Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 8460–8477.
DOI:10.3934/math.2023426
Received: 20 November 2022
Revised: 22 December 2022
Accepted: 02 January 2023
Published: 03 February 2023

Research article

Numerical algorithms for solutions of nonlinear problems in some distance spaces

Junaid Ahmad[1,*], **Kifayat Ullah**[2] **and Reny George**[3,*]

[1] Department of Mathematics and Statistics, International Islamic University, Islamabad 44000, Pakistan

[2] Department of Mathematical Sciences, University of Lakki Marwat, Lakki Marwat 28420, Khyber Pakhtunkhwa, Pakistan

[3] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia

* **Correspondence:** Email: ahmadjunaid436@gmail.com, renygeorge02@yahoo.com.

Abstract: This paper introduces some numerical algorithms for finding solutions of nonlinear problems like functional equations, split feasibility problems (SFPs) and variational inequality problems (VIPs) in the setting of Hilbert and Banach spaces. Our approach is based on the Thakur-Thakur-Postolache (TTP) iterative algorithm and the class of mean nonexpansive mappings. First we provide some convergence results (including weak and strong convergence) in the setting of Banach space. To support these results, we provide a numerical example and prove that our TTP algorithm in this case converges faster to fixed point compared to other iterative algorithms of the literature. After that, we consider two new TTP type projection iterative algorithms to solve SFPs and VIPs on the Hilbert space setting. Our result are new in analysis and suggest new type effective numerical algorithms for finding approximate solutions of some nonlinear problems.

Keywords: algorithm; fixed point; numerical solution; Hilbert space; Banach space
Mathematics Subject Classification: 47H09, 47H10

1. Introduction

Suppose $\mathcal{B}$ is a Banach space and $\emptyset \neq C \subseteq \mathcal{B}$. Now we many set an operator $T : C \to C$. T is known as contraction if $\|Tv - Tv'\| \leq \alpha\|v - v'\|$, whenever $v, v' \in C$ and $\alpha \in (0,1)$. We call T a nonexpansive operator if $\alpha = 1$. When $Tv^* = v^*$ for some $v^* \in C$, then this element v^* is known as a fixed point for T and in this case we denote the set $\{v^* \in C : Tv^* = v^*\}$ simply by F_T. However, if $Gv^* = 0$, for an operator $G : C \to C$, then it is called a zero of G and in this case we denote the set

$\{v^* \in C : Gv^* = 0\}$ by $\mathcal{S}$. The class of nonexpansive operators is widely considered by lot of authors in different frame of works. In particular, Browder [6] and Gohde [10] separately provided the existence of fixed point result for these operators in a uniformly convex Banach space (UCBS) setting. Precisely, they noted that if C is bounded convex and closed in a UCBS, then a nonexpansive operator $T : C \to C$ essentially admits a fixed point. In 1975, Zhang [24] noticed a new notion of nonlinear operators as follows: an operator $T : C \to C$ on a subset C is essentially called mean nonexpansive if one can find two nonnegative reals a, b satisfying $a + b \leq 1$ such that

$$\|Tv - Tv'\| \leq a\|v - v'\| + b\|v - Tv'\| \text{ for every choice of } v, v' \in C.$$

Remark 1.1. *The class of mean nonexpansive operators is one of the important class of nonlinear mappings because it includes properly the class of all nonexpansive operators, that is, if $T : C \to C$ is nonexpansive then T satisfies the requirement of a mean nonexpansive operator with $a = 1$ and $b = 0$. However, the converse is not valid in general as shown by the following example (see also Examples 5.1 and 5.2 in the last section of this paper).*

Example 1.1. *If $C = [0, 4]$, then we can set an operator $T : C \to C$ by the following formula*

$$Tv = \begin{cases} 1, & 0 \leq v < 4, \\ 0, & v = 4. \end{cases}$$

Here, T is discontinuous at $v = 4$ and hence not nonexpansive. On the other hand, it is straightforward to show that T is a mean nonexpansive operator.

The existence of fixed points of mean nonexpansive operators in Banach spaces has been studied by some authors. In particular, Zhang [24] suggested a unique fixed point result for the class of mean nonexpansive operators in a Banach space endowed with a normal structure. Moreover, Wu and Zhang [23] and Zuo [25] provided some related properties and fixed point theorems for these operators in a Banach space. The first purpose of this research is to study the computation of fixed points for these maps under an appropriate algorithm. Secondly, we apply these results to solve some problems under new algorithms.

After the existence of a fixed point for a given operator, it is very natural to construct an iterative scheme, which approximate the value of this fixed point. We know that in general, that Picard iterates $v_{k+1} = Tv_k$ converges for Banach contractions but not for nonexpansive maps. Thus to find the value of the fixed point for nonexpansive and accordingly of generalized nonexpansive maps and also to obtain a relatively better rate of convergence, many schemes are available in the literature given below.

Mann [14] provided the following algorithm:

$$\begin{cases} v_1 \in C, \\ v_{k+1} = (1 - \mu_k)v_k + \mu_k T v_k, \\ k \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\mu_k \in (0, 1)$.

Ishikawa [11] constructed a new iterative algorithm as follows:

$$\begin{cases} v_1 \in C, \\ s_k = (1 - \eta_k)v_k + \eta_k T v_k, \\ v_{k+1} = (1 - \mu_k)v_k + \mu_k T s_k, \\ k \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\mu_k, \eta_k \in (0, 1)$.

In 2000, Noor [16] first time introduced a three-step iterative algorithm, which is more general than that of Mann and Ishikawa iterative algorithms:

$$\begin{cases} v_1 \in C, \\ e_k = (1-\theta_k)v_k + \theta_k T v_k, \\ s_k = (1-\eta_k)v_k + \eta_k T e_k, \\ v_{k+1} = (1-\mu_k)v_k + \mu_k T s_k, \\ k \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\mu_k, \eta_k, \theta_k \in (0, 1)$.

In 2007, Agarwal et al. [4] provided the S iterative algorithm, which is independent of but better than the both the Mann and Ishikawa iterative algorithm for many nonlinear operators:

$$\begin{cases} v_1 \in C, \\ s_k = (1-\eta_k)v_k + \eta_k T e_k, \\ v_{k+1} = (1-\mu_k)T v_k + \mu_k T s_k, \\ k \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\mu_k, \eta_k \in (0, 1)$.

In 2014, Abbas and Nazir [2] considered another three-step iterative algorithm, which gives better approximation results as compared Mann, Ishikawa, Noor and S iterative algorithm. Their algorithm reads as follows:

$$\begin{cases} v_1 \in C, \\ e_k = (1-\theta_k)v_k + \theta_k T v_k, \\ s_k = (1-\eta_k)T v_k + \eta_k T e_k, \\ v_{k+1} = (1-\mu_k)T s_k + \mu_k T e_k, \\ k \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\mu_k, \eta_k, \theta_k \in (0, 1)$.

In 2016, Thakur et al. [21] (TTP) constructed a novel three-step iterative algorithm as follows:

$$\begin{cases} v_1 \in C, \\ e_k = (1-\theta_k)v_k + \theta_k T v_k, \\ s_k = (1-\eta_k)e_k + \eta_k T e_k, \\ v_{k+1} = (1-\mu_k)T e_k + \mu_k T s_k, \\ k \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\mu_k, \eta_k, \theta_k \in (0, 1)$.

By means of the iterative algorithm [21], they approximate fixed points for nonexpansive operators through weak and strong convergence on a Banach space setting. They also compared the high accuracy of this algorithm with the other well-known algorithms in the setting of nonexpansive operators. Maniu [15] proved that this algorithm is stable with respect to weak contractions. Here, we want to show that the main results of [21] can be extended to the setting of mean nonexpansive operators. We also give an example of mean nonexpansive operators which fails to hold the nonexpansiveness condition. We connect the three-step algorithm (1.6) and the other three-step algorithms with this example and show that this algorithm provide a high accuracy as compared the others three-step algorithms in the general setting of mean nonexpansive operators.

2. Preliminaries

To obtain the required aim, we first provide some elementary facts and results.

Consider a Banach space $\mathcal{B}$ and $\emptyset \neq C \subseteq \mathcal{B}$. Select an element $q_0 \in \mathcal{B}$ and choose a bounded sequence, namely, $\{v_k\} \subseteq \mathcal{B}$. We may set $r(q_0, \{v_k\})$ as

$$r(q_0, \{v_k\}) := \limsup_{k\to\infty} \|q_0 - v_k\|.$$

The asymptotic radius of the sequence $\{v_k\}$ connected with the set V will be denoted by $r(C, \{v_k\})$ and given by

$$r(C, \{v_k\}) := \inf\{r(q_0, \{v_k\}) : q_0 \in C\}.$$

The asymptotic center of the sequence $\{v_k\}$ connected with the set C will be denoted by $A(C, \{v_k\})$ and given by

$$A(C, \{v_k\}) := \{q_0 \in C : r(q_0, \{v_k\}) = r(C, \{v_k\})\}.$$

Remark 2.1. *The set $A(C, \{v_k\})$ some-times does not has any element. But in the setting of UCBS, it is always a singleton set. The convexity of the $A(C, \{v_k\})$ is also known in the case when C is a weakly compact and convex set, (see, e.g., [3, 20] and others).*

If a Banach space $\mathcal{B}$ is given. Then it is said to be endowed with the Opial's condition [17] in the case, when each $\{v_k\}$ in $\mathcal{B}$ converges in the weak sense to $v \in \mathcal{B}$ enjoys the following strict inequality:

$$\liminf_{k\to\infty} \|v_k - v\| < \liminf_{k\to\infty} \|v_k - u\| \text{ for each choice of } u \in \mathcal{B} - \{v\}.$$

The following result is known from [25].

Lemma 2.1. *Assume that C is a nonempty convex and closed subset of a reflexive Banach space (RBS) $\mathcal{B}$ and $T : C \to C$ a mean nonexpansive operator. If a sequence $\{v_k\}$ converges in the weak sense to v^* and $\lim_{k\to\infty} \|v_k - Tv_k\| = 0$, then $v^* \in F_T$ provided that $\mathcal{B}$ has the Opial's property.*

Now we take an important property of a UCBS from [18].

Lemma 2.2. *If $\mathcal{B}$ is a UCBS such that for any two sequences $\{e_k\}$ and $\{v_k\}$ in $\mathcal{B}$ with $\limsup_{k\to\infty} \|e_k\| \leq z$, $\limsup_{k\to\infty} \|v_k\| \leq z$ and $\lim_{k\to\infty} \|\mu_k e_k + (1 - \mu_k)v_k\| = z$, for $0 < c \leq \mu_k \leq d < 1$ and some $z \geq 0$. Then $\lim_{k\to\infty} \|e_k - v_k\| = 0$.*

3. Convergence theorems of mean nonexpansive operators

Notice that, throughout the section, $\mathcal{B}$ denotes a UCBS. The main outcome of this section is begun with the following key lemma. It should be noted that this lemma extends and improves [21, Lemma 4.1] from the setting of nonexpansive operators to the setting of mean nonexpansive operators.

Lemma 3.1. *If C is a nonempty convex and closed subset of $\mathcal{B}$ such that $T : C \to C$ is a mean nonexpansive operator having $F_T \neq \emptyset$. Then $\lim_{k\to\infty} \|v_k - v^*\|$ exists for every $v^* \in F_T$, where $\{v_k\}$ is a sequence generated by (1.6).*

Proof. For, $v^* \in F_T$, we have $v^* = Tv^*$. Then using (1.6), we have

$$\begin{aligned}
\|e_k - v^*\| &\leq (1-\theta_k)\|v_k - v^*\| + \theta_k\|Tv_k - Tv^*\| \\
&\leq (1-\theta_k)\|v_k - v^*\| + \theta_k(a\|v_k - v^*\| + b\|v_k - Tv^*\|) \\
&= (1-\theta_k)\|v_k - v^*\| + \theta_k(a\|v_k - v^*\| + b\|v_k - v^*\|) \\
&= (1-\theta_k)\|v_k - v^*\| + \theta_k((a+b)\|v_k - v^*\|) \\
&\leq (1-\theta_k)\|v_k - v^*\| + \theta_k((1)\|v_k - v^*\|) \\
&= \|v_k - v^*\|,
\end{aligned}$$

and

$$\begin{aligned}
\|s_k - v^*\| &\leq (1-\eta_k)\|e_k - v^*\| + \eta_k\|Te_k - Tv^*\| \\
&\leq (1-\eta_k)\|e_k - v^*\| + \eta_k(a\|e_k - v^*\| + b\|e_k - Tv^*\|) \\
&= (1-\eta_k)\|e_k - v^*\| + \eta_k(a\|e_k - v^*\| + b\|e_k - v^*\|) \\
&= (1-\eta_k)\|e_k - v^*\| + \eta_k((a+b)\|e_k - v^*\|) \\
&\leq (1-\eta_k)\|e_k - v^*\| + \eta_k((1)\|e_k - v^*\|) \\
&= \|e_k - v^*\|.
\end{aligned}$$

While using the above inequilities, we have

$$\begin{aligned}
\|v_{k+1} - v^*\| &\leq (1-\mu_k)\|Te_k - Tv^*\| + \mu_k\|Ts_k - Tv^*\| \\
&\leq (1-\mu_k)(a\|e_k - v^*\| + b\|e_k - Tv^*\|) + \mu_k(a\|s_k - v^*\| + b\|s_k - Tv^*\|) \\
&= (1-\mu_k)(a\|e_k - v^*\| + b\|e_k - v^*\|) + \mu_k(a\|s_k - v^*\| + b\|s_k - v^*\|) \\
&= (1-\mu_k)((a+b)\|e_k - v^*\|) + \mu_k((a+b)\|s_k - v^*\|) \\
&\leq (1-\mu_k)\|e_k - v^*\| + \mu_k\|e_k - v^*\| \\
&= \|e_k - v^*\| \leq \|v_k - v^*\|.
\end{aligned}$$

It has been observed that $\{\|v_k - v^*\|\}$ is non-increasing and bounded. It follows that $\lim_{k\to\infty}\|v_k - v^*\|$ exists for all fixed point v^* of T. □

The next theorem will be helpful in proving the main results of the sequel.

Theorem 3.1. *If C is a nonempty convex and closed subset of $\mathcal{B}$ such that $T : C \to C$ is a mean nonexpansive operator having $F_T \neq \emptyset$ and $\{v_k\}$ is a sequence defined in (1.6). Then the sequence $\{v_k\}$ is bounded and* $\lim_{k\to\infty}\|v_k - Tv_k\| = 0$.

Proof. If we select any element $v^* \in F_T$, then it is clear from the Lemma 3.1, that, $\{v_k\}$ is bounded. We want to show that $\lim_{k\to\infty}\|v_k - Tv_k\| = 0$. Now, according to Lemma 3.1, $\lim_{k\to\infty}\|v_k - v^*\|$ exists. We set

$$\lim_{k\to\infty}\|v_k - v^*\| = z. \tag{3.1}$$

Now,

$$
\begin{aligned}
\|e_k - r\| &\le (1-\theta_k)\|v_k - v^*\| + \theta_k\|Tv_k - Tv^*\| \\
&\le (1-\theta_k)\|v_k - v^*\| + \theta_k(a\|v_k - v^*\| + b\|v_k - Tv^*\|) \\
&= (1-\theta_k)\|v_k - v^*\| + \theta_k(a\|v_k - v^*\| + b\|v_k - v^*\|) \\
&= (1-\theta_k)\|v_k - v^*\| + \theta_k((a+b)\|v_k - v^*\|) \\
&\le (1-\theta_k)\|v_k - v^*\| + \theta_k((1)\|v_k - v^*\|) \\
&= \|v_k - v^*\|
\end{aligned}
$$

$$
\implies \limsup_{k\to\infty}\|e_k - v^*\| \le \limsup_{k\to\infty}\|v_k - v^*\| = z. \tag{3.2}
$$

Also,

$$
\begin{aligned}
\|Tv_k - v^*\| &= \|Tv_k - Tv^*\| \\
&\le a\|v_k - v^*\| + b\|v_k - Tv^*\| \\
&= a\|v_k - v^*\| + b\|v_k - v^*\| \\
&= (a+b)\|v_k - v^*\| \\
&\le \|v_k - v^*\|
\end{aligned}
$$

$$
\implies \limsup_{k\to\infty}\|Tv_k - v^*\| \le \limsup_{k\to\infty}\|v_k - v^*\| = z. \tag{3.3}
$$

Similarly,

$$
\begin{aligned}
\|v_{k+1} - v^*\| &\le (1-\mu_k)\|Te_k - Tv^*\| + \mu_k\|Ts_k - Tv^*\| \\
&\le (1-\mu_k)(a\|e_k - v^*\| + b\|e_k - Tv^*\|) + \mu_k(a\|s_k - v^*\| + b\|s_k - Tv^*\|) \\
&= (1-\mu_k)(a\|e_k - v^*\| + b\|e_k - v^*\|) + \mu_k(a\|s_k - v^*\| + b\|s_k - v^*\|) \\
&= (1-\mu_k)((a+b)\|e_k - v^*\|) + \mu_k((a+b)\|s_k - v^*\|) \\
&\le (1-\mu_k)\|e_k - v^*\| + \mu_k\|e_k - v^*\| \\
&= \|e_k - v^*\|
\end{aligned}
$$

$$
\implies z = \liminf_{k\to\infty}\|v_{k+1} - v^*\| \le \liminf_{k\to\infty}\|e_k - v^*\|. \tag{3.4}
$$

From (3.2) and (3.4), we get

$$
z = \lim_{k\to\infty}\|e_k - v^*\|. \tag{3.5}
$$

From (3.5), we have

$$
z = \lim_{k\to\infty}\|e_k - v^*\| = \lim_{k\to\infty}\|(1-\theta_k)(v_k - v^*) + \theta_k(Tv_k - v^*)\|.
$$

Applying Lemma 2.2, we obtain

$$
\lim_{k\to\infty}\|Tv_k - v_k\| = 0.
$$

□

Now we want to establish a strong convergence theorem on a compact domain. This result includes the result if one considers a nonexpansive operator.

Theorem 3.2. *If C is a nonempty convex and closed subset of $\mathcal{B}$ such that $T : C \to C$ is a mean nonexpansive operator having $F_T \neq \emptyset$. Then $\{v_k\}$ defined in (1.6) converges in the strong sense to a point of F_T if C is compact.*

Proof. Thanks to the convexity of C, we can say that $\{v_k\} \subseteq C$. Also remembering the compactness of C, one can choose a strongly convergent subsequence $\{v_{k_m}\}$ of $\{v_k\}$ such that $v_{k_m} \to v_0$. Now we show that $Tv_0 = v_0$. For this

$$\begin{aligned} \|v_0 - Tv_0\| &\leq \|v_0 - v_{k_m}\| + \|v_{k_m} - Tv_{k_m}\| + \|Tv_{k_m} - Tv_0\| \\ &\leq \|v_0 - v_{k_m}\| + \|v_{k_m} - Tv_{k_m}\| + (a\|v_{k_m} - v_0\| + b\|v_{k_m} - Tv_0\|) \\ &\leq \|v_0 - v_{k_m}\| + \|v_{k_m} - Tv_{k_m}\| + (a\|v_{k_m} - v_0\| + b\|v_0 - v_{k_m}\| \\ &\quad +b\|v_{k_m} - Tv_{k_m}\|) \\ &= (a + b + 1)\|v_0 - v_{k_m}\| + (b + 1)\|v_{k_m} - Tv_{k_m}\|. \end{aligned}$$

Consequently, we obtain

$$\|v_0 - Tv_0\| \leq (a + b + 1)\|v_0 - v_{k_m}\| + (b + 1)\|v_{k_m} - Tv_{k_m}\|. \tag{3.6}$$

According to Theorem 3.1, we have $\lim_{k\to\infty} \|v_{k_m} - Tv_{k_m}\| = 0$, so applying $m \to \infty$ on (3.6), we obtain $Tv_0 = v_0$. This shows that $v_0 \in F_T$. By Lemma 3.1, $\lim_{k\to\infty} \|v_k - v_0\|$ exists. Consequently, v_0 is the strong limit of $\{v_k\}$ and element of F_T. □

If we drop the compactness assumption, we have the following result. It should be noted that this theorem extends and improves [21, Theorem 4.4] from the setting of nonexpansive operators to the setting of mean nonexpansive operators.

Theorem 3.3. *If C is a nonempty convex and closed subset of $\mathcal{B}$ such that $T : C \to C$ is a mean nonexpansive operator having $F_T \neq \emptyset$. Then $\{v_k\}$ defined in (1.6) converges in the strong sense to a point of F_T, if and only if* $\liminf_{k\to\infty} d(v_k, F_T) = 0$.

Proof. The necessity is straight forward.

Conversely, we may assume that $\liminf_{k\to\infty} d(v_k, F_T) = 0$ and choose $v^* \in F_T$. From the Lemma 3.1, $\lim_{k\to\infty} \|v_k - v^*\|$ exists. By assumption, we conclude that $\lim_{k\to\infty} d(v_k, F_T) = 0$. We want to show that $\{v_k\}$ form a Cauchy sequence in the set C. As we have proved that $\lim_{k\to\infty} d(v_k, F_T) = 0$, so for a given $\varepsilon > 0$, one can choose $r_0 \in \mathbb{N}$ in such a way that for each $k \geq r_0$,

$$d(v_k, F_T) < \frac{\varepsilon}{2} \implies \inf\{\|v_k - v^*\| : v^* \in F_T\} < \frac{\varepsilon}{2}.$$

In particular, $\inf\{\|v_{k_0} - v^*\| : v^* \in F_T\} < \frac{\varepsilon}{2}$. This suggests the existence of $v^* \in F_T$ such that

$$\|v_{k_0} - v^*\| < \frac{\varepsilon}{2}.$$

Now for $r, m \geq k_0$,

$$\begin{aligned} \|v_{k+r} - v_k\| &\leq \|v_{k+r} - v^*\| + \|v_k - v^*\| \\ &\leq \|v_{k_0} - v^*\| + \|v_{k_0} - v^*\| \\ &= 2\|v_{k_0} - v^*\| < \varepsilon. \end{aligned}$$

Hence, we observe that $\{v_k\}$ form a Cauchy sequence in the closed set V and so one can choose some $v^* \in C$ such that $\lim_{k\to\infty} v_k = v^*$. Now $\lim_{k\to\infty} d(v_k, F_T) = 0$ gives that $d(v^*, F_T) = 0$. The closeness of F_T follows from the mean nonexpansiveness of T. Hence $v^* \in F_T$. □

We want to show a strong convergence theorem under the following condition.

Definition 3.1. *[19] On a nonempty subset C of $\mathcal{B}$, an operator $T : C \to C$ is said to have a condition (I) in the case, when there is a selfmap $R : [0,\infty) \to [0,\infty)$ such that $R(i) = 0$, if and only if $i = 0$, $R(i) > 0$ for all real constants $i \in (0,\infty)$ and $\|v - Tv'\| \geq R(d(v, F_T))$ for all $v \in C$.*

The following theorem extends and improves [21, Theorem 4.5] from the setting of nonexpansive operators to the setting of mean nonexpansive operators.

Theorem 3.4. *If C is a nonempty convex and closed subset of $\mathcal{B}$ such that $T : C \to C$ is a mean nonexpansive operator having $F_T \neq \emptyset$. Then $\{v_k\}$ defined in (1.6) converges in the strong sense to a point of F_T if T has a condition (I).*

Proof. In the view of Theorem 3.1, we conclude the following

$$\liminf_{k\to\infty} \|v_k - Tv_k\| = 0. \tag{3.7}$$

By combining condition (I) with (3.7), we get

$$\liminf_{k\to\infty} R(d(v_k, F_T)) = 0.$$

Since the selfmap R is such that $R(0) = 0$ and $R(a) > 0$ for all $i > 0$. It follows that

$$\liminf_{k\to\infty} d(v_k, F_T) = 0.$$

It has been observed that all the requirements of Theorem 3.3 are present, one concludes that T converges strongly to a fixed point of T. □

This section we close by providing a weak convergence theorem. It should be noted that this theorem extends and improves [21, Theorem 4.3] from the setting of nonexpansive operators to the setting of mean nonexpansive operators.

Theorem 3.5. *If C is a nonempty convex and closed subset of $\mathcal{B}$ such that $T : C \to C$ is a mean nonexpansive operator having $F_T \neq \emptyset$. Then $\{v_k\}$ defined in (1.6) converges in the weak sense to a point of F_T if $\mathcal{B}$ has the Opial's property.*

Proof. We can write from Theorem 3.1 that $\{v_k\}$ is bounded and $\lim_{k\to\infty} \|v_k - Tv_k\| = 0$. It is known that $\mathcal{B}$ is RBS in the case when $\mathcal{B}$ is UCBS. Thanks to the reflexiveness of the space, the generated sequence $\{v_k\}$ eventually possess a subsequence which we may denote by $\{v_{k_s}\}$ equiped with a weak limit $v_1 \in C$. Now we may apply Lemma 2.1, and obtain $v_1 \in F_T$. Next we want to show that the element v_1 is also a weak limit of $\{v_k\}$. If the element v_1 is not the weak limit of the sequence $\{v_k\}$, then we may set another weakly convergent subsequence $\{v_{k_t}\}$ of $\{v_k\}$ equiped with a weak limit $v_2 \in C$ in such a way that $v_2 \neq v_1$. Applying Lemma 2.1, $v_2 \in F_T$. Now applying Lemma 3.1 and also the Opial condition, one has

$$\begin{aligned}
\lim_{k\to\infty} \|v_k - v_1\| &= \lim_{s\to\infty} \|v_{k_s} - v_1\| < \lim_{s\to\infty} \|v_{k_s} - v_2\| \\
&= \lim_{k\to\infty} \|v_k - v_2\| = \lim_{t\to\infty} \|v_{k_t} - v_2\| \\
&< \lim_{t\to\infty} \|v_{k_t} - v_1\| = \lim_{k\to\infty} \|v_k - v_1\|.
\end{aligned}$$

The above observations give a contradiction. Thus, we must accept that the element v_1 is the weak limit of $\{v_k\}$. This finishes the proof. □

4. Some applications

If a linear or nonlinear equation has a solution then some-times it is either very hard or impossible to compute the value of such a solution under ordinary analytical approaches [1, 12, 22]. For instance, see the following equations,

$$v^2 - \sin v = 0 \text{ and } v^3 \ln v - e^v = 0,$$

which are not easy to solve by applying available analytical approaches of the literature. In such a case, the approximate value of such a solution is desirable. To find an approximate value of a solution, we must rearrange the given equation in the form of fixed point equation $v = Tv$. Notice that here the operator T should be set on a certian space $\mathcal{B}$. In this case, fixed point set of T is same as the solution set of the given equation. Fixed point theorems provides the existence and uniqueness of a fixed point for T, while iterative algorithms finds the value of this fixed point by imposing some conditions (on T, the domain of T or any other). Banach Contraction Principle (BCP) [5] offers a unique fixed point for T if T is a contraction and $\mathcal{B}$ is a complete metric space and suggest a basic iterative algorithm due to Picard for computing its value. However, for nonexpansive operators, Picard algorithm in general fails to converge. The class of nonexpansive operators includes contractions and has important applications in many areas of applied sciences, that is, in image reconstruction and signal processing problems, the operator T is essentially nonexpansive, when the method of an averaging operators is used, (see e.g., Byrne [7] and others).

Accordingly, here we propose algorithms that are the modifications of the TTP algorithm (1.6) (which we call here TTP type algorithms) and using our main results, we show that these algorithms eventually solves SFPs and VIPs, respectively.

4.1. Split feasibility problems

If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two given real Hilbert spaces. Then the concept of a SFP mathematically, a SFP [8] reads as follows:

$$\text{Find } v^* \in C\ Av^* \in Q. \tag{4.1}$$

The subset $C \subseteq \mathcal{B}_1$ as well as the subset $Q \subseteq \mathcal{B}_2$ are assumed as closed, convex and $A : \mathcal{B}_1 \to \mathcal{B}_2$ is an operator which is assumed at the same time linear and bounded. From [13], we know that, many problems arise in the research of signal processing as well as the design of any provided nonlinear synthetic discriminant filter as concerns optical pattern recognition, can be set in the form of SFPs.

We may suppose that the SFP (4.1) has a nonempty solution set, and we denote it by $\mathcal{S}$. Now, from [13], we know that, $v^* \in C$ solves (4.1) if and only if it solves the below provided fixed point equation:

$$v = P_C(I_{id} - \xi A^*(I_{id} - P_Q)A)v.$$

The above used notations P_C as well as P_Q are used for the nearest point projection (NPP) onto the already choosen sets C and Q, respectively. While $\xi > 0$ and A^* denotes the adjoint operator of corresponding to the operator A. In [7], Byrne was the first researcher among other things, who noted that if η is a scalar that denotes a spectral radius of A^*A and suppose $0 < \xi < \frac{2}{\eta}$, we can say that

$$T = P_C(I_{id} - \xi A^*(I - P_Q)A)$$

is essentially nonexpansive and the weak convergence of the below CQ iterative algorithm

$$v_{k+1} = P_C(I_{id} - \xi A^*(I_{id} - P_Q)A)v_k, k \in \mathbb{N},$$

is confirm in the set $\mathcal{S}$.

The improvement and extension the above weak convergence to case of strong convergence gained the attention of many authors. However to do this, one needs some more assumptions, (see e.g., [13] and others) to study a recent survey on the Halpern type algorithms.

In this research, our approach is to consider mean nonexpansive operators, which are generally not continuous on the domain on which they are defined (as shown by examples in this paper), instead of nonexpansive operators, that are essentially throughout continuous on the domain on which they are defined. In this case, we assume that a SFP has a solution, and prove that the proposed iterative algorithm converges weakly and strongly to its solution.

Theorem 4.1. *Let the SFP (4.1) be consistent, that is,* $\mathcal{S} \neq \emptyset$, $0 < \xi < \frac{2}{\eta}$ *and* $P_C(I_{id} - \xi A^*(I_{id} - P_Q)A)$ *be a mean nonexpansive operator. Then there exists* $\mu_k, \eta_k, \theta_k \in (0, 1)$ *in a way that the suggested iterative algorithm sequence* $\{v_k\}$ *produced as*

$$\begin{cases} v_1 \in C, \\ e_k = (1 - \theta_k)v_k + \theta_k P_C(I - \xi A^*(I_{id} - P_Q)A)v_k, \\ s_k = (1 - \eta_k)e_k + \eta_k P_C(I - \xi A^*(I_{id} - P_Q)A)e_k, \\ v_{k+1} = (1 - \mu_k)P_C(I_{id} - \xi A^*(I_{id} - P_Q)A)e_k + \mu_k \\ P_C(I_{id} - \xi A^*(I_{id} - P_Q)A)s_k, \\ k \in \mathbb{N}, \end{cases}$$

accordingly weakly convergent to v^* *which is a solution of SFP problem (4.1).*

Proof. Put $T = P_C(I_{id} - \xi A^*(I_{id} - P_Q)A)$. Then T is a mean nonexpansive operator. According to Theorem 3.5, $\{v_k\}$ converges weakly to a point of F_T. But $F_T = \mathcal{S}$, it follows that $\{v_k\}$ converges weakly to a solution v^* of the SFP problem (4.1). □

The strong convergence is the following.

Theorem 4.2. *Let the SFP (4.1) be consistent, that is,* $\mathcal{S} \neq \emptyset$, $0 < \xi < \frac{2}{\eta}$ *and* $P_C(I_{id} - \xi A^*(I_{id} - P_Q)A)$ *be a mean nonexpansive operator. Then there exists* $\mu_k, \eta_k, \theta_k \in (0, 1)$ *in a way that the suggested iterative algorithm sequence* $\{v_k\}$ *produced as*

$$\begin{cases} v_1 \in C, \\ e_k = (1 - \theta_k)v_k + \theta_k P_C(I - \xi A^*(I_{id} - P_Q)A)v_k, \\ s_k = (1 - \eta_k)e_k + \eta_k P_C(I - \xi A^*(I_{id} - P_Q)A)e_k, \\ v_{k+1} = (1 - \mu_k)P_C(I_{id} - \xi A^*(I_{id} - P_Q)A)e_k + \mu_k \\ P_C(I_{id} - \xi A^*(I_{id} - P_Q)A)s_k, \\ k \in \mathbb{N}, \end{cases}$$

accordingly strongly convergent to v^* *which is a solution of SFP problem (4.1) provided that* $\liminf_{k\to\infty} d(v_k, \mathcal{S}) = 0$.

Proof. Put $T = P_C(I_{id} - \xi A^*(I_{id} - P_Q)A)$. Then T is mean nonexpansive. According to Theorem 3.3, $\{v_k\}$ converges strongly to a point of F_T. But $F_T = \mathcal{S}$, it follows that $\{v_k\}$ converges strongly to a solution v^* of the SFP problem (4.1). □

4.2. Variational inequality problems

Suppose a Hilbert space $\mathcal{B}$ and $\emptyset \neq C \subset \mathcal{B}$ is closed and convex. The operator $M : \mathcal{B} \to \mathcal{B}$ is known as monotone if

$$\langle Mv - Mv', v - v' \rangle \geq 0, \forall v, v' \in \mathcal{B}.$$

Now we give the concept of a VIP. Mathematically, a VIP reads as follows:

$$\text{Find } v^* \in C \text{ such that } \langle Mv^*, v - v^* \rangle \geq 0 \ \forall v \in \mathcal{B}. \tag{4.2}$$

In [7], the author noted that if $\xi > 0$, then $v^* \in C$ is always a solution for the VIP (4.2) if and only if v^* is a solution of the below given fixed point equation:

$$v = P_C(I_{id} - \xi M)v,$$

where P_C denotes the nearest point projection onto the set C.

In [7], the author noted that if $I_{id} - \xi M$ and $P_C(I_{id} - \xi M)$ are nonexpansive operators, then, the sequence $\{v_k\}$ generated by the following iterative algorithm:

$$v_{k+1} = P_C(I_{id} - \xi M)v_k, k \in \mathbb{N}$$

converges weakly to a solution of the VIP (4.2), provided that such solutions essentially exist.

In this research, our approach is to consider mean nonexpansive operators, which are generally not continuous on the domain on which they are defined (as shown by examples in this paper), instead of nonexpansive operators, that are essentially throughout continuous on the domain on which they are defined. In this case, we assume that a VIP has a solution, and prove that the proposed iterative algorithm converges weakly and strongly to its solution.

Theorem 4.3. *Let the VIP (4.2) be consistent, that is,* $\mathcal{S} \neq \emptyset$, $\xi > 0$. *If* $P_C(I_{id} - \xi M)$ *is a mean nonexpansive operator. Then there exists* $\mu_k, \eta_k, \theta_k \in (0,1)$ *in a way that the suggested iterative algorithm sequence* $\{v_k\}$ *produced as*

$$\begin{cases} v_1 \in C, \\ e_k = (1-\theta_k)v_k + \theta_k P_C(I_{id} - \xi M)v_k, \\ s_k = (1-\eta_k)e_k + \eta_k P_C(I_{id} - \xi M)e_k, \\ v_{k+1} = (1-\mu_k)P_C(I_{id} - \xi M)e_k + \mu_k \\ P_C(I_{id} - \xi M)s_k, \\ k \in \mathbb{N}, \end{cases}$$

accordingly weakly convergent to v^* *which is a solution of VIP problem (4.2).*

Proof. Put $T = P_C(I_{id} - \xi M)$. Then T is a mean nonexpansive operator. According to Theorem 3.5, $\{v_k\}$ converges weakly to a point of F_T. But $F_T = \mathcal{S}$, it follows that $\{v_k\}$ converges weakly to a solution v^* of the VIP problem (4.2). □

Theorem 4.4. *Let the VIP (4.2) be consistent, that is,* $\mathcal{S} \neq \emptyset$, $\xi > 0$. *If* $P_C(I_{id} - \xi M)$ *is a mean nonexpansive operator. Then there exists* $\mu_k, \eta_k, \theta_k \in (0,1)$ *in a way that the suggested iterative algorithm sequence* $\{v_k\}$ *produced as*

$$\begin{cases} v_1 \in C, \\ e_k = (1-\theta_k)v_k + \theta_k P_C(I_{id} - \xi M)v_k, \\ s_k = (1-\eta_k)e_k + \eta_k P_C(I_{id} - \xi M)e_k, \\ v_{k+1} = (1-\mu_k)P_C(I_{id} - \xi M)e_k + \mu_k \\ P_C(I_{id} - \xi M)s_k, \\ k \in \mathbb{N}, \end{cases}$$

accordingly strongly convergent to v^* *which is a solution of VIP problem (4.2) provided that* $\liminf_{k\to\infty} d(v_k, \mathcal{S}) = 0$.

Proof. Put $T = P_C(I_{id} - \xi M)$. Then T is mean nonexpansive. According to Theorem 3.3, $\{v_k\}$ converges weakly to a point of F_T. But $F_T = \mathcal{S}$, it follows that $\{v_k\}$ converges strongly to a solution v^* of the VIP problem (4.2). □

5. Examples

We first construct an example of discontinuous mean nonexpansive maps having a unique fixed point. We show that this example does not belong to the class of nonexpansive operators as.

Example 5.1. *If* $C = [1, 5]$, *then* C *is clearly close and convex. We now set* $T : C \to C$ *that is defined as follows:*

$$Tv = \begin{cases} 5, & 1 \le v < 2, \\ \frac{v+5}{2}, & 2 \le v \le 5. \end{cases}$$

We want to show that T *is mean nonexpansive on* C. *We need to find some positive real constants* a, b *with* $a + b \le 1$, *such that* $\|Tv - Tv'\| \le a\|v - v'\| + b\|v - Tv'\|$, *for every* $v, v' \in C$. *Choose* $a = \frac{1}{2} = b$, *then* $a + b \le 1$. *We suggest the following cases.*

(i) Suppose that $1 \le v, v' < 2$. *Then*

$$\|Tv - Tv'\| = 0 \le a\|v - v'\| + b\|v - Tv'\|.$$

(ii) Suppose that $2 \le v, v' < 5$. *Then*

$$\begin{aligned} a\|v - v'\| + b\|v - Pv'\| &= \frac{1}{2}\|v - v'\| + b\left\|v - (\frac{v' + 5}{2})\right\| \\ &\ge \frac{1}{2}\|v - v'\| \\ &= \|Tv - Tv'\|. \end{aligned}$$

(iii) Suppose that $1 \le v' < 2$ *and* $2 < v \le 5$.

$$\begin{aligned} a\|v - v'\| + b\|v - Tv'\| &= a\|v - v'\| + \frac{1}{2}\|v - 5\| \\ &\ge \frac{1}{2}\|v - 5\| \\ &= \|\frac{v - 5}{2}\| \\ &= \|Tv - Tv'\|. \end{aligned}$$

(iv) Suppose that $1 \le v < 2$ *and* $2 \le v' \le 5$.

$$\begin{aligned} a\|v - v'\| + b\|v - Tv'\| &= \frac{1}{2}\|v - v'\| + \frac{1}{2}\left\|v - \left(\frac{v' + 5}{2}\right)\right\| \\ &\ge \frac{1}{2}\left\|(v - v') - \left(v - \left(\frac{v' + 5}{2}\right)\right)\right\| \\ &= \frac{1}{2}\left\|\frac{-2v' + v' + 5}{2}\right\| \\ &= \left\|\frac{v' - 5}{2}\right\| \\ &= \|Tv - Tv'\|. \end{aligned}$$

It has been observed that the operator T *is mean nonexpansive on the set* C. *Notice that* T *is being discontinuous and hence not nonexpansive. Precisely, if* $v = 1.9$ *and* $v' = 2$, *then* $\|Tv - Tv'\| > \|v - v'\|$. *Using this example, we provide some values obtained from different iterative algorithms in the Table 1 and the behaviors of there iterates can be viewed in the Figure 1.*

Table 1. Iterates of Picard , Mann (1.1), Ishikawa (1.2), Noor (1.3), S (1.4), Abbas (1.5), and TTP (1.6) using the operator T of Example 5.1.

k	Picard	Mann	Ishikawa	Noor	S	Abbas and Nazir	TTP
1	4.3	4.3	4.3	4.3	4.3	4.3	4.3
2	4.6500	4.5975	4.6942	4.7159	4.7467	4.7470	4.8037
3	4.8250	4.7686	4.8664	4.8847	4.9083	4.9086	4.9449
4	4.9125	4.8669	4.9416	4.9532	4.9668	4.9670	4.9846
5	4.9563	4.9235	4.9745	4.9810	4.9880	4.9881	4.9957
6	4.9781	4.9560	4.9889	4.9923	4.9957	4.9957	4.9988
7	4.9891	4.9747	4.9951	4.9969	4.9984	4.9984	4.9997
8	4.9945	4.9855	4.9979	4.9987	4.9994	4.9994	5.9999
9	4.9973	4.9916	4.9991	4.9995	4.9998	4.9998	**5.0000**
10	4.9986	4.9952	4.9996	4.9998	4.9999	4.9999	5.0000
11	4.9993	4.9972	4.9998	4.9999	**5.0000**	**5.0000**	5.0000
12	4.9997	4.9984	4.9999	**5.0000**	5.0000	5.0000	5.0000
13	4.9998	4.9991	**5.0000**	5.0000	5.0000	5.0000	5.0000
14	4.9999	4.9995	5.0000	5.0000	5.0000	5.0000	5.0000
15	**5.0000**	4.9997	5.0000	5.0000	5.0000	5.0000	5.0000
16	5.0000	4.9998	5.0000	5.0000	5.0000	5.0000	5.0000
17	5.0000	4.9999	5.0000	5.0000	5.0000	5.0000	5.0000
18	5.0000	**5.0000**	5.0000	5.0000	5.0000	5.0000	5.0000

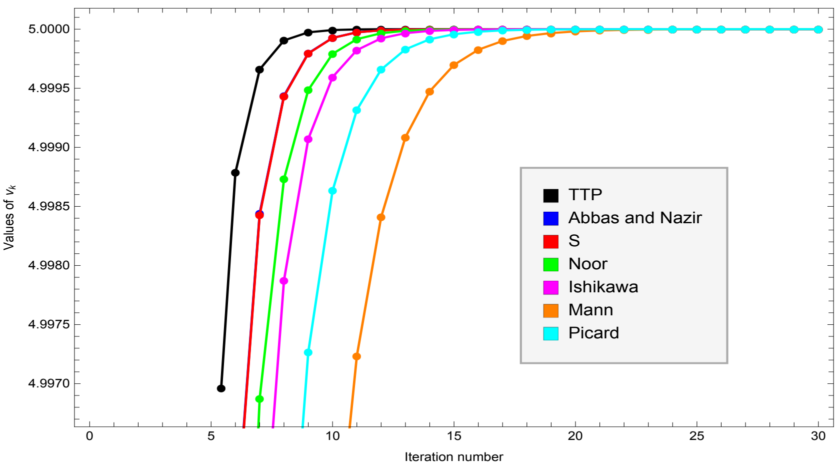

Figure 1. Behaviors of iterates of the Table 1.

Now, we show some furhter high accuracy of the TTP iterative algorithm. We use Example 5.2, and set $\|v_k - v^*\| < 10^{-15}$ our stopping cretrian. Observations are provided in the Tables 2–4 and Figures 2–4.

Table 2. $\mu_k = \frac{k}{k+3}$, $\eta_k = \frac{k}{\sqrt{(k+7)}}$ and $\theta_k = \frac{2k}{5k+2}$.

Number of iterative steps to reach the fixed point.			
v_1	Noor (1.3)	Abbas and Nazir (1.5)	TTP(1.6)
0.12	17	13	**10**
0.28	17	14	**10**
0.50	17	14	**11**
0.75	18	14	**11**
0.99	18	16	**11**

Table 3. $\mu_k = \frac{k}{k+1}$, $\eta_k = \frac{k}{k+7}$ and $\theta_k = \sqrt{\left(\frac{1}{3k+4}\right)}$.

Number of iterative steps to reach the fixed point.			
v_1	Noor (1.3)	Abbas and Nazir (1.5)	TTP(1.6)
0.12	15	16	**14**
0.28	16	17	**14**
0.50	16	17	**15**
0.75	16	17	**15**
0.99	16	17	**15**

Table 4. $\mu_k = \frac{k}{(7k+25)^{\frac{1}{7}}}$, $\eta_k = 1 - \left(\frac{1}{k+7}\right)$ and $\theta_k = \frac{k}{k+25}$.

Number of iterative steps to reach the fixed point.			
v_1	Noor (1.3)	Abbas and Nazir (1.5)	TTP(1.6)
0.12	26	15	**12**
0.28	27	15	**13**
0.50	27	16	**13**
0.75	27	16	**13**
0.99	28	16	**13**

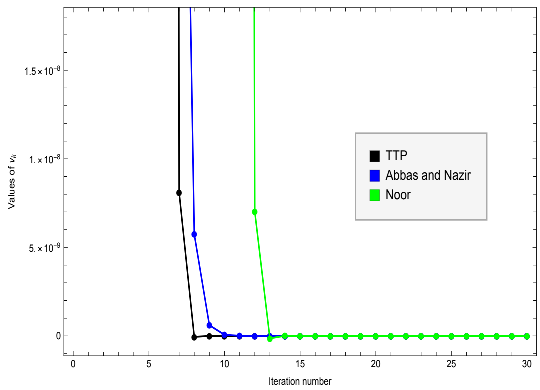

Figure 2. Behaviors comparison of the three-step methods by putting the parameters as in the Table 2 and the starting point $v_1 = 0.14$.

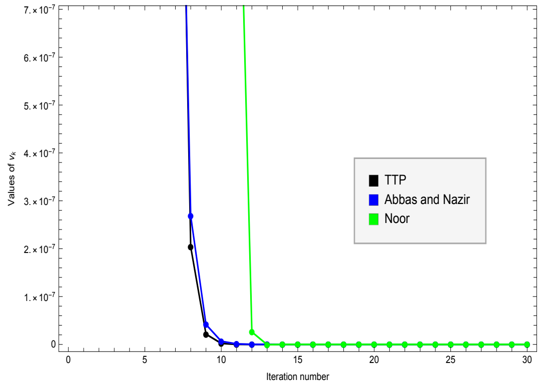

Figure 3. Behaviors comparison of the three-step methods by putting the parameters as in the Table 3 and the starting point $v_1 = 0.52$.

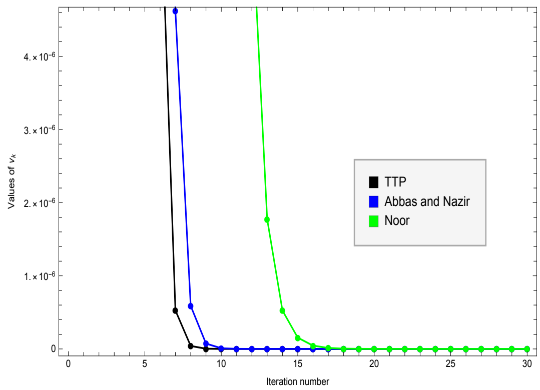

Figure 4. Behaviors comparison of the three-step methods by putting the parameters as in the Table 4 and the starting point $v_1 = 0.98$.

We finish our paper with the following example that illustrates our main results.

Example 5.2. *If $C = [0, 1]$, then we can set an operator $T : C \to C$ by the following formula*

$$Tv = \begin{cases} \frac{v}{5}, & 0 \leq v < \frac{1}{2}, \\ \frac{v}{6}, & \frac{1}{2} \leq v \leq 1. \end{cases}$$

Here, T is discontinuous and hence not nonexpansive. On the other hand, T is a mean nonexpansive operator. Moreover, the domain of T is closed convex subset of a UCBS, so the sequence of TTP iteration (1.6) converges to its fixed point.

6. Conclusions

The paper provided a three-step iterative approach to compute fixed points of mean nonexpansive maps in a Banach space setting. Weak and strong convergence on compact and noncompact domains are essentially established. We have showed by examples that mean nonexpansive mappings are in general discontinuous and include all nonexpansive mappings. Accordingly, we have improved the main results of Thakur et al. [21] in two ways:

(i) From nonexpansive operators to the wider setting of mean nonexpansive operators.
(ii) From continuous operators to the general setting of discontinuous operators.

As applications of the main outcome, we have suggested two new three-step TTP type projection algorithms to find a solution for SFP and VIP in the context of mean nonexpansive mappings. We have performed some numerical experiments to provide the high accuracy of the studied three-step algorithm corresponding to the other three-step algorithms in context of mean nonexpansive operators.

7. Open problem

We now leave an intersting open problem for the readers as follows.
Open Problem. Can we extend the results of this paper to the setting of hyperbolic spaces?

Acknowledgments

This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444).

Conflict of interest

The authors declare no conflict of interest.

References

1. H. A. Abass, C. Izuchukwu, O. T. Mewomo, On split equality generalized equilibrium problems and fixed point problems of Bregman relatively nonexpansive semigroups, *J. Appl. Numer. Optim.*, **4** (2022), 357–380. 10.23952/jano.4.2022.3.04
2. M. Abbas, T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, *Mat. Vestn.*, **66** (2014), 223–234.
3. R. P. Agarwal, D. O'Regan, D. R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings with Applications Series*, New York: Springer, **6** (2009).
4. R. P. Agarwal, D. O'Regon, D. R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, *J. Nonlinear Convex Anal.*, **8** (2007), 61–79.
5. S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, *Fund. Math.*, (1922), 133–181.
6. F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Natl. Acad. Sci.*, **54** (1965), 1041–1044. https://doi.org/10.1073/pnas.54.4.1041
7. C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Probl.*, **20** (2004), 103–120. https://doi.org/10.1088/0266-5611/20/1/006
8. Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numer. Algorithms*, **8** (1994), 221–239.
9. J. A. Clarkson, Uniformly convex spaces, *Trans. Amer. Math. Soc.*, **40** (1936), 396–414.

10. D. Gohde, Zum Prinzip der Kontraktiven Abbildung, *Math. Nachr.*, **30** (1965), 251–258. https://doi.org/10.1002/mana.19650300312

11. S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, **44** (1974), 147–150. https://doi.org/10.1090/S0002-9939-1974-0336469-5

12. L. Liu, B. Tan, A. Latif, Approximation of fixed points for a semigroup of Bregman quasi-nonexpansive mappings in Banach spaces, *J. Nonlinear Var. Anal.*, **5** (2021), 9–22. https://doi.org/10.23952/jnva.5.2021.1.02

13. G. Lopez, V. Martín-Ma Themrquez, H. K. Xu, Halpern's iteration for nonexpansive mappings, In: *Nonlinear Analysis and Optimization I: Nonlinear Analysis*, **513** (2010), 211–231. http://doi.org/10.1090/conm/513/10085

14. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506–510. https://doi.org/10.2307/2032162

15. G. Maniu, On a three-step iteration process for Suzuki mappings with qualitative study, *Numer. Funct. Anal. Optim.*, **41** (2020), 929–949. https://doi.org/10.1080/01630563.2020.1719415

16. M. A. Noor, New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.*, **251** (2000), 217–229. https://doi.org/10.1006jmaa.2000.70

17. Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73** (1967), 591–597.

18. J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, *Bull. Aust. Math. Soc.*, **43** (1991), 153–159. https://doi.org/10.1017/S0004972700028884

19. H. F. Senter, W. G. Dotson, Approximating fixed points of non-expansive mappings, *Proc. Amer. Math. Soc.*, **44** (1974), 375–380.

20. W. Takahashi, *Nonlinear Functional Analysis*, Yokohoma: Yokohoma Publishers, 2000.

21. B. S. Thakur, D. Thakur, M. Postolache, New iteration scheme for approximating fixed point of nonexpansive mappings, *Filomat*, **30** (2016), 2711–2720. https://doi.org/10.2298/FIL1610711T

22. T. Thianwan, A new iteration scheme for mixed type asymptotically nonexpansive mappings in hyperbolic spaces, *J. Nonlinear Funct. Anal.*, **2021** (2021), 21. https://doi.org/10.23952/jnfa.202

23. C. X. Wu, L. J. Zhang, Fixed points for mean non-expansive mappings, *Acta Math. Appl. Sin.*, **23** (2007), 489–494.

24. S. Zhang, About fixed point theory for mean nonexpansive mapping in Banach spaces. *J. Sichuan Univ.*, **2** (1975), 67–78.

25. Z. F. Zuo, Fixed-point theorems for mean nonexpansive mappings in Banach spaces, *Abstr. Appl. Anal.* **2014** (2014), 746291. https://doi.org/10.1155/2014/746291