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Research article

Towards coupled coincidence theorems of generalized admissible types of mappings on partial satisfactory cone metric spaces and some applications

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Abstract: This paper introduces a novel class of generalized α-admissible contraction types of mappings in the framework of θ-complete partial satisfactory cone metric spaces and proves the existence and uniqueness of coincidence points for such mappings. In this setting, the topology generated and induced by the partial satisfactory cone metric is associated with semi-interior points rather than interior points of the underlying cone. In addition, some applications of the paper's main coincidence point theorems are given. The results of this paper unify, extend and generalize some previously proved theorems in this generalized setting.

Keywords: normal cones; solid cones; partial satisfactory cone metric spaces; semi-interior points; admissible mappings; fixed-points; coincidence points; coupled coincidence points; coupled fixed-points
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1. Introduction

In 1922, the mathematician Banach [1] proved the well-known fixed-point theorem named the Banach contraction principle in the settings of complete metric spaces. He proved that every contraction mapping on a complete metric space has a unique fixed-point. Thereafter, various generalizations and fixed-point results have been proved by many authors and some are recently appeared in [2–4].

In 2007, Huang and Zhang [5] introduced the concept of cone metric spaces by replacing real numbers with a cone in a normed space. They also defined the convergence and Cauchyness concepts of sequences in terms of the interior points of the given cone. Moreover, they proved some important fixed-point theorems and extended the well-known Banach contraction principle to the settings of cone metric spaces where the respective cones are normal and solid. Such a line allows for investigating lots of studies and results in fixed-point theory without assuming the normality property of the underlying cone. The results of Huang and Zhang were generalized by Rezapour and Hamlbarani [6] by eliminating the normality assumption of the underlying cone. There are actually lots of generalizations of metric spaces in which the distance function takes values in ordered cones.

Unfortunately, when cones are assumed to be normal and solid, these generalizations become impractical due to the equivalent of the topology induced by a metric and the topology induced by a cone metric. In fact; the equivalent characterizations have been shown by many authors, see for instance [7–14] and references cited therein. We also mention that Azam and Mehmood [15] introduced the notion of tvs-valued cone metric space to present the same notions in more general settings.

In 2006, Bhaskar and Lakshmikantham [16] studied the existence and uniqueness of coupled fixed-point theorems for maps with mixed monotonic properties in metric spaces with a partial order. The obtained results were investigated using the assumption of weak contraction type.

In 2009, Lakshmikantham and Ćirić [17] introduced more notations of mixed g-monotone maps and proved coupled coincidence and coupled common fixed-point theorems for such types of contractive maps in the case of partially ordered complete metric spaces. These presented results are a generalization of the results given in [16].

In 2011, Janković et al. [18] showed that all fixed-point results in cone metric spaces wherein the underlying cone is normal and solid are proper copies of classical results in metric spaces. Therefore, any generalizations of fixed-point from metric space to cone metric space are repeated.

In 2012, Sönmez [19, 20] defined a partial cone metric space and studied its topological properties. In the same paper, fixed-point results for some contractive types of operators are proved in the generalized complete partial cone metric spaces.

In 2012, Samet et al. [21] initially considered the notion of α-admissible mappings in metric spaces and they gave some examples to elucidate and support the concept. Furthermore, they presented some relevant fixed-point results for such a class of mappings in this space. Subsequently, a number of authors have exploited the concept of α-admissible contraction types of mappings to study the existence of fixed-points in many generalized spaces.

In 2013, Malhotra et al. [22], Jiang and Li [23] extended the results of [19] and [20] to θ-complete partial cone metric spaces without using the normality condition of the ordered cones. In all the results listed before, the given Banach space is considered to be with a solid cone.

In 2017, Basile et al. [24] defined the notion of semi-interior point as a partial treatment of the non-solidness problem of cones and hence solved many equilibrium and computer problems in this setting.

In 2018, Aleksi´ et al. [25] gave a survey on some properties and results of (non)-normal and (non)-solid cones. On the other hand, they showed that any solid cone in a topological vector space can be replaced by a solid and normal cone in a normed space. Consequently, most of the problems in (TVS) cone metric spaces can be reduced to their standard metric counterparts.

In 2019, Mehmood et al. [26] defined a new concept of convergence by means of semi-interior points of the positive cone in the settings of E-metric spaces. The authors proved some generalizations of fixed point theorem of contraction, Kannan and Chatterjea types of mappings in the context of E-metric spaces where the underlying positive cone of a real normed space E is non-solid and possibly non-normal.

After that, Huang et al. [27] explored some topological properties and fixed point results in cone metric spaces over Banach algebras. Also, Huang [28] gave some fixed points theorems with some applications in E-metric spaces using the concept of semi-interior points.

In 2020, 2021 and 2022 Sahar Mohamed Ali Abou Bakr [29–32] studied various types of cone metric spaces and made some generalizations in the case of b-cone metric spaces, cone metric spaces, θ-cone metric spaces and b-cone metric spaces and applied these generalizations to some fixed-point and coupled fixed-point theorems.

In 2022 Sahar Mohamed Ali Abou Bakr [33] considered non-normal and non-solid cones and proved more generalized fixed-point theorems in the case of generalized b-cone metric spaces over Banach algebras.

Motivated by the preceding studies, most of our efforts in this research are directed to study the topological structure of partially satisfactory cone metric spaces when cones in normed spaces fail to have interior points but have semi-interior points and the cones possibly non-normal. As a sequel, in the settings of partial satisfactory cone metric spaces, we generalize many concepts of α-admissible types of mappings and define improved wider categories of these generalized functions of α-admissibility types. Since the class of α-admissible mappings are a special case of these improvements, we find results valid to a wider range of contraction classes of mappings. Further, we investigate a new aspect of fixed-point theory where the real contraction constant of the fundamental contraction inequality is replaced by a suitable control sequence of positive real numbers backed by a certain condition to make the generalized inequality more general. Besides that, we will stick to looking for coincidence points, coupled coincidence points, coupled fixed-point and fixed-points of the contraction mapping in a small set of points rather than the whole domain of the mapping. In fact; we precisely combine all of the above trends in our obtained main results.

2. Basic definitions and mathematical preliminaries

For the sake of simplicity in notation, here and in what follows, let E be a real Banach space, θ be the zero vector in E, C be a cone in E, $\operatorname{Int} C$ denotes the set of all interior points of C, $U := \{x \in E : \|x\| \leq 1\}$ denotes the closed unit ball of E and the set $U_+ := U \cap C$ denotes the positive part of the unit ball of E defined by C.

Whenever misunderstandings might occur, we write U^E to confirm that U^E is the closed unit ball in the space E and we denote by U_+^E, the positive part of U^E.

Any cone $C \subset E$ defines the following partial ordered relations:

$$x \leq y \text{ if and only if } y - x \in C,$$

$$x \prec y \text{ if and only if } y - x \in C \text{ and } x \neq y,$$

and

$$x \ll y \text{ if and only if } y - x \in \operatorname{Int} C.$$

The following basic definitions and facts are mostly presented in [5–10,12–15,19,20,22–24,29–34].

Definition 2.1. *A cone C of a real Banach space E is solid if and only if Int $C \neq \emptyset$ and it is normal if and only if there exists a real number $M > 0$ such that $\|x\| \leq M\|y\|$ for every $x, y \in E$ with $\theta \leq x \leq y$.*

The smallest positive constant M for which the above inequality holds is called the normal constant of C.

Lemma 2.2. *Let C be a solid cone of the normed space $(E, \|.\|)$ and $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in E. Then, $u_n \xrightarrow{\|.\|} \theta$ implies that for each $c \in Int\ C$, there exists a positive integer n_0 such that $u_n \ll c$ for all $n \geq n_0$.*

Definition 2.3. *A partial cone metric on a non-empty set X is a mapping $p : X \times X \to C$ such that for all $x, y, z \in X$, the following conditions are satisfied:*
$(\mathrm{PCM}_1) : \theta \preceq p(x, x) \preceq p(x, y);$
(PCM_2) : *If $p(x, x) = p(x, y) = p(y, y)$, then $x = y$;*
$(\mathrm{PCM}_3) : p(x, y) = p(y, x);$
$(\mathrm{PCM}_4) : p(x, y) \preceq p(x, z) + p(z, y) - p(z, z).$
The quadruple (X, E, C, p) in this case is said to be partial cone metric space.

Theorem 2.4. *Any partial cone metric space (X, E, C, p) is a topological space. If C is a normal cone, then (X, E, C, p) is T_0-space.*

Definition 2.5. *Let (X, E, C, p) be a partial cone metric space over a solid cone C and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then, we have the following:*
(1) *$\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$ if and only if for each $c \gg \theta$, there exists a positive integer n_0 such that $p(x_n, x) \ll p(x, x) + c$ for all $n \geq n_0$. This type of convergence denoted by $x_n \xrightarrow{\tau_p} x$.*
(2) *$\{x_n\}_{n\in\mathbb{N}}$ strongly converges to $x \in X$, if $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = p(x, x)$, the limit is taken with respect to the norm $\|.\|$ on E. This type of convergence denoted by $x_n \xrightarrow{s-\tau_p} x$.*
(3) *$\{x_n\}_{n\in\mathbb{N}}$ is a θ-Cauchy if and only if for each $c \gg \theta$, there exists a positive integer n_0 such that $p(x_n, x_m) \ll c$ for all $n, m \geq n_0$.*
(4) *The partial cone metric space (X, E, C, p) is a θ-complete, if each θ-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ of X converges to some point $x \in X$ with $p(x, x) = \theta$.*
(5) *$\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, if there exists $u \in C$ such that $\lim_{n\to\infty} p(x_n, x_m) = u$.*
(6) *The partial cone metric space (X, E, C, p) is complete if and only if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ of X strongly converges to some point $x \in X$ with $p(x, x) = u$.*

Remark 2.6. *Every complete partial cone metric space (X, E, C, p) is θ-complete, but the converse is not generally true. In fact; example (3) in [32] is an example of θ-complete partial cone metric space with a Cauchy sequence which is not strongly convergent to any element in X. Consequently, it represents an example of θ-complete partial cone metric space which is not complete.*

Now, we recall a class of cones in an ordered normed spaces $(E, \|.\|)$ defined by means of points in C, cones with semi-interior points, that are weaker than the one of interior points of C. The concept of semi-interior point of cone C and some of its characteristics can be found in [24, 28].

Definition 2.7. *The vector $x_0 \in C$ is called semi-interior point of C if there exists a positive real number $\rho > 0$ such that $x_0 - \rho U_+ \subseteq C$.*

The set of semi-interior points of C is generally denoted by $C^{\ominus}$. The partial ordered relation $\lll$ can be defined for $x, y \in E$ as follows:

$$x \lll y \text{ if and only if } y - x \in C^{\ominus}.$$

In particular; we have

$$\theta \lll x \ \text{ if and only if } x \in C^{\ominus}.$$

Remark 2.8. (1) *We have the following relations:*

(a) $C^{\ominus} + C^{\ominus} \subset C^{\ominus}$.
(b) $C^{\ominus} + C \subset C^{\ominus}$.
(c) $\alpha C^{\ominus} \subset C^{\ominus}$ *for any real number* $\alpha > 0$.

(2) *Any interior point of a cone C is a semi-interior point of C with respect to the norm* $\|.\|$ *on E, while the converse is not true. Some examples of non-solid cones in normed and Banach spaces having some semi-interior points can be found in [24].*

By the help of example (2.5) in [24], we rebuild an illustrative example to support definition (2.7) in the following way:

Example 2.9. *Let* $X_n := \left(\mathbb{R}^2, \|.\|_n\right)$ *be the Banach space* $\mathbb{R}^2$ *ordered by the point-wise ordering and equipped with the norm* $\|.\|_n$ *defined by the following formula:*

$$\left\|(x_1^n, x_2^n)\right\|_n = \begin{cases} \left|x_1^n\right| + \left|x_2^n\right|, & x_1^n x_2^n \geq 0, \\ \max\left\{\left|x_1^n\right|, \left|x_2^n\right|\right\} - \left(\frac{n-1}{n}\right)\min\left\{\left|x_1^n\right|, \left|x_2^n\right|\right\}, & x_1^n x_2^n < 0. \end{cases}$$

Figure 1 sketches the closed unit ball of X_n *whose vertices are the points* $(1,0)$*,* $(0,1)$*,* $(-n,n)$*,* $(-1,0)$*,* $(0,-1)$ *and* $(n,-n)$*.*

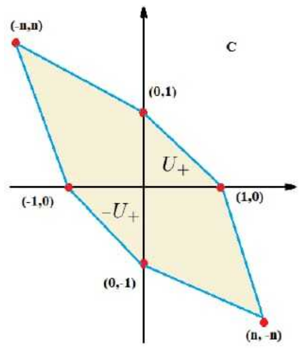

Figure 1. The closed unit ball in X_n.

Let $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ *be any sequence of positive numbers,* $\beta_n > 0$ *for all* $n \in \mathbb{N}$*. Denote by* $E := \left(\bigoplus_{n\in\mathbb{N}} X_n\right)_{\ell_\infty(\beta)}$*, the linear space of all sequences defined by:*

$$E := \left(\bigoplus_{n\in\mathbb{N}} X_n\right)_{\ell_\infty(\beta)} = \left\{\{x_n\}_{n\in\mathbb{N}} : x_n = (x_1^n, x_2^n) \in X_n,\ \{\|\beta_n x_n\|_n\}_{n\in\mathbb{N}} \in \ell_\infty\right\}.$$

Endow the space $E := \left(\bigoplus_{n\in\mathbb{N}} X_n\right)_{\ell_\infty(\beta)}$ *with the following norm:*

$$\|x\|_{\ell_\infty(\beta)} = \sup_{n\in\mathbb{N}} \{\beta_n \|x_n\|_n\} \ \textit{for every} \ \ x = \{x_n\}_{n\in\mathbb{N}} \in E.$$

Assume that the space E is ordered by the cone

$$C := \{x = \{x_n\}_{n\in\mathbb{N}} \in E : x_n = (x_1^n, x_2^n), x_i^n \geq 0, i \in \{1,2\}, n \in \mathbb{N}\}.$$

Choose in particular the weighted sequence $\beta = \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ *and let* $x = \{(n,n)\}_{n\in\mathbb{N}}$. *Since* $\|x\|_{\ell_\infty\left(\{\frac{1}{n}\}_{n\in\mathbb{N}}\right)} = \sup_{n\in\mathbb{N}}\left\{2n \cdot \frac{1}{n}\right\} = 2 < \infty$, *we see that* $x \in C$ *is not an interior point of* C.

Now, take any $y = \left\{\left(y_1^n, y_2^n\right)\right\}_{n\in\mathbb{N}} \in U_+^E$. *Then, we have*

$$\frac{1}{n}\left\|(y_1^n, y_2^n)\right\|_n \leq \|y\|_{\ell_\infty\left(\{\frac{1}{n}\}_{n\in\mathbb{N}}\right)} \leq 1 \text{ for all } n \in \mathbb{N}.$$

Clearly, $x - y = \left\{\left(n - y_1^n, n - y_2^n\right)\right\}_{n\in\mathbb{N}}$. *It is easy to check that*

$$n - y_1^n \geq \left(n - \frac{1}{n}\right) + y_2^n \geq y_2^n \geq 0 \text{ for all } n \in \mathbb{N}.$$

Similarly, we have $n - y_2^n \geq 0$ *for all* $n \in \mathbb{N}$. *Thus, we find a real number* $\rho = 1 > 0$ *such that* $x - y \in C$ *for all* $y \in U_+^E$. *Therefore,* $x = \{(n,n)\}_{n\in\mathbb{N}} \in C^\ominus$.

Depending on Huaping Huang results [28] in 2019, the following results are based on the assumption that the cone C has a semi-interior points.

Definition 2.10. *A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in C is called s-sequence, if for each* $c \in C^\ominus$, *there exists* $n_0 \in \mathbb{N}$ *such that* $x_n \lll c$ *for all* $n \geq n_0$.

Lemma 2.11. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in E and* $x_n \to \theta$ *as* $n \to \infty$. *Then,* $\{x_n\}_{n\in\mathbb{N}}$ *is s−sequence.*

Proposition 2.12. *Let* $x, y, z \in E$. *Then,* $x \lll z$ *if one of the following holds:*

$$\text{either } x \leq y \lll z, \quad x \lll y \leq z, \quad \text{or } x \lll y \lll z.$$

Proposition 2.13. *If* $\theta \leq u \lll c$ *holds for any* $c \in C^\ominus$, *then* $u = \theta$.

Proposition 2.14. *Let* (X, E, C, p) *be a partial cone metric space. Then, some topology* τ_p *is generated on X and defined by:*

$$\tau_p = \left\{U \subseteq X : \forall x \in U, \exists c \in C^\ominus, B_p(x,c) \subseteq U\right\} \cup \{\phi\}, \ X = \bigcup_{B_p \in \beta_p} B_p.$$

The base of this topology is given by $\beta_p = \left\{B_p(x,c) : (x,c) \in X \times C^\ominus\right\}$, *where the set* $B_p(x,c) := \{y \in X : p(x,y) \lll c + p(x,x)\}$ *is the neighborhood of x with radius c.*

In the following, redefined versions of the convergent and Cauchy sequences in our space are given by exchanging roles of $\lll$ and $\ll$. Therefore, the new definitions are controlled by $C^\ominus$ instead of Int C.

Definition 2.15. *Let (X, E, C, p) be a partial cone metric space, $x \in X$ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then,*
(1) $\{x_n\}_{n\in\mathbb{N}}$ *is convergent to x, we denote this by $x_n \overset{\tau_p}{\to} x$, whenever for every $c \in E$ with $c \ggg \theta$, there is $n_0 \in \mathbb{N}$ such that $p(x_n, x) \lll p(x, x) + c$ for all $n \geq n_0$.*
(2) $\{x_n\}_{n\in\mathbb{N}}$ *is strongly convergent to x, we denote this by $x_n \overset{s-\tau_p}{\to} x$, if* $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = p(x, x)$.
(3) $\{x_n\}_{n\in\mathbb{N}}$ *is a θ-Cauchy whenever for every $c \in E$ with $c \ggg \theta$, there is $n_0 \in \mathbb{N}$ such that $p(x_n, x_m) \lll c$ for all $n, m \geq n_0$. That is; a sequence $\{x_n\}_{n\in\mathbb{N}}$ is a θ-Cauchy if and only if $\|p(x_n, x_m)\| \to \theta$ as $n, m \to \infty$.*
(4) *The partial cone metric space (X, E, C, p) is said to be θ-complete, if each θ-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ of X converges to x in X such that $p(x, x) = \theta$.*
(5) $\{x_n\}_{n\in\mathbb{N}}$ *is Cauchy, if there is $u \in C$ such that* $\lim_{n,m\to\infty} p(x_n, x_m) = u$.
(6) *The partial cone metric space (X, E, C, p) is complete, if each Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is strongly convergent to $x \in X$ such that $p(x, x) = u$.*

Remark 2.16. (1) *A sequence $\{x_n\}_{n\in\mathbb{N}}$ is a θ-Cauchy if and only if $\{p(x_m, x_n)\}_{m,n\in\mathbb{N}}$ is s-sequence in E.*
(2) *For $s-$sequence which is not convergent, one can see example* (2) *in [23].*
(3) *Each strongly convergent sequence of a partial cone metric space (X, E, C, p) is convergent with respect to τ_p. However, the converse of this fact need not hold. In particular; the converse is true if C is a normal cone. In fact; example* (3) *in [23] showed the existence of some sequences of a partial cone metric space which are convergent, but not strongly convergent if the cone C is non-normal.*

Now, we are going to highlight two new classes of cones in normed spaces, namely; semi-solid cones and satisfactory cones. These classes will play a key role in our results and enable moving the roles from interior points to semi-interior points of cones.

Definition 2.17. *A cone C in the normed space E is called semi-solid if and only if it has a non-empty set of semi-interior points, $C^{\ominus} \neq \emptyset$, and it is called a satisfactory cone if and only if cone C satisfies any one of the following:*
(1) *C is normal and solid,*
(2) *C is not-normal and solid,*
(3) *C is normal and semi-solid,*
(4) *C is not-normal and semi-solid.*
A partial cone metric space (X, E, C, p) is said to be a partial satisfactory cone metric space if and only if the cone C is satisfactory.

With this notion, the above-mentioned conclusions are still working with non-normal semi-solid cones and hence generally for partial (satisfactory) cone metric spaces. Particularly, the following remark is a direct consequence of Lemma (2.11) and part (3) of Definition (2.15).

Remark 2.18. *Let (X, E, C, p) be a partial (satisfactory) cone metric space. Then,*
(1) *A complete partial (satisfactory) cone metric space is a subcategory of a θ-complete partial (satisfactory) cone metric space. In particular; if C is a normal cone of the normed space $(E, \|.\|)$, then every $\theta-$Cauchy sequence in (X, E, C, p) is a Cauchy sequence and every complete partial (satisfactory) cone metric space is $\theta-$complete.*
(2) *If $\{y_n\}_{n\in\mathbb{N}}$ is s-sequence in E satisfying $p(x_n, x_m) \leq y_n$ for all $m, n \in \mathbb{N}$ with $m > n$, then $\{x_n\}_{n\in\mathbb{N}}$ is a*

θ-Cauchy sequence in X.
(3) *If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X, $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence in E that converges to θ and satisfying $p(x_n, x_m) \preceq \alpha_n$ for all $m, n \in \mathbb{N}$ with $m > n$, then $\{x_n\}_{n\in\mathbb{N}}$ is a θ-Cauchy sequence.*
(4) *The limit of a convergent sequence in a partial (satisfactory) cone metric space may not be unique. In fact; the partial (satisfactory) cone metric space (X, E, C, p) need not be T_1-space. Actually, Example* (3.1) *in [35] and Examples* (3), (11) *in [32] showed that the limit of convergent sequence in (X, E, C, p) is not necessarily unique.*
(5) *The partial (satisfactory) cone metric p is not always continuous mapping, in the sense of $x_n \overset{\tau_p}{\to} x$ and $y_n \overset{\tau_p}{\to} y$ imply that $p(x_n, y_n) \overset{\|.\|}{\to} p(x, y)$. In other words; the fact that $p(x_n, y_n) \overset{\|.\|}{\to} p(x, y)$ if $x_n \overset{\tau_p}{\to} x$ and $y_n \overset{\tau_p}{\to} y$, is not guaranteed. See example* (11) *in [32].*

Now, we are going to display the concept of α-admissible mappings defined by Samet [21] and review the essential definition of generalized α-admissible mappings given by Zhu [36].

Definition 2.19. *Let X be a non-empty set, $\alpha : X \times X \to [0, \infty)$ be a mapping and $S, T : X \to X$ be two self-mappings. Then,*
(1) T is said to be an α-admissible, if for every $x, y \in X$,

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1.$$

(2) S and T are called generalized α-admissible, if for every $x, y \in X$,

$$\alpha(Sx, Sy) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1.$$

Before starting the core results, we need to recall some standard terminology from fixed-point theory.

Definition 2.20. *[37] Let X be a non-empty set and $T, S : X \to X$ be mappings such that $TX \subseteq SX$. If $v = Tu = Su$ for some $u \in X$, then u is a coincidence point of T and S, and v is a point of coincidence of T and S. Furthermore, if $Tv = Sv = v$, then v is a common fixed-point of T and S. Finally, if $TSw = STw$, whenever $Tw = Sw$ for some $w \in X$, then T and S are said to be weakly compatible. That is; if they commute at their coincidence points.*

For simplicity, we use the notation Λ to denote the set of coincidence points of T and S.

Proposition 2.21. *[37] Let T and S be coincidentally commuting self-mappings on a set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed-point of T and S.*

We need to consider the followings which will be effectively used in the proof of our next main results.

Definition 2.22. *[16] Let $(X, \leqslant)$ be an ordered set and $T : X \times X \to X$. Then, T is said to have the mixed monotone property in X, if for any $x, y \in X$,*

$$\begin{cases} x_1, x_2 \in X, \quad x_1 \leqslant x_2 \text{ implies } T(x_1, y) \leqslant T(x_2, y), \\ \\ y_1, y_2 \in X, \quad y_1 \leqslant y_2 \text{ implies } T(x, y_1) \geqslant T(x, y_2). \end{cases}$$

Definition 2.23. *[16] An element* $(x, y) \in X \times X$ *is said to be a coupled fixed-point of the mapping* $T : X \times X \to X$, *if* $T(x, y) = x$ *and* $T(y, x) = y$.

Definition 2.24. *[17] Let* $T : X \times X \to X$ *and* $S : X \to X$ *be two mappings. An element* $(x, y) \in X \times X$ *is called a coupled coincidence point of the mappings* T *and* S, *if* $T(x, y) = Sx$ *and* $T(y, x) = Sy$, *and* (Sx, Sy) *is called coupled point of coincidence.*

Definition 2.25. *[17] Let* $(X, \leqslant)$ *be a partially ordered set,* $T : X \times X \to X$ *and* $S : X \to X$ *be two mappings. Then,* T *is said to have the mixed* S*-monotone property, if* T *is monotone* S*-non-decreasing in its first argument and is monotone* S*-non-increasing in its second argument. That is; for any* $x, y \in X$

$$\begin{cases} x_1, x_2 \in X, \quad Sx_1 \leqslant Sx_2 \text{ implies } T(x_1, y) \leqslant T(x_2, y), \\ \\ y_1, y_2 \in X, \quad Sy_1 \leqslant Sy_2 \text{ implies } T(x, y_1) \geqslant T(x, y_2). \end{cases}$$

The following definitions are part of the main topics in our work.

Remark 2.26. *Suppose that* $(X, \leqslant)$ *is a partially ordered set and let* (X, E, C, p) *be a partial (satisfactory) cone metric space. Then, a partial ordered relation* $\leqslant$ *on* X *can be induced on* $X \times X$ *in the following way: for every* (x, y) *and* $(u, v) \in X \times X$,

$$(x, y) \lesssim (u, v) \text{ if and only if } x \leqslant u \text{ and } y \geqslant v.$$

The element (x, y) *is said to be comparable to* (u, v), *if either* $(x, y) \lesssim (u, v)$, *or* $(x, y) \gtrsim (u, v)$ *and the sequence* $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset X \times X$ *is non-decreasing with respect to* $\lesssim$, *if* $(x_n, y_n) \lesssim (x_{n+1}, y_{n+1})$ *for all* n.

Definition 2.27. *Let* (X, E, C, p) *be a* $\theta-$*complete partial (satisfactory) cone metric space ordered with the relation* $\leqslant$. *Then,* $(X, E, C, p, \leqslant)$ *is said to be regular, if* X *has the following properties:*
(1) *If for every non-decreasing sequence* $\{x_n\}$ *in* X *such that* $x_n \xrightarrow{\tau_p} x$, *then* $x_n \leqslant x$ *for all* n.
(2) *If for every non-increasing sequence* $\{y_n\}$ *in* X *such that* $y_n \xrightarrow{\tau_p} y$, *then* $y_n \geqslant y$ *for all* n.

Definition 2.28. *[38] Let* $T : X \times X \to X$ *and* $\alpha : X^2 \times X^2 \to [0, \infty)$ *be given mappings. Then,* T *is said to be an* α*-admissible mapping, if for all* $(x, y), (u, v) \in X \times X$, *the following is satisfied*

$$\alpha((x, y), (u, v)) \geq 1 \text{ implies } \alpha(T(x, y), T(y, x), T(u, v), T(v, u)) \geq 1.$$

Definition 2.29. *[39] Let* $T : X \times X \to X, S : X \to X$ *and* $\alpha : X^2 \times X^2 \to [0, \infty)$ *be mappings. Then,* T *and* S *are said to be* $\alpha-$*admissible, if*

$$\alpha((Sx, Sy), (Su, Sv)) \geq 1 \text{ implies } \alpha(T(x, y), T(y, x), T(u, v), T(v, u)) \geq 1$$

for all $x, y, u, v \in X$.

3. Main results: Generalized classes of admissible mappings

Let $T : X \to X$ be a given self-mapping. The set of all fixed-points of the mapping T is denoted by $\mathrm{Fix}(T) = \{x \in X : Tx = x\}$.

Lemma 3.1. *Every contraction mapping on a metric space (X, d) is an α-admissible mapping for some mapping $\alpha : X \times X \to [0, \infty)$. However, not every α-admissible mapping is a contraction mapping.*

Proof. Let T be a contraction mapping on a metric space (X, d) . Then, there exists a constant $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for every $x, y \in X$. Consider the mapping $\alpha : X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} \frac{1}{d(x,y)}, & \text{if } x \neq y, \\ 1, & \text{otherwise.} \end{cases}$$

Then, T is an α-admissible mapping. More exactly, we need here to think over two situations as follows:
Case (1) : For any $x \neq y$ in X, we have $\frac{1}{k}\frac{1}{d(x,y)} \leq \frac{1}{d(Tx,Ty)}$. This implies $\alpha(x, y) \leq k\alpha(Tx, Ty)$. Since $0 < k < 1$, then we have $\alpha(x, y) < \alpha(Tx, Ty)$. It is fairly simple to see that $\alpha(Tx, Ty) \geq 1$ whenever $\alpha(x, y) \geq 1$.
Case (2) : Otherwise, we know that $x = y$ implies $d(x, y) = 0$. Imposing that T is a contraction mapping on X, it yields $d(Tx, Ty) = 0$ and so $Tx = Ty$. Eventually, the conclusion that $\alpha(Tx, Ty) = \alpha(x, y) = 1$ is valid for $x = y$. In both cases, the contraction mapping T is an $\alpha-$admissible mapping, but not conversely in general.

We demonstrate that the converse of Lemma (3.1) is not true as in the following example.

Example 3.2. *Let X be the metric space $([0, \infty), d)$ with the absolute value metric function $d(x, y) = |x - y|$ for all $x, y \in [0, \infty)$. Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be defined by*

$$Tx = \sqrt{x} \text{ for all } x \in X \quad \text{and} \quad \alpha(x, y) = \begin{cases} e^{x-y}, & \text{for } x \geq y, \\ 0, & \text{otherwise.} \end{cases}$$

Then, T is an α-admissible mapping, but it is not contraction because T has two fixed-points on the given complete metric space, $Fix(T) = \{0, 1\}$.

□

Remark 3.3. *By virtue of Lemma (3.1) and Example (3.2), we can understand that the class of $\alpha-$admissible mappings is effectively more generalized than the class of contraction mappings.*

In the sequel, we will continue to modify the concept of α-admissible operators by generalizing a new function class of such mappings in more general conditions.

We state Definition (3.4) in the line of Definition (2.19) as follows:

Definition 3.4. *Let (X, E, C, p) be a partial (satisfactory) cone metric space and C be a cone of a normed space $(E, \|.\|)$. In a non-empty set X, define $T : X \to X$ and $\alpha : X \times X \to C$. Assume that c runs through $C - \{\theta\}$. Then,*
(1) *T is said to be $\alpha - c-$admissible mapping if and only if*

$$\alpha(Tx, Ty) \geq c \text{ whenever } \alpha(x, y) \geq c.$$

(2) *(X, E, C, p) is $\alpha - c-$regular, if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\alpha(x_n, x_{n+1}) \geq c$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow{\tau_p} x^* \in X$, we have $\alpha(x_n, x^*) \geq c$ for sufficiently large n.*

Inspired by Definition (3.4), we went further, defining a new class of $\alpha-$admissible mappings which is different from and stronger than the one introduced in Definition (3.4). The refinement version of these mappings will be crucial in our main results.

Definition 3.5. *Let (X, E, C, p) be a partial (satisfactory) cone metric space and C be a cone of a normed space $(E, \|.\|)$. In a non-empty set X, define $T : X \to X$ and $\alpha : X \times X \to C$. Assume that $\{c_n\}_{n\in\mathbb{N}}$ be a non-zero sequence in C. Then,*
(1) *T is said to be $\alpha-$sequentially admissible mapping if and only if*

$$\alpha(Tx, Ty) \geq c_{n+1} \text{ whenever } \alpha(x, y) \geq c_n \text{ for every } n \in \mathbb{N}.$$

(2) *(X, E, C, p) is said to be $\alpha-$sequentially regular, if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\alpha(x_n, x_{n+1}) \geq c_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \overset{\tau_p}{\to} x^* \in X$, we have $\alpha(x_n, x^*) \geq c_{n+1}$ for sufficiently large n.*

Remark 3.6. (1) *Note that the class of all $\alpha-$admissible operators described in Definition (2.19), is included in two classes of all $\alpha - c-$admissible and $\alpha-$sequentially admissible operators. Indeed; in Definition (3.4), let $(E, \|.\|) := (\mathbb{R}, |.|)$, where the normed space E is endowed with the usual ordering of real numbers $\leq$ and ordered by the cone $C := [0, \infty)$. Further, if $c = 1$, then T is an α-admissible mapping. Similarly, in Definition (3.5), let $(E, \|.\|)$, C and $\preceq$ be the same ones as those stipulated above. Moreover, let $\{c_n\}_{n\in\mathbb{N}}$ be the constant sequence $c_n = 1$ for all $n \in \mathbb{N}$. Thus, T is an α-admissible mapping.*
(2) *The class of all $\alpha - c-$admissible operators is included in the class of all $\alpha-$sequentially admissible operators. In fact; suppose that T is $\alpha - c-$admissible operator. In Definition (3.5), we can take $\{c_n\}_{n\in\mathbb{N}}$ equals the constant sequence $c_n = c$ for all $n \in \mathbb{N}$. Thus, T is $\alpha-$sequentially admissible operator. It is obvious that the last category is the widest.*
(3) *If T is $\alpha-$sequentially admissible and $\{c_n\}_{n\in\mathbb{N}}$ is an increasing sequence starting with the element $c_{n_0} \neq \theta$, then T is $\alpha - c_n-$admissible mapping for every $n \geq n_0$. This is true in particular for arithmetic sequences with base c^* belonging to C, where $c_1 = c_1,\ c_2 = c_1 + c^*, \ldots, c_n = c_1 + (n-1)c^*,\ n \in \mathbb{N}$.*
(4) *If T is $\alpha-$sequentially admissible mapping and $\{c_n\}_{n\in\mathbb{N}}$ is a decreasing sequence bounded below by $\theta \neq c^* \in C$, then for every $\theta \neq c \in C$, there is $n(c) \in \mathbb{N}$ such that $c^* \preceq c_{n(c)} \prec c^* + c$. Now, the inequality $\alpha(x, y) \geq c^* + c$ implies $\alpha(x, y) \geq c_{n(c)}$, and the later one suggests that $\alpha(Tx, Ty) \geq c_{n(c)+1}$. Using the lower bound c^*, it follows that $\alpha(Tx, Ty) \geq c^*$.*

Taking inspiration from Definition (2.19), we shall establish our newly corresponding generalizations in the following way:

Definition 3.7. *Let (X, E, C, p) be a partial (satisfactory) cone metric space and C be a cone of a normed space $(E, \|.\|)$. In a non-empty set X, define $T, S : X \to X$ and $\alpha : X \times X \to C$. Assume that c runs through $C - \{\theta\}$ and $\{c_n\}_{n\in\mathbb{N}}$ be a non-zero sequence in C. Then,*
(1) *The mapping T is called $\alpha_S - c-$admissible, if*

$$\alpha(Sx, Sy) \geq c \text{ implies } \alpha\,(Tx, Ty) \geq c.$$

(2) *T is called α_S-sequentially admissible if and only if*

$$\alpha(Sx, Sy) \geq c_n \text{ implies } \alpha(Tx, Ty) \geq c_{n+1} \text{ for all } n \in \mathbb{N}.$$

Remark 3.8. *Every $\alpha - c-$admissible is $\alpha_I - c-$admissible, where I denotes the identity mapping on X. Similarly, every $\alpha-$sequentially admissible mapping is α_I-sequentially admissible.*

To ensure clarity, we will deal particularly with partial satisfactory cone metric spaces in which the cone C is semi-solid and need not be normal. The results in the case of ordering solid cones will be the same as those concerning the case of semi-solid cones. It is important to mention that our results are valid in all cases of the satisfactory cone C.

We begin with the following main generalized theorem.

Theorem 3.9. *Suppose that (X, E, C, p) is a θ-complete partial satisfactory cone metric space. Let $\alpha : X \times X \to C$ be a symmetric mapping and $T, S : X \to X$ be two self-mappings. Presume that $\{c_n\}_{n\in\mathbb{N}}$ is a non-zero sequence in C. Also, assume that the following assumptions are fulfilled:*
(1) *$TX \subseteq SX$ and SX is a closed subset of X;*
(2) *T is α_S-sequentially admissible mapping;*
(3) *There exists $x_0 \in X$ such that $\alpha(Sx_0, Tx_0) \succeq c_1$;*
(4) *(X, E, C, p) is α_S-sequentially regular;*
(5) *There is a sequence of positive real numbers $\{k_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} k_n < 1$ and satisfying the following condition:*

$$p(Tx, Ty) \preceq k_n p(Sx, Sy) \textit{ for every } x, y \in X \textit{ with } \alpha(Sx, Sy) \succeq c_n, n \in \mathbb{N}.$$

Then, T and S have coincidence points. Moreover, if T and S are weakly compatible such that for all $x, y \in \Lambda$ we have $\alpha(Sx, Sy) \succeq c_1$, then T and S have a unique common fixed-point in X.

Proof. From assumption (3), there exists $x_0 \in X$ such that

$$\alpha(Sx_0, Tx_0) \succeq c_1. \tag{3.1}$$

Since $TX \subseteq SX$, we get an element $x_1 \in X$ such that $Sx_1 = Tx_0$. Again, we set $Sx_2 = Tx_1$. In a similar manner, we define two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ as follows: $y_{n+1} = Sx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$.

First, if we can find some $N \in \mathbb{N}$ such that $y_N = y_{N+1}$, then we have

$$Tx_N = Sx_{N+1} = y_{N+1} = y_N = Sx_N.$$

Thus, x_N is a coincidence point of T and S and the conclusion is checked. Without any loss of generality, we consider that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$.

Since $\alpha(Sx_0, Tx_0) = \alpha(Sx_0, Sx_1)$, inequality (3.1) gives the following:

$$\alpha(Sx_0, Sx_1) \succeq c_1. \tag{3.2}$$

Since T is α_S-sequentially admissible, inequality (3.2) implies $\alpha(Tx_0, Tx_1) \succeq c_1$. Consequently, we have $\alpha(Sx_1, Sx_2) \succeq c_2$ and so $\alpha(Tx_1, Tx_2) \succeq c_2$. By repetition of the above procedure, we get $\alpha(Sx_n, Sx_{n+1}) \succeq c_{n+1}$ which implies $\alpha(Tx_n, Tx_{n+1}) \succeq c_{n+1}$. Equivalently; we get

$$\alpha(y_n, y_{n+1}) \succeq c_{n+1} \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

This in turns implies the following:

$$\alpha\left(Sx_n, Sx_{n+1}\right) \geq c_{n+1}. \tag{3.4}$$

Taking advantage of the given generalized contractive condition (5), we arrive at

$$p\left(y_{n+1}, y_{n+2}\right) \leq k_{n+1} p\left(Sx_n, Sx_{n+1}\right) = k_{n+1} p\left(y_n, y_{n+1}\right).$$

Using a similar way of the above process up to n times, we obtain

$$\begin{aligned} p\left(y_n, y_{n+1}\right) &\leq k_n p\left(y_{n-1}, y_n\right) \leq \left[k_n \times k_{n-1}\right] p\left(y_{n-2}, y_{n-1}\right) \\ &\leq \cdots \leq \left[\prod_{i=1}^{n} k_i\right] p\left(y_0, y_1\right). \end{aligned}$$

Consider the sequence

$$s_1 := \{k_1, k_1 \times k_2, k_1 \times k_2 \times k_3, \ldots, \prod_{j=1}^{n} k_j, \ldots\}$$

with $a_n := \prod_{j=1}^{n} k_j$, we have $\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \lim_{n\to\infty} k_n < 1$. Hence, the sequence s_1 should converge to zero sequence, $\lim_{n\to\infty} a_n = 0$, and we have

$$p\left(y_n, y_{n+1}\right) \leq a_n p\left(y_0, y_1\right) \underset{n\to\infty}{\longrightarrow} \theta. \tag{3.5}$$

For any $n, p \in \mathbb{N}$, we have

$$\begin{aligned} p(y_n, y_{n+p}) &\overset{(\mathrm{PCM}_4)}{\leq} \sum_{i=n}^{n+p-1} p(y_i, y_{i+1}) \leq \sum_{i=n}^{n+p-1} \Big[\prod_{j=1}^{i} k_j\Big] p(y_0, y_1) \\ &= \prod_{i=1}^{n} k_i \Big[1 + k_{n+1} + k_{n+1} \times k_{n+2} + k_{n+1} \times k_{n+2} \times k_{n+3} \\ &\quad + \cdots + \prod_{i=n+1}^{n+p-1} k_i\Big] p(y_0, y_1). \end{aligned}$$

Consider the sequence

$$s_2 := \{1, k_{n+1}, k_{n+1} \times k_{n+2}, k_{n+1} \times k_{n+2} \times k_{n+3}, \ldots, \prod_{i=n+1}^{n+p-1} k_i, \ldots\}$$

with $b_n := \prod_{i=n+1}^{n+p-1} k_i$, we have $\lim_{n\to\infty} \frac{b_n}{b_{n-1}} = \lim_{n\to\infty} k_{n+p-1} < 1$. Hence, using the usual form of the Ratio test of series, the sequence of partial sums of s_2 should converge to some number (say) k such that

$$k := 1 + k_{n+1} + k_{n+1} \times k_{n+2} + k_{n+1} \times k_{n+2} \times k_{n+3} + \cdots + \prod_{i=n+1}^{n+p-1} k_i + \ldots.$$

Thus, one can see that $\{\sum_{m=1}^{l}\left[\prod_{j=n+1}^{n+1+m} k_j\right]\}_{l\in\mathbb{N}}$ is convergent to some number (say) K such that

$$K := \lim_{l\to\infty}\sum_{m=1}^{l}\left[\prod_{j=n+1}^{n+1+m} k_j\right] = \sum_{m=1}^{\infty}\left[\prod_{j=n+1}^{n+1+m} k_j\right].$$

In conclusion, we proved the following:

$$\begin{aligned} p(y_n, y_{n+p}) &\preceq a_n\Big[1 + k_{n+1} + k_{n+1}\times k_{n+2} + \cdots + \prod_{i=n+1}^{n+p-1} k_i\Big]p(y_0, y_1) \\ &\preceq a_n K\ p(y_0, y_1). \end{aligned}$$

Since $\{a_n K\ p(y_0, y_1)\}_{n\in\mathbb{N}}$ is convergent to zero, $\lim_{n\to\infty} a_n K\ p(y_0, y_1) \overset{\|.\|}{\to} \theta$, it is $s-$sequence. Now, let $c \in E$ with $c \ggg \theta$, then there exists $n_0 \in \mathbb{N}$ such that

$$a_n K\ p(y_0, y_1) \lll c \text{ for all } n \geq n_0.$$

Hence, for any $n, p \in \mathbb{N}$, we have

$$p\left(y_n, y_{n+p}\right) \preceq a_n K\ p(y_0, y_1) \lll c \text{ for all } n \geq n_0.$$

This concluded that for any $n, p \in \mathbb{N}$ and any $c \ggg \theta$, there exists $n_0 \in \mathbb{N}$ such that

$$p\left(y_n, y_{n+p}\right) \lll c \text{ for all } n \geq n_0.$$

Owing to the above arguments, we find that $\{y_n\}_{n\in\mathbb{N}}$ is a θ-Cauchy sequence in (X, E, C, p).

Regarding the θ-completeness of the space, there exists an element (say) $y' \in X$ such that $y_n \overset{\tau_p}{\to} y'$ and $p(y', y') = \theta$. Since $\{y_n\}_{n\in\mathbb{N}} \subseteq SX$ and SX is closed set in X, it leads that $y' \in SX$. Then, there exists $z \in X$ such that $y' = Sz$.

Now, we wish to show that $Tz = Sz$. Employing (PCM$_4$), we have

$$p(Tz, Sz) \preceq p(Tz, Tx_n) + p(y_{n+1}, y').$$

Since $\alpha(y_n, y_{n+1}) \succeq c_{n+1}$ for all $n \in \mathbb{N}$ and $y_n \overset{\tau_p}{\to} y'$, by making use of condition (4), we obtain $\alpha(y_n, y') = \alpha(Sx_n, Sz) \succeq c_{n+1}$ for sufficiently large n.

Accordingly, we find $p(Tz, Sz) \preceq k_{n+1} p(y', y_n) + p(y_{n+1}, y')$. Since $y_n \overset{\tau_p}{\to} y'$, then for $c \in E$ with $c \ggg \theta$ and for all $m \in \mathbb{N}$, choose $n_3 \in \mathbb{N}$ such that

$$k_{n+1} p(y_n, y') \lll \frac{c}{2m} \text{ and } p(y_{n+1}, y') \lll \frac{c}{2m} \text{ for all } n \geq n_3.$$

Hence, for all $c \ggg \theta$ and for all $m \in \mathbb{N}$, it follows that $p(Tz, Sz) \lll \frac{c}{m}$. Taking the limit as $m \to \infty$, we get $p(Tz, Sz) = \theta$ and so $Tz = Sz$.

Therefore, T and S have a coincidence point in X. As a last step, we claim that T and S possess a unique point of coincidence. In order to obtain the claim, consider that $Tw = Sw$ be another point of coincidence of T and S. So, we assume that $Tw = Sw \neq Tz = Sz$. By the hypothesis $\alpha(Sw, Sz) \succeq c_1$, we have $p(Tw, Tz) \leq k_1 p(Sw, Sz) = k_1 p(Tw, Tz)$. As $0 < k_1 < 1$, we get $p(Tw, Tz) = \theta$ and so $Tw = Tz$. This contradicts the assumption that $Tw = Sw \neq Tz = Sz$. Thus, the point of coincidence is uniquely determined. Bearing the assertion that the mappings T and S are weakly compatible in mind, we deduce that $Sv = STz = TSz = Tv$. Regarding to the uniqueness of the point of coincidence of T and S, we get $Tv = Sv = v$.

As a consequence, v is the unique common fixed-point of T and S and so the proof is done. □

Once again, we can here replace the condition $C^{\ominus} \neq \emptyset$ with the other states of the satisfactory cone C, if we wish.

Remark 3.10. *As a special case, if we replace the mapping S with I, the identity mapping on X, in the statement of Theorem (3.9), we conclude that any mapping with these prescribed conditions has fixed-points in X.*

Now our purpose is to determine sufficient conditions to acquire the uniqueness of the fixed-point of the mapping T stipulated in Theorem (3.9) with $S = I$.

Proposition 3.11. *Assume that all the hypothesis of Theorem (3.9) are verified with $S = I$. Furthermore, suppose that the following properties are hold:*

Let $c^* \in C$ such that $c^* \geq c_{i_0}$ for some $c_{i_0} \in C - \{\theta\}$ and $i_0 \in \mathbb{N}$. Let the set $\{\alpha(x, y) : x, y \in Fix(T)\}$ be bounded below by c^*. Under these conditions, we obtain that the fixed-point of T is uniquely determined.

Proof. Since T satisfies the hypothesis of Theorem (3.9), then the fixed-point of T exists. We show that the set $\mathrm{Fix}(T)$ is in fact reduced to a single point. For this, if possible, let $x, y \in \mathrm{Fix}(T)$. Then, $\alpha(x, y) \geq c^*$ and so $\alpha(x, y) \geq c_{i_0}$. Making use of condition (2) in Theorem (3.9), we guarantee that $\alpha(Tx, Ty) = \alpha(x, y) \geq c_{i_0+1}$. Continuing in this way, we derive that $\alpha(x, y) \geq c_n$ for all $n \geq i_0$. We can now apply assumption (4), which leads to $p(Tx, Ty) \leq k_n p(x, y)$ for all $n \geq i_0$. That is; $p(x, y) \leq k_n p(x, y)$ for all $n \geq i_0$. On taking the limit as $n \to \infty$ of the sequence $\{(k_n - 1) p(x, y)\}_{n \geq i_0}$ gives us $-p(x, y) \in C$. Thus, we have $p(x, y) \in -C$, but $p(x, y) \in C$ and so $p(x, y) \in C \cap -C = \{\theta\}$. Then, $p(x, y) = \theta$ implies $x = y$. Therefore, the set $\mathrm{Fix}(T)$ should be singleton. □

As a usual relationship between more and less general theorem, we have the following one:

Corollary 3.12. *Suppose that (X, E, C, p) is a θ-complete partial satisfactory cone metric space. Let $\alpha : X \times X \to C$ be a symmetric mapping and $T : X \to X$ be a self-mapping. Presume that $\{c_n\}_{n \in \mathbb{N}}$ is a non-zero sequence in C. Also, assume that the following assumptions are fulfilled:*
(1) *There exists $N \in \mathbb{N}$ such that T^N is $\alpha-$sequentially admissible;*
(2) *There exists $x_0 \in X$ such that $\alpha(x_0, T^N x_0) \geq c_1$;*
(3) *(X, E, C, p) is $\alpha-$sequentially regular;*
(4) *There is a sequence of positive real numbers $\{k_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} k_n < 1$ and satisfying the following condition:*

$$p(T^N x, T^N y) \leq k_n p(x, y) \text{ for every } x, y \in X \text{ with } \alpha(x, y) \geq c_n, n \in \mathbb{N}.$$

Then, the mapping T has fixed-points in X.

Theorem 3.13. *Suppose that (X,E,C,p) is a θ-complete partial satisfactory cone metric space. Let $\alpha : X\times X \to C$ be a symmetric mapping and $T : X \to X$ be a bijective self-mapping. Presume that $\{c_n\}_{n\in\mathbb{N}}$ is a non-zero sequence in C. Also, assume that the following assumptions are fulfilled:*
(1) T^{-1} is $\alpha-$sequentially admissible mapping;
(2) There exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \succeq c_1$;
(3) (X,E,C,p) is $\alpha-$sequentially regular;
(4) There is a sequence of positive real numbers $\{k_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}(k_n)^{-1} < 1$ and satisfying the following condition:

$$p(Tx,Ty) \succeq k_n p(x,y) \text{ for every } x,y \in X \text{with } \alpha(x,y) \succeq c_{n+1}, n\in\mathbb{N}.$$

Then, the mapping T has fixed-points in X.

Proof. Since T is bijective, then it is an invertible mapping, say $T^{-1} : X \longrightarrow X$ is the inverse mapping of T. Let $x_0 \in X$ be a chosen point and define the sequence $x_1 = T^{-1}x_0$, $x_2 = T^{-1}x_1 = \left(T^{-1}\right)^2 x_0, \ldots,$ $x_n = T^{-1}x_{n-1} = \left(T^{-1}\right)^n x_0$ for all $n\in\mathbb{N}$. Since $\alpha\left(x_0, T^{-1}x_0\right) \succeq c_1$, we conclude that

$$\alpha\left(x_0, T^{-1}x_0\right) = \alpha(x_0,x_1) \succeq c_1 \overset{(1)}{\Rightarrow} \alpha\left(T^{-1}x_0, T^{-1}x_1\right) = \alpha(x_1,x_2) \succeq c_2.$$

Inductively, we get $\alpha(x_n,x_{n+1}) \succeq c_{n+1}$ for all $n\in\mathbb{N}$.

We can employ condition (4) as follows: $p(x_{n-1},x_n) = p(Tx_n, Tx_{n+1}) \succeq k_n p(x_n,x_{n+1})$ for all $n\in\mathbb{N}$. This is equivalent to $p(x_n,x_{n+1}) \preceq (k_n)^{-1} p(x_{n-1},x_n)$ for all $n\in\mathbb{N}$. Set $\eta_n = (k_n)^{-1}$ for all $n\in\mathbb{N}$, we infer that

$$p(x_n,x_{n+1}) \preceq \eta_n p(x_{n-1},x_n) \preceq \cdots \preceq \left[\prod_{j=1}^{n}\eta_j\right] p(x_0,x_1).$$

For $n,p\in\mathbb{N}$, consider

$$p\left(x_n,x_{n+p}\right) \preceq \sum_{i=n}^{n+p-1} p(x_i,x_{i+1}) \preceq \sum_{i=n}^{\infty}\left[\prod_{j=1}^{i}\eta_j\right] p(x_0,x_1).$$

The next step is easily obtained by following the related lines from the proof of Theorem (3.9). Hence, we assure that $\{x_n\}_{n\in\mathbb{N}} = \left\{\left(T^{-1}\right)^n x_0\right\}_{n\in\mathbb{N}}$ is a θ-Cauchy sequence. For the sake of θ-completeness of the space, there exists $x\in X$ such that $x_n \overset{\tau_p}{\to} x$ with $p(x,x) = \theta$. Now, we show that x is a fixed-point of T. Since T is onto, there exists $u\in X$ such that $x = Tu$. Since we have $x_n \overset{\tau_p}{\to} x$ and $\alpha(x_n,x_{n+1}) \succeq c_{n+1}$ for all $n\in\mathbb{N}$, then it follows that $\alpha(x_n,x) \succeq c_{n+1}$ for sufficiently large n.

Thereafter, by using assumption (1), we get $\alpha(x_{n+1},u) \succeq c_{n+2}$. Suppose now that condition (4) takes place, we conclude that

$$p(x_n,x) = p\left(T\left(T^{-1}x_n\right), Tu\right) \succeq k_{n+1} p(x_{n+1},u).$$

Hence, $p(x_{n+1},u) \preceq \eta_{n+1} p(x_n,x)$. Since $x_n \overset{\tau_p}{\to} x$, then for any $c \ggg \theta$, there exists $n_0\in\mathbb{N}$ such that

$$\eta_{n+1} p(x_n,x) \lll \frac{c}{2} \text{ and } p(x_{n+1},x) \lll \frac{c}{2} \text{ for all } n \geq n_0.$$

For all $n \geq n_0$ and for any $c \ggg \theta$, consider that

$$p(u,x) \overset{(\mathrm{PCM}_4)}{\leq} p(u,x_{n+1}) + p(x_{n+1},x) \leq \eta_{n+1} p(x_n,x) + p(x_{n+1},x) \lll c.$$

In conclusion, we arrive at

$$\theta \preceq p(u,x) \lll c \text{ holds for any } c \in C^{\ominus} \text{ implies } p(u,x) = \theta.$$

Which leads us to $u = x = Tu$. Since T is injective mapping, then $Tx = Tu = x$. Therefore, $x \in X$ is a fixed-point of T and $\mathrm{Fix}(T) \neq \emptyset$. □

Now, let us introduce our newly major concepts.

Definition 3.14. *Let (X,E,C,p) be a partial satisfactory cone metric space. In a non-empty set X, define $T : X \times X \to X$, $S : X \to X$ and $\beta : X^2 \times X^2 \to C$. Let $\{c_n\}_{n\in\mathbb{N}}$ be a non-zero sequence in C. Then,*
(1) *T is called β_S−sequentially admissible mapping if and only if*

$$\beta((Sx,Sy),(Su,Sv)) \succeq c_n \text{ implies } \beta((T(x,y),T(y,x)),(T(u,v),T(v,u))) \succeq c_{n+1}$$

for all $n \in \mathbb{N}$ and for all $x,y,u,v \in X$.
(2) *(X,E,C,p) is said to be β_S-sequentially regular, if $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that*

$$\begin{cases} \beta((Sx_n,Sy_n),(Sx_{n+1},Sy_{n+1})) \succeq c_{n+1}, \\ \beta((Sy_n,Sx_n),(Sy_{n+1},Sx_{n+1})) \succeq c_{n+1} \end{cases}$$

for all $n \in \mathbb{N}$ and $Sx_n \xrightarrow{\tau_p} x^*, Sy_n \xrightarrow{\tau_p} y^*$, then

$$\beta((Sx_n,Sy_n),(Sx^*,Sy^*)) \succeq c_{n+1} \text{ and } \beta((Sy_n,Sx_n),(Sy^*,Sx^*)) \succeq c_{n+1} \text{ for sufficiently large } n.$$

Theorem 3.15. *Let $(X,\preccurlyeq)$ be a partially ordered set induced with partial satisfactory cone metric p such that (X,E,C,p) is a θ-complete partial satisfactory cone metric space. Let $T : X \times X \to X$ and $S : X \to X$ be such that T has S-mixed monotone property. Presume that $\beta : X^2 \times X^2 \to C$ is a symmetric mapping. Also, assume that the following assertions are fulfilled:*
(1) *T is β_S−sequentially admissible mapping;*
(2) *There exist $x_0,y_0 \in X$ such that*

$$\begin{cases} \beta((Sx_0,Sy_0),(T(x_0,y_0),T(y_0,x_0))) \succeq c_1, \\ \beta((Sy_0,Sx_0),(T(y_0,x_0),T(x_0,y_0))) \succeq c_1; \end{cases}$$

(3) *$T(X \times X) \subseteq SX$ and SX is closed subset of X;*
(4) *(X,E,C,p) is β_S − sequentially regular and $(X,E,C,p,\preccurlyeq)$ is regular;*
(5) *There is a sequence of positive real numbers $\{k_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} k_n < 1$ and satisfying the following condition:*

$$p(T(x,y),T(u,v)) \preceq \frac{k_n}{2}\Big[p(Sx,Su) + p(Sy,Sv)\Big] \text{ for all } x,y,u,v \in X$$

with $(Sx, Sy) \lesssim (Su, Sv)$ *and* $\beta((Sx, Sy), (Su, Sv)) \geq c_n, n \in \mathbb{N}$. *Under these conditions, if there exist* $x_0, y_0 \in X$ *such that* $Sx_0 \leqslant T(x_0, y_0)$ *and* $Sy_0 \geqslant T(y_0, x_0)$, *then* T *and* S *have coupled coincidence points.*

Proof. By starting from arbitrary points $x_0, y_0 \in X$ such that $Sx_0 \leqslant T(x_0, y_0)$ and $Sy_0 \geqslant T(y_0, x_0)$. Since $T(X \times X) \subseteq SX$ and $x_0, y_0 \in X$, there exist $x_1, y_1 \in X$ such that $Sx_1 = T(x_0, y_0)$ and $Sy_1 = T(y_0, x_0)$. Let $x_2, y_2 \in X$ be such that $Sx_2 = T(x_1, y_1)$ and $Sy_2 = T(y_1, x_1)$.

Inductively, we construct the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X by

$$\begin{cases} Sx_{n+1} = T(x_n, y_n) = T^{n+1}(x_0, y_0) = T(T^n(x_0, y_0), T^n(y_0, x_0)), \\ \\ Sy_{n+1} = T(y_n, x_n) = T^{n+1}(y_0, x_0) = T(T^n(y_0, x_0), T^n(x_0, y_0)) \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$. By Mathematical Induction, we verify that $Sx_n \leqslant Sx_{n+1}$ and $Sy_n \geqslant Sy_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Given that $Sx_0 \leqslant T(x_0, y_0)$ and $Sy_0 \geqslant T(y_0, x_0)$. Thus, the statement is true for $n = 0$. Suppose that the claim is true for some fixed $n = k$. That is; $Sx_k \leqslant Sx_{k+1}$ and $Sy_k \geqslant Sy_{k+1}$. By S-mixed monotone property of T, we obtain $Sx_{k+1} \leqslant Sx_{k+2}$ and $Sy_{k+1} \geqslant Sy_{k+2}$. Thus, the statement is true for $n = k + 1$. The later lines guarantee that

$$(Sx_n, Sy_n) \lesssim (Sx_{n+1}, Sy_{n+1}) \text{ and } (Sy_{n+1}, Sx_{n+1}) \lesssim (Sy_n, Sx_n) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Without loss of generality, we assume that $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \in \mathbb{N} \cup \{0\}$. From assumption (2), we have

$$\beta((Sx_0, Sy_0), (Sx_1, Sy_1)) = \beta((Sx_0, Sy_0), (T(x_0, y_0), T(y_0, x_0))) \geq c_1.$$

Due to the fact that T is β_S-sequentially admissible, it follows that

$$\beta((T(x_0, y_0), T(y_0, x_0)), (T(x_1, y_1), T(y_1, x_1))) = \beta((Sx_1, Sy_1), (Sx_2, Sy_2)) \geq c_2.$$

By continuing this procedure, we have $\beta((Sx_n, Sy_n), (Sx_{n+1}, Sy_{n+1})) \geq c_{n+1}$ for all $n \in \mathbb{N}$. Similarly, we obtain that $\beta((Sy_n, Sx_n), (Sy_{n+1}, Sx_{n+1})) \geq c_{n+1}$ for all $n \in \mathbb{N}$. Now, we can apply condition (5) as follows:

$$\begin{cases} p(Sx_n, Sx_{n+1}) \leq \frac{k_n}{2}\Big[p(Sx_{n-1}, Sx_n) + p(Sy_{n-1}, Sy_n)\Big], \\ \\ p(Sy_n, Sy_{n+1}) \leq \frac{k_n}{2}\Big[p(Sy_{n-1}, Sy_n) + p(Sx_{n-1}, Sx_n)\Big]. \end{cases}$$

On adding the previous two inequalities, one has

$$p(Sx_n, Sx_{n+1}) + p(Sy_n, Sy_{n+1}) \leq k_n\left[p(Sx_{n-1}, Sx_n) + p(Sy_{n-1}, Sy_n)\right].$$

Repeating the above process, we deduce

$$p(Sx_n, Sx_{n+1}) + p(Sy_n, Sy_{n+1}) \leq \prod_{i=1}^{n} k_i\left[p(Sx_0, Sx_1) + p(Sy_0, Sy_1)\right] \text{ for all } n \in \mathbb{N}.$$

For any $n, p \in \mathbb{N}$, we infer

$$p(Sx_n, Sx_{n+p}) \overset{(\mathrm{PCM}_4)}{\leq} \sum_{i=n}^{n+p-1} p(Sx_i, Sx_{i+1}) \leq \sum_{i=n}^{n+p-1} \Big[\prod_{i=1}^{n} \frac{k_i}{2}\Big] p(Sx_0, Sx_1)$$

$$= \Big[\prod_{i=1}^{n}\frac{k_i}{2}\Big]\Big[\sum_{m=1}^{p-1}\prod_{i=n+1}^{m}\frac{k_i}{2}\Big]p(Sx_0, Sx_1).$$

By a similar manner, we obtain

$$p(Sy_n, Sy_{n+p}) \overset{(\mathrm{PCM}_4)}{\leq} \sum_{i=n}^{n+p-1} p(Sy_i, Sy_{i+1}) \leq \sum_{i=n}^{n+p-1}\Big[\prod_{i=1}^{n}\frac{k_i}{2}\Big]p(Sy_0, Sy_1)$$

$$= \Big[\prod_{i=1}^{n}\frac{k_i}{2}\Big]\Big[\sum_{m=1}^{p-1}\prod_{i=n+1}^{m}\frac{k_i}{2}\Big]p(Sy_0, Sy_1).$$

Therefore, we arrive at

$$p(Sx_n, Sx_{n+p}) + p(Sy_n, Sy_{n+p}) \leq \Big[\prod_{i=1}^{n}k_i\Big]\Big[\sum_{m=1}^{p-1}\prod_{i=n+1}^{m}k_i\Big]\Big[p\,(Sx_0, Sx_1) + p\,(Sy_0, Sy_1)\Big].$$

Since $\lim_{n\to\infty} k_n < 1$, using the usual form of the Ratio test of series, one can easily see that $\{\sum_{i=n}^{m}\Big[\prod_{i=1}^{n}k_i\Big]\Big[p\,(Sx_0, Sx_1) + p\,(Sy_0, Sy_1)\Big]\}_{m\in\mathbb{N}}$ is convergent to the limit $\sum_{i=n}^{\infty}\Big[\prod_{i=1}^{n}k_i\Big]\Big[p\,(Sx_0, Sx_1) + p\,(Sy_0, Sy_1)\Big]$ and its n's term $\{\prod_{i=1}^{n}k_i\}_{n\in\mathbb{N}}$ tends to zero. Therefore, for given $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$\sum_{i=n}^{\infty}\left[\prod_{j=1}^{i}k_j\right] < \frac{\varepsilon}{\big\|[p\,(Sx_0, Sx_1) + p\,(Sy_0, Sy_1)]\big\|} \quad \text{for all } n \geq n_1.$$

For all $n \geq n_1$, we consider

$$\Big\|\sum_{i=n}^{n+p-1}\Big[\prod_{j=1}^{i}k_j\Big]\Big[p\,(Sx_0, Sx_1) + p\,(Sy_0, Sy_1)\Big]\Big\| \leq \sum_{i=n}^{n+p-1}\Big\|\Big[\prod_{j=1}^{i}k_j\Big]\Big[p\,(Sx_0, Sx_1) + p\,(Sy_0, Sy_1)\Big]\Big\|$$

$$\leq \sum_{i=n}^{\infty}\Big\|\Big[\prod_{j=1}^{i}k_j\Big]\Big[p\,(Sx_0, Sx_1) + p\,(Sy_0, Sy_1)\Big]\Big\|$$

$$\leq \sum_{i=n}^{\infty}\prod_{j=1}^{i}k_j\,\Big\|\Big[p\,(Sx_0, Sx_1) + p\,(Sy_0, Sy_1)\Big]\Big\|$$

$$< \varepsilon.$$

Thus, $\sum_{i=n}^{n+p-1}\Big[\prod_{j=1}^{i}k_j\Big]\Big[p\,(Sx_0, Sx_1) + p\,(Sy_0, Sy_1)\Big] \xrightarrow{\|.\|} \theta$. It follows that, for any $c \in E$ with $c \ggg \theta$, there exists $n_2 \in \mathbb{N}$ such that

$$p\left(y_n, y_{n+p}\right) \leq \sum_{i=n}^{n+p-1}\Big[\prod_{j=1}^{i}k_j\Big]\Big[p\,(Sx_0, Sx_1) + p\,(Sy_0, Sy_1)\Big] \lll c \text{ for all } n \geq n_2.$$

Owing to the above arguments, we deduce that the sequences $\{Sx_n\}_{n\in\mathbb{N}}$ and $\{Sy_n\}_{n\in\mathbb{N}}$ are θ-Cauchy in the θ-complete partial satisfactory cone metric space (X, E, C, p). Then, there exist $x, y \in X$ such that

$Sx_n \xrightarrow{\tau_p} x$ and $Sy_n \xrightarrow{\tau_p} y$ with $p(x,x)=\theta$ and $p(y,y)=\theta$. Since $\{Sx_n\}_{n\in\mathbb{N}} \subseteq SX$ and SX is closed, it leads that $x \in SX$. So, there must be some $x' \in X$ such that $Sx' = x$. Similarly, $Sy' = y$ for some $y' \in X$.

Now, since $\{Sx_n\}$ is a non-decreasing sequence that converges to Sx', we get $Sx_n \leqslant Sx'$ for all n.

Similarly, we have $Sy' \leqslant Sy_n$ for all n. That is; $(Sx_n, Sy_n) \lesssim (Sx', Sy')$ for all n. Since $(X, E, C, p, \leqslant)$ is β_S − regular, we obtain

$$\beta\left((Sx_n, Sy_n),(Sx', Sy')\right) \geq c_{n+1} \text{ and } \beta\left((Sy_n, Sx_n),(Sy', Sx')\right) \geq c_{n+1}.$$

Since $Sx_n \xrightarrow{\tau_p} Sx'$ and $Sy_n \xrightarrow{\tau_p} Sy'$, then for $c \in E$ with $c \ggg \theta$ and for all $m \in \mathbb{N}$, choose $n_3 \in \mathbb{N}$ such that $\frac{k_{n+1}}{2} p(Sx_n, Sx') \lll \frac{c}{3m}$, $\frac{k_{n+1}}{2} p(Sy_n, Sy') \lll \frac{c}{3m}$ and $p(Sx_{n+1}, Sx') \lll \frac{c}{3m}$ for all $n \geq n_3$.

For all $n \geq n_3$, we have

$$\begin{aligned} p\left(T(x', y'), Sx'\right) &\overset{(\mathrm{PCM}_4)}{\preceq} p\left(T(x', y'), T(x_n, y_n)\right) + p\left(Sx_{n+1}, Sx'\right) \\ &\overset{(5)}{\preceq} \frac{k_{n+1}}{2}\Big[p\left(Sx_n, Sx'\right) + p\left(Sy_n, Sy'\right)\Big] + p\left(Sx_{n+1}, Sx'\right) \\ &\lll \tfrac{c}{m}. \end{aligned}$$

Proceeding limit as $m \to \infty$, we get $p\left(T(x', y'), Sx'\right) = \theta$ and so $T(x', y') = Sx'$. In such a similar way, one can easily get $T(y', x') = Sy'$. Therefore, we reach that (x', y') is a coupled coincidence point of T, S and (Sx', Sy') is a coupled point of coincidence of T, S. □

Remark 3.16. *If we replace the mapping S by I, the identity mapping on X, in the statements of Theorem (3.15), we have a new type of mappings for which we proved the existence of the coupled fixed-point of the mapping T.*

4. Applications of fixed-point theory to ordinary differential equations

In this section, we apply Theorem (3.9) with $S = I$ to study the existence of a unique solution for the following two-point boundary value problem of the second-order differential equation:

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t, x(t)), \quad t \in [0,1]; \\ \\ \qquad x(0) = x(1) = 0, \end{cases} \tag{4.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Theorem 4.1. *Presume that the following hypotheses hold:*
(1) *For all $s \in [0,1]$, $f(s,.)$ is a non-decreasing function;*
(2) *If $f(s, x(s)) = x(s)$ for all $s \in [0,1]$, then we have*

$$f\left(s, \int_0^s x(w)\,dw\right) = \int_0^s f(w, x(w))\,dw \text{ for all } s \in [0,1];$$

(3) *There exists a continuous function $K : [0,1] \to \mathbb{R}^+$ such that*

$$f(s, y(s)) - f(s, x(s)) \leq K(s)\left[y(s) - x(s)\right]$$

for all $s \in [0,1]$ *with* $x \preceq y$;
(4) *There exists* $L \in [0,1)$ *such that* $\sup_{s\in[0,1]} K(s) \leq \frac{L}{2}$;
(5) *There exists a symmetric function* $\varphi : \mathbb{R}^2 \to \mathbb{R}$ *with the following properties:*
(φ_1) *There exists* $x_0 \in C_{\mathbb{R}}[0,1]$ *such that for all* $t \in [0,1]$, *we have*

$$\varphi\left(x_0(t), \int_0^1 G(t,s) f(s, x_0(s))\, ds\right) \geq 0;$$

(φ_2) *For all* $t \in [0,1]$ *and for any* $x, y \in C_{\mathbb{R}}[0,1]$, $\varphi(x(t), y(t)) \geq 0$ *implies*

$$\varphi\left(\int_0^1 G(t,s) f(s, x(s))\, ds, \int_0^1 G(t,s) f(s, y(s))\, ds\right) \geq 0;$$

(φ_3) *If* $\{x_n\}_{n\in\mathbb{N}}$ *in* $C_{\mathbb{R}}[0,1]$ *such that* $\varphi(x_n, x_{n+1}) \geq 0$ *for all* $n \in \mathbb{N}$ *and* $x_n \xrightarrow{\tau_p} x^* \in C_{\mathbb{R}}[0,1]$, *we have* $\varphi(x_n, x^*) \geq 0$ *for sufficiently large* n.

Then, the mentioned second-order differential equation (4.1) *has a unique solution in* $C_{\mathbb{R}}[0,1]$.

Proof. Clearly, the problem (4.1) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s) f(s, x(s))\, ds \text{ for all } t \in [0,1], \tag{4.2}$$

where $G(t,s)$ is the Green function defined by

$$G(t,s) = \begin{cases} t(1-s), & \text{if } 0 \leq t \leq s \leq 1; \\ s(1-t), & \text{if } 0 \leq s \leq t \leq 1. \end{cases}$$

It is clear that the existence of a solution of (4.1) is equivalent to the existence of the integral equation (4.2).

Let $X = E := C_{\mathbb{R}}[0,1]$ be the Banach space of all real continuous functions on the closed unit interval $[0,1]$ and $C := \{u \in E : u(t) \geq 0, t \in [0,1]\}$, which is a solid (semi-solid) cone.

Define $p : X \times X \to C$ by

$$p(x,y)(t) = \left(\sup_{t\in[0,1]} \{|x(t) - y(t)|\}\right) \psi(t),$$

where $\psi(t) = e^t \in E$. Then, easily one can verify that (X, E, C, p) is a $\theta-$complete partial satisfactory cone metric space.

We endow X with the partial order $\preceq$ given by

$$x \preceq y \text{ if and only if } x(t) \leq y(t) \text{ for all } t \in [0,1].$$

Let $T : C_{\mathbb{R}}[0,1] \to C_{\mathbb{R}}[0,1]$ be defined by

$$Tx(t) = \int_0^1 G(t,s) f(s, x(s))\, ds \text{ for all } t \in [0,1].$$

Obviously, the fixed-point of T is a solution of (4.1) or, equivalently; a solution of the problem (4.2).

We will check that the mapping T satisfies all the conditions of Theorem (3.9) with $S = I$.

First, we show that T is non-decreasing with regards to $\preceq$. Since f is non-decreasing with respect to its second variable, then for any $x, y \in X$ with $x \preceq y$ and for any $t \in [0,1]$, we have

$$Tx(t) = \int_0^1 G(t,s) f(s, x(s))\, ds \leq \int_0^1 G(t,s) f(s, y(s))\, ds = Ty(t),$$

since $G(x,y) \geq 0$ for any $t, s \in [0,1]$. Thus, we have $Tx \preceq Ty$.

Define $\alpha : X \times X \to C$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } \varphi(x(t), y(t)) \geq 0 \text{ or } x \preceq y, \\ \lambda, & \text{otherwise,} \end{cases}$$

where $\lambda \in (0,1)$ and $t \in [0,1]$. If $x, y \in X, \alpha(x,y) \succeq 1$, then $x \preceq y$ or $\varphi(x(t), y(t)) \geq 0$. Observe that $\alpha(x,y) \succeq 1 \implies x \preceq y \implies Tx \preceq Ty \implies \alpha(Tx, Ty) \succeq 1$. If $x, y \in X, \alpha(x,y) \succeq 1$ implies $\varphi(x(t), y(t)) \geq 0$, then by condition (φ_2), we get $\varphi(Tx(t), Ty(t)) \geq 0$ and thus $\alpha(Tx, Ty) \succeq 1$. Therefore, T is $\alpha-$sequentially admissible mapping with $c_n = 1$ for any $n \in \mathbb{N}$.

Since there exists $x_0 \in C_{\mathbb{R}}[0,1]$ such that

$$\varphi\left(x_0(t), \int_0^1 G(t,s) f(s, x_0(s))\, ds\right) = \varphi(x_0(t), Tx_0(t)) \geq 0$$

for all $t \in [0,1]$, then $\alpha(x_0, Tx_0) \succeq 1$.

From condition (φ_3) , it is easy to verify that (X, E, C, p) is α_S-sequentially regular.

Now, let $x, y \in X$ such that $x \preceq y$. Then, we have

$$\begin{aligned}
p(Tx, Ty)(t) &= \left(\sup_{t\in[0,1]} \{|Tx(t) - Ty(t)|\}\right)\psi(t) \\
&= \left(\sup_{t\in[0,1]} \left\{\left[Ty(t) - Tx(t)\right]\right\}\right)\psi(t) \\
&= \left(\sup_{t\in[0,1]} \left[\int_0^1 G(t,s) f(s, y(s))\, ds - \int_0^1 G(t,s) f(s, x(s))\, ds\right]\right)\psi(t) \\
&= \left(\sup_{t\in[0,1]} \left\{\int_0^1 G(t,s)\left[f(s, y(s)) - f(s, x(s))\right]ds\right\}\right)\psi(t) \\
&\leq \left(\sup_{t\in[0,1]} \left\{\int_0^1 G(t,s)\left(K(s)\left[y(s) - x(s)\right]\right)ds\right\}\right)\psi(t) \\
&\leq \left(\sup_{t\in[0,1]} \left\{\sup_{s\in[0,1]} K(s) \int_0^1 G(t,s)\left[y(s) - x(s)\right]ds\right\}\right)\psi(t) \\
&\leq \tfrac{L}{2}\left(\sup_{t\in[0,1]} \left\{\int_0^1 G(t,s)\left[y(s) - x(s)\right]ds\right\}\right)\psi(t) \\
&\leq \tfrac{L}{2}\left(\sup_{t\in[0,1]} \left[y(t) - x(t)\right]\left\{\sup_{t\in[0,1]} \int_0^1 G(t,s)\, ds\right\}\right)\psi(t) \\
&= \frac{L}{2}\left(\left\{\sup_{t\in[0,1]} |x(t) - y(t)|\right\} \sup_{t\in[0,1]} \left\{-\frac{t^2}{2} + \frac{t}{2}\right\}\right)\psi(t)
\end{aligned}$$

$$= \frac{L}{16}\left(\sup_{t\in[0,1]}\{|x(t)-y(t)|\}\right)\psi(t)$$

$$= \left(\frac{L}{16}\right)p(x,y)(t).$$

Therefore,

$$p(Tx,Ty)(t) \le \left(\frac{L}{16}\right)p(x,y)(t)$$

for any $t \in [0,1]$ and for certain $L \in [0,1)$.

Hence, we find a constant sequence of positive real numbers $k_n = \frac{L}{16}$ for any $n \in \mathbb{N}$ such that $\lim_{n\to\infty} k_n = \frac{L}{16} < 1$ and satisfying

$$p(Tx,Ty) \le k_n p(x,y)$$

for any $x, y \in X$ with $\alpha(x,y) \ge 1$. Therefore, all the conditions of Theorem (3.9) hold with $S = I$ and thus T has a unique fixed-point in $C_{\mathbb{R}}[0,1]$. Thus, there is a unique solution of problem (4.1). □

5. Applications of coupled fixed-point to integral equations

In this section, we study the existence of solutions for the following system of integral equations:

$$\begin{cases} x(r) = \int_a^b (H_1(r,s) + H_2(r,s))\,[f(s,x(s)) + g(s,y(s))]\,ds + k(r); \\ y(r) = \int_a^b (H_1(r,s) + H_2(r,s))\,[f(s,y(s)) + g(s,x(s))]\,ds + k(r), \end{cases} \tag{5.1}$$

where $r \in [a,b]$; $H_1, H_2 \in C_{\mathbb{R}}([a,b]\times[a,b])$; $f, g \in C_{\mathbb{R}}([a,b]\times\mathbb{R})$ and $k \in C_{\mathbb{R}}[a,b]$.

Theorem 5.1. *Assume that the following conditions hold:*
(1) $H_1(r,s) \ge 0$ *and* $H_2(r,s) \le 0$ *for all* $r, s \in [a,b]$;
(2) $\sup_{t\in[a,b]} \int_a^b [H_1(r,s) - H_2(r,s)]\,ds \le \frac{1}{4\rho}$ *for some positive real number* ρ;
(3) *There exists a symmetric function* $\vartheta : C^2_{\mathbb{R}}[a,b] \times C^2_{\mathbb{R}}[a,b] \to \mathbb{R}$ *with the following properties:*
(ϑ_1) *For all* $(x,y), (u,v) \in C^2_{\mathbb{R}}[a,b]$, $\vartheta((x,y),(u,v)) \ge 0$ *implies*

$$\vartheta((T(x,y), T(y,x)), (T(u,v), T(v,u))) \ge 0;$$

(ϑ_2) *There exist* $x_0, y_0 \in C_{\mathbb{R}}[a,b]$ *such that* $(x_0,y_0) \lesssim (T(x_0,y_0), T(y_0,x_0))$,

$$\begin{cases} \vartheta((x_0,y_0),(T(x_0,y_0),T(y_0,x_0))) \ge 0, \\ \vartheta((y_0,x_0),(T(y_0,x_0),T(x_0,y_0))) \ge 0; \end{cases}$$

(ϑ_3) *For all* $x, y, u, v \in X$ *with* $(x,y) \lesssim (u,v)$ *and* $\vartheta((x,y),(u,v)) \ge 0$, *the following Lipschitzian-type conditions hold:*

$$\begin{cases} 0 \le f(r,x) - f(r,u) \le (x-u), \\ 0 \le f(r,v) - f(r,y) \le (v-y), \\ -(v-y) \le g(r,v) - g(r,y) \le 0, \\ -(x-u) \le g(r,x) - g(r,u) \le 0; \end{cases}$$

(ϑ_4) *If* $\{x_n\}$ *and* $\{y_n\}$ *are two sequences in* $C_{\mathbb{R}}[a,b]$ *such that*

$$\begin{cases} \vartheta\left((x_n,y_n),(x_{n+1},y_{n+1})\right) \geq 1, \\ \vartheta\left((y_n,x_n),(y_{n+1},x_{n+1})\right) \geq 1 \end{cases}$$

for all $n \in \mathbb{N}$ *and* $x_n \stackrel{\tau_p}{\to} x^*$, $y_n \stackrel{\tau_p}{\to} y^*$, *then*

$$\vartheta\left((x_n,y_n),(x^*,y^*)\right) \geq 1 \text{ and } \vartheta\left((y_n,x_n),(y^*,x^*)\right) \geq 1 \text{ for sufficiently large } n.$$

With these conditions, the system of integral equations (5.1) *has at least one solution in* $C_{\mathbb{R}}[a,b] \times C_{\mathbb{R}}[a,b]$.

Proof. Let $X = E := C_{\mathbb{R}}[a,b]$ be the Banach space of all real continuous functions on $[a,b]$ and $C := \{u \in E : u(t) \geq 0, t \in [a,b]\}$, which is a solid (semi-solid) cone. Define $p : X \times X \to C$ by

$$p(x,y)(t) = \left(\sup_{t\in[a,b]} \{|x(t) - y(t)|\}\right)\psi(t),$$

where $\psi(t) = e^t \in E$. Then, (X,E,C,p) is a $\theta-$complete partial satisfactory cone metric space. Suppose that $C_{\mathbb{R}}[a,b]$ is endowed with the natural partial ordered relation, that is; for all $x,y \in C_{\mathbb{R}}[a,b]$, $x \preceq y$ if and only if $x(t) \leq y$ for all $t \in [a,b]$.

The set $X \times X = C_{\mathbb{R}}[a,b] \times C_{\mathbb{R}}[a,b]$ is partially ordered under the following ordered relation:

$$(x,y) \lesssim (u,v) \text{ if and only if } x(r) \leq u(r) \text{ and } y(r) \geq v(r) \text{ for all } r \in [a,b].$$

For any $x,y \in X$, $\max_{r\in[a,b]}\{x(r),y(r)\}$, $\min_{r\in[a,b]}\{x(r),y(r)\} \in X$ are the upper and lower bounds of x and y, respectively. Therefore, for every $(x,y),(u,v) \in X \times X$, there is $(\max\{x,u\},\min\{y,v\}) \in X \times X$ that comparable to (x,y) and (u,v).

Define $T : X \times X \to X$ by

$$T(x,y)(r) = \int_a^b H_1(r,s)\Big[f(s,x(s)) + g(s,y(s))\Big]ds + \int_a^b H_2(r,s)\Big[f(s,y(s)) + g(s,x(s))\Big]ds + k(r)$$

for all $r \in [a,b]$.

First, we show that T has the mixed monotone property. If $(t_1,u) \lesssim (t_2,u)$, then for all $r \in [a,b]$, we have

$$\begin{aligned} T(t_1,u)(r) &= \int_a^b H_1(r,s)\Big[f(s,t_1(s))+g(s,u(s))\Big]ds+\int_a^b H_2(r,s)\Big[f(s,u(s))+g(s,t_1(s))\Big]ds+k(r) \\ &\leq \int_a^b H_1(r,s)\Big[f(s,t_2(s))+g(s,u(s))\Big]ds+\int_a^b H_2(r,s)\Big[f(s,u(s))+g(s,t_2(s))\Big]ds+k(r) \\ &= T(t_2,u)(r). \end{aligned}$$

Thus, $T(t_1,u) \leq T(t_2,u)$. Similarly, $T(t,u_1) \leq T(t,u_2)$ whenever $(t,u_1) \lesssim (t,u_2)$.

Let $\beta : X^2 \times X^2 \to C$ be defined by

$$\beta\left(\left(x,y\right),\left(u,v\right)\right)=\begin{cases}1, & \text{if } \vartheta\left(\left(x\left(t\right),y\left(t\right)\right),\left(u\left(t\right),v\left(t\right)\right)\right)\geq 0,\\ 0, & \text{otherwise.}\end{cases}$$

If $x,y,u,v\in X, \beta\left(\left(x,y\right),\left(u,v\right)\right)\geq 1$, then $\vartheta\left(\left(x\left(t\right),y\left(t\right)\right),\left(u\left(t\right),v\left(t\right)\right)\right)\geq 0$.

By condition (ϑ_1), we get $\varphi\left(Tx\left(t\right),Ty\left(t\right)\right)\geq 0$ and thus $\beta\left(\left(x,y\right),\left(u,v\right)\right)\geq 1$. Therefore, T is β-sequentially admissible mapping with $c_n=1$ for any $n\in\mathbb{N}$.

From condition (ϑ_2), it follows that there exist $x_0,y_0 \in C_{\mathbb{R}}\left[a,b\right]$ with $\left(x_0,y_0\right) \lesssim \left(T\left(x_0,y_0\right),T\left(y_0,x_0\right)\right)$ such that $\beta\left(\left(x_0,y_0\right),\left(T\left(x_0,y_0\right),T\left(y_0,x_0\right)\right)\right)\geq 0$ and $\beta\left(\left(y_0,x_0\right),\left(T\left(y_0,x_0\right),T\left(x_0,y_0\right)\right)\right)\geq 0$. Thus, condition (2) in Theorem (3.15) is satisfied with $S=I$.

The property (X,E,C,p) is $\beta-$sequentially regular follows trivially from the corresponding condition of the mapping ϑ.

Next, suppose that $\{t_n\}$ is a monotone non-decreasing sequence in X that converges to a point $t\in X$. Then, for any $r\in[a,b]$, the sequence of real numbers

$$t_1\left(t\right)\leq t_2\left(t\right)\leq\cdots\leq t_n\left(t\right)\leq\cdots$$

converges to $t\left(r\right)$. Thus, for all $t\left(r\right)\in[a,b]$, we have $t_n\left(r\right)\leq t\left(r\right)$ and thus $t_n\leq t$ for all $n\in\mathbb{N}$. Similarly, if $u\left(r\right)$ is a limit of a monotone non-increasing sequence $\{u_n\}$ in X, then $u_n\left(r\right)\geq u\left(r\right)$ and thus $u_n\geq u$ for all $n\in\mathbb{N}$. Therefore, $(X,E,C,p,\preceq)$ is regular.

For all $x,y,u,v\in X$ with $\left(x,y\right)\lesssim\left(u,v\right)$ and $\vartheta\left(\left(x,y\right),\left(u,v\right)\right)\geq 0$, it follows that

$$\begin{aligned}
&p\left(T\left(x,y\right),T\left(u,v\right)\right)\left(r\right)\\
&=\Big(\sup\nolimits_{r\in[a,b]}\left|T\left(x,y\right)\left(r\right)-T\left(u,v\right)\left(r\right)\right|\Big)\psi\left(t\right)\\
&=\Big(\sup_{r\in[a,b]}\left|\int_a^b H_1\left(r,s\right)\Big[\left(f\left(s,x\left(s\right)\right)-f\left(s,u\left(s\right)\right)\right)-\left(g\left(s,v\left(s\right)\right)-g\left(s,y\left(s\right)\right)\right)\Big]ds\right.\\
&\qquad\left.-\int_a^b H_2\left(r,s\right)\Big[\left(f\left(s,v\left(s\right)\right)-f\left(s,y\left(s\right)\right)\right)-\left(g\left(s,x\left(s\right)\right)-g\left(s,u\left(s\right)\right)\right)\Big]ds\right|\Big)\psi\left(t\right)\\
&\leq\Big(\sup\nolimits_{r\in[a,b]}\left|\int_a^b H_1\left(r,s\right)\Big[\left(u\left(s\right)-x\left(s\right)\right)+\left(y\left(s\right)-v\left(s\right)\right)\Big]ds\right.\\
&\qquad\left.-\int_a^b H_2\left(r,s\right)\Big[\left(y\left(s\right)-v\left(s\right)\right)+\left(u\left(s\right)-x\left(s\right)\right)\Big]ds\right|\Big)\psi\left(t\right)\\
&=\Big(\sup_{r\in[a,b]}\left|\int_a^b\left(H_1\left(r,s\right)-H_2\left(r,s\right)\right)\Big[\left(x\left(s\right)-u\left(s\right)\right)+\left(v\left(s\right)-y\left(s\right)\right)\Big]ds\right|\Big)\psi\left(t\right)\\
&\leq\Big(\sup_{r\in[a,b]}\left|\int_a^b\left(H_1\left(r,s\right)-H_2\left(r,s\right)\right)\Big[\sup_{s\in[a,b]}\left(x\left(s\right)-u\left(s\right)\right)+\sup_{s\in[a,b]}\left(v\left(s\right)-y\left(s\right)\right)\Big]ds\right|\Big)\psi\left(t\right)\\
&=\Big(\Big[\sup_{s\in[a,b]}\left|x\left(s\right)-u\left(s\right)\right|+\sup_{s\in[a,b]}\left|v\left(s\right)-y\left(s\right)\right|\Big]\sup_{r\in[a,b]}\int_a^b\left(H_1\left(r,s\right)-H_2\left(r,s\right)\right)ds\Big)\,\psi\left(t\right)
\end{aligned}$$

$$\leq \frac{1}{4\rho}\Big[\sup_{s\in[a,b]}|x(s)-u(s)|+\sup_{s\in[a,b]}|v(s)-y(s)|\Big]\psi(t)$$

$$\leq \frac{1}{4\rho}\Big[p(x,u)+p(y,v)\Big](r),$$

for any $r\in[a,b]$. Therefore, we can find a constant sequence of positive real numbers $k_n=\frac{1}{2\rho}$ for all $n\in\mathbb{N}$ such that $\lim_{n\to\infty}k_n=\frac{1}{2\rho}<1$ and satisfying

$$p(T(x,y),T(u,v))\leq\frac{k_n}{2}\Big[p(x,u)+p(y,v)\Big]$$

for all $(x,y)\lesssim(u,v)$ and $\beta((x,y),(u,v))\geq 1$. Which is just the contractive condition in Theorem (3.15). All the hypotheses of Theorem (3.15) with $S=I$, are satisfied. Therefore, T has a coupled fixed-points in $C_{\mathbb{R}}[a,b]\times C_{\mathbb{R}}[a,b]$. □

6. Conclusions

The study of this article along with our defined distance structure represents a new research direction that included updated versions of some abstract results and some methods in fixed-point lectures. Many of the previously known results found in fixed-point theory consider direct generalizations and special occurrences of the results of this article. In the present paper, there are multiple appearances of various types of generalized admissible mappings and mappings have mixed monotone property associated with several interesting conditions. In this regard, we discussed the problem of finding coincidence points, coupled coincidence points, coupled fixed-point and fixed-points of such mappings. For showing efficiency of the obtained main results we gave some applications. In one approach, we introduced a novel fixed-point technique to ordinary differential equations in partial satisfactory cone metric spaces, and in another approach we studied the existence of solutions in a system including non-linear integral equations.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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