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Research article

Note on complete convergence and complete moment convergence for negatively dependent random variables under sub-linear expectations

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Abstract: In this article, we study the complete convergence and the complete moment convergence for negatively dependent (ND) random variables under sub-linear expectations. Under proper conditions of the moment of random variables, we establish the complete convergence and the complete moment convergence. As applications, we obtain the Marcinkiewicz-Zygmund type strong law of large numbers of ND random variables under sub-linear expectations. The results here generalize the corresponding ones in classic probability space to those under sub-linear expectations.

Keywords: negatively dependent random variables; complete convergence; complete moment convergence; sub-linear expectations
Mathematics Subject Classification: 60F15, 60F05

1. Introduction

Peng [1,2] introduced seminal concepts of the sub-linear expectations space to study the uncertainty in probability. The works of Peng [1, 2] stimulate many scholars to investigate the results under sub-linear expectations space, extending those in classic probability space. Zhang [3, 4] got exponential inequalities and Rosenthal's inequality under sub-linear expectations. For more limit theorems under sub-linear expectations, the readers could refer to Zhang [5], Xu and Zhang [6, 7], Wu and Jiang [8], Zhang and Lin [9], Zhong and Wu [10], Chen [11], Chen and Wu [12], Zhang [13], Hu et al. [14], Gao and Xu [15], Kuczmaszewska [16], Xu and Cheng [17–19], Xu et al. [20] and references therein.

In probability space, Shen et al. [21] obtained equivalent conditions of complete convergence and complete moment convergence for extended negatively dependent random variables. For references on complete moment convergence and complete convergence in probability space, the reader could refer to Hsu and Robbins [22], Chow [23], Ko [24], Meng et al. [25], Hosseini and Nezakati [26], Meng et al. [27] and refercences therein. Inspired by the work of Shen et al. [21], we try to investigate complete convergence and complete moment convergence for negatively dependent (ND) random variables under sub-linear expectations, and the Marcinkiewicz-Zygmund type result for ND random variables under sub-linear expectations, which complements the relevant results in Shen et al. [21].

Recently, Srivastava et al. [28] introduced and studied comcept of statistical probability convergence. Srivastava et al. [29] investigated the relevant results of statistical probability convergence via deferred Nörlund summability mean. For more recent works, the interested reader could refer to Srivastava et al. [30–32], Paikary et al. [33] and references therein. We conjecture the relevant notions and results of statistical probability convergence could be extended to that under sub-linear expectation.

We organize the remainders of this article as follows. We cite relevant basic notions, concepts and properties, and present relevant lemmas under sub-linear expectations in Section 2. In Section 3, we give our main results, Theorems 3.1 and 3.2, the proofs of which are given in Section 4.

2. Preliminary

In this article, we use notions as in the works by Peng [2], Zhang [4]. Suppose that $(\Omega, \mathcal{F})$ is a given measurable space. Assume that $\mathcal{H}$ is a collection of all random variables on $(\Omega, \mathcal{F})$ satisfying $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for $X_1, \cdots, X_n \in \mathcal{H}$, and each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ represents the space of φ fulfilling

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \le C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)(|\mathbf{x} - \mathbf{y}|), \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

for some $C > 0$, $m \in \mathbb{N}$ relying on φ.

Definition 2.1. *A sub-linear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \mapsto \bar{\mathbb{R}} := [-\infty, \infty]$ fulfilling the following: for every $X, Y \in \mathcal{H}$,*

(a) *$X \ge Y$ yields $\mathbb{E}[X] \ge \mathbb{E}[Y]$;*

(b) *$\mathbb{E}[c] = c$, $\forall c \in \mathbb{R}$;*

(c) *$\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall \lambda \ge 0$;*

(d) *$\mathbb{E}[X + Y] \le \mathbb{E}[X] + \mathbb{E}[Y]$ whenever $\mathbb{E}[X] + \mathbb{E}[Y]$ is not of the form $\infty - \infty$ or $-\infty + \infty$.*

We name a set function $V : \mathcal{F} \mapsto [0, 1]$ a capacity if

(a) $V(\emptyset) = 0$, $V(\Omega) = 1$;

(b) $V(A) \le V(B)$, $A \subset B$, $A, B \in \mathcal{F}$.
Moreover, if V is continuous, then V obey

(c) $A_n \uparrow A$ concludes $V(A_n) \uparrow V(A)$;

(d) $A_n \downarrow A$ concludes $V(A_n) \downarrow V(A)$.

V is named to be sub-additive when $V(A + B) \le V(A) + V(B)$, $A, B \in \mathcal{F}$.

Under the sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, set $\mathbb{V}(A) := \inf\{\mathbb{E}[\xi] : I_A \le \xi, \xi \in \mathcal{H}\}$, $\forall A \in \mathcal{F}$ (cf. Zhang [3, 4, 9, 13], Chen and Wu [12], Xu et al. [20]). $\mathbb{V}$ is a sub-additive capacity. Set

$$\mathbb{V}^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mathbb{V}(A_n) : A \subset \bigcup_{n=1}^{\infty} A_n\right\}, A \in \mathcal{F}.$$

By Definition 4.2 and Lemma 4.3 of Zhang [34], if $\mathbb{E} = \mathbf{E}$ is linear expectation, $\mathbb{V}^*$ coincide with the probability measure introduced by the linear expectation $\mathbf{E}$. As in Zhang [3], $\mathbb{V}^*$ is countably sub-additive, $\mathbb{V}^*(A) \le \mathbb{V}(A)$. Hence, in Theorem 3.1, Corollary 3.1, $\mathbb{V}$ could be replaced by $\mathbb{V}^*$, implying that the results here could be considered as natural extensions of the corresponding ones in classic probability space. Write

$$C_{\mathbb{V}}(X) := \int_0^\infty \mathbb{V}(X > x)\mathrm{d}x + \int_{-\infty}^0 (\mathbb{V}(X > x) - 1)\mathrm{d}x.$$

Assume $\mathbf{X} = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ and $\mathbf{Y} = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ are two random vectors on $(\Omega, \mathcal{H}, \mathbb{E})$. $\mathbf{Y}$ is called to be negatively dependent to $\mathbf{X}$, if for ψ_1 on $C_{l,Lip}(\mathbb{R}^m)$, ψ_2 on $C_{l,Lip}(\mathbb{R}^n)$, we have $\mathbb{E}[\psi_1(\mathbf{X})\psi_2(\mathbf{Y})] \le \mathbb{E}[\psi_1(\mathbf{X})]\mathbb{E}[\psi_2(\mathbf{Y})]$ whenever $\psi_1(\mathbf{X}) \ge 0$, $\mathbb{E}[\psi_2(\mathbf{Y})] \ge 0$, $\mathbb{E}[|\psi_1(\mathbf{X})\psi_2(\mathbf{Y})|] < \infty$, $\mathbb{E}[|\psi_1(\mathbf{X})|] < \infty$, $\mathbb{E}[|\psi_2(\mathbf{Y})|] < \infty$, and either ψ_1 and ψ_2 are coordinatewise nondecreasing or ψ_1 and ψ_2 are coordinatewise nonincreasing (cf. Definition 2.3 of Zhang [3], Definition 1.5 of Zhang [4]).

$\{X_n\}_{n=1}^\infty$ is called to be negatively dependent, if X_{n+1} is negatively dependent to $(X_1, \cdots, X_n)$ for each $n \ge 1$. The existence of negatively dependent random variables $\{X_n\}_{n=1}^\infty$ under sub-linear expectations could be yielded by Example 1.6 of Zhang [4] and Kolmogorov's existence theorem in classic probabililty space. We below give an concrete example.

Example 2.1. *Let $\mathcal{P} = \{Q_1, Q_2\}$ be a family of probability measures on $(\Omega, \mathcal{F})$. Suppose that $\{X_n\}_{n=1}^\infty$ are independent, identically distributed under each Q_i, $i = 1, 2$ with $Q_1(X_1 = -1) = Q_1(X_1 = 1) = 1/2$, $Q_2(X_1 = -1) = 1$. Define $\mathbb{E}[\xi] = \sup_{Q\in\mathcal{P}} \mathbf{E}_Q[\xi]$, for each random variable ξ. Here $\mathbb{E}[\cdot]$ is a sub-linear expectation. By the discussion of Example 1.6 of Zhang [4], we see that $\{X_n\}_{n=1}^\infty$ are negatively dependent random variables under $\mathbb{E}$.*

Assume that $\mathbf{X}_1$ and $\mathbf{X}_2$ are two n-dimensional random vectors in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$ repectively. They are named identically distributed if for every $\psi \in C_{l,Lip}(\mathbb{R}^n)$,

$$\mathbb{E}_1[\psi(\mathbf{X}_1)] = \mathbb{E}_2[\psi(\mathbf{X}_2)].$$

$\{X_n\}_{n=1}^\infty$ is called to be identically distributed if for every $i \ge 1$, X_i and X_1 are identically distributed.

In this article we assume that $\mathbb{E}$ is countably sub-additive, i.e., $\mathbb{E}(X) \le \sum_{n=1}^\infty \mathbb{E}(X_n)$ could be implied by $X \le \sum_{n=1}^\infty X_n$, $X, X_n \in \mathcal{H}$, and $X \ge 0$, $X_n \ge 0$, $n = 1, 2, \ldots$. Write $S_n = \sum_{i=1}^n X_i$, $n \ge 1$. Let C denote a positive constant which may vary in different occasions. $I(A)$ or I_A represent the indicator function of A. The notion $a_x \approx b_x$ means that there exist two positive constants C_1, C_2 such that $C_1|b_x| \le |a_x| \le C_2|b_x|$.

As in Zhang [4], by definition, if $X_1, X_2, \ldots, X_n$ are negatively dependent random variables and f_1, $f_2, \ldots, f_n$ are all non increasing (or non decreasing) functions, then $f_1(X_1)$, $f_2(X_2), \ldots, f_n(X_n)$ are still negatively dependent random variables.

We cite the useful inequalities under sub-linear expectations.

Lemma 2.1. *(See Lemma 4.5 (iii) of Zhang [3]) If $\mathbb{E}$ is countably sub-additive under $(\Omega, \mathcal{H}, \mathbb{E})$, then for $X \in \mathcal{H}$,*

$$\mathbb{E}|X| \le C_{\mathbb{V}}(|X|).$$

Lemma 2.2. *(See Lemmas 2.3, 2.4 of Xu et al. [20] and Theorem 2.1 of Zhang [4]) Assume that $p \geq 1$ and $\{X_n; n \geq 1\}$ is a sequence of negatively dependent random varables under $(\Omega, \mathcal{H}, \mathbb{E})$. Then there exists a positive constant $C = C(p)$ relying on p such that*

$$\mathbb{E}\left[\left|\sum_{j=1}^{n} X_j\right|^p\right] \leq C\left\{\sum_{i=1}^{n} \mathbb{E}\,|X_i|^p + \left(\sum_{i=1}^{n} [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^p\right\},\ 1 \leq p \leq 2, \tag{2.1}$$

$$\mathbb{E}\left[\max_{1\leq i\leq n}\left|\sum_{j=1}^{i} X_j\right|^p\right] \leq C(\log n)^p\left\{\sum_{i=1}^{n} \mathbb{E}\,|X_i|^p + \left(\sum_{i=1}^{n} [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^p\right\},\ 1 \leq p \leq 2, \tag{2.2}$$

$$\mathbb{E}\left[\max_{1\leq i\leq n}\left|\sum_{j=1}^{i} X_i\right|^p\right] \leq C\left\{\sum_{i=1}^{n} \mathbb{E}\,|X_i|^p + \left(\sum_{i=1}^{n} \mathbb{E}X_i^2\right)^{p/2} + \left(\sum_{i=1}^{n} [|\mathbb{E}(-X_i)| + |\mathbb{E}(X_i)|]\right)^p\right\},\ p \geq 2. \tag{2.3}$$

Lemma 2.3. *Assume that $X \in \mathcal{H}$, $\alpha > 0$, $\gamma > 0$, $C_{\mathbb{V}}(|X|^\alpha) < \infty$. Then there exists a positive constant C relying on α, γ such that*

$$\int_0^\infty \mathbb{V}\{|X| > \gamma y\}\, y^{\alpha-1}\mathrm{d}y \leq CC_{\mathbb{V}}(|X|^\alpha) < \infty.$$

Proof. By the method of substitution of definite integral, letting $\gamma y = z^{1/\alpha}$, we get

$$\int_0^\infty \mathbb{V}\{|X| > \gamma y\}\, y^{\alpha-1}\mathrm{d}y \leq \int_0^\infty \mathbb{V}\{|X|^\alpha > z\}\,(z^{1/\alpha}/\gamma)^{\alpha-1} z^{1/\alpha-1}/\gamma \mathrm{d}z \leq CC_{\mathbb{V}}(|X|^\alpha) < \infty.$$

□

Lemma 2.4. *Let $Y_n, Z_n \in \mathcal{H}$. Then for any $q > 1$, $\varepsilon > 0$ and $a > 0$,*

$$\mathbb{E}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y_i + Z_i)\right| - \varepsilon a\right)^+ \leq \left(\frac{1}{\varepsilon^q} + \frac{1}{q-1}\right)\frac{1}{a^{q-1}}\mathbb{E}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Y_i\right|^q\right) + \mathbb{E}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Z_i\right|\right). \tag{2.4}$$

Proof. By Markov' inequality under sub-linear expectations, Lemma 2.1, and the similar proof of Lemma 2.4 of Sung [35], we could finish the proof. Hence, the proof is omitted here. □

3. Main results

Our main results are below.

Theorem 3.1. *Suppose $\alpha > \frac{1}{2}$ and $\alpha p > 1$. Assume that $\{X_n, n \geq 1\}$ is a sequence of negatively dependent random variables, and for each $n \geq 1$, X_n is identically distributed as X under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Moreover, assume $\mathbb{E}(X) = \mathbb{E}(-X) = 0$ if $p \geq 1$. Suppose $C_{\mathbb{V}}(|X|^p) < \infty$. Then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^\alpha\right\} < \infty. \tag{3.1}$$

Remark 3.1. *By Example 2.1, the assumption $\mathbb{E}(X) = \mathbb{E}(-X) = 0$ if $p \geq 1$ in Theorem 3.1 can not be weakened to $\mathbb{E}(X) = 0$ if $p \geq 1$. In fact, in the case of Example 2.1, if $\frac{1}{2} < \alpha \leq 1$, $\alpha p > 1$, then for any $0 < \varepsilon < 1$,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right\} \geq \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| \geq n\right\} = \sum_{n=1}^{\infty} n^{\alpha p-2} = +\infty,$$

which implies that Theorems 3.1, 3.2, Corollary 3.1 do not hold. However, by Example 1.6 of Zhang [4], the assumptions of Theorem 3.3, Corollary 3.2 hold for random variables in Example 2.1, hence Theorem 3.3, Corollary 3.2 are valid in this example.

By Theorem 3.1, we could get the Marcinkiewicz-Zygmund strong law of large numbers for negatively dependent random variables under sub-linear expectations below.

Corollary 3.1. *Let $\alpha > \frac{1}{2}$ and $\alpha p > 1$. Assume that under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, $\{X_n\}$ is a sequence of negatively dependent random variables and for each n, X_n is identically distributed as X. Moreover, assume $\mathbb{E}(X) = \mathbb{E}(-X) = 0$ if $p \geq 1$. Assume that $\mathbb{V}$ induced by $\mathbb{E}$ is countably sub-additive. Suppose $C_{\mathbb{V}}\{|X|^p\} < \infty$. Then*

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{1}{n^{\alpha}}\left|\sum_{i=1}^{n} X_i\right| > 0\right) = 0. \tag{3.2}$$

Theorem 3.2. *If the assumptions of Theorem 3.1 hold for $p \geq 1$ and $C_{\mathbb{V}}\{|X|^p \log^{\theta}|X|\} < \infty$ for some $\theta > \max\{\frac{\alpha p-1}{\alpha-\frac{1}{2}}, p\}$, then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\mathbb{E}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| - \varepsilon n^{\alpha}\right)^{+} < \infty. \tag{3.3}$$

By the similar proof of Theorem 3.1, with Theorem 2.1 (b) for negative dependent random variables of Zhang [4] (cf. the proof of Theorem 2.1 (c) there) in place of Lemma 2.2 here, we could obtain the following result.

Theorem 3.3. *Suppose $\alpha > \frac{1}{2}$, $p \geq 1$, and $\alpha p > 1$. Assume that X_k is negatively dependent to $(X_{k+1}, \ldots, X_n)$, for each $k = 1, \ldots, n$, $n \geq 1$. Suppose for each n, X_n is identically distributed as X under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Suppose $C_{\mathbb{V}}(|X|^p) < \infty$. Then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\leq j\leq n}\sum_{i=1}^{j}[X_i - \mathbb{E}(X_i)] > \varepsilon n^{\alpha}\right\} < \infty,$$

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\leq j\leq n}\sum_{i=1}^{j}[-X_i - \mathbb{E}(-X_i)] > \varepsilon n^{\alpha}\right\} < \infty.$$

By the similar proof of Corollary 3.1, with Theorem 3.3 in place of Theorem 3.1, we get the following result.

Corollary 3.2. *Let $\alpha > \frac{1}{2}$, $p \geq 1$, and $\alpha p > 1$. Assume that X_k is negatively dependent to $(X_{k+1}, \ldots, X_n)$, for each $k = 1, \ldots, n$, $n \geq 1$. Suppose for each n, X_n is identically distributed as X under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Assume that $\mathbb{V}$ induced by $\mathbb{E}$ is countably sub-additive. Suppose $C_{\mathbb{V}}\{|X|^p\} < \infty$. Then*

$$\mathbb{V}\left(\left\{\limsup_{n\to\infty} \frac{1}{n^\alpha}\sum_{i=1}^{n}[X_i - \mathbb{E}(X_i)] > 0\right\} \bigcup \left\{\limsup_{n\to\infty} \frac{1}{n^\alpha}\sum_{i=1}^{n}[-X_i - \mathbb{E}(-X_i)] > 0\right\}\right) = 0.$$

By the similar proof of Theorem 3.1 and Corollary 3.1, and adapting the proof of (4.10), we could obtain the following result.

Corollary 3.3. *Suppose $\alpha > 1$ and $p \geq 1$. Assume that $\{X_n, n \geq 1\}$ is a sequence of negatively dependent random variables, and for each $n \geq 1$, X_n is identically distributed as X under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Suppose $C_{\mathbb{V}}(|X|^p) < \infty$. Then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\leq j\leq n}\sum_{i=1}^{j}[X_i - \mathbb{E}(X_i)] > \varepsilon n^\alpha\right\} < \infty,$$

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\leq j\leq n}\sum_{i=1}^{j}[-X_i - \mathbb{E}(-X_i)] > \varepsilon n^\alpha\right\} < \infty.$$

Moreover assume that $\mathbb{V}$ induced by $\mathbb{E}$ is countably sub-additive. Then

$$\mathbb{V}\left(\left\{\limsup_{n\to\infty} \frac{1}{n^\alpha}\sum_{i=1}^{n}[X_i - \mathbb{E}(X_i)] > 0\right\} \bigcup \left\{\limsup_{n\to\infty} \frac{1}{n^\alpha}\sum_{i=1}^{n}[-X_i - \mathbb{E}(-X_i)] > 0\right\}\right) = 0.$$

By the discussion below Definition 4.1 of Zhang [34], and Corollary 3.2, we conjecture the following.

Conjecture 3.1. *Suppose $\frac{1}{2} < \alpha \leq 1$ and $\alpha p > 1$. Assume that $\{X_n, n \geq 1\}$ is a sequence of negatively dependent random variables, and for each $n \geq 1$, X_n is identically distributed as X under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Assume that $\mathbb{V}$ induced by $\mathbb{E}$ is continuous. Suppose $C_{\mathbb{V}}\{|X|^p\} < \infty$. Then*

$$\mathbb{V}\left(\left\{\limsup_{n\to\infty} \frac{1}{n^\alpha}\sum_{i=1}^{n}[X_i - \mathbb{E}(X_i)] > 0\right\} \bigcup \left\{\limsup_{n\to\infty} \frac{1}{n^\alpha}\sum_{i=1}^{n}[-X_i - \mathbb{E}(-X_i)] > 0\right\}\right) = 0.$$

4. Proofs of main results, Theorems 3.1, 3.2, Corollary 3.1

Proof of Theorem 3.1. We investigate the following cases.
Case 1. $0 < p < 1$.

For fixed $n \geq 1$, for $1 \leq i \leq n$, write

$$Y_{ni} = -n^\alpha I\{X_i < -n^\alpha\} + X_i I\{|X_i| \leq n^\alpha\} + n^\alpha I\{X_i > n^\alpha\},$$

$$Z_{ni} = (X_i - n^\alpha)I\{X_i > n^\alpha\} + (X_i + n^\alpha)I\{X_i < -n^\alpha\},$$

$$Y_n = -n^\alpha I\{X < -n^\alpha\} + XI\{|X| \le n^\alpha\} + n^\alpha I\{X > n^\alpha\},$$

$$Z_n = X - Y_n.$$

Observing that $X_i = Y_{ni} + Z_{ni}$, we see that for all $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^\alpha\right\} \\
&\le \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_{ni}\right| > \varepsilon n^\alpha/2\right\} + \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j} Z_{ni}\right| > \varepsilon n^\alpha/2\right\} \\
&=: I_1 + I_2. \qquad (4.1)
\end{aligned}$$

By Markov's inequality under sub-linear expectations, C_r inequality, and Lemmas 2.1, 2.3, we conclude that

$$\begin{aligned}
I_1 &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\sum_{i=1}^{n}\mathbb{E}|Y_{ni}| = C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}\mathbb{E}|Y_n| \le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}C_{\mathbb{V}}(|Y_n|) \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}\int_0^{n^\alpha}\mathbb{V}\{|Y_n| > x\}\,\mathrm{d}x = C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}\sum_{k=1}^{n}\int_{(k-1)^\alpha}^{k^\alpha}\mathbb{V}\{|X| > x\}\,\mathrm{d}x \\
&= C\sum_{k=1}^{\infty}\int_{(k-1)^\alpha}^{k^\alpha}\mathbb{V}\{|X| > x\}\,\mathrm{d}x\sum_{n=k}^{\infty} n^{\alpha p-1-\alpha} = C\sum_{k=1}^{\infty} k^{\alpha-1}\mathbb{V}\{|X| > (k-1)^\alpha\}k^{\alpha p-\alpha} \\
&\le C\sum_{k=1}^{\infty} k^{\alpha p-1}\mathbb{V}\{|X| > k^\alpha\} + C \\
&\le C\int_0^{\infty} x^{\alpha p-1}\mathbb{V}\{|X| > x^\alpha\}\,\mathrm{d}x + C \le CC_{\mathbb{V}}\{|X|^p\} + C < \infty, \qquad (4.2)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha p/2}\sum_{i=1}^{n}\mathbb{E}|Z_{ni}|^{p/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p/2-1}\mathbb{E}|Z_n|^{p/2} \le C\sum_{n=1}^{\infty} n^{\alpha p/2-1}C_{\mathbb{V}}\left\{|Z_n|^{p/2}\right\} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p/2-1}C_{\mathbb{V}}\left\{|X|^{p/2}I\{|X| > n^\alpha\}\right\} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p/2-1}\left[\int_0^{n^\alpha}\mathbb{V}\{|X| > n^\alpha\}s^{p/2-1}\mathrm{d}s + \int_{n^\alpha}^{\infty}\mathbb{V}\{|X| > s\}s^{p/2-1}\mathrm{d}s\right] \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{V}\{|X| > n^\alpha\} + C\sum_{n=1}^{\infty} n^{\alpha p/2-1}\sum_{k=n}^{\infty}\int_{k^\alpha}^{(k+1)^\alpha}\mathbb{V}\{|X| > s\}s^{p/2-1}\mathrm{d}s \\
&\le CC_{\mathbb{V}}\{|X|^p\} + C\sum_{k=1}^{\infty}\int_{k^\alpha}^{(k+1)^\alpha}\mathbb{V}\{|X| > s\}s^{p/2-1}\mathrm{d}s\sum_{n=1}^{k} n^{\alpha p/2-1}
\end{aligned}$$

$$\leq CC_{\mathbb{V}}\{|X|^p\} + C\sum_{k=1}^{\infty}\mathbb{V}\{|X|>k^{\alpha}\}k^{\alpha p-1}$$
$$\leq CC_{\mathbb{V}}\{|X|^p\} + CC_{\mathbb{V}}\{|X|^p\} < \infty. \tag{4.3}$$

Therefore, by (4.1)–(4.3), we deduce that (3.1) holds.
Case 2. $p \geq 1$.

Observing that $\alpha p > 1$, we choose a suitable q such that $\frac{1}{\alpha p} < q < 1$. For fixed $n \geq 1$, for $1 \leq i \leq n$, write

$$X_{ni}^{(1)} = -n^{\alpha q}I\{X_i < -n^{\alpha q}\} + X_i I\{|X_i| \leq n^{\alpha q}\} + n^{\alpha q}I(X_i > n^{\alpha q}),$$
$$X_{ni}^{(2)} = (X_i - n^{\alpha q})I\{X_i > n^{\alpha q}\}, X_{ni}^{(3)} = (X_i + n^{\alpha q})I\{X_i < -n^{\alpha q}\},$$

and $X_n^{(1)}$, $X_n^{(2)}$, $X_n^{(3)}$ is defined as $X_{ni}^{(1)}$, $X_{ni}^{(2)}$, $X_{ni}^{(3)}$ only with X in place of X_i above. Observing that $\sum_{i=1}^{j} X_i = \sum_{i=1}^{j} X_{ni}^{(1)} + \sum_{i=1}^{j} X_{ni}^{(2)} + \sum_{i=1}^{j} X_{ni}^{(3)}$, for $1 \leq j \leq n$, we see that for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right\}$$
$$\leq \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}^{(1)}\right| > \varepsilon n^{\alpha}/3\right\} + \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}^{(2)}\right| > \varepsilon n^{\alpha}/3\right\}$$
$$+ \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_{ni}^{(3)}\right| > \varepsilon n^{\alpha}/3\right\} =: II_1 + II_2 + II_3. \tag{4.4}$$

Therefore, to establish (3.1), it is enough to prove that $II_1 < \infty$, $II_2 < \infty$, $II_3 < \infty$.

For II_1, we first establish that

$$n^{-\alpha}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\mathbb{E}X_{ni}^{(1)}\right| \to 0, \text{ as } n\to\infty. \tag{4.5}$$

By $\mathbb{E}(X) = 0$, Markov's inequality under sub-linear expectations, Lemma 2.1, we conclude that

$$n^{-\alpha}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\mathbb{E}X_{ni}^{(1)}\right| \leq n^{-\alpha}\sum_{i=1}^{n}\left|\mathbb{E}(X_n^{(1)})\right| \leq n^{1-\alpha}|\mathbb{E}(X_n^{(1)}) - \mathbb{E}(X)|$$
$$\leq n^{1-\alpha}\mathbb{E}|X_n^{(1)} - X| \leq n^{1-\alpha}C_{\mathbb{V}}\left(|X_n^{(1)} - X|\right)$$
$$\leq Cn^{1-\alpha}\left[\int_0^{\infty}\mathbb{V}\{|X|I\{|X|>n^{\alpha q}\}>x\}\,\mathrm{d}x\right]$$
$$\leq Cn^{1-\alpha}\left[\int_0^{n^{\alpha q}}\mathbb{V}\{|X|>n^{\alpha q}\}\mathrm{d}x + \int_{n^{\alpha q}}^{\infty}\mathbb{V}\{|X|>y\}\mathrm{d}y\right]$$
$$\leq Cn^{1-\alpha+\alpha q}\mathbb{V}\{|X|>n^{\alpha q}\} + Cn^{1-\alpha}\int_{n^{\alpha q}}^{\infty}\frac{\mathbb{V}\{|X|>y\}y^{p-1}}{n^{\alpha q(p-1)}}\mathrm{d}y$$
$$\leq Cn^{1-\alpha+\alpha q}\frac{\mathbb{E}|X|^p}{n^{\alpha qp}} + Cn^{1-\alpha+\alpha q-\alpha qp}C_{\mathbb{V}}\{|X|^p\}$$
$$\leq Cn^{1-\alpha qp-\alpha+\alpha q}C_{\mathbb{V}}\{|X|^p\},$$

which results in (4.5) by $C_{\mathbb{V}}\{|X|^p\} < \infty$ and $1/(\alpha p) < q < 1$. Thus, from (4.5), it follows that

$$II_1 \le C\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(X_{ni}^{(1)} - \mathbb{E}X_{ni}^{(1)}\right)\right| > \frac{\varepsilon n^{\alpha}}{6}\right\}. \tag{4.6}$$

For fixed $n \ge 1$, we note that $\{X_{ni}^{(1)} - \mathbb{E}X_{ni}^{(1)}, 1 \le i \le n\}$ are negatively dependent random variables. By (4.6), Markov's inequality under sub-linear expectations, and Lemma 2.2, we see that for any $\beta \ge 2$,

$$\begin{aligned} II_1 &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta}\mathbb{E}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(X_{ni}^{(1)} - \mathbb{E}X_{ni}^{(1)}\right)\right|^{\beta}\right) \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta}\left[\sum_{i=1}^{n}\mathbb{E}|X_{ni}^{(1)}|^{\beta} + \left(\sum_{i=1}^{n}\mathbb{E}|X_{ni}^{(1)}|^2\right)^{\beta/2} + \left(\sum_{i=1}^{n}[|\mathbb{E}X_{ni}^{(1)}| + |\mathbb{E}(-X_{ni}^{(1)})|]\right)^{\beta}\right] \\ &=: II_{11} + II_{12} + II_{13}. \end{aligned} \tag{4.7}$$

Taking $\beta > \max\left\{\frac{\alpha p-1}{\alpha-1/2}, 2, p, \frac{\alpha p-1}{\alpha qp-\alpha q+\alpha-1}\right\}$, we obtain

$$\alpha p - \alpha\beta + \alpha q\beta - \alpha pq - 1 = \alpha(p-\beta)(1-q) - 1 < -1,$$

$$\alpha p - 2 - \alpha\beta + \beta/2 < -1,$$

and

$$\alpha p - 2 - \alpha\beta + \beta - \alpha q(p-1)\beta < -1.$$

By C_r inequality, Markov's inequality under sub-linear expectations, Lemma 2.1, we see that

$$\begin{aligned} II_{11} &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta}\sum_{i=1}^{n}\mathbb{E}|X_{ni}^{(1)}|^{\beta} \le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\beta}\mathbb{E}|X_{n}^{(1)}|^{\beta} \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\beta}C_{\mathbb{V}}\left\{|X_n^{(1)}|^{\beta}\right\} \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\beta}\int_0^{n^{\alpha q\beta}}\mathbb{V}\{|X|^{\beta} > x\}\mathrm{d}x \le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\beta}\int_0^{n^{\alpha q}}\mathbb{V}\{|X| > x\}x^{\beta-1}\mathrm{d}x \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\beta}\int_0^{n^{\alpha q}}\mathbb{V}\{|X| > x\}x^{p-1}n^{\alpha q(\beta-p)}\mathrm{d}x \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\beta+\alpha q\beta-\alpha qp}C_{\mathbb{V}}\{|X|^p\} < \infty, \end{aligned} \tag{4.8}$$

$$\begin{aligned} II_{12} &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta}\left(\sum_{i=1}^{n}\mathbb{E}|X_{n}^{(1)}|^2\right)^{\beta/2} \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta+\beta/2}\left(C_{\mathbb{V}}\{|X_n^{(1)}|^2\}\right)^{\beta/2} \end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta+\beta/2}\left(\int_0^{n^{\alpha q}} \mathbb{V}\{|X|>x\}x\mathrm{d}x\right)^{\beta/2}\\
&\leq \begin{cases} C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta+\beta/2}\left(C_{\mathbb{V}}\{|X|^2\}\right)^{\beta/2}, & \text{if } p\geq 2;\\ C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta+\beta/2}\left(\int_0^{n^{\alpha q}} \mathbb{V}\{|X|>x\}x^{p-1}n^{\alpha q(2-p)}\mathrm{d}x\right)^{\beta/2}, & \text{if } 1\leq p<2,\end{cases}\\
&\leq \begin{cases} C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta+\beta/2}\left(C_{\mathbb{V}}\{|X|^2\}\right)^{\beta/2}<\infty, & \text{if } p\geq 2;\\ C\sum_{n=1}^{\infty} n^{(\alpha p-1)(1-\beta/2)-1}\left(C_{\mathbb{V}}(|X|^p)\right)^{\beta/2}<\infty, & \text{if } 1\leq p<2,\end{cases} \qquad (4.9)
\end{aligned}
$$

and

$$
\begin{aligned}
II_{13} &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta}\left(\sum_{i=1}^{n}[|\mathbb{E}X_n^{(1)}|+|\mathbb{E}(-X_n^{(1)})|]\right)^{\beta}\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta+\beta}\left(\mathbb{E}|X_n^{(1)}-X|\right)^{\beta}\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta+\beta}\frac{(\mathbb{E}|X|^p)^{\beta}}{n^{\alpha q(p-1)\beta}}\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta+\beta-\alpha q(p-1)\beta}\left(C_{\mathbb{V}}(|X|^p)\right)^{\beta}<\infty. \qquad (4.10)
\end{aligned}
$$

Therefore, combining (4.7)–(4.10) results in $II_1<\infty$.

Next, we will establish that $II_2<\infty$. Let $g_\mu(x)$ be a non-increasing Lipschitz function such that $I\{x\leq\mu\}\leq g_\mu(x)\leq I\{x\leq 1\}$, $\mu\in(0,1)$. Obviously, $I\{x>\mu\}>1-g_\mu(x)>I\{x>1\}$. For fixed $n\geq 1$, for $1\leq i\leq n$, write

$$
X_{ni}^{(4)}=(X_i-n^{\alpha q})I\left(n^{\alpha q}<X_i\leq n^{\alpha}+n^{\alpha q}\right)+n^{\alpha}I\left(X_i>n^{\alpha}+n^{\alpha q}\right),
$$

and

$$
X_{n}^{(4)}=(X-n^{\alpha q})I\left(n^{\alpha q}<X\leq n^{\alpha}+n^{\alpha q}\right)+n^{\alpha}I\left(X>n^{\alpha}+n^{\alpha q}\right).
$$

We see that

$$
\left(\max_{1\leq j\leq n}\left|\sum_{j=1}^{i}X_{ni}^{(2)}\right|>\frac{\varepsilon n^{\alpha}}{3}\right)\subset\left(\max_{1\leq i\leq n}|X_i|>n^{\alpha}\right)\bigcup\left(\max_{1\leq j\leq n}\left|\sum_{j=1}^{i}X_{ni}^{(4)}\right|>\frac{\varepsilon n^{\alpha}}{3}\right),
$$

which results in

$$
\begin{aligned}
II_2 &\leq \sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{V}\left\{|X_i|>n^{\alpha}\right\}+\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\max_{1\leq j\leq n}\left|\sum_{j=1}^{i}X_{ni}^{(4)}\right|>\frac{\varepsilon n^{\alpha}}{3}\right\}\\
&=:II_{21}+II_{22}. \qquad (4.11)
\end{aligned}
$$

By $C_{\mathbb{V}}\{|X|^p\}<\infty$, we conclude that

$$
II_{21} \leq C\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{E}[1-g_\mu(|X_i|)]=C\sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{E}[1-g_\mu(|X|)]
$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-1}\mathbb{V}\{|X|>\mu n^{\alpha}\}\\
&\leq C\int_{0}^{\infty}x^{\alpha p-1}\mathbb{V}\{|X|>\mu x^{\alpha}\}\mathrm{d}x\leq CC_{\mathbb{V}}(|X|^{p})<\infty. \qquad (4.12)
\end{aligned}
$$

Observing that $\frac{1}{\alpha p}<q<1$, from the definition of $X_{ni}^{(2)}$, follows that

$$
\begin{aligned}
n^{-\alpha}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\mathbb{E}X_{ni}^{(4)}\right| &\leq Cn^{1-\alpha}\mathbb{E}|X_{n}^{(4)}|\leq Cn^{1-\alpha}C_{\mathbb{V}}(|X_{n}^{(4)}|)\\
&\leq Cn^{1-\alpha}\left[\int_{0}^{n^{\alpha q}}\mathbb{V}\left\{|X|I\{|X|>n^{\alpha q}\}>x\right\}\mathrm{d}x+\int_{n^{\alpha q}}^{\infty}\mathbb{V}\{|X|>x\}\mathrm{d}x\right]\\
&\leq Cn^{1-\alpha+\alpha q}\frac{\mathbb{E}|X|^{p}}{n^{\alpha pq}}+Cn^{1-\alpha}\int_{n^{\alpha q}}^{\infty}\mathbb{V}\{|X|>x\}\frac{x^{p-1}}{n^{\alpha q(p-1)}}\mathrm{d}x\\
&\leq Cn^{1-\alpha+\alpha q-\alpha pq}C_{\mathbb{V}}\{|X|^{p}\}\to 0 \text{ as } n\to\infty. \qquad (4.13)
\end{aligned}
$$

By $X_{ni}^{(4)}>0$, (4.11)–(4.13), we see that

$$
II_{2}\leq C\sum_{n=1}^{\infty}n^{\alpha p-2}\mathbb{V}\left\{\left|\sum_{i=1}^{n}[X_{ni}^{(4)}-\mathbb{E}X_{ni}^{(4)}]\right|>\frac{\varepsilon n^{\alpha}}{6}\right\}. \qquad (4.14)
$$

For fixed $n\geq 1$, we know that $\{X_{ni}^{(4)}-\mathbb{E}X_{ni}^{(4)}, 1\leq i\leq n\}$ are negatively dependent random variables under sub-linear expectations. By Markov's inequality under sub-linear expectations, C_r-inequality, Lemma 2.2, we obtain

$$
\begin{aligned}
II_{2}&\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha\beta}\mathbb{E}\left(\left|\sum_{i=1}^{n}[X_{ni}^{(4)}-\mathbb{E}X_{ni}^{(4)}]\right|^{\beta}\right)\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha\beta}\left[\sum_{i=1}^{n}\mathbb{E}\left|X_{ni}^{(4)}\right|^{\beta}+\left(\sum_{i=1}^{n}\mathbb{E}\left(X_{ni}^{(4)}\right)^{2}\right)^{\beta/2}+\left(\sum_{i=1}^{n}[|\mathbb{E}X_{ni}^{(4)}|+|\mathbb{E}(-X_{ni}^{(4)})|]\right)^{\beta}\right]\\
&=: II_{21}+II_{22}+II_{23}. \qquad (4.15)
\end{aligned}
$$

By C_r inequality, Lemma 2.3, we have

$$
\begin{aligned}
II_{21} &\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha\beta}\sum_{i=1}^{n}\mathbb{E}\left|X_{n}^{(4)}\right|^{\beta}\leq C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha\beta}C_{\mathbb{V}}\left\{\left|X_{n}^{(4)}\right|^{\beta}\right\}\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha\beta}C_{\mathbb{V}}\left\{|X|^{\beta}I\{n^{\alpha q}<X\leq n^{\alpha}+n^{\alpha q}\}+n^{\alpha q\beta}I\{X>n^{\alpha}+n^{\alpha q}\}\right\}\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha\beta}\int_{0}^{2n^{\alpha}}\mathbb{V}\left\{|X|>x\right\}x^{\beta-1}\mathrm{d}x\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha\beta}\sum_{k=1}^{n}\int_{2(k-1)^{\alpha}}^{2k^{\alpha}}\mathbb{V}\left\{|X|>x\right\}x^{\beta-1}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{k=1}^{\infty}\mathbb{V}\{|X|>2(k-1)^{\alpha}\}k^{\alpha\beta-1}\sum_{n=k}^{\infty}n^{\alpha p-1-\alpha\beta}\\
&\le C\sum_{k=1}^{\infty}\mathbb{V}\{|X|>2(k-1)^{\alpha}\}k^{\alpha p-1}\\
&\le C\int_0^{\infty}\mathbb{V}\{|X|>2x^{\alpha}\}x^{\alpha p-1}\mathrm{d}x\le CC_{\mathbb{V}}\{|X|^p\}<\infty. \qquad (4.16)
\end{aligned}
$$

As in the proof of (4.9) and (4.16), we can deduce that $II_{22}<\infty$.

By Lemma 2.1, we see that

$$
\begin{aligned}
II_{23} &\le C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha\beta}n^{\beta}\left(\mathbb{E}|X_n^{(4)}|\right)^{\beta}\le C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha\beta+\beta}\left(\frac{\mathbb{E}|X|^p}{n^{\alpha q(p-1)}}\right)^{\beta}\\
&\le C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha\beta+\beta-\alpha q(p-1)}\left(C_{\mathbb{V}}\{|X|^p\}\right)^{\beta}<\infty. \qquad (4.17)
\end{aligned}
$$

By (4.15)–(4.17), we deduce that $II_2<\infty$.

As in the proof of $II_2<\infty$, we also can obtain $II_3<\infty$. Therefore, combining (4.5), $II_1<\infty$, $II_2<\infty$, and $II_3<\infty$ results in (3.1). This finishes the proof. □

Proof of Corollary 3.1. By $C_{\mathbb{V}}\{|X|^p\}<\infty$, and Theorem 3.1, we deduce that for all $\varepsilon>0$,

$$
\sum_{n=1}^{\infty}n^{\alpha p-2}\mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_i\right|>\varepsilon n^{\alpha}\right)<\infty. \qquad (4.18)
$$

By (4.18), we conclude that for any $\varepsilon>0$,

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty}n^{\alpha p-2}\mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_i\right|>\varepsilon n^{\alpha}\right)\\
&= \sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1}n^{\alpha p-2}\mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_i\right|>\varepsilon n^{\alpha}\right)\\
&\ge \begin{cases}\sum_{k=0}^{\infty}(2^k)^{\alpha p-2}2^k\mathbb{V}\left(\max_{1\le j\le 2^k}\left|\sum_{i=1}^{j}X_i\right|>\varepsilon 2^{(k+1)\alpha}\right), & \text{if } \alpha p\ge 2,\\ \sum_{k=0}^{\infty}(2^{k+1})^{\alpha p-2}2^k\mathbb{V}\left(\max_{1\le j\le 2^k}\left|\sum_{i=1}^{j}X_i\right|>\varepsilon 2^{(k+1)\alpha}\right), & \text{if } 1<\alpha p<2,\end{cases}\\
&\ge \begin{cases}\sum_{k=0}^{\infty}\mathbb{V}\left(\max_{1\le j\le 2^k}\left|\sum_{i=1}^{j}X_i\right|>\varepsilon 2^{(k+1)\alpha}\right), & \text{if } \alpha p\ge 2,\\ \sum_{k=0}^{\infty}\frac{1}{2}\mathbb{V}\left(\max_{1\le j\le 2^k}\left|\sum_{i=1}^{j}X_i\right|>\varepsilon 2^{(k+1)\alpha}\right), & \text{if } 1<\alpha p<2,\end{cases}
\end{aligned}
$$

which, combined with Borel-Cantell lemma under sub-linear expectations, yields that

$$
\mathbb{V}\left(\limsup_{n\to\infty}\frac{\max_{1\le j\le 2^k}\left|\sum_{i=1}^{j}X_i\right|}{2^{(k+1)\alpha}}>0\right)=0. \qquad (4.19)
$$

For all positive integers n, $\exists$ a positive integer k satisfying $2^{k-1}\le n<2^k$, we see that

$$
n^{-\alpha}\left|\sum_{i=1}^{n}X_i\right|\le\max_{2^{k-1}\le n\le 2^k}n^{-\alpha}\left|\sum_{i=1}^{n}X_i\right|\le\frac{2^{2\alpha}\max_{1\le j\le 2^k}|\sum_{i=1}^{j}X_i|}{2^{(k+1)\alpha}},
$$

which yields (3.2). This completes the proof. □

Proof of Theorem 3.2. For fixed $n \ge 1$, for $1 \le i \le n$, write

$$Y_{ni} = -n^{\alpha} I\{X_i < -n^{\alpha}\} + X_i I\{|X_i| \le n^{\alpha}\} + n^{\alpha} I\{X_i > n^{\alpha}\},$$

$$Z_{ni} = X_i - Y_{ni} = (X_i - n^{\alpha}) I\{X_i > n^{\alpha}\} + (X_i + n^{\alpha}) I\{X_i < -n^{\alpha}\},$$

and

$$Y_n = -n^{\alpha} I\{X < -n^{\alpha}\} + X I\{|X| \le n^{\alpha}\} + n^{\alpha} I\{X > n^{\alpha}\},$$

$$Z_n = X - Y_n = (X - n^{\alpha}) I\{X > n^{\alpha}\} + (X + n^{\alpha}) I\{X < -n^{\alpha}\}.$$

From Lemma 2.4 follows that for any $\beta > 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \mathbb{E}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right| - \varepsilon n^{\alpha}\right)^{+} \\
&\quad \le C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\beta} \mathbb{E}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}(Y_{ni} - \mathbb{E}Y_{ni})\right|^{\beta}\right) \\
&\qquad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \mathbb{E}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}(Z_{ni} - \mathbb{E}Z_{ni})\right|\right) =: III_1 + III_2.
\end{aligned} \tag{4.20}$$

Noticing that $Z_n \le (|X| - n^{\alpha}) I(|X| > n^{\alpha}) \le |X| I(|X| > n^{\alpha})$, by Lemma 2.3, we see that

$$\begin{aligned}
III_2 &\le C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^{n} \mathbb{E}|Z_{ni}| \le C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \mathbb{E}|Z_n| \\
&\le C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} C_{\mathbb{V}}\{|Z_n|\} \le C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} C_{\mathbb{V}}\{|X| I(|X| > n^{\alpha})\} \\
&\le C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \left[\int_0^{n^{\alpha}} \mathbb{V}\{|X| > n^{\alpha}\} \mathrm{d}x + \int_{n^{\alpha}}^{\infty} \mathbb{V}\{|X| > x\} \mathrm{d}x\right] \\
&\le C \sum_{n=1}^{\infty} n^{\alpha p-1} \mathbb{V}\{|X| > n^{\alpha}\} + C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \sum_{k=n}^{\infty} \int_{k^{\alpha}}^{(k+1)^{\alpha}} \mathbb{V}\{|X| > x\} \mathrm{d}x \\
&\le C C_{\mathbb{V}}\{|X|^p\} + C \sum_{k=1}^{\infty} \mathbb{V}\{|X| > k^{\alpha}\} k^{\alpha-1} \sum_{n=1}^{k} n^{\alpha p-1-\alpha} \\
&\le \begin{cases} C C_{\mathbb{V}}\{|X|^p\} + C \sum_{k=1}^{\infty} \mathbb{V}\{|X| > k^{\alpha}\} k^{\alpha-1} \log(k), & \text{if } p = 1, \\ C C_{\mathbb{V}}\{|X|^p\} + C \sum_{k=1}^{\infty} \mathbb{V}\{|X| > k^{\alpha}\} k^{\alpha p-1}, & \text{if } p > 1, \end{cases} \\
&\le \begin{cases} C C_{\mathbb{V}}\{|X| \log |X|\} < \infty, & \text{if } p = 1, \\ C C_{\mathbb{V}}\{|X|^p\} < \infty, & \text{if } p > 1. \end{cases}
\end{aligned} \tag{4.21}$$

Now, we will establish $III_1 < \infty$. Observing that $\theta > p \ge 1$, we can choose $\beta = \theta$. We analysize the following two cases.

Case 1. $1 < \theta \le 2$. From (2.2) of Lemma 2.2, Lemma 2.1, $\mathbb{E}(Y) = \mathbb{E}(-Y) = 0$, and Markov's inequality under sub-linear expectations follows that

$$\begin{aligned}
III_1 &= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta}\mathbb{E}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}(Y_{ni}-\mathbb{E}Y_{ni})\right|^{\theta}\right) \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta}\log^{\theta} n\left[\sum_{i=1}^{n}\mathbb{E}|Y_{ni}|^{\theta}+\left(\sum_{i=1}^{n}[|\mathbb{E}(Y_{ni})|+|\mathbb{E}(-Y_{ni})|]\right)^{\theta}\right] \\
&= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta}\log^{\theta} n\left[\sum_{i=1}^{n}\mathbb{E}|Y_{n}|^{\theta}+\left(\sum_{i=1}^{n}[|\mathbb{E}(Y_{n})|+|\mathbb{E}(-Y_{n})|]\right)^{\theta}\right] \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\theta}\log^{\theta} n C_{\mathbb{V}}\{|Y_n|^{\theta}\}+C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta+\theta}\log^{\theta} n\,(\mathbb{E}|Y_n-X|)^{\theta} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\theta}\log^{\theta} n\int_0^{n^{\alpha}}\mathbb{V}\{|X|>x\}x^{\theta-1}\mathrm{d}x \\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta+\theta}\log^{\theta} n\,(C_{\mathbb{V}}\{|Y_n-X|\})^{\theta} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\theta}\log^{\theta} n\sum_{k=1}^{n}\int_{(k-1)^{\alpha}}^{k^{\alpha}}\mathbb{V}\{|X|>x\}x^{\theta-1}\mathrm{d}x \\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta+\theta}\log^{\theta} n\,(C_{\mathbb{V}}\{|X|I\{|X|>n^{\alpha}\}\})^{\theta} \\
&\le C\sum_{k=1}^{\infty}\mathbb{V}\{|X|>(k-1)^{\alpha}\}k^{\alpha\theta-1}\sum_{n=k}^{\infty} n^{\alpha p-1-\alpha\theta} \\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta+\theta}\log^{\theta} n\left(\int_0^{\infty}\mathbb{V}\{|X|I\{|X|>n^{\alpha}\}>y\}\mathrm{d}y\right)^{\theta} \\
&\le C\sum_{k=1}^{\infty}\mathbb{V}\{|X|>(k-1)^{\alpha}\}k^{\alpha p-1}\log^{\theta} k+C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta+\theta}\log^{\theta} n\left(n^{\alpha}\frac{\mathbb{E}\{|X|^{p}\log^{\theta}|X|\}}{n^{\alpha p}\log^{\theta} n}\right)^{\theta} \\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta+\theta}\log^{\theta} n\left(\int_{n^{\alpha}}^{\infty}\mathbb{V}\{|X|^{p}\log^{\theta}|X|>y^{p}\log^{\theta} y\}\mathrm{d}(y^{p}\log^{\theta} y)/(n^{\alpha(p-1)}\log^{\theta} n)\right)^{\theta} \\
&\le C\int_0^{\infty}\mathbb{V}\{|X|>x^{\alpha}\}x^{\alpha p-1}\log^{\theta} x\mathrm{d}x+C\left(C_{\mathbb{V}}\{|X|^{p}\log^{\theta}|X|\}\right)^{\theta} \\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2+\theta-\alpha p\theta}\log^{\theta-\theta^{2}} n\left(C_{\mathbb{V}}\{|X|^{p}\log^{\theta}|X|\}\right)^{\theta} \\
&\le CC_{\mathbb{V}}\{|X|^{p}\log^{\theta}|X|\}+C\left(C_{\mathbb{V}}\{|X|^{p}\log^{\theta}|X|\}\right)^{\theta}<\infty. \qquad (4.22)
\end{aligned}$$

Case 2. $\theta > 2$. Observe that $\theta > \frac{\alpha p-1}{\alpha-\frac{1}{2}}$, we conclude that $\alpha p - 2 - \alpha\theta + \frac{\theta}{2} < -1$. As in the proof of (4.22), by Lemma 2.2, and C_r inequality, we see that

$$\begin{aligned} III_1 &= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta}\mathbb{E}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}(Y_{ni}-\mathbb{E}(Y_{ni}))\right|^{\theta}\right) \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta}\left[\sum_{i=1}^{n}\mathbb{E}|Y_{ni}|^{\theta}+\left(\sum_{i=1}^{n}\mathbb{E}|Y_{ni}|^{2}\right)^{\theta/2}+\left(\sum_{i=1}^{n}[|\mathbb{E}(Y_{ni})|+|\mathbb{E}(-Y_{ni})|]\right)^{\theta}\right] \\ &=: III_{11}+III_{12}+III_{13}. \end{aligned} \tag{4.23}$$

By Lemma 2.3, we see that

$$\begin{aligned} III_{11} &\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\theta}\mathbb{E}|Y_n|^{\theta}\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\theta}C_{\mathbb{V}}\{|Y_n|^{\theta}\} \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\theta}\int_0^{n^{\alpha}}\mathbb{V}\{|X|>x\}x^{\theta-1}\mathrm{d}x \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\theta}\sum_{k=1}^{n}\int_{(k-1)^{\alpha}}^{k^{\alpha}}\mathbb{V}\{|X|>x\}x^{\theta-1}\mathrm{d}x \\ &\le C\sum_{k=1}^{\infty}\mathbb{V}\{|X|>(k-1)^{\alpha}\}k^{\alpha\theta-1}\sum_{n=k}^{\infty} n^{\alpha p-1-\alpha\theta}\le C\sum_{k=1}^{\infty}\mathbb{V}\{|X|>(k-1)^{\alpha}\}k^{\alpha p-1} \\ &\le C\int_0^{\infty}\mathbb{V}\{|X|>x^{\alpha}\}x^{\alpha p-1}\mathrm{d}x\le CC_{\mathbb{V}}\{|X|^{p}\}<\infty. \end{aligned}$$

By Lemma 2.1, we deduce that

$$\begin{aligned} III_{12} &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta}\left(\sum_{i=1}^{n}\mathbb{E}|Y_n|^{2}\right)^{\theta/2} \\ &\le \begin{cases} C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta+\theta/2}(\mathbb{E}|X|^2)^{\theta/2}, & \text{if } p\ge 2,\\ C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta+\theta/2}\left(\mathbb{E}|X|^{p}n^{\alpha(2-p)}\right)^{\theta/2}, & \text{if } 1\le p<2, \end{cases} \\ &\le \begin{cases} CC_{\mathbb{V}}\{|X|^2\}<\infty, & \text{if } p\ge 2,\\ C\sum_{n=1}^{\infty} n^{\alpha p-2+\theta/2-\alpha p\theta/2}\left(C_{\mathbb{V}}\{|X|^{p}\}\right)^{\theta/2}<\infty, & \text{if } 1\le p<2. \end{cases} \end{aligned}$$

And the proof of $III_{13}<\infty$ is similar to that of (4.22). This finishes the proof. □

5. Conclusions

We have obtained new results about complete convergence and complete moment convergence for maximum partial sums of negatively dependent random variables under sub-linear expectations. Results obtained in our article extend those for negatively dependent random variables under classical probability space, and Theorems 3.1, 3.2 here are different from Theorems 3.1, 3.2 of Xu et al. [20], and the former can not be deduced from the latter. Corollary 3.1 complements Theorem 3.1 in Zhang [9] in the case $p\ge 2$, Corollaries 3.2, 3.3 complement Theorem 3.3 in Zhang [4] in the case $p>1$ in some sense.

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Conflict of interest

All authors state no conflicts of interest in this article.

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