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Research article

An optimal eighth order derivative free multiple root finding scheme and its dynamics

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Abstract: The problem of solving a nonlinear equation is considered to be one of the significant domain. Motivated by the requirement to achieve more optimal derivative-free schemes, we present an eighth-order optimal derivative-free method to find multiple zeros of the nonlinear equation by weight function approach in this paper. This family of methods requires four functional evaluations. The technique is based on a three-step method including the first step as a Traub-Steffensen iteration and the next two as Traub-Steffensen-like iterations. Our proposed scheme is optimal in the sense of Kung-Traub conjecture. The applicability of the proposed schemes is shown by using different nonlinear functions that verify the robust convergence behavior. Convergence of the presented family of methods is demonstrated through the graphical regions by drawing basins of attraction.

Keywords: nonlinear equations; multiple roots; derivative-free methods; basins of attraction
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1. Introduction

Many practical problems are nonlinear in nature, therefore, the problem of solving a nonlinear equation is considered to be one of the significant domain. In addition, the construction of higher order optimal iterative methods for multiple roots having prior knowledge of multiplicity ($\sigma > 1$) has remained one of the most important and challenging tasks in computational mathematics.

Modified Newton's method is a one-point scheme used to find multiple roots ω of a nonlinear equation $f(x) = 0$, with known multiplicity σ. Its iterative expression is

$$\theta_{k+1} = \theta_k - \sigma \frac{f(\theta_k)}{f'(\theta_k)},\ k = 0, 1, 2 \dots,$$

where f is an analytic function in a neighborhood of the zero ω. Sometimes, the derivative $f'(x)$ may be expensive to calculate or may indeed be unavailable. To overcome this problem, Traub–Steffensen [18] replaced the derivative of the function in the modified Newton's method by the divided difference

$$f'(x) \approx f[\mu_k, \theta_k] = \frac{f(\mu_k) - f(\theta_k)}{\mu_k - \theta_k},$$

where $\mu_k = \theta_k + \gamma f(\theta_k)$. Therefore, the modified Newton's method becomes

$$\theta_{k+1} = \theta_k - \sigma \frac{f(\theta_k)}{f[\mu_k, \theta_k]}\ k = 0, 1, 2 \ldots$$

In the literature, there exist many iterative procedures to find the multiple roots of $f(x) = 0$ with derivatives (see, for example [2–4, 12, 13, 15]). The motivation for constructing high-order methods is closely related to the Kung–Traub conjecture [10]. It establishes an upper bound for the order of convergence $\rho \leq 2^{d-1}$, where ρ is the order of convergence and d is the number of functional evaluations. Any iterative method without memory attaining the maximum bound of the Kung-Traub conjecture is called optimal method. However, there are few optimal derivative-free schemes [1, 14, 16, 17], the iterative expression of some of them will be used in the numerical section and are shown below.

In 2019, Sharma et al. [16] proposed multiple root finding method with known multiplicity $\sigma > 1$ given as

$$\begin{aligned}
\mu_k &= \theta_k + \gamma f(\theta_k), \text{ where } \gamma \in \mathbb{R} - \{0\}, \\
\upsilon_k &= \theta_k - \sigma \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \\
\xi_k &= \upsilon_k - \sigma r_k V(r_k) \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \\
\theta_{k+1} &= \xi_k - \sigma s_k L(r_k, t_k) \frac{f(\theta_k)}{f[\theta_k, \mu_k]},\ k = 0, 1, 2, \ldots
\end{aligned} \tag{1.1}$$

being

$$\begin{aligned}
r_k &= \left(\frac{f(\upsilon_k)}{f(\theta_k)}\right)^{\frac{1}{\sigma}}, \\
s_k &= \left(\frac{f(\xi_k)}{f(\theta_k)}\right)^{\frac{1}{\sigma}}, \\
t_k &= \left(\frac{f(\xi_k)}{f(\upsilon_k)}\right)^{\frac{1}{\sigma}},
\end{aligned}$$

where θ_0 is the initial estimation and $V : \mathbb{C} \to \mathbb{C}$ and $L : \mathbb{C}^2 \to \mathbb{C}$ are analytic in the neighborhood of 0 and $(0, 0)$ respectively such that, conditions on V and L are as follows:

$$\begin{aligned}
V(0) &= 1, \quad V'(0) = 2, \quad V''(0) = -2, \quad \text{and } \left|V'''(0)\right| < \infty \\
L(0,0) &= 1, \quad L_{10}(0,0) = 2, \quad L_{01}(0,0) = 1, \quad L_{20}(0,0) = 0
\end{aligned}$$

$$\text{and} \quad |L_{11}(0,0)| < \infty, \text{ where } L_{ij}(0,0) = \left.\frac{\partial^{i+j}}{\partial r^i t^j} L(r,t)\right|_{(0,0),} \quad \text{for } i,j \in \mathbb{N}.$$

Furthermore in 2019, Sharma et al. designed in [17] an optimal eighth order scheme to find the multiple root of the nonlinear equation with known multiplicity $\sigma > 1$:

$$\begin{aligned}
\mu_k &= \theta_k + \gamma f(\theta_k), \text{ where } \gamma \in \mathbb{R} - \{0\}, \\
\upsilon_k &= \theta_k - \sigma \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \\
\xi_k &= \upsilon_k - \sigma h_k (\alpha_1 + \alpha_2 h_k) \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \\
\theta_{k+1} &= \xi_k - \sigma r_k s_k L(h_k, s_k) \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \ k = 0, 1, 2, \ldots
\end{aligned} \tag{1.2}$$

where

$$\begin{aligned}
r_k &= \left(\frac{f(\upsilon_k)}{f(\theta_k)}\right)^{\frac{1}{\sigma}}, \\
s_k &= \left(\frac{f(\xi_k)}{f(\upsilon_k)}\right)^{\frac{1}{\sigma}}, \\
h_k &= \frac{r_k}{1 + r_k},
\end{aligned}$$

and $L : \mathbb{C}^2 \to \mathbb{C}$ is analytic in the neighborhood of $(0,0)$ such that, conditions on α_1, α_2 and L are given as:

$$\begin{aligned}
\alpha_1 &= 1, \ \alpha_2 = 3, \ L_{00} = 1, \\
L_{01} &= 1, L_{10} = 2, L_{20} = -4, \ L_{11} = 4, \\
L_{30} &= -72, \ \mid L_{02} \mid < \infty \text{ and } \mid L_{21} \mid < \infty, \\
\text{where} \quad L_{ij} &= \left.\frac{\partial^{i+j}}{\partial h^i \partial s^j} L(h_k, s_k)\right|_{(0,0)}, \ i,j \in \{0,1,2,3,4\}.
\end{aligned}$$

Recently, Sharma and Kumar [14] presented another eighth order derivative-free multiple root finding scheme with multiplicity $\sigma > 1$. This three-step iterative scheme is described as follows:

$$\begin{aligned}
\mu_k &= \theta_k + \gamma f(\theta_k), \text{ where } \gamma \in \mathbb{R} - \{0\}, \\
\upsilon_k &= \theta_k - \sigma \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \\
\xi_k &= \upsilon_k - G(r_k, s_k) \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \\
\theta_{k+1} &= \xi_k - u_k H(r_k, s_k, t_k) \frac{f(\theta_k)}{f[\theta_k, \mu_k]},
\end{aligned}$$

where, $G : \mathbb{C}^2 \to \mathbb{C}$ and $H : \mathbb{C}^3 \to \mathbb{C}$ are holomorphic in the neighborhood of $(0,0)$ and $(0,0,0)$ respectively. Here,

$$r_k = \left(\frac{f(\upsilon_k)}{f(\theta_k)}\right)^{\frac{1}{\sigma}},$$

$$\begin{aligned} s_k &= \left(\frac{f(\mu_k)}{f(\theta_k)}\right)^{\frac{1}{\sigma}}, \\ t_k &= \left(\frac{f(\xi_k)}{f(\upsilon_k)}\right)^{\frac{1}{\sigma}}, \\ u_k &= \left(\frac{f(\xi_k)}{f(\theta_k)}\right)^{\frac{1}{\sigma}}. \end{aligned}$$

A drawback of this scheme is that the conditions on weight functions G and H varies with the changing value of multiplicity σ.

Based on the requirement to develop efficient derivative-free multiple root schemes, we give a derivative-free optimal eighth order convergent scheme to find the repeated roots with multiplicity $\sigma > 1$ (Section 2). This proposed scheme has four functional evaluations and is based on the first-order divided differences and involvement of two weight functions. We compare our methods in Section 3 with two of the recent derivative free methods of seventh [16] and eighth order [17] using physical problems of chemistry, physics and biology [6, 8]. The performance of our family of methods along with the demonstration of their basins of attraction is also discussed in Section 4.

2. Construction of optimal eighth-order scheme

Let us give a three-step derivative free scheme to find multiple zeros of the nonlinear equations, having a positive integer multiplicity $\sigma \geqslant 1$. If this multiplicity is unknown, it can be estimated by different techniques that appear in [11].

$$\begin{aligned} \mu_k &= \theta_k + \gamma f(\theta_k), \text{ where } \gamma \in \mathbb{R} - \{0\}, \\ \upsilon_k &= \theta_k - \sigma \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \\ \xi_k &= \upsilon_k - \sigma r_k V(r_k) \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \\ \theta_{k+1} &= \xi_k - \sigma s_k P(r_k, s_k, t_k) \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \ k = 0, 1, 2, \ldots \end{aligned} \tag{2.1}$$

where $r_k = \left(\frac{f(\upsilon_k)}{f(\theta_k)}\right)^{\frac{1}{\sigma}}$, $s_k = \left(\frac{f(\xi_k)}{f(\theta_k)}\right)^{\frac{1}{\sigma}}$ and $t_k = \left(\frac{f(\xi_k)}{f(\upsilon_k)}\right)^{\frac{1}{\sigma}}$. Let $V : \mathbb{C} \to \mathbb{C}$ and $P : \mathbb{C}^3 \to \mathbb{C}$ be analytic functions in the neighborhood of 0 and $(0, 0, 0)$ respectively.

The investigation on the convergence analysis of the proposed family (2.1) and the conditions on weight functions $V(r_k)$ and $P(r_k, s_k, t_k)$ are apparent from the following result.

2.1. Convergence analysis

Theorem 1. *Let function $f : \mathbb{C} \to \mathbb{C}$ be analytic in a region that contains the multiple root ω of f with known multiplicity σ. Let θ_0 be an initial guess which is sufficiently close to the repeated root. Then, scheme* (2.1) *possess eighth order of convergence in case it satisfies the following conditions:*

$$V(0) = 1, \ V'(0) = 2, \ V''(0) = -2 \textit{ and } V'''(0) = 36,$$

$$\begin{aligned} P_{000} &= 1,\ P_{100}=2,\ P_{001}=1,\ P_{101}=4-P_{010}, \\ |P_{110}| &< \infty,\ |P_{002}|<\infty, \\ where,\quad P_{ijl} &= \frac{\partial^{i+j+l}}{\partial r^i \partial s^j \partial t^l} P(r,s,t)\Big|_{(0,0,0)}. \end{aligned} \tag{2.2}$$

The error equation of the proposed scheme is given by:

$$\begin{aligned} e_{k+1} &= -\frac{1}{48\sigma^7}(c_1((11+\sigma)c_1^2-2\sigma c_2)(-24(1+\sigma)^2c_1^3+ \\ &(3P_{002}(11+\sigma)^2+2(-665-84\sigma+5\sigma^2+6P_{110}(11+\sigma)))c_1^4 \\ &-12\sigma(P_{002}(11+\sigma)+2(-10+P_{110}+4\sigma))c_1^2c_2+12(-2+P_{002})\sigma^2c_2^2 \\ &+120\sigma^2c_1c_3))e_k^8+O\left(e_k^9\right), \end{aligned} \tag{2.3}$$

where

$$c_i = \frac{\sigma!}{(\sigma+i)!}\frac{f^{(\sigma+i)}(\omega)}{f^{(\sigma)}(\omega)},\ i\in\mathbb{N}.$$

Proof. Let ω be the multiple root of $f(x)=0$ and $e_k=\theta_k-\omega$ be error in the *kth* iteration. Considering that $f^{(m)}(\omega)=0$, $m=0,1,2,...,\sigma-1$ and $f^{(\sigma)}(\omega)\neq 0$, the Taylor's expansion of f around ω, gives:

$$\begin{aligned} f(\theta_k) &= \frac{f^{(\sigma)}(\omega)}{\sigma!}e_k^\sigma+\frac{f^{(\sigma+1)}(\omega)}{(\sigma+1)!}e_k^{\sigma+1}+\frac{f^{(\sigma+2)}(\omega)}{(\sigma+2)!}e_k^{\sigma+2}+\frac{f^{(\sigma+3)}(\omega)}{(\sigma+3)!}e_k^{\sigma+3} \\ &+\frac{f^{(\sigma+4)}(\omega)}{(\sigma+4)!}e_k^{\sigma+4}+\frac{f^{(\sigma+5)}(\omega)}{(\sigma+5)!}e_k^{\sigma+5}+\frac{f^{(\sigma+6)}(\omega)}{(\sigma+6)!}e_k^{\sigma+6}+\frac{f^{(\sigma+7)}(\omega)}{(\sigma+7)!}e_k^{\sigma+7} \\ &+\frac{f^{(\sigma+8)}(\omega)}{(\sigma+8)!}e_k^{\sigma+8}+O(e_k^{\sigma+9}), \end{aligned} \tag{2.4}$$

which can be written as:

$$f(\theta_k)=\frac{f^{(\sigma)}(\omega)}{\sigma!}e_k^\sigma\left(1+c_1e_k+c_2e_k^2+\ldots+c_7e_k^7+c_8e_k^8+O\left(e_k^9\right)\right), \tag{2.5}$$

where,

$$c_i = \frac{\sigma!}{(\sigma+i)!}\frac{f^{(\sigma+i)}(\omega)}{f^{(\sigma)}(\omega)},$$

for $i\in\mathbb{N}$.

Next let us consider $\mu_k=\theta_k+\gamma f(\theta_k)$ and $e_k=\theta_k-\omega$, given as:

$$\mu_k-\omega=\theta_k-\omega+\gamma f(\theta_k),$$

$$\mu_k-\omega=e_k+\gamma f(\theta_k), \tag{2.6}$$

such that from (2.5),

$$\mu_k=e_k+\frac{\gamma f^{(\sigma)}(\omega)}{\sigma!}e_k^\sigma\left(1+c_1e_k+c_2e_k^2+\ldots+c_7e_k^7+c_8e_k^8+O\left({e_k}^9\right)\right). \tag{2.7}$$

Expanding $f(\mu_k)$ around ω, we have

$$\begin{aligned} f(\mu_k) &= \frac{f^{(\sigma)}(\omega)}{\sigma!}(\mu_k-\omega)^{\sigma}\Big(1+c_1(\mu_k-\omega)+c_2(\mu_k-\omega)^2+\ldots \\ &\quad +c_7(\mu_k-\omega)^7+c_8(\mu_k-\omega)^8\Big). \end{aligned} \tag{2.8}$$

Upon substituting the values from (2.5) and (2.7) in the first step of (2.1) and simplifying yields:

$$\begin{aligned} \upsilon_k &= \frac{c_1}{\sigma}e_k^2+\frac{2\sigma c_2-(\sigma+1)c_1^2}{\sigma^2}e_k^3 \\ &\quad +\frac{1}{\sigma^3}\Big((\sigma+1)^2c_1^2+\sigma(4+3\sigma)c_1c_2-3\sigma^2c_3\Big)e_k^4+\sum_{i=1}^{4}a_ie_k^{i+4}+O\left(e_k^9\right), \end{aligned} \tag{2.9}$$

where, $a_i=a_i(\sigma,c_1,\ldots,c_8)$.

Next, the expansion of $f(\upsilon_k)$ around ω is:

$$\begin{aligned} f(\upsilon_k) &= \left(e_k^2\frac{c_1}{\sigma}\right)^{\sigma}\left(\frac{1}{\sigma!}+\frac{2\sigma c_2-(1+\sigma)c_1^2}{\sigma!c_1}\right)e_k+\frac{1}{2\sigma\sigma!c_1^2}(2(1+\sigma)^2c_1^3 \\ &\quad +\left(\sigma^3+\sigma^2-\sigma+1\right)c_1^4+2\sigma\left(6+3\sigma-2\sigma^2\right)c_1^2c_2+4(\sigma-1)\sigma^2c_2^2 \\ &\quad -6\sigma^2c_1c_3)e_k^2+\sum_{i=1}^{6}d_ie_k^{i+2}+O\left(e_k^9\right), \end{aligned}$$

which can also be written as,

$$\begin{aligned} f(\upsilon_k) &= \frac{f^{(\sigma)}(\omega)}{\sigma!}(\frac{c_1}{\sigma})^{\sigma}e_k^{2\sigma}\Bigg(1+\frac{2c_2\sigma-c_1^2(\sigma+1)}{c_1}e_k+\frac{1}{2\sigma c_1^2} \\ &\quad ((3+3\sigma+3\sigma^2+\sigma^3)c_1^4-2\sigma(2+3\sigma+2\sigma^2)c_1^2c_2 \\ &\quad +4(-1+\sigma)\sigma^2c_2^2+6\sigma^2c_1c_3)e_k^2+\sum_{i=1}^{6}d_ie_k^{i+2}+O\left(e_k^9\right)\Bigg), \end{aligned} \tag{2.10}$$

where, $d_i=d_i(\sigma,c_1,\ldots,c_8)$.

Using (2.5) and (2.10) in $r_k=\left(\frac{f(\upsilon_k)}{f(\theta_k)}\right)^{\frac{1}{\sigma}}$,

$$\begin{aligned} r_k &= \frac{c_1}{\sigma}e_k+\frac{(2\sigma c_2-(2+\sigma)c_1^2)}{\sigma^2}e_k^2+\frac{1}{2\sigma^3}(2(1+\sigma)^2c_1^2+ \\ &\quad (5+3\sigma)c_1^3+2\sigma(1+3\sigma)c_1c_2-6\sigma^2c_3)e_k^3+\sum_{i=1}^{5}h_ie_k^{i+3}+O\left(e_k^9\right), \end{aligned} \tag{2.11}$$

where, $h_i=h_i(\sigma,c_1,\ldots,c_8)$ are given in the terms of σ and $c_j's$, $j=1,\ldots,8$.

Developing the expansion of the weight function $V(r_k)$ in the neighbourhood of 0 implies,

$$V(r_k)\approx V(0)+r_kV'(0)+\frac{1}{2}r_k^2V''(0)+\frac{1}{6}r_k^3V'''(0)+\ldots. \tag{2.12}$$

So, using (2.9)–(2.12) in the second step of the method,

$$
\begin{aligned}
\xi_k &= -\frac{(V(0)-1)c_1e_k^2}{\sigma} - \frac{(1+V'(0)+\sigma-V(0)(3+\sigma))c_1^2+2(V(0)-1)\sigma c_2)e_k^3}{\sigma^2} \\
&+\frac{1}{2\sigma^3}(-2(V(0)-1)(1+\sigma)^2c_1^2-(V''(0)-2V'(0)(5+2\sigma) \\
&+V(0)(11+7\sigma)c_1^3+2\sigma(4-4V'(0)-3V(0)(\sigma-1)+3\sigma)c_1c_2 \\
&+6(V(0)-1)\sigma^2c_3)e_k^4+\sum_{i=1}^{4}w_ie_k^{i+4}+O\left(e_k^9\right),
\end{aligned}
\tag{2.13}
$$

where, $w_i = w_i(\sigma, c_1, \dots, c_8)$. If we choose the values of $V(0)$ and $V'(0)$, given as:

$$
V(0) = 1, \quad V'(0) = 2, \tag{2.14}
$$

then we achieve fourth order for the second step as:

$$
\begin{aligned}
\xi_k &= \frac{((9-V''(0)+\sigma)c_1^3-2\sigma c_1c_2)e_k^4}{2\sigma^3} \\
&- \frac{1}{6\sigma^4}(6(1+\sigma)^2c_1^3+(119+V'''(0)+72\sigma \\
&+\sigma^2-3V''(0)(7+3\sigma))c_1^4+12\sigma^2c_2^2-24\sigma^2c_1c_3 \qquad (2.15)\\
&+6(-20+3V''(0)+2\sigma)\sigma c_1^2c_2)e_k^5+\sum_{i=1}^{3}w_i'e_k^{5+i}+O\left(e_k^9\right), \qquad (2.16)
\end{aligned}
$$

where, $w_i' = w_i'(\sigma, c_1, \dots, c_8)$. Subsequently, $f(\xi_k)$ around ω results in:

$$
\begin{aligned}
f(\xi_k) &= e_k^{4\sigma}\left(\frac{2^{-\sigma}}{\sigma!}\left(\frac{(11+\sigma)c_1^3-2\sigma c_1c_2}{\sigma^3}\right)^{\sigma}\right. \\
&-\frac{1}{3(\sigma^3\sigma!)}(2^{-\sigma}\left(\frac{(11+\sigma)c_1^3-2\sigma c_1c_2}{\sigma^3}\right)^{-1+\sigma}(6(1+\sigma)^2c_1^3 \\
&+(161+V'''(0)+90\sigma+\sigma^2)c_1^4+12(\sigma-13)\sigma c_1^2c_2 \\
&\left.+12\sigma^2c_2^2-24\sigma^2c_1c_3))e_k\right)+\sum_{i=1}^{7}u_ie_k^{1+i}+O\left(e_k^9\right),
\end{aligned}
\tag{2.17}
$$

where, $u_i = u_i(\sigma, c_1, \dots, c_8)$. By using (2.5) and (2.17), $s_k = \left(\frac{f(\xi_k)}{f(\theta_k)}\right)^{\frac{1}{\sigma}}$ it becomes:

$$
\begin{aligned}
s_k &= \frac{\left((9-V''(0)+\sigma)c_1^3-2\sigma c_1c_2\right)e_k^3}{2\sigma^3}-\frac{1}{6\sigma^4}(6(1+\sigma)^2c_1^3 \\
&+(194+V'''(0)+93\sigma+\sigma^2)c_1^4+6\sigma(-27+2\sigma)c_1^2c_2+12\sigma^2c_2^2 \\
&-24\sigma^2c_1c_3)e_k^4+\sum_{i=1}^{4}u_i'e_k^{4+i}+O\left({e_k}^9\right),
\end{aligned}
\tag{2.18}
$$

where $u'_i = u'_i(\sigma, c_1, \ldots, c_8)$ and from (2.10) and (2.17) $t_k = \left(\frac{f(\xi_k)}{f(\upsilon_k)}\right)^{\frac{1}{\sigma}}$, implies:

$$\begin{aligned} t_k \;=\; & \frac{\left((9 - V''(0) + \sigma) c_1^2 - 2\sigma c_2\right) e_k^2}{2\sigma^2} + \frac{1}{6\sigma^3}(-6(1+\sigma)^2 c_1^2 \\ & -(128 + V'''(0) + 54\sigma - 2\sigma^2) c_1^3 + 12(7 - 2\sigma)\sigma c_1 c_2 \\ & +24\sigma^2 c_3) e_k^3 + \sum_{i=1}^{5} v_i e_k^{3+i} + O\left(e_k{}^9\right), \end{aligned} \tag{2.19}$$

where, $v_i = v_i(\sigma, c_1, \ldots, c_8)$. The expansion of the weight function $P(r, s, t)$ in the neighborhood of $(0, 0, 0)$ is given by,

$$\begin{aligned} e_{k+1} \;=\; & \xi_k - s_k(P_{000} + r_k P_{100} + t_k P_{001} + s_k P_{010} + r_k t_k P_{101} \\ & +r_k s_k P_{110} + s_k t_k P_{011} + \frac{t_k^2}{2} P_{002})(e_k - \upsilon_k). \end{aligned} \tag{2.20}$$

Applying the values of r_k, s_k and t_k from (2.11), (2.18) and (2.19) in (2.20), we obtain:

$$\begin{aligned} e_{k+1} \;=\; & -\frac{1}{2\sigma^3}\left((-1 + P_{000}) c_1 \left((9 - V''(0) + \sigma) c_1^2 - 2\sigma c_2\right)\right) e_k^4 \\ & +\frac{1}{6\sigma^4}(6(1+\sigma)^2(-1 + P_{000}) c_1^3 + (-161 + V'''(0)(-1 + P_{000}) \\ & +\sigma^2(-1 + P_{000}) + 227P_{000} + \sigma(-90 + 96P_{000} - 3P_{100}) - 33P_{100}) c_1^4 \\ & +6\sigma(26 + 2\sigma(-1 + P_{000}) - 28P_{000} + P_{100}) c_1^2 c_2 + 12\sigma^2(-1 + P_{000}) c_2^2 \\ & -24\sigma^2(-1 + P_{000}) c_1 c_3) e_k^5 + \sum_{i=1}^{3} v'_i e_k^{5+i} + O\left(e_k{}^9\right), \end{aligned} \tag{2.21}$$

where, $v'_i = v'_i(\sigma, c_1, \ldots, c_8)$. To remove the lower order terms, we use the values of $V''(0)$, P_{000}, P_{100}, P_{001}, P_{101} as:

$$V''(0) = -2, \tag{2.22}$$

$$P_{000} = 1, \quad P_{100} = 2, \tag{2.23}$$

$$P_{001} = 1, \quad P_{101} = 4 - P_{010}, \tag{2.24}$$

so that (2.22) and (2.23) yields:

$$\begin{aligned} e_{k+1} \;=\; & -\frac{1}{4\sigma^5}\left((-1 + P_{001}) c_1 \left((11 + \sigma) c_1^2 - 2\sigma c_2\right)^2\right) e_k^6 \\ & -\frac{1}{12\sigma^6}\left(\left((11 + \sigma) c_1^2 - 2\sigma c_2\right)(-12(1+\sigma)^2(-1 + P_{001}) c_1^3\right. \\ & +(259 + V'''(0) + \sigma^2(-1 + P_{001}) - 355P_{001} - 2V'''(0) P_{001} + 33P_{010} \\ & +33P_{101} + 3\sigma(46 - 50P_{001} + P_{010} + P_{101})) c_1^4 - 6\sigma(38 \\ & +6\sigma(-1 + P_{001}) - 42P_{001} + P_{010} + P_{101}) c_1^2 c_2 - 12\sigma^2(-1 + P_{001}) c_2^2 \\ & \left.+48\sigma^2(-1 + P_{001}) c_1 c_3)\right) e_k^7 + \sum_{i=1}^{2} k_i e_k^{7+i} + O\left(e_k{}^9\right), \end{aligned} \tag{2.25}$$

where, $k_i = k_i(\sigma, c_1, \ldots, c_8)$. and (2.24) gives:

$$\begin{aligned} e_{k+1} &= \frac{1}{12\sigma^6}\left(-36 + V'''(0)\right) c_1^4 \left((11+\sigma) c_1^2 - 2\sigma c_2\right) e_k^7 \\ &-\frac{1}{144\sigma^7}(c_1(24(1+\sigma)^2 (V'''(0) - 3(23+\sigma)) c_1^5 + (4(V'''(0))^2 \\ &+4V'''(0)\left(323 + 207\sigma + 10\sigma^2\right) + 3(-31862 + 3993P_{002} + \sigma^3(10 + 3P_{002}) \\ &+1452P_{110} + \sigma^2(-538 + 99P_{002} + 12P_{110}) + \sigma(-13114 + 1089P_{002} + 264P_{110})))c_1^6 \\ &-6\sigma(-12010 + 4V'''(0)(65 + 4\sigma) + 1089P_{002} + \sigma^2(58 + 9P_{002}) + 264P_{110} \\ &+6\sigma(-56 + 33P_{002} + 4P_{110}))c_1^4 c_2)) + O\left({e_k}^8\right). \end{aligned} \tag{2.26}$$

If $V'''(0) = 36$ in (2.26), the required eighth order of convergence is achieved and the error equation is:

$$\begin{aligned} e_{k+1} &= -\frac{1}{48\sigma^7}(c_1((11+\sigma)c_1^2 - 2\sigma c_2)(-24(1+\sigma)^2 c_1^3 \\ &+(3P_{002}(11+\sigma)^2 + 2(-665 - 84\sigma + 5\sigma^2 + 6P_{110}(11+\sigma)))c_1^4 \\ &-12\sigma(P_{002}(11+\sigma) + 2(-10 + P_{110} + 4\sigma))c_1^2 c_2 + 12(-2 + P_{002})\sigma^2 c_2^2 \\ &+120\sigma^2 c_1 c_3))e_k^8 + O\left(e_k^9\right). \end{aligned} \tag{2.27}$$

□

We can observe that from Theorem 1, several repeated root-finding schemes can be obtained by merely changing the cases of $V(r_k)$ and $P(r_k, s_k, t_k)$ according to the condition set (2.2). It is noteworthy that the selection of specific values of parameter γ can be made under the point of view of improvement in the stability of the scheme and a widening of the set of converging initial estimations.

2.2. Particular cases of weight functions

As mentioned earlier, we can generate many cases of scheme (2.1) by using different kind of weight functions $V(r)$ and $P(r, s, t)$ that satisfy the conditions stated in Theorem 1. The discussion of some of these special cases is given as follows:

Case 1. Let the weight function be the polynomial of degree three satisfying the condition set (2.2) and is defined as,

$$V(r) = a + br + cr^2 + dr^3.$$

From the described conditions on V, the expression for $V(r)$ is:

$$V(r) = 1 + 2r - r^2 + 6r^3.$$

Let us take another weight function $P(r, s, t)$, to be a linear polynomial:

$$P(r, s, t) = a + br + cs + dt.$$

Upon applying the conditions described in Theorem 1, we get,

$$P(r, s, t) = 1 + 2r + 4s + t.$$

Therefore,

$$\begin{aligned}
\mu_k &= \theta_k + \gamma f(\theta_k), \text{ where } \gamma \in \mathbb{R} - \{0\}, \\
\upsilon_k &= \theta_k - \sigma \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \\
\xi_k &= \upsilon_k - \sigma r_k (1 + 2r_k - r_k^2 + 6r_k^3) \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \\
\theta_{k+1} &= \xi_k - \sigma s_k (1 + 2r_k + 4s_k + t_k) \frac{f(\theta_k)}{f[\theta_k, \mu_k]}.
\end{aligned}$$

Case 2. Here, we take $V(r)$ as an improper rational function as:

$$V(r) = \frac{1 - 9r^2}{1 - 2r - 4r^2}$$

and taking $P(r, s, t)$ as a polynomial function:

$$P(r, s, t) = 1 + 2r + t + 4rt$$

where both of them are satisfying the conditions of Theorem 1. Consequently, we get the following:

$$\begin{aligned}
\mu_k &= \theta_k + \gamma f(\theta_k), \text{ where } \gamma \in \mathbb{R} - \{0\}, \\
\upsilon_k &= \theta_k - \sigma \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \\
\xi_k &= \upsilon_k - \sigma r_k \left(\frac{1 - 9r_k^2}{1 - 2r_k - 4r_k^2} \right) \frac{f(\theta_k)}{f[\theta_k, \mu_k]}, \\
\theta_{k+1} &= \xi_k - \sigma s_k (1 + 2r_k + t_k + 4r_k t_k) \frac{f(\theta_k)}{f[\theta_k, \mu_k]}.
\end{aligned}$$

Case 3. Further, let $V(r)$ has the improper rational form as,

$$V(r) = \frac{1 + ar + br^2 + cr^3}{1 + dr}.$$

Applying the conditions to this function results in:

$$V(r) = \frac{1 + 3r + r^2 + 5r^3}{1 + r}$$

and $P(r, s, t)$ is same as that of Case 1,

$$P(r, s, t) = 1 + 2r + 4s + t,$$

so that, the family of methods presented in (2.1) becomes:

$$\begin{aligned}
\mu_k &= \theta_k + \gamma f(\theta_k), \text{ where } \gamma \in \mathbb{R} - \{0\}, \\
\upsilon_k &= \theta_k - \sigma \frac{f(\theta_k)}{f[\theta_k, \mu_k]},
\end{aligned}$$

$$\begin{aligned}\xi_k &= \upsilon_k - \sigma r_k \left(\frac{1+3r_k+r_k^2+5r_k^3}{1+r_k}\right)\frac{f(\theta_k)}{f[\theta_k,\mu_k]},\\ \theta_{k+1} &= \xi_k - \sigma s_k(1+2r_k+4s_k+t_k)\frac{f(\theta_k)}{f[\theta_k,\mu_k]}.\end{aligned}$$

Case 4. Similarly, let V be given by,

$$V(r) = \frac{1+8r+11r^2}{1+6r},$$

where the weight function P is the same as that of Case 2,

$$P(r,s,t) = 1+2r+t+4rt.$$

Then, it results in the following new scheme:

$$\begin{aligned}\mu_k &= \theta_k + \gamma f(\theta_k),\ \text{where } \gamma \in \mathbb{R}-\{0\},\\ \upsilon_k &= \theta_k - \sigma\frac{f(\theta_k)}{f[\theta_k,\mu_k]},\\ \xi_k &= \upsilon_k - \sigma r_k \left(\frac{1+8r_k+11r_k^2}{1+6r_k}\right)\frac{f(\theta_k)}{f[\theta_k,\mu_k]},\\ \theta_{k+1} &= \xi_k - \sigma s_k(1+2r_k+t_k+4r_kt_k)\frac{f(\theta_k)}{f[\theta_k,\mu_k]}.\end{aligned}$$

3. Numerical results

We investigate the performance and convergence behavior of our proposed eighth order methods given by Cases 1–4, we denote the cases as $DZ1$, $DZ2$, $DZ3$, and $DZ4$, respectively. Our test functions involve some physical problems of physics, chemistry and biology. We compare the methods with the recent derivative-free methods of seventh order of Sharma et al. [16] (Case I(a), Case I(b), Case II(c)) denoted by $SH1$, $SH2$ and eighth order schemes of Sharma et al. [17] (M-1, M-4) denoted as $SH3$, $SH4$. We take the value of $\gamma = 0.001$.

For numerical tests, all computations have been performed in computer algebra software Maple 16 using 300 significant digits of precision. Tables show per step numerical errors of approximating real root $|\theta_k - \theta_{k-1}|$ of the first three iterations, the absolute residual error of the test function at the third iteration and the computational order of convergence [9] defined as:

$$COC \approx \frac{\ln|f(\theta_{k+2})/f(\theta_{k+1})|}{\ln|f(\theta_{k+1})/f(\theta_k)|}, \quad k = 1,2,...$$

Let us explicitly give the schemes $SH1$, $SH2$, $SH3$ and $SH4$. First consider the seventh order scheme of Sharma et al. [16] as stated in (1.1). The special cases of the scheme denoted by $SH1$ and $SH2$, are given as,

$$\begin{aligned}SH1: \mu_k &= \theta_k + \gamma f(\theta_k),\ \text{where } \gamma \in \mathbb{R}-\{0\},\\ \upsilon_k &= \theta_k - \sigma\frac{f(\theta_k)}{f[\theta_k,\mu_k]},\end{aligned}$$

$$
\begin{aligned}
\xi_k &= \upsilon_k - \sigma r_k\left(1+2r_k-r_k^2\right)\frac{f(\theta_k)}{f[\theta_k,\mu_k]},\\
\theta_{k+1} &= \xi_k - \sigma s_k\left(1+2r_k+t_k+t_k^2\right)\frac{f(\theta_k)}{f[\theta_k,\mu_k]},
\end{aligned}
$$

and

$$
\begin{aligned}
SH2:\mu_k &= \theta_k+\gamma f(\theta_k),\ \text{where } \gamma\in\mathbb{R}-\{0\},\\
\upsilon_k &= \theta_k-\sigma\frac{f(\theta_k)}{f[\theta_k,\mu_k]},\\
\xi_k &= \upsilon_k - \sigma r_k\left(\frac{2+5r_k}{2+r_k}\right)\frac{f(\theta_k)}{f[\theta_k,\mu_k]},\\
\theta_{k+1} &= \xi_k - \sigma s_k\left(1+2r_k+t_k+t_k^2\right)\frac{f(\theta_k)}{f[\theta_k,\mu_k]},
\end{aligned}
$$

The performance of this newly presented optimal eighth order family of methods can also be compared with the eighth order methods (see [17], M-1, M-4) given by (1.2) The special cases denoted by $SH3$ and $SH4$ are:

$$
\begin{aligned}
SH3:\mu_k &= \theta_k+\gamma f(\theta_k),\ \text{where } \gamma\in\mathbb{R}-\{0\},\\
\upsilon_k &= \theta_k-\sigma\frac{f(\theta_k)}{f[\theta_k,\mu_k]},\\
\xi_k &= \upsilon_k - \sigma h_k(1+3h_k)\frac{f(\theta_k)}{f[\theta_k,\mu_k]},\\
\theta_{k+1} &= \xi_k - \sigma r_k s_k\left(1+2h_k+t_k-2h_k^2+4h_kt_k-12h_k^3\right)\frac{f(\theta_k)}{f[\theta_k,\mu_k]},
\end{aligned}
$$

and

$$
\begin{aligned}
SH4:\mu_k &= \theta_k+\gamma f(\theta_k),\ \text{where } \gamma\in\mathbb{R}-\{0\},\\
\upsilon_k &= \theta_k-\sigma\frac{f(\theta_k)}{f[\theta_k,\mu_k]},\\
\xi_k &= \upsilon_k - \sigma h_k(1+3h_k)\frac{f(\theta_k)}{f[\theta_k,\mu_k]},\\
\theta_{k+1} &= \xi_k - \sigma r_k s_k\left(\frac{1+3h_k+2t_k+8h_kt_k-14h_k^3}{(1+h_k)(1+t_k)}\right)\frac{f(\theta_k)}{f[\theta_k,\mu_k]}.
\end{aligned}
$$

Next, we have considered the following physical problems.

Example 1. Continuous stirred tank reactor.

Consider an isothermal CST reactor. Let U and Γ be the components taken in the reactor then the following reaction scheme develops in the reactor (see [5]):

$$
\begin{aligned}
U+\Gamma &\rightarrow W.\\
W+\Gamma &\rightarrow X.\\
X+\Gamma &\rightarrow Y.
\end{aligned}
$$

$$Y + \Gamma \quad \rightarrow \quad Z.$$

Douglas (see [6]) described this phenomena as a simple feedback control system. For the transfer function of the reactor, the following equation was considered:

$$\kappa_c \times \frac{2.98(\theta + 2.25)}{(\theta^4 + 11.50\theta^3 + 47.49\theta^2 + 83.063250\theta + 51.23266875)} = -1,$$

where κ_c is the gain of proportional controller. For the stability of the control system, we have to take the values of κ_c that result in the zeros of the transfer function possessing a negative real part. Let us consider that $\kappa_c = 0$, then the roots of the nonlinear equation are obtained from the singularities of the open-loop transfer function:

$$f_1(\theta) = (\theta^4 + 11.50\theta^3 + 47.49\theta^2 + 83.063250\theta + 51.23266875),$$

where, $\omega = -1.45$, $\omega = -2.85$, $\omega = -2.85$ and $\omega = -4.35$. We take $\omega = -2.85$ with multiplicity $\sigma = 2$. Taking the initial guess $\theta_0 = -3.13$ gives the numerical calculations presented in Table 1.

Table 1. Comparison of multiple root finding methods for $f_1(\theta)$.

$Scheme$	$\lvert\theta_1 - \theta_0\rvert$	$\lvert\theta_2 - \theta_1\rvert$	$\lvert\theta_3 - \theta_2\rvert$	$\lvert f_1(\theta)\rvert$	COC
$SH1$	0.4506	0.22799	5.7457×10^{-2}	3.4471×10^{-8}	5.65
$SH2$	0.4506	0.22800	5.7461×10^{-2}	3.4527×10^{-8}	5.65
$SH3$	0.3670	0.09130	4.2327×10^{-3}	9.9881×10^{-27}	8.22
$SH4$	0.3351	0.05596	8.4514×10^{-4}	1.3030×10^{-27}	5.80
$DZ1$	0.3676	0.09191	4.3049×10^{-3}	3.5910×10^{-27}	8.43
$DZ2$	0.3676	0.09191	4.3050×10^{-3}	3.5772×10^{-27}	8.43
$DZ3$	0.3676	0.09191	4.3048×10^{-3}	3.5996×10^{-27}	8.43
$DZ4$	0.3676	0.09191	4.3046×10^{-27}	3.6355×10^{-27}	8.42

Example 2. Isentropic supersonic flow.

Hoffman and Zucrow [17] derived a relation between the Mach number, before and after the corner, represented by m_1 and m_2 respectively. Then along a sharp extension corner, the isentropic supersonic flow is given by .

$$\phi = a^{\frac{1}{2}} \left(\tan^{-1} \left(\frac{m_2^2 - 1}{a} \right)^{\frac{1}{2}} - \tan^{-1} \left(\frac{m_1^2 - 1}{a} \right)^{\frac{1}{2}} \right) - ((\tan^{-1} \left(m_2^2 - 1 \right)^{\frac{1}{2}}) - (\tan^{-1} \left(m_1^2 - 1 \right)^{\frac{1}{2}})),$$

$$a = \frac{\beta + 1}{\beta - 1},$$

where β is the specific heat ratio of the gas. For specific values of $\beta = 1.4$, $m_1 = 1.5$ and $\phi = 10^0$, we solve the equation for $m_2 = \theta$ and get,

$$f_2(\theta) = \tan^{-1} \left(\frac{\sqrt{5}}{2} \right) - \tan^{-1} \left(\sqrt{\theta^2 - 1} \right) + \sqrt{6} \left[\tan^{-1} \left(\frac{\sqrt{\theta^2 - 1}}{6} \right) - \tan^{-1} \left(\frac{1}{2} \sqrt{\frac{5}{6}} \right) \right] - \frac{11}{63}.$$

This yields the simple root $\omega = -1.8411$ with multiplicity $\sigma = 1$. Taking the initial guess $\theta_0 = -0.315$ gives the computational results as shown in Table 2.

Table 2. Comparison of multiple root finding methods for $f_2(\theta)$.

$Scheme$	$\lvert\theta_1 - \theta_0\rvert$	$\lvert\theta_2 - \theta_1\rvert$	$\lvert\theta_3 - \theta_2\rvert$	$\lvert f_2(\theta)\rvert$	COC
$SH1$	4.3699	39.13936	3.7074×10^{3}	1.8937	0.14
$SH2$	3.8452	18.03197	6.5655×10^{2}	1.8932	0.75
$SH3$	3.8043	16.11592	5.3614×10^{2}	1.8858	−0.46
$SH4$	1.8896	1.77053	0.8179	1.1978×10^{-4}	4.23
$DZ1$	1.8607	1.84020	0.0707	1.6545×10^{-13}	7.06
$DZ2$	2.7685	0.98545	4.1725×10^{-3}	2.0833×10^{-22}	8.07
$DZ3$	1.2552	1.05743	1.3499×10^{-2}	3.8592×10^{-19}	8.31
$DZ4$	2.9098	0.92201	2.0697×10^{-3}	3.0161×10^{-24}	7.87

Example 3. Van Der Waals equation of state.

The Van Der Waals equation of state [19], is defined as:

$$p = \frac{RT}{v-b} - \frac{a}{v^2},$$

where p is the pressure, v is the volume, R is the gas constant, T is the temperature, a is the force of attraction between the molecules and b is the molecular size. The alternate form the Van Der Waals equation of state is given as,

$$\left(p + \frac{an^2}{v^2}\right)(v - nb) = nRT,$$

that explains the behavior of a real gas, by introducing two parameters, a and b specific for each gas in the ideal gas equation, where n is the number of moles. Determination of the volume V of the gas in terms of the remaining parameters requires the solution of the nonlinear equation in terms of V.

$$pv^3 - (nbp + nRT)v^2 + an^2v = abn^3.$$

Let us suppose that $n = 0.1807$ mole of gas has a pressure of 1 atmospheres and a temperature of $313K$. For this gas, $a = 278.3 atm \cdot L^2/mol^2$ and $b = 3.2104 L/mol$. The universal gas constant has the value $R = 0.08206 atm \cdot L/mol \cdot K$. Hence, we obtain following equation, which is cubic in v:

$$f_3(\theta) = \theta^3 - 5.22\theta^2 + 9.0825\theta - 5.2675,$$

where $\theta = v$, yielding the multiple roots $\omega = 1.75$ with multiplicity $\sigma = 2$. Taking the initial guess $\theta_0 = 2.05$ implies the analytical results of Table 3.

Table 3. Comparison of multiple root finding methods for $f_3(\theta)$.

$Scheme$	$\lvert\theta_1-\theta_0\rvert$	$\lvert\theta_2-\theta_1\rvert$	$\lvert\theta_3-\theta_2\rvert$	$\lvert f_3(\theta)\rvert$	COC
$SH1$	0.2828	1.7129×10^{-2}	1.8135×10^{-5}	9.5468×10^{-50}	6.18
$SH2$	0.2828	1.7124×10^{-2}	1.8216×10^{-5}	1.0675×10^{-49}	6.18
$SH3$	0.2820	1.7951×10^{-2}	2.0742×10^{-5}	8.1167×10^{-54}	6.94
$SH4$	0.2795	2.0368×10^{-2}	5.3306×10^{-5}	1.1791×10^{-46}	6.65
$DZ1$	0.2847	1.5319×10^{-2}	5.7302×10^{-6}	6.6723×10^{-63}	7.13
$DZ2$	0.2847	1.5293×10^{-2}	6.1453×10^{-6}	1.9309×10^{-62}	7.13
$DZ3$	0.2847	1.5319×10^{-2}	5.5644×10^{-6}	4.1702×10^{-63}	7.13
$DZ4$	0.2847	1.5311×10^{-2}	5.0567×10^{-6}	8.2275×10^{-64}	7.13

Example 4. Kepler's equation.

In celestial mechanics, Kepler's equation possesses a fundamental importance. As it is a transcendental equation, it cannot be inverted directly into simpler form of the function to determine the position of the planet at a certain time. Therefore, considering its importance, many algorithms were generated to solve this equation. The relation between the polar coordinates of the celestial body and the time taken from the initial point is described by the Kepler's equation. Here, for an orbiting body about an ellipse having eccentricity E, θ represents the "eccentric anomaly" (polar angle parametrization) and M represents the mean anomaly (time parametrization). Let us consider the conventional form of the Kepler's equation, given as $f(\theta) = \theta - E\sin(\theta) - M$. In [7], Danby et al. described the behaviour of this equation on many specific values of the parameters E and M. In particular, let the value of $E = \frac{1}{4}$ and $M = \frac{\pi}{5}$, that gives $f(\theta) = \theta - \frac{\sin(\theta)}{4} - \frac{\pi}{5}$. Taking four times the Kepler's equation on the same values of the parameters, implies,

$$f_4(\theta) = \left(\theta - \frac{\sin(\theta)}{4} - \frac{\pi}{5}\right)^4.$$

This gives us the multiple root $\omega \approx 1.833$ with multiplicity $\sigma = 4$ and taking the initial guess $\theta_0 = 1$, results in the numerical computations that are presented in Table 4.

Table 4. Comparison of multiple root finding methods for $f_4(\theta)$.

$Scheme$	$\lvert\theta_1-\theta_0\rvert$	$\lvert\theta_2-\theta_1\rvert$	$\lvert\theta_3-\theta_2\rvert$	$\lvert f_4(\theta)\rvert$	COC
$SH1$	0.2879	7.8645×10^{-2}	8.7165×10^{-13}	1.2881×10^{-356}	7.01
$SH2$	0.2879	7.8645×10^{-2}	9.2896×10^{-13}	1.0363×10^{-355}	7.01
$SH3$	0.2488	3.9591×10^{-2}	2.6372×10^{-16}	2.5232×10^{-517}	8.00
$SH4$	0.2346	2.5389×10^{-2}	2.3460×10^{-17}	8.0997×10^{-549}	8.00
$DZ1$	0.2489	3.9693×10^{-2}	4.6641×10^{-17}	1.2535×10^{-544}	8.00
$DZ2$	0.2489	3.9693×10^{-2}	7.9380×10^{-17}	3.2987×10^{-536}	8.00
$DZ3$	0.2489	3.9693×10^{-2}	2.2725×10^{-17}	3.0380×10^{-556}	8.01
$DZ4$	0.2489	3.9693×10^{-2}	9.3809×10^{-17}	1.6458×10^{-129}	1.09

Example 5. Predator prey model.

Let us consider the predator-prey model with ladybugs as predators and aphids as prey. Let θ be the number of aphids eaten by the ladybugs per unit time per unit area, called the predation rate denoted by $J(\theta)$ (see [20]). Usually, prey density is the factor on which the predation rate relies:

$$J(\theta) = K\left(\frac{\theta^n}{\theta^n + a^n}\right)$$

for $a, K > 0$ where K is the predation constant. Let the growth of the aphids is controlled by Malthusian Model; therefore, the growth rate G of the aphids per hour is:

$$G(\theta) = \theta s$$

for $s > 0$ where s is the growth constant per hour. The problem is to find aphid density θ for which $J(\theta) = G(\theta)$ gives,

$$-s\theta^{n+1} + K\theta^n - sa^n\theta = 0.$$

Let for $n = 2$, $s = 0.5$ per hour, $K = 20$ aphids eaten per hour and $a = 20$ aphids, we get

$$f_5(\theta) = -0.5\theta^3 + 20\theta^2 - 200\theta.$$

This gives us the roots $\{0, 20, 20\}$. We take the multiple roots $\omega = 20$ with multiplicity $\sigma = 2$ and the initial approximation $\theta_0 = 20.07$, that yields the computations presented in Table 5.

Table 5. Comparison of multiple root finding methods for $f_5(\theta)$.

$Scheme$	$\lvert\theta_1 - \theta_0\rvert$	$\lvert\theta_2 - \theta_1\rvert$	$\lvert\theta_3 - \theta_2\rvert$	$\lvert f_5(\theta)\rvert$	COC
$SH1$	0.0699	5.4694×10^{-14}	–	2.1561×10^{-147}	5.00
$SH2$	0.0699	5.4741×10^{-14}	–	2.1747×10^{-147}	5.00
$SH3$	0.0700	1.2586×10^{-14}	2.5346×10^{-29}	2×10^{-296}	8.14
$SH4$	0.0700	7.3348×10^{-15}	5.7386×10^{-30}	1×10^{-296}	7.89
$DZ1$	0.0700	1.2530×10^{-14}	2.5121×10^{-29}	1×10^{-296}	8.15
$DZ2$	0.0700	1.2529×10^{-14}	2.5116×10^{-29}	2×10^{-296}	8.14
$DZ3$	0.0700	1.2530×10^{-14}	2.5124×10^{-29}	1×10^{-296}	8.15
$DZ4$	0.0700	1.2534×10^{-14}	2.5139×10^{-29}	2×10^{-296}	8.14

4. Dynamical analysis

The complex dynamical analysis of the presented eighth order family of methods to solve the multiple zeros of the nonlinear equations, is discussed in this section. The analysis entirely depends on the graphical representations called basins of attraction. Here, we elaborate that to which intensity the functional convergence towards the exact root depends on the choice of the initial estimate. The basic idea about the convergence and divergence region of the iterative schemes is presented by this dynamical behavior of the function.

4.1. Attraction basins

Consider a function $f_k(\theta)$ such that $\theta \in \mathbb{C}$ and the root of the function is ω_k. The schemes $DZ1$–$DZ4$ and the existing methods named $SH1$–$SH4$ are compared in terms of the attraction basins of the test functions on which the corresponding methods are applied. The attraction basins are drawn in MATLAB. Considering the parameter $\beta = 0.001$, grid points of 1000×1000 in the complex plane $[a,\ b] \times [c,\ d]$ where the values of a, b, c and d are selected based on the zero of the function. The maximum value of the number of iterations is taken as 15 with tolerance value 10^{-5}. 'Hot' is the selected color-map and black color is allocated to the divergence region. The hues are interpreted based on the number of iterations taken by the family of iterative methods.

In this example, we assume the function

$$f_1(\theta) = (\theta^4 + 11.50\theta^3 + 47.49\theta^2 + 83.063325\theta + 51.23266875)$$

having zeros at $\{-2.85, -1.45, -4.35\}$. We observe the root -2.85 having multiplicity two. The basins of attraction obtained for the methods $SH1$–$SH4$ and $DZ1$–$DZ4$ are shown in Figures 1 and 2. Upon observing the dynamical view for the region $[-4, 0]\times[-1, 1]$, we notice that $SH1$–$SH4$ take a minimum of 2 iterations to converge to the root and use a maximum of 15 iterations, $DZ1$–$DZ4$ take a minimum of 2 iterations and use a maximum 10 iterations to converge to the root. Furthermore, if we compare Figures 1 and 2 of $SH4$ and $DZ4$ respectively, the convergence regions for $SH4$ are as follows:

$$[-3.93, -1.85] \times [0.359, 1],\quad [-3.65, -2.09] \times [-0.247, 0.359],\quad [-3.78, -1.93] \times [-0.351, -0.247],$$

$$[-4, -1.84] \times [-1, -0.351],\quad [-1.76, -0.89] \times [-1, -0.236].$$

Similarly, the convergence regions for $DZ4$ are:

$$[-0.40, -0.394] \times [0.452, 1],\quad [-3.74, -2.14] \times [0.24, 0.452],\quad [-1.41, -1.26] \times [0.24, 0.452],$$

$$[-1.03, -0.042]\times[0.24, 0.452],\quad [-1.48, -0.855]\times[-0.24, 0.24],\quad [-0.508, -0.0991]\times[-0.247, 0.247].$$

In this example, we assume the function

$$f_2(\theta) = \tan^{-1}\frac{\sqrt{5}}{2} - \tan^{-1}\sqrt{\theta^2 - 1} + \sqrt{6}\left(\tan^{-1}\frac{\sqrt{\theta^2 - 1}}{6} - \tan^{-1}\left(\frac{1}{2}\sqrt{\frac{5}{6}}\right)\right)$$

having a zero at $\omega \approx -1.8411$ of multiplicity one. The basins of attraction obtained for the methods SH1–SH4 and DZ1–DZ4 are shown in Figures 3 and 4. For the region $[-4, 0] \times [-1, 1]$, we notice that SH1–SH4 take a minimum of 1 iteration to converge to the root and use a maximum of 6 iterations, while DZ1–DZ4 take a minimum of 1 iteration and use a maximum 6 iterations to converge to the root. Also, we compare Figures 3 and 4 of $SH3$ and $DZ3$ respectively, to locate the convergence regions for $SH3$ which are as follows:

$$[-2.89, -1.12] \times [-1, 1],\quad [-3.23, -2.89] \times [0.459, 0.697],\quad [-3.23, -2.89] \times [-0.459, -0.697],$$

$$[-3.05, -2.89] \times [-0.459, -0.0911],\quad [-3.19, -2.89] \times [-0.0911, 0.0911].$$

The convergence regions for $DZ3$ are as follows:

$$[-3.09, -1.09] \times [-1, 1],\quad [-3.23, -3.09] \times [0.723, 0.898],\quad [-3.35, -3.09] \times [0.723, 0.541],$$

$$[-3.25, -3.09] \times [0.541, 0.0911], \quad [-3.34, -3.09] \times [-0.0911, 0.0911],$$

$$[-3.25, -3.09] \times [-0.541, -0.0911].$$

In this example, we assume function

$$f_3(\theta) = \theta^3 - 5.22\theta^2 + 9.08250\theta - 5.2675$$

possessing the multiple root $\omega = 1.75$ having multiplicity two. The basins of attraction obtained for the methods SH1–SH4 and DZ1–DZ4 are shown in Figures 5 and 6. By drawing the attraction basins in the region $[1, 3] \times [-0.5, 0.5]$, we notice that SH1–SH4 take a minimum of 1 iteration to converge to the root and use a maximum of 7 iterations. On the other hand, DZ1–DZ4 take a minimum of 1 iteration and use a maximum of 6 iterations with darker hues to converge to the root. To further check the performance of our newly proposed method, we compare the convergence regions. Let us compare Figures 5 and 6 of $SH3$ and $DZ3$ respectively, the convergence regions for $SH3$ are:

$$[1, 1.74] \times [-0.5, 0.5], \quad [1.73, 3] \times [-0.5, 0.5],$$

$$[1.73, 1.74] \times [-0.5, -0.00418], \quad [1.73, 1.74] \times [0.5, 0.00418]$$

and the regions of convergence of $DZ3$ are:

$$[1, 1.72] \times [-0.5, 0.5], \quad [1.74, 3] \times [-0.5, 0.5],$$

$$[1.72, 1.74] \times [-0.5, -0.0079], \quad [1.72, 1.74] \times [0.5, 0.0079].$$

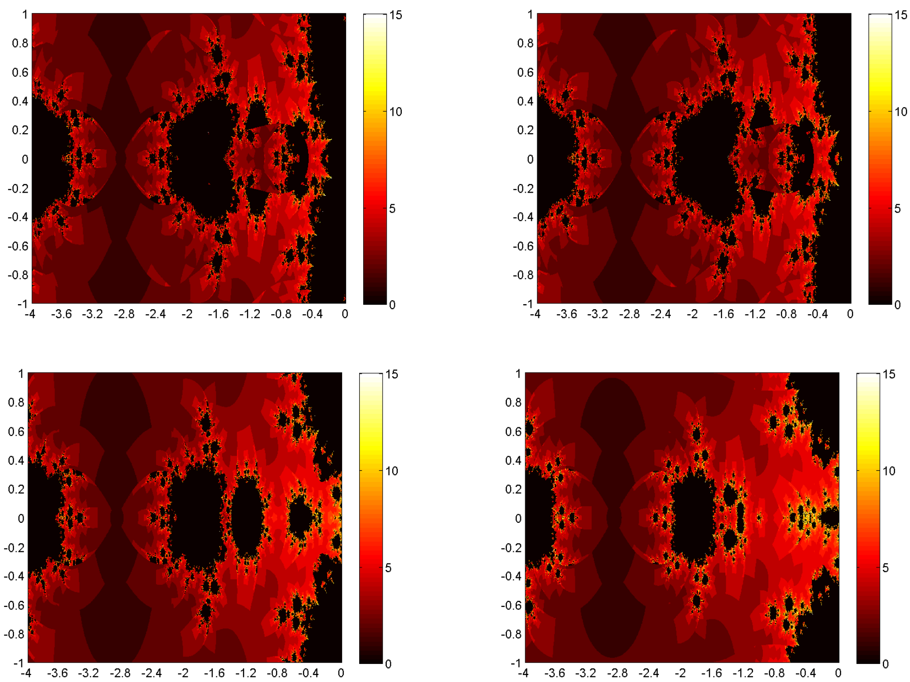

Figure 1. Attraction basins of SH1–SH4 of $f_1(\theta)$.

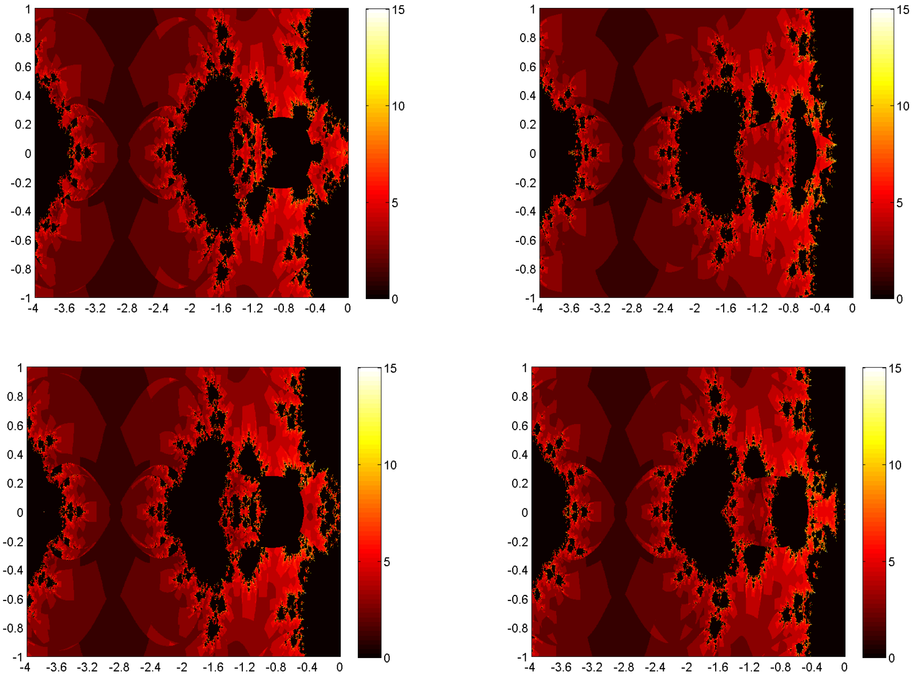

Figure 2. Attraction basins of DZ1–DZ4 of $f_1(\theta)$.

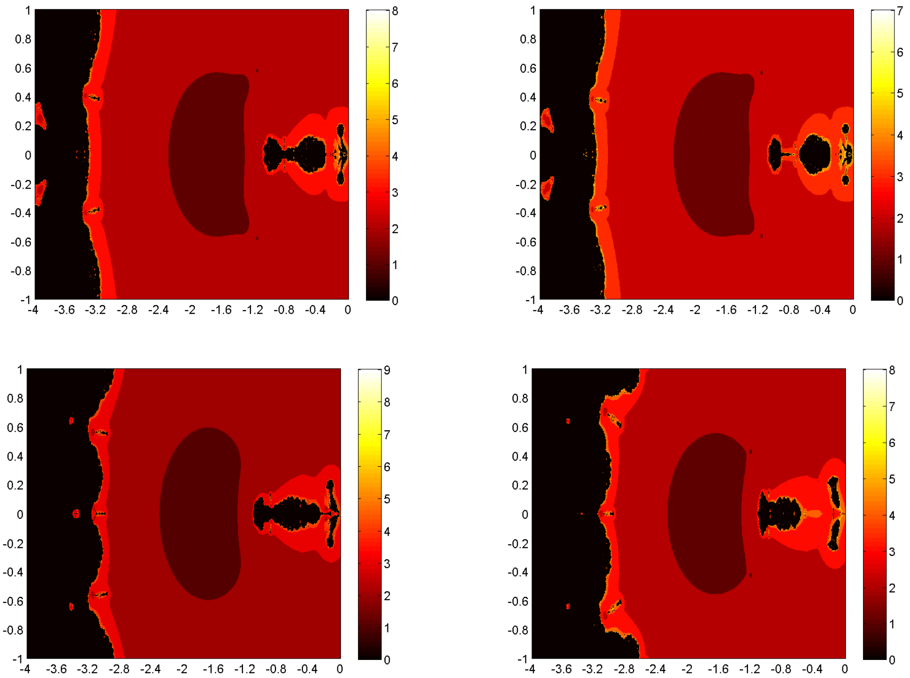

Figure 3. Attraction basins of SH1–SH4 of $f_2(\theta)$.

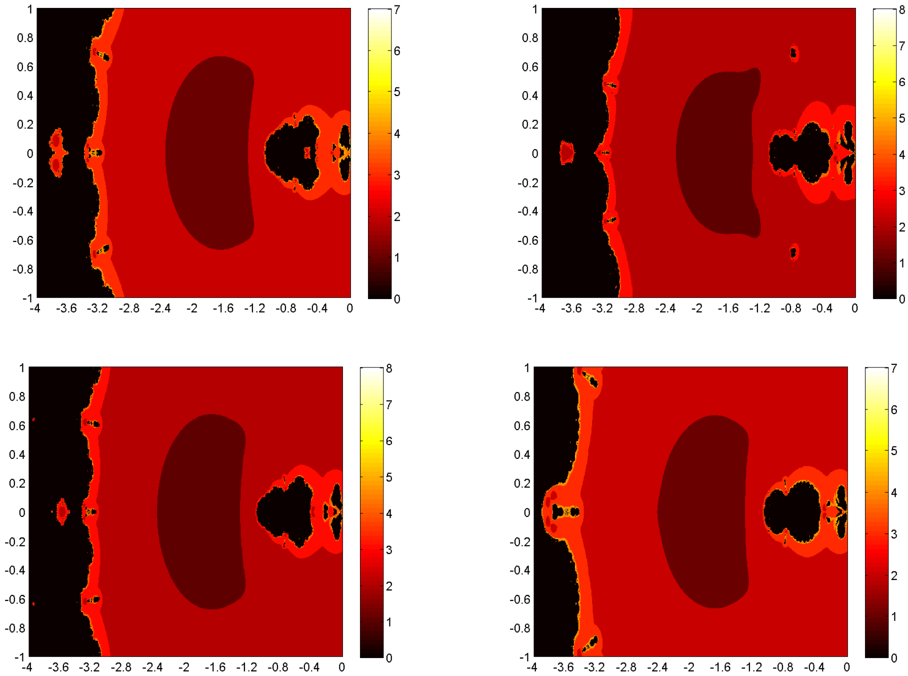

Figure 4. Attraction basins of DZ1–DZ4 of $f_2(\theta)$.

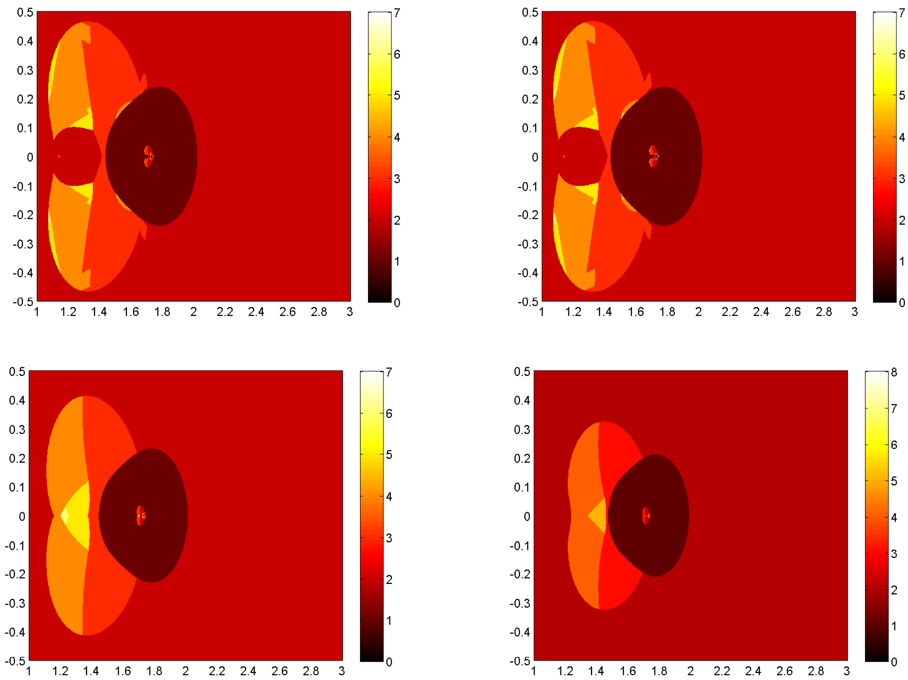

Figure 5. Attraction Basins of SH1–SH4 of $f_3(\theta)$.

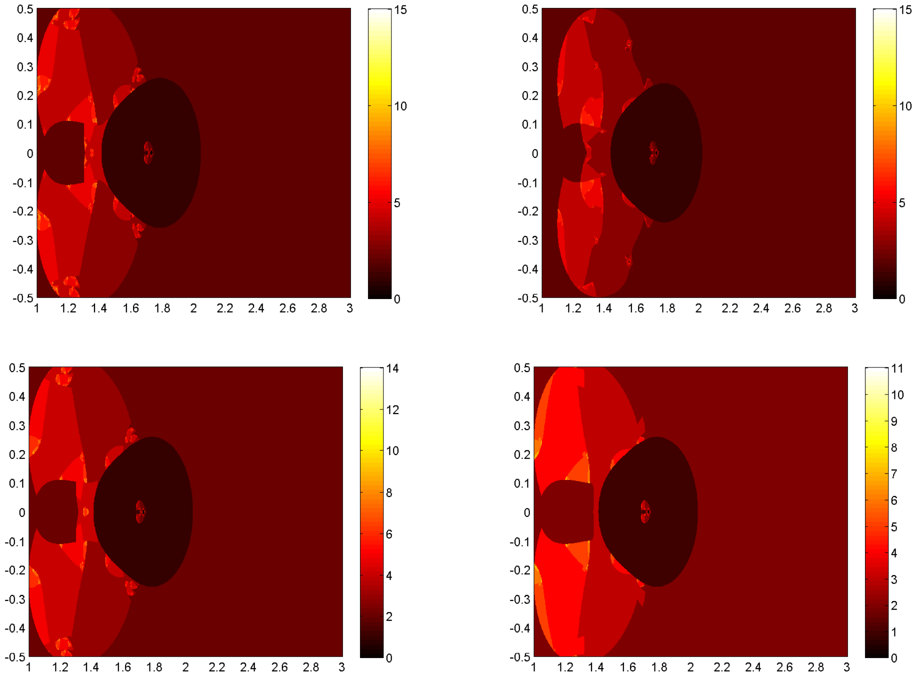

Figure 6. Attraction basins of DZ1–DZ4 of $f_3(\theta)$.

Let us consider the nonlinear function

$$f_4(\theta) = \left(\theta - \frac{\sin(\theta)}{4} - \frac{\pi}{5}\right)^4$$

which has a multiple root $\omega \approx 1.833$ with multiplicity four. The basins of attraction for the methods SH1–SH4 and DZ1–DZ4 are shown in Figures 7 and 8. On observing the dynamical view in the region $[0,2] \times [-1,1]$, we notice that H1–SH4 take a minimum of 1 iteration to converge to the root and use a maximum of 11 iterations, DZ1–DZ4 take a minimum of 1 iteration and use a maximum of 10 iterations to converge to the root. To check further, we compare the convergence regions. Let us take Figures 7 and 8 of $SH4$ and $DZ3$, the convergence regions for $SH4$ are as follows:

$$[0.561,2] \times [-1,1], \quad [0.43,0.561] \times [-0.946,0.946], \quad [0,0.43] \times [-0.794,0.794],$$

$$[0,0.225] \times [-0.794,-1], \quad [0,0.225] \times [0.794,1].$$

The regions of convergence of $DZ3$ are as follows:

$$[1.26,2]\times[-1,1], \quad [0.96,1.26]\times[-0.872,0.872], \quad [0.842,0.96]\times[-1,1], \quad [0.79,0.842]\times[-0.831,0.831],$$

$$[0.7,0.79] \times [-0.92,0.92], \quad [0.45,0.7] \times [-0.857,0.857], \quad [0.381,0.45] \times [-1,-0.757].$$

In this example we take the function

$$f_5(\theta) = -0.5\theta^3 + 20\theta^2 - 200\theta$$

yielding the roots $\{20, 20, 0\}$. We take the root $\omega = 20$ having multiplicity two. The basins of attraction obtained for the methods SH1–SH4 and DZ1–DZ4 are shown in Figures 9 and 10. Observing the dynamical view in the region $[10, 30] \times [-10, 10]$, we notice that SH1–SH4 take a minimum of 1 iteration to converge to the root and use a maximum of 15 iterations, DZ1–DZ4 take a minimum of 1 iteration and use a maximum of 11 iterations to converge to the root. To further check the performance of our newly proposed method, we compare the convergence regions. Let us take Figures 9 and 10 of $SH3$ and $DZ1$ respectively. The convergence regions for $SH3$ are as follows:

$$[17, 30] \times [-10, 10], \quad [12.8, 17] \times [-10, -4.63], \quad [11.5, 12.8] \times [-8.16, -5.63],$$

$$[10, 12.8] \times [-8.35, -10], \quad [15.2, 17] \times [-3.1, 3.1], \quad [13, 15.2] \times [0.836, 2.29].$$

and the convergence regions for $DZ1$ are as follows:

$$[17, 30] \times [-10, 10], \quad [10, 17] \times [-10, -4.55], \quad [13.8, 17] \times [-3.18, -4.55],$$

$$[12.6, 13.8] \times [-4, -1.84], \quad [15.7, 17] \times [-3.84, 3.84],$$

$$[14.7, 15.7] \times [-0.502, -2.62], \quad [14.7, 15.7] \times [0.502, 2.62].$$

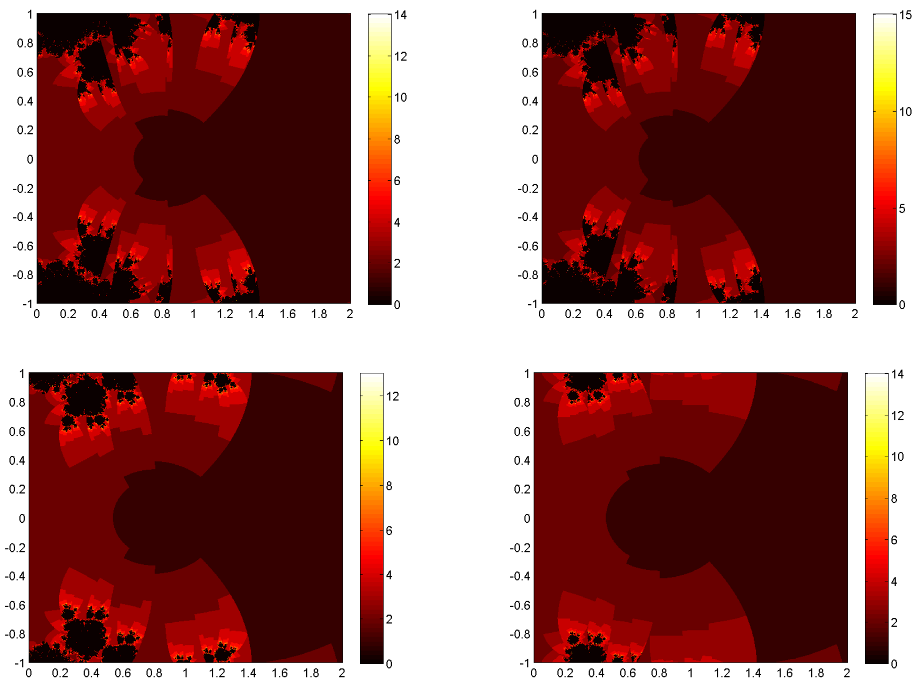

Figure 7. Attraction basins of SH1–SH4 of $f_4(\theta)$.

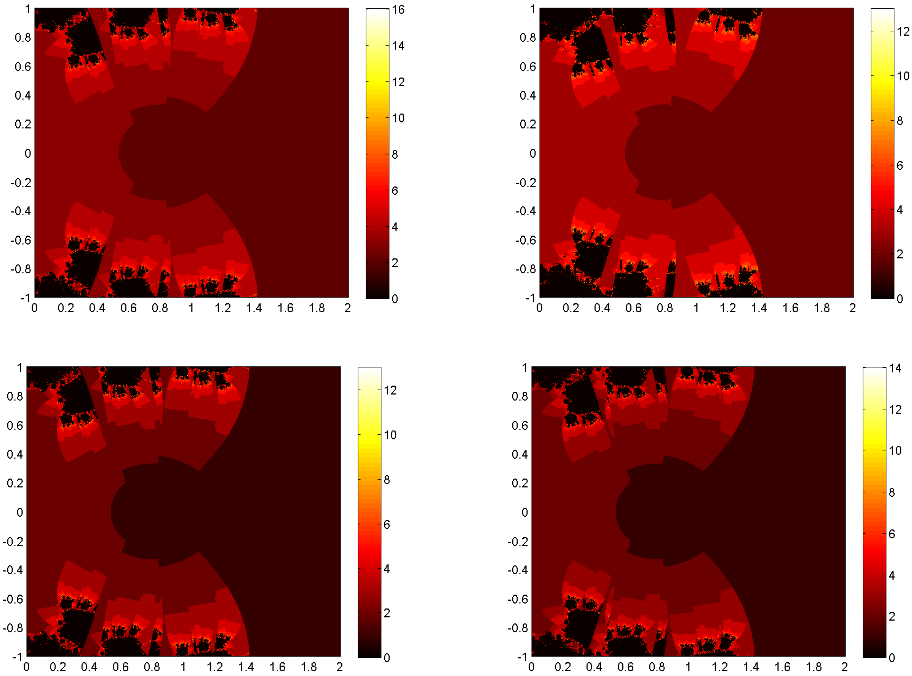

Figure 8. Attraction basins of DZ1–DZ4 of $f_4(\theta)$.

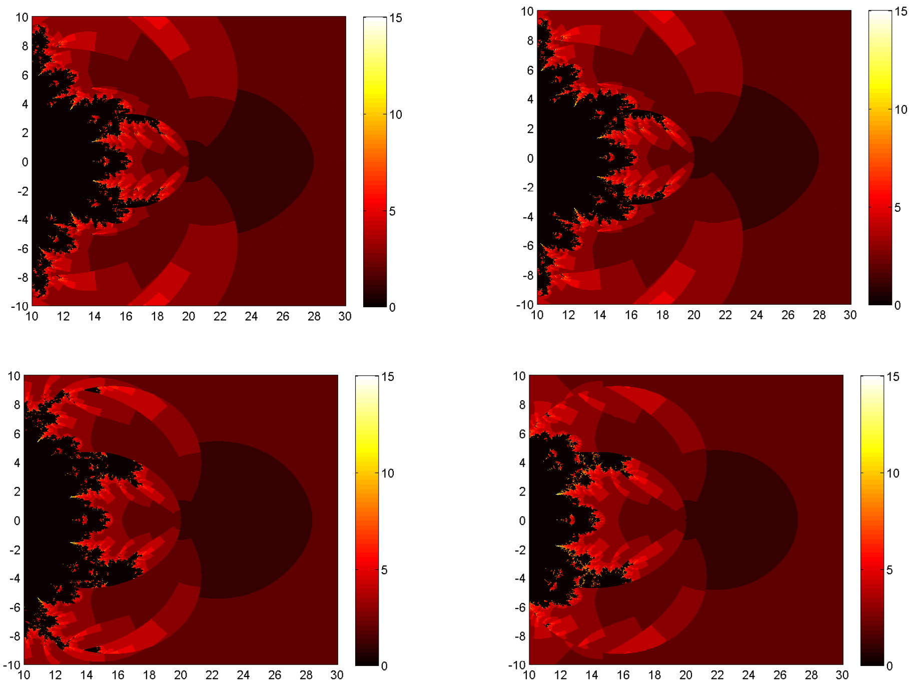

Figure 9. Attraction basins of SH1–SH4 of $f_5(\theta)$.

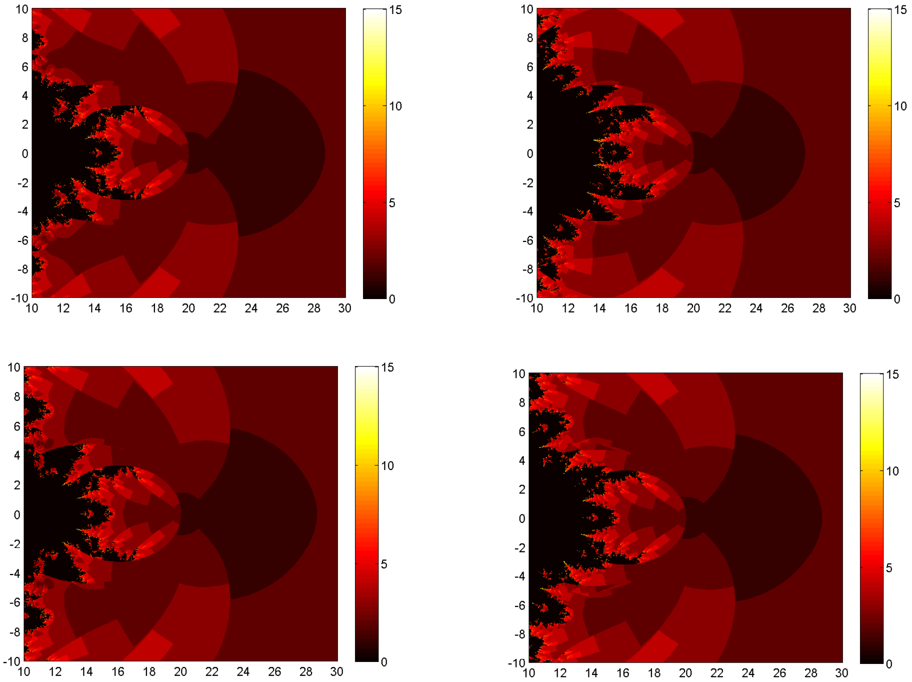

Figure 10. Attraction basins of DZ1–DZ4 of $f_5(\theta)$.

5. Conclusions

There are many high order numerical root-solvers established in the past, which are used to compute multiple roots, and evaluations of the derivative are mandatory in them. But, the high order derivative-free root-finders for multiple roots are hard to accomplish. These kinds of methods are infrequent and therefore, it is necessary to explore them. The current paper describes the newly introduced derivative-free approximate iterative methods having eighth order of convergence to find multiple zeros with a known multiplicity of the nonlinear equations. It includes two weighted functions, one of which is univariate, and the other is multivariate. The basins of attractions present the dynamical behaviour of the schemes.

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Conflict of interest

There is no conflict of interest regarding publication of this manuscript.

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