $q_{\min} \leq q_{\beta_i} \leq q_{\max}$ and $r_{\min} \leq r_{\gamma_i} \leq r_{\max}$, we have $\prod_{i=1}^{n} p_{\min} \leq \prod_{i=1}^{n} p_{\alpha_i} \leq \prod_{i=1}^{n} p_{\max}$, $\prod_{i=1}^{n} q_{\min} \leq \prod_{i=1}^{n} q_{\beta_i} \leq \prod_{i=1}^{n} q_{\max}$ and $\prod_{i=1}^{n} r_{\min} \leq \prod_{i=1}^{n} r_{\gamma_i} \leq \prod_{i=1}^{n} r_{\max}$. Let $\mathrm{GPPHFWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) = \tilde{n} = \left(\tilde{\mu}|p_{\tilde{\mu}}, \tilde{\eta}|q_{\tilde{\eta}}, \tilde{\nu}|r_{\tilde{\nu}}\right)$. By Definition 4.3, we obtain $s(\tilde{n}^-) \leq s(\tilde{n}) \leq s(\tilde{n}^+)$. Hence, we have $\tilde{n}^- \leq \mathrm{GPPHFWA}(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_n) \leq \tilde{n}^+$.

(2) Similar to that of (1).

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